Pre-Announcements

Blockchain event is today 7 - 10 PM at International House at Chevron House.

- Lots of fancy people from fancy places will be there.
- Blockchain is a topical concept.
- Pizza is a topical concept.

Flyers available.

Announcements

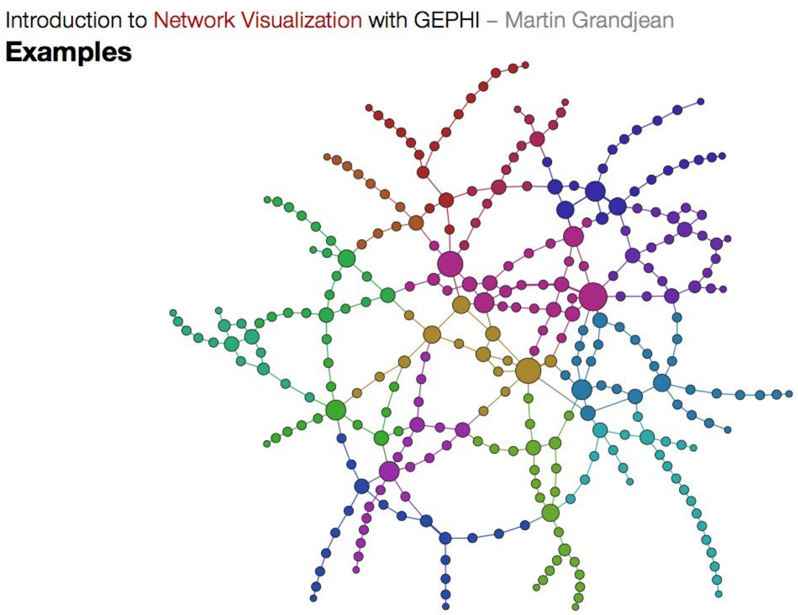
Exam solution exists, not sure why it’s not posted, but will be posted soon.

- Exam was really hard, but that’s how exams go.
- More later.

CS61B

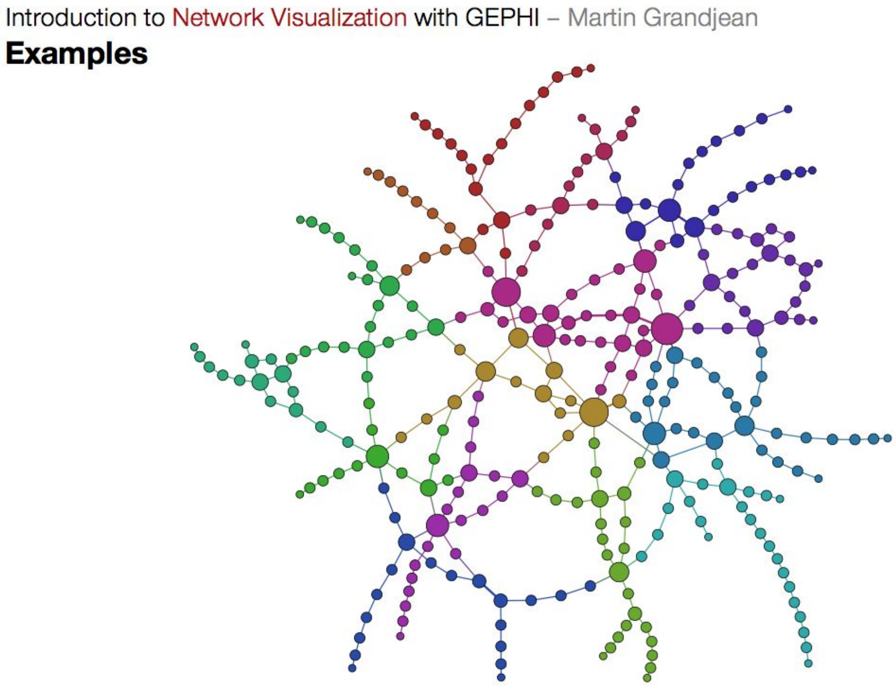
Lecture 26: Graphs

- Intro
- Graph Implementations
- Depth First Traversal



Graph

Graph: A set of nodes (a.k.a. vertices) connected pairwise by edges.

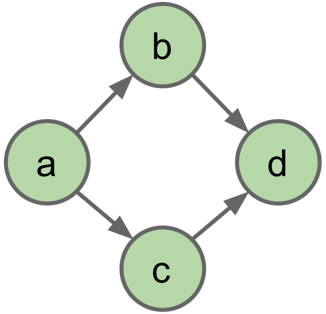


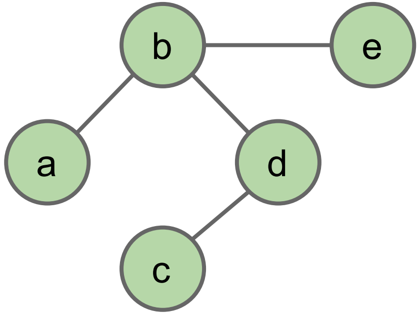
Graph Types

Directed

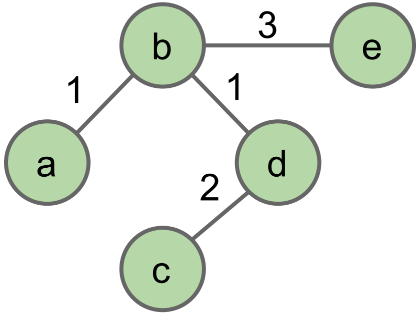
Undirected

Acyclic:

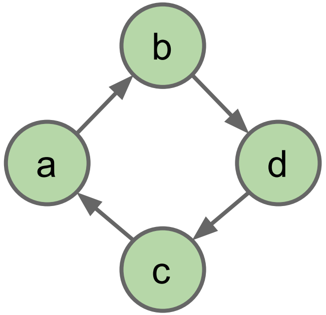


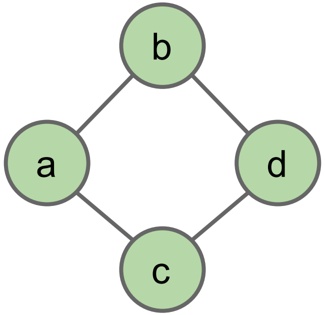


With Edge Labels



Cyclic:





Graph Terminology

- Graph:
  - Set of ***vertices***, a.k.a. ***nodes***.
  - Set of ***edges***: Pairs of vertices.
  - Vertices with an edge between are ***adjacent***.
  - Optional: Vertices or edges may have ***labels*** (or ***weights***).
- A ***path*** is a sequence of vertices connected by edges.
- A ***cycle*** is a path whose first and last vertices are the same.
  - A graph with a cycle is 'cyclic'.
- Two vertices are ***connected*** if there is a path between them. If all vertices are connected, we say the graph is connected.

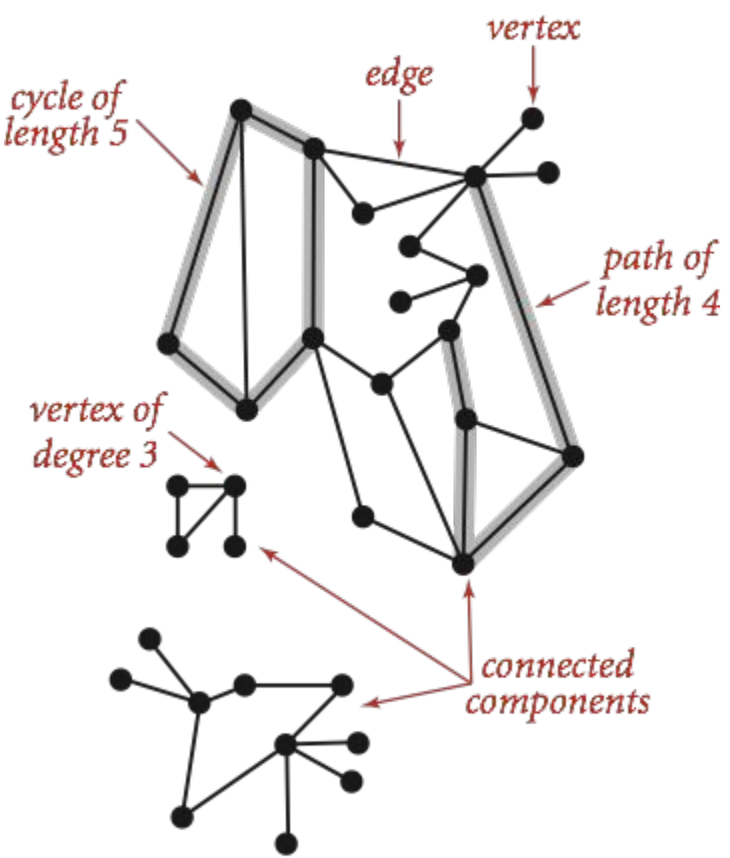


Figure from Algorithms 4th Edition

Some Graph-Processing Problems

s-t Path. Is there a path between vertices s and t?

Shortest s-t Path. What is the shortest path between vertices s and t?

Cycle. Does the graph contain any cycles?

Euler Tour. Is there a cycle that uses every edge exactly once?

Hamilton Tour. Is there a cycle that uses every vertex exactly once?

Connectivity. Is the graph connected, i.e. is there a path between all vertex pairs?

Biconnectivity. Is there a vertex whose removal disconnects the graph?

Planarity. Can you draw the graph on a piece of paper with no crossing edges?

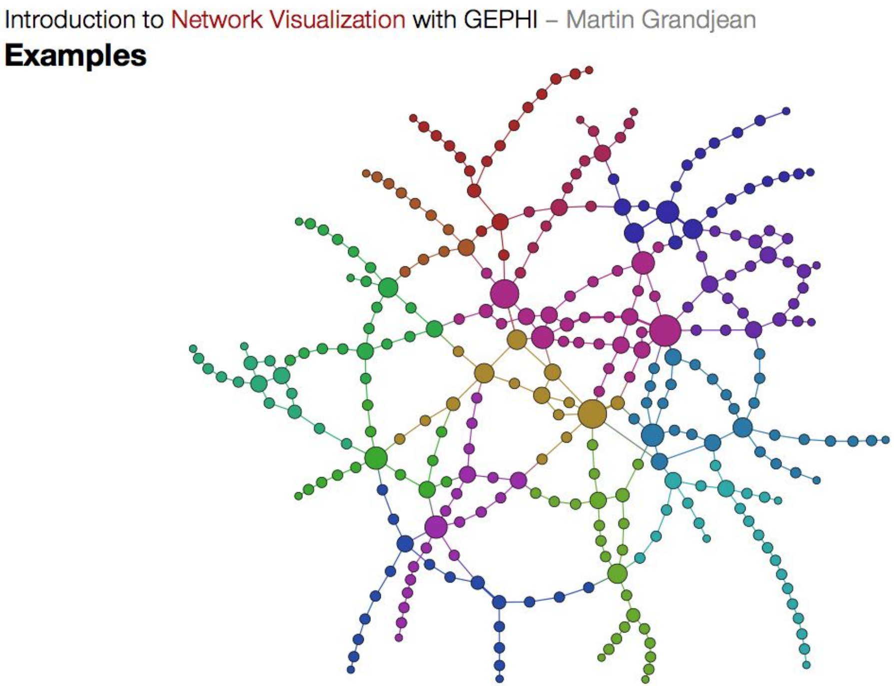
Isomorphism. Are two graphs isomorphic (the same graph in disguise)?

Graph problems: Unobvious which are easy, hard, or computationally intractable.

Graph Example: The Paris Metro

This subway map of Paris is:

- Undirected
- Connected
- Cyclic (not a tree!)
- Vertex-labeled



Graph Example: BART

Is the BART graph a tree?

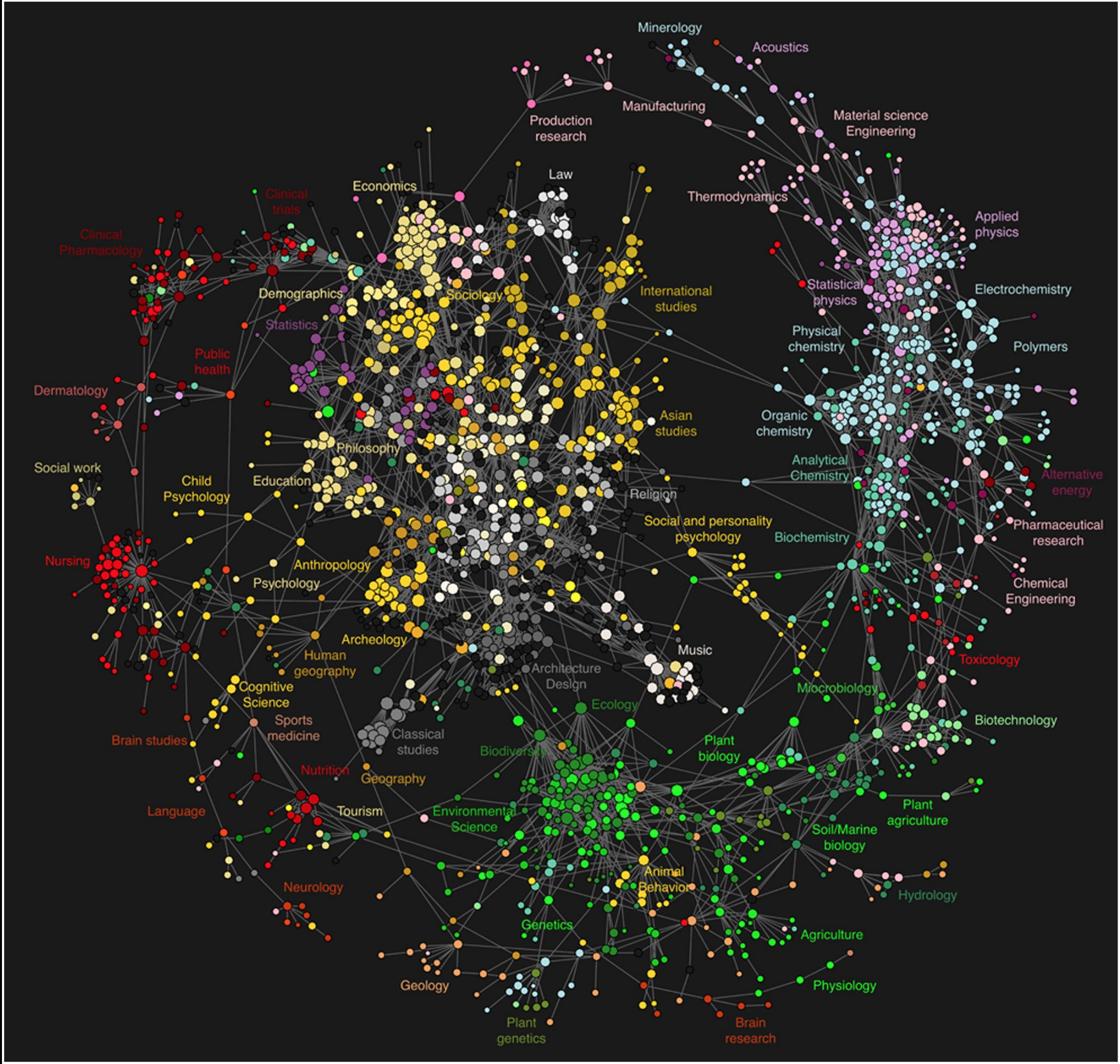
Nodes: Cities.	Edge Weights: ~Number of friends between cities

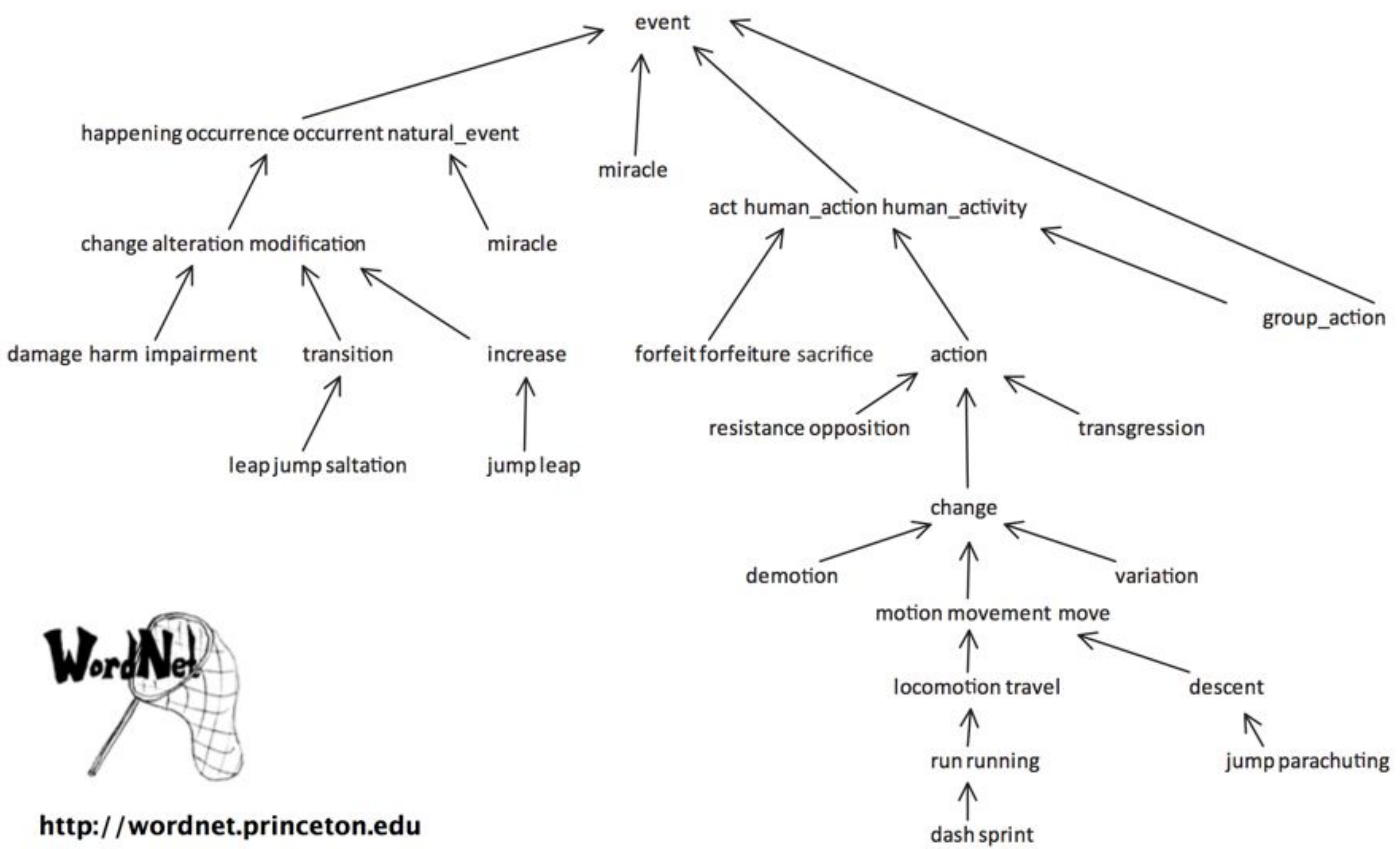
Nodes: Scientific Journals.

- Label: AAT classification (the topic that it covers)

Edges:

- Based on clickthrough data.
- Clickthrough from v to w means that someone reading an article in journal v clicked on a link to an article in journal w.
- Edge assigned from v to w if clickthrough rate from v to w is above some arbitrary threshold.



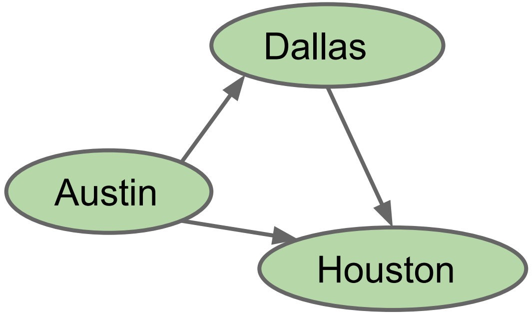


Edge captures 'is-a-type-of' relationship. Example: descent is-a-type-of movement.

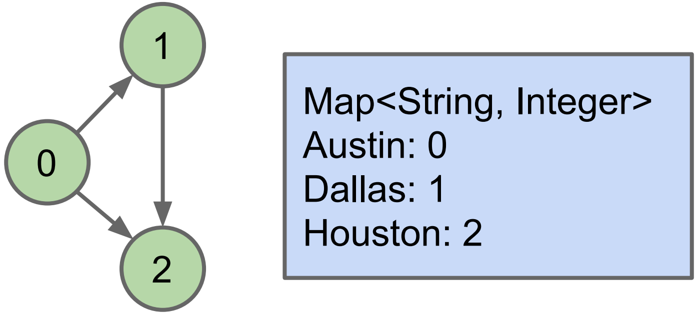
Graph Representations

Common Simplification: Integer Vertices

Common convention: Number nodes irrespective of label, and use number throughout the graph implementation. To lookup a vertex by label, use a Map<Label, Integer>.



Intended graph.



What you get.

Graph API

Using a graph in Java:

```
public class Graph {
  public Graph(int V):                        Create empty graph with v vertices
  public void addEdge(int v, int w): add an edge v-w
  Iterable<Integer> adj(int v):         vertices adjacent to v
  int V():                                          number of vertices
  int E():                                          number of edges
...
```

Graph API

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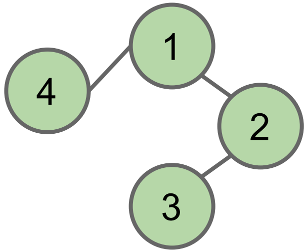
public class Graph {
  public Graph(int V):                   Create empty graph with v vertices
  public void addEdge(int v, int w):     add an edge v-w
  Iterable<Integer> adj(int v):          vertices adjacent to v
  int V():                               number of vertices
  int E():                               number of edges
...
```

Example client:



degree(G, 2) = 2

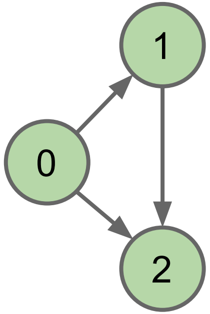
```
/** degree of vertex v in graph G */
public static int degree(Graph G, int v) {
    int degree = 0;
    for (int w : G.adj(v)) {
        degree += 1;
    }
    return degree; }
```

(degree = # edges)

Graph Representations

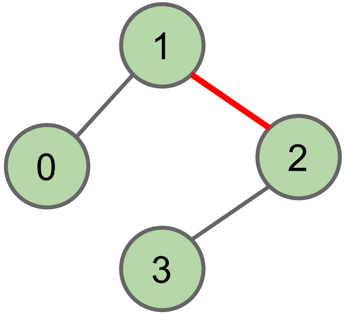
- Representation 1: Adjacency Matrix.

s \ t	0	1	2
0	0	1	1
1	0	0	1
2	0	0	0



For undirected graph: Each edge is represented twice in the matrix. Simplicity at the expense of space.

v \ w	0	1	2	3
0	0	1	0	0
1	1	0	**1**	0
2	0	**1**	0	1
3	0	0	1	0



Graph Printing Runtime: http://yellkey.com/paper

What is the order of growth of the running time of the following code if the graph uses an adjacency-matrix representation, where V is the number of vertices, and E is the total number of edges?

A. $\Theta(V)$
B. $\Theta(V + E)$
C. $\Theta(V^2)$
D. $\Theta(V*E)$

```java
for (int v = 0; v < G.V(); v++) {
    for (int w : G.adj(v)) {
        System.out.println(v + "-" + w);
    }
}
```

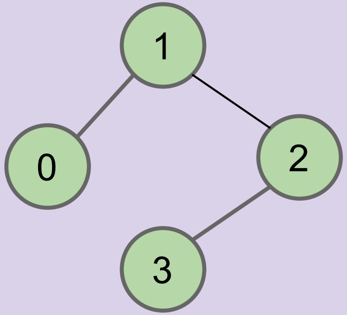
What is the runtime of the for-each?

-

How many times is the for-each run?

-

	0	1	2	3
0	0	1	0	0
1	1	0	1	0
2	0	1	0	1
3	0	0	1	0



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        System.out.println(v + "-" + w);
    }
}
```

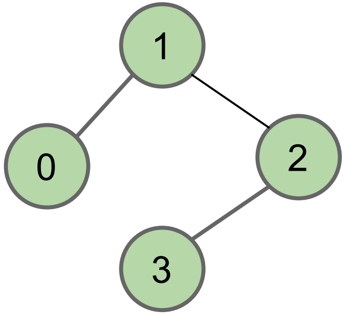
What is the runtime of the for-each?

- $\Theta(V)$.

How many times is the for-each run?

- V times.

	0	1	2	3
0	0	1	0	0
1	1	0	1	0
2	0	1	0	1
3	0	0	1	0



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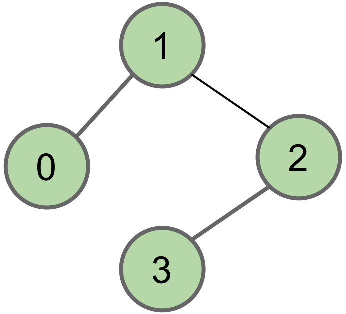
B. $\Theta(V + E)$
C. $\boldsymbol{\Theta(V^2)}$
D. $\Theta(V^*E)$

```java
for (int v = 0; v < G.V(); v++) {
    for (int w : G.adj(v)) {
        System.out.println(v + "-" + w);
    }
}
```

What does G.adj(1) return?

- An iterator with next() = 0, then next() = 2.

	0	1	2	3
0	0	1	0	0
1	1	0	1	0
2	0	1	0	1
3	0	0	1	0

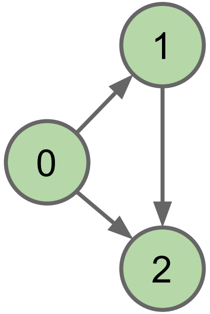


More Graph Representations

Representation 2: Edge Sets: Collection of all edges.

- Example: HashSet<Edge>, where each Edge is a pair of ints.

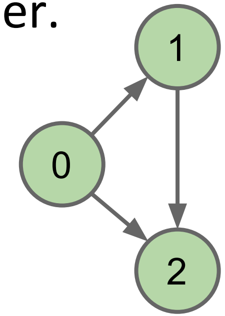
```
{(0, 1), (0, 2), (1, 2)}
```

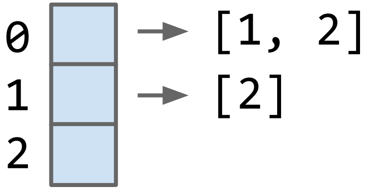


More Graph Representations

Representation 3: Adjacency lists.

- Common approach: Maintain array of lists indexed by vertex number.
- Most popular approach for representing graphs.





Graph Printing Runtime: http://shoutkey.com/laugh

What is the order of growth of the running time of the following code if the graph uses an ***adjacency-list*** representation, where V is the number of vertices, and E is the total number of edges?

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B. $\Theta(V + E)$
C. $\Theta(V^2)$
D. $\Theta(V*E)$

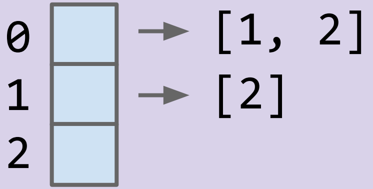
```java
for (int v = 0; v < G.V(); v++) {
    for (int w : G.adj(v)) {
        System.out.println(v + "-" + w);
    }
}
```

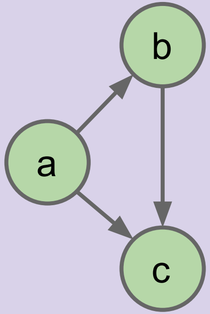
What is the runtime of the for-each?

-

How many times is the for-each run?

-





Graph Printing Runtime: http://shoutkey.com/laugh

What is the order of growth of the running time of the following code if the graph uses an ***adjacency-list*** representation, where V is the number of vertices, and E is the total number of edges? **Best case: Θ(V)** **Worst case: Θ(V²)**

A. $\Theta(V)$
B. $\Theta(V + E)$
C. $\Theta(V^2)$
D. $\Theta(V*E)$

```
for (int v = 0; v < G.V(); v++) {
    for (int w : G.adj(v)) {
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    }
}
```
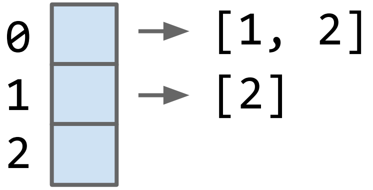
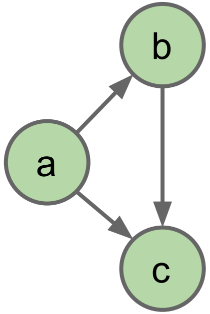

What is the runtime of the for-each? List can be between 1 and V items.

- $\Omega(1)$, $O(V)$.





How many times is the for-each run?

- V.

Graph Printing Runtime: http://shoutkey.com/ready

What is the order of growth of the running time of the following code if the graph uses an ***adjacency-list*** representation, where V is the number of vertices, and E is the total number of edges? **Best case: Θ(V)** **Worst case: Θ(V²)**

A. Θ(V)
B. Θ(V + E) ?
C. Θ(V²) ?
D. Θ(V*E) ?

```
for (int v = 0; v < G.V(); v++) {
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    }
}
```

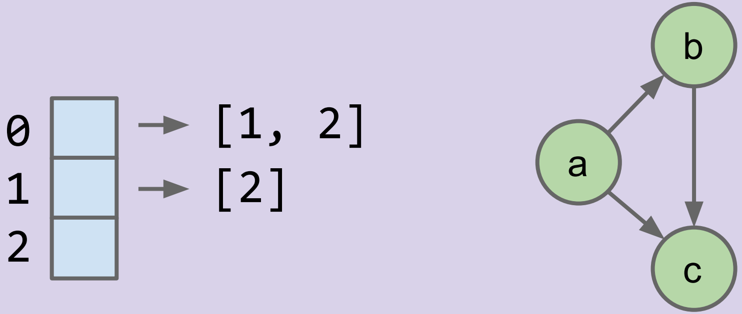
What is the runtime of the for-each? List can be between 1 and V items. ?

- Ω(1), O(V).

?

How many times is the for-each run?

- V.



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```
for (int v = 0; v < G.V(); v++) {
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    }
}
```
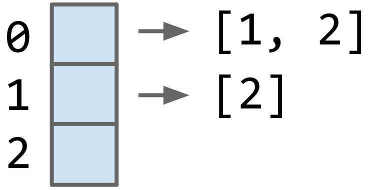
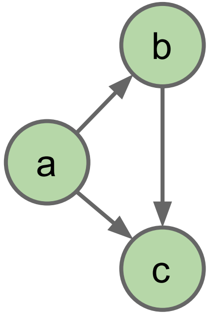

Best case: $\Theta(V)$ Worst case: $\Theta(V^2)$

All cases: $\Theta(V + E)$





Graph Printing Runtime: http://shoutkey.com/ready

Runtime: $\Theta(V + E)$

V is total number of vertices.

E is total number of edges in the entire graph.

```java
for (int v = 0; v < G.V(); v++) {
    for (int w : G.adj(v)) {
        System.out.println(v + "-" + w);
    }
}
```

How to interpret: No matter what "shape" of increasingly complex graphs we generate, as V and E grow, the runtime will always grow exactly as $\Theta(V + E)$.

- Example shape 1: Very sparse graph where E grows very slowly, e.g. every vertex is connected to its square: 2 - 4, 3 - 9, 4 - 16, 5 - 25, etc.
    - E is $\Theta(\text{sqrt}(V))$. Runtime is $\Theta(V + \text{sqrt}(V))$, which is just $\Theta(V)$.
- Example shape 2: Very dense graph where E grows very quickly, e.g. every vertex connected to every other.
    - E is $\Theta(V^2)$. Runtime is $\Theta(V + V^2)$, which is just $\Theta(V^2)$.

Graph Representations

Runtime of some basic operations for each representation:

idea	addEdge(s, t)	for(w : adj(v))	printgraph()	hasEdge(s, t)	space used
adjacency matrix	$\Theta(1)$	$\Theta(V)$	$\Theta(V^2)$	$\Theta(1)$	$\Theta(V^2)$
list of edges	$\Theta(1)$	$\Theta(E)$	$\Theta(E)$	$\Theta(E)$	$\Theta(E)$
adjacency list	$\Theta(1)$	$\Theta(1)$ to $\Theta(V)$	$\Theta(V+E)$	$\Theta(\text{degree}(v))$	$\Theta(E+V)$

Note: These operations are not part of the `Graph` class's API.

In practice, adjacency lists are most common.

- Many graph algorithms rely heavily on `adj(s)`.
- Most graphs are sparse (not many edges in each bucket).

Bare-Bones Undirected Graph Implementation

```
public class Graph {
     private final int V;  private List<Integer>[] adj;

     public Graph(int V) {
          this.V = V;
          adj = (List<Integer>[]) new ArrayList[V];
          for (int v = 0; v < V; v++) {
               adj[v] = new ArrayList<Integer>();
          }
     }

     public void addEdge(int v, int w) {
          adj[v].add(w);   adj[w].add(v);
     }

     public Iterable<Integer> adj(int v) {
         return adj[v];
     }
}
```

Cannot create array of anything involving generics, so have to use weird cast as with project 1A.

Depth-First Traversal

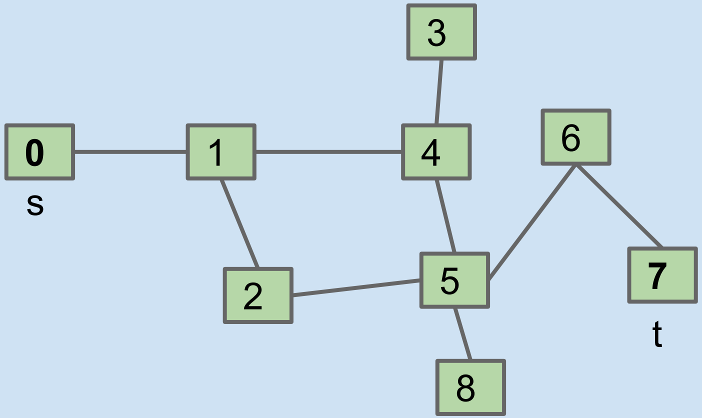
Maze Traversal / s-t Path

Suppose we want to know if there exists a path from vertex s=0 to vertex t=7.
What is wrong with the following recursive algorithm for connected(s, t)?

- Does s == t? If so, return true.
- Otherwise, check all of s's children for connectivity to t.

Example:

- connected(0, 7):
  - Does 0 == 7? No, so...
  - if (connected(1, 7)) return true;
  - return false;



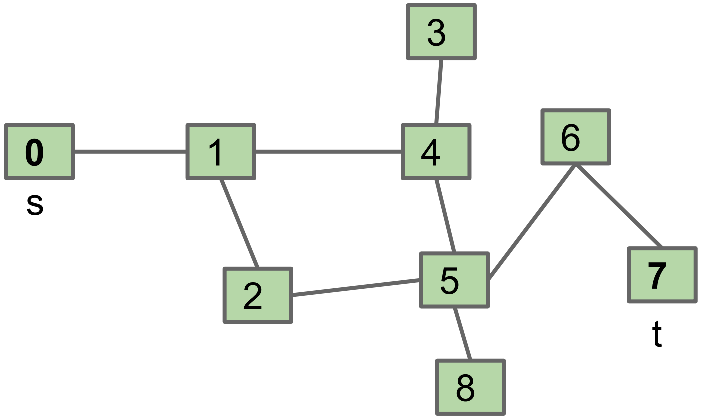
- connected(1, 7): ...

Improving Our Connectivity Algorithm

Goal: Search for a path from s to t, but visit each vertex at most once. To do this, we can mark each vertex as we search. Resulting algorithm for connected(s, t) is as follows:

- Mark s.
- Does s == t? If so, return true.
- Check all of s's unmarked neighbors for connectivity to t.

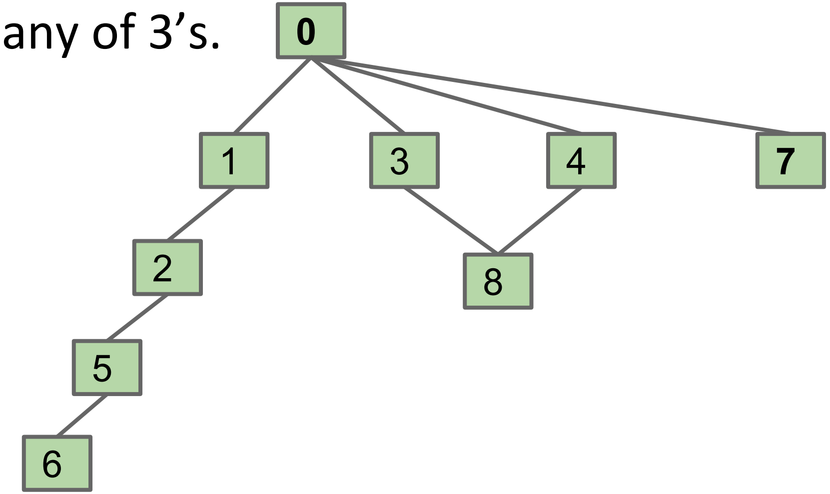
[Recursive connectivity demo.]

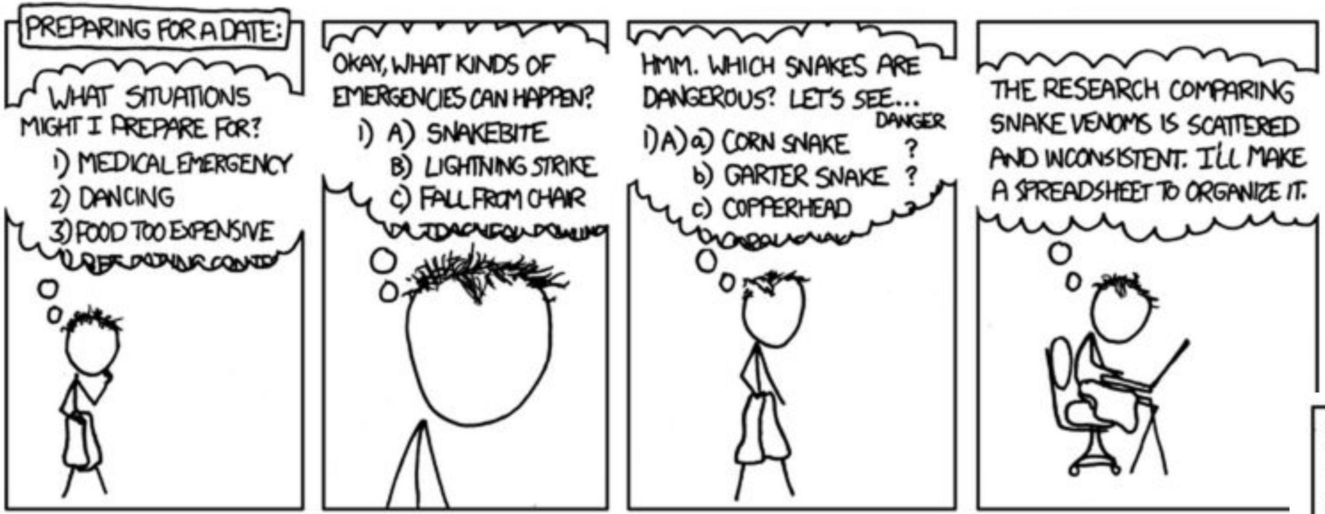


Depth First Traversal

This idea of exploring the entire subgraph for each child is known as Depth First Traversal.

- Ex. Visit all of 1's children before we visit any of 3's.



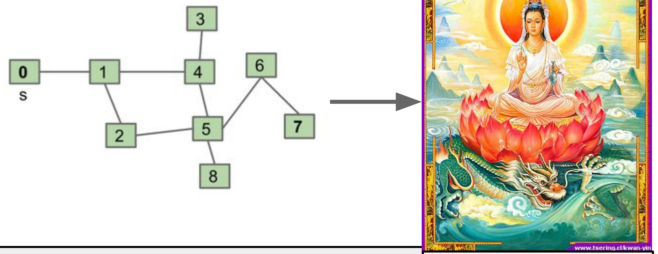


Or a more visceral example: https://xkcd.com/761/

Depth First Search Implementation

Common design pattern in graph algorithms: Decouple type from processing algorithm.

- Create a graph object.
- Pass the graph to a graph-processing method (or constructor) in a client class.
- Query the client class for information.



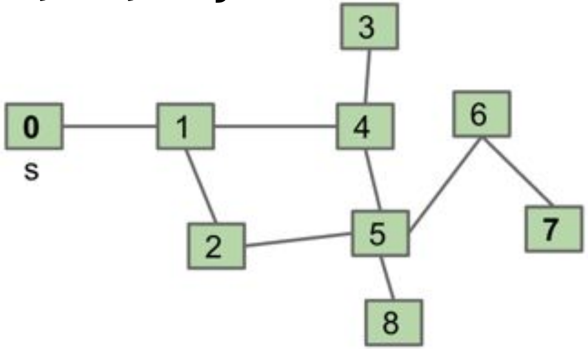
Paths.java

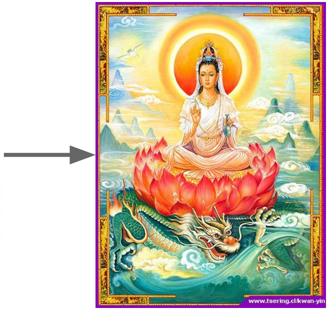
```
public class Paths {
    public Paths(Graph G, int s):    Find all paths from G
    boolean hasPathTo(int v):        is there a path from s to v?
    Iterable<Integer> pathTo(int v): path from s to v (if any)
}
```

Example Usage

Start by calling: `Paths P = new Paths(G, 0);`

- `P.hasPathTo(3); //returns true`
- `P.pathTo(3); //returns {0, 1, 4, 3}`





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    boolean hasPathTo(int v):        is there a path from s to v?
    Iterable<Integer> pathTo(int v): path from s to v (if any)
}
```

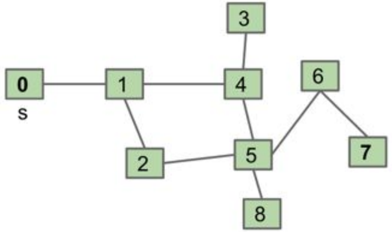
Implementing Paths With Depth First Search

To visit a vertex v:

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.

Data Structures:

- `boolean[] marked`
- `int[] edgeTo`
  - `edgeTo[4] = 1`, means we went from 1 to 4.



Paths.java

```
public class Paths {
    public Paths(Graph G, int s):    Find all paths from G
    boolean hasPathTo(int v):        is there a path from s to v?
    Iterable<Integer> pathTo(int v): path from s to v (if any)
}
```

DepthFirstPaths

Demo: DepthFirstPaths

DepthFirstPaths, Recursive Implementation

```
public class DepthFirstPaths {
    private boolean[] marked;
    private int[] edgeTo;
    private int s;

    public DepthFirstPaths(Graph G, int s) {
        ...
        dfs(G, s);
    }

    private void dfs(Graph G, int v) {
        marked[v] = true;
        for (int w : G.adj(v)) {
        if (!marked[w]) {
            edgeTo[w] = v;
            dfs(G, w);
        }
    }
}
```

marked[v] is true iff v connected to s

edgeTo[v] is previous vertex on path from s to v

not shown: data structure initialization

find vertices connected to s.

recursive routine does the work and stores results in an easy to query manner!

Question: How would we write hasPathTo(v)?

DepthFirstPaths Summary

Demo: DepthFirstPaths

Properties of Depth First Search:

- Guaranteed to reach every node.
- Runs in O(V + E) time.
  - Analysis next time, but basic idea is that every edge is used at most once, and total number of vertex considerations is equal to number of edges.
  - Runtime may be faster than Θ(V+E) for problems which quit early on some stopping condition (for example connectivity).