MA5701: Statistical Methods

Chapter 4 : Inferences on a Single Population

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Introduction

- The examples used in Chapter 3 to introduce the concepts of statistical inference were not very practical, because they required outside knowledge of the population variance. We did this to avoid distractions from issues that were irrelevant to the principles we were introducing.

Introduction

- In this chapter, we will present procedures for:
  - Making inferences on the mean of a normally distributed population where the variance is unknown.
  - Making inferences on the proportion of successes in a binomial population.

Inference on Population Mean

- In Chapter 3, we use the statistic $z = \frac{(\bar{y}-\mu)}{\sigma/\sqrt{n}}$, which has the standard normal distribution, to make inferences about the population mean.
- This statistic has limited practical value because, if the population mean is unknown, it is also likely that the variance of the population is unknown.
- The idea is to use the estimate of population variance in the statistic.

One Sample *t*-test – Test Statistic

- The test statistic is:

$$T = \frac{\bar{Y} - \mu_0}{S/\sqrt{n}}$$

- Where S is the sample standard deviation.
- When the null hypothesis is true ($\mu = \mu_0$),

$$T = \frac{\bar{Y}-\mu_0}{S/\sqrt{n}} \sim t_{n-1}$$

- **Question:** when $\mu \neq \mu_0$, what is the distribution of *T*?

Hypothesis Test on Population Mean

- The test is called ***t*-test.**
- To test the hypothesis $H_0: \mu = \mu_0$ with the significance level of α.
  - The test statistic is: $t = \frac{(\bar{y} - \mu_0)}{s/\sqrt{n}}$.
  - Here we use the sample standard deviation instead of the population standard deviation.
  - Again, the rejection region and p-values depends on H_a.

One Sample *t*-test – Rejection Region

- For two-sided test ($H_a\colon\ \mu \neq \mu_0$), the rejection region is:

$$|t| > t_{n-1,\alpha/2}$$

- For upper-tailed (right-tailed) test ($H_a\colon\ \mu > \mu_0$), the rejection region:

$$t > t_{n-1,\alpha}$$

- For lower-tailed (left-tailed) test ($H_a\colon\ \mu < \mu_0$), the rejection region is:

$$t < -t_{n-1,\alpha}$$

- Here the statistic is $t = \frac{(\bar{y}-\mu_0)}{s/\sqrt{n}}$.

One Sample *t*-test – *p*-value Approach

- For two-sided test (H_a: $\mu \neq \mu_0$):

$$p\text{-value} = 2\Pr(T_{n-1} > |t|)$$

- For upper-tailed (right-tailed) test (H_a: $\mu > \mu_0$):

$$p\text{-value} = \Pr(T_{n-1} > t)$$

- For lower-tailed (left-tailed) test (H_a: $\mu < \mu_0$):

$$p\text{-value} = \Pr(T_{n-1} < t)$$

- Here T_{n-1} represents a random variable with t-distribution of $n-1$ degrees of freedom.

One Sample *t*-test – Confidence Interval

- For two-sided test ($H_a\!: \ \mu \neq \mu_0$), construct a $100(1-\alpha)\%$ two-sided confidence interval $\left(\bar{y} - t_{n-1,\frac{\alpha}{2}} \frac{s}{\sqrt{n}}, \bar{y} + t_{n-1,\frac{\alpha}{2}} \frac{s}{\sqrt{n}}\right)$.
- For upper-tailed (right-tailed) test ($H_a\!: \ \mu > \mu_0$), construct a $100(1-\alpha)\%$ lower confidence interval $\left(\bar{y} - t_{n-1,\alpha} \frac{s}{\sqrt{n}}, \infty\right)$.
- For lower-tailed (left-tailed) test ($H_a\!: \ \mu < \mu_0$), construct a $100(1 -$

One Sample t-test - Example 4.2

- **Example 4.2** - In **Example 3.3** we presented a quality control problem in which we tested the hypothesis that the mean weight of peanuts being put in jars was the required 8 oz. We assumed that we knew the population standard deviation, possibly from experience. We now relax that assumption and estimate both mean and variance from the sample. Data are presented on the next slide.

One Sample t-test - Example 4.2

Example 4.2 – Some basics:

- Test if $H_a: \mu \neq 8$.
- Sample size: $n = 16$
- Sample mean: $\bar{y} = 7.8925$;
- Sample variance: $s^2 = 0.03174$; $s = 0.1782$
- Still, follow 5-steps of hypothesis testing.
- You can use either rejection region, p-value, or confidence interval approach.

One Sample *t*-test – Example 4.2

- **Step 1:** Specify appropriate H_0, H_a, and an acceptable level of α.

$$H_0: \mu = 8; H_a: \mu \neq 8; \alpha = 0.05$$

- **Step 2:** Define a sample-based test statistic:

$$t = \frac{\bar{y} - \mu_0}{s/\sqrt{n}} = \frac{\bar{y} - 8}{0.1782/\sqrt{16}}$$

- **Step 3:** Find the rejection region for the specified H_a and α.

The rejection is $|t| > t_{n-1,\alpha/2} = t_{15,0.025} = 2.1315$.

One Sample *t*-test – Example 4.2

- **Step 4:** Collect the sample data and calculate the test statistic.

$$t = \frac{\bar{y} - 8}{s/\sqrt{16}} = \frac{7.8925 - 8}{0.1782/\sqrt{16}} = -2.4130$$

- **Step 5:** Decide to either reject or fail to reject H_0.
  - At the 5% significance level, the test statistic $t = -2.4130$ falls in the rejection region. Therefore, we reject the null hypothesis. The data provides sufficient evidence that the mean weight from filling process is different from the targeted weight of 8 oz.

One Sample *t*-test – Example 4.2

- **Step 4:** we have $t = -2.4130$, so the *p*-value is

$$p\text{-value} = 2\Pr(T_{n-1} > |t|) = 2\Pr(T_{15} > 2.4130) = 0.0291$$

- **Step 5:** Make a decision to either reject or fail to reject H_0.
  - The p-value is 0.0291 and is less than the significance level of 0.05. Therefore, we reject the null hypothesis. The data provides sufficient evidence that the mean weight from filling process is different from the targeted weight of 8 oz.

One Sample *t*-test – Example 4.2

- **Step 4:** The 95% confidence interval of mean concentricity is

$$\left(\bar{y} \pm t_{n-1,\alpha/2}\,\frac{s}{\sqrt{n}}\right) = \left(7.8912 \pm 2.1315 * \frac{0.1782}{\sqrt{16}}\right) = (7.7962, 7.9862)$$

- **Step 5:** Make a decision to either reject or fail to reject H_0.
  - The targeted mean weight of 8 oz does not fall in the 95% confidence interval of mean weight $(7.7962, 7.9862)$. Therefore, we reject the null hypothesis. The data provides sufficient evidence that the mean weight from filling process is different from the targeted weight of 8 oz.

Example 4.2 – R Program and Output

```
t.test(x = peanuts$weight, mu = 8)
t.test(x = peanuts$weight, , mu = 8,
  alternative = "two.sided", conf.level = 1 - alpha)

alternative = c("two.sided“, “less”, “greater”)
```

Example 4.2 – R Program and Output

One Sample t-test

data: peanuts$weight
t = -2.4136, df = 15, p-value = 0.02904
alternative hypothesis: true mean is not equal to 8
95 percent confidence interval:
 7.797567 7.987433
sample estimates:
mean of x
   7.8925

Example 4.2 – R Program and Output

One Sample t-test

data: peanuts$weight

t = -2.4136, df = 15, p-value = 0.01452

alternative hypothesis: true mean is less than 8

95 percent confidence interval:

-Inf 7.97058

sample estimates:

mean of x

7.8925

One Sample *t*-test – Exercise

- **Exercise (Porosity of Battery)** – Nickel-hydrogen (Ni-H) batteries use a nickel plate as the anode. A critical quality characteristic is the plate's porosity. The manufacturer has set a target porosity of 80%. The production have felt that there is no sufficient porous due to the equipment. They collected 10 samples to investigate the problem.

79.1	79.5	79.3	79.3	78.8
79.0	79.2	79.7	79.0	79.2

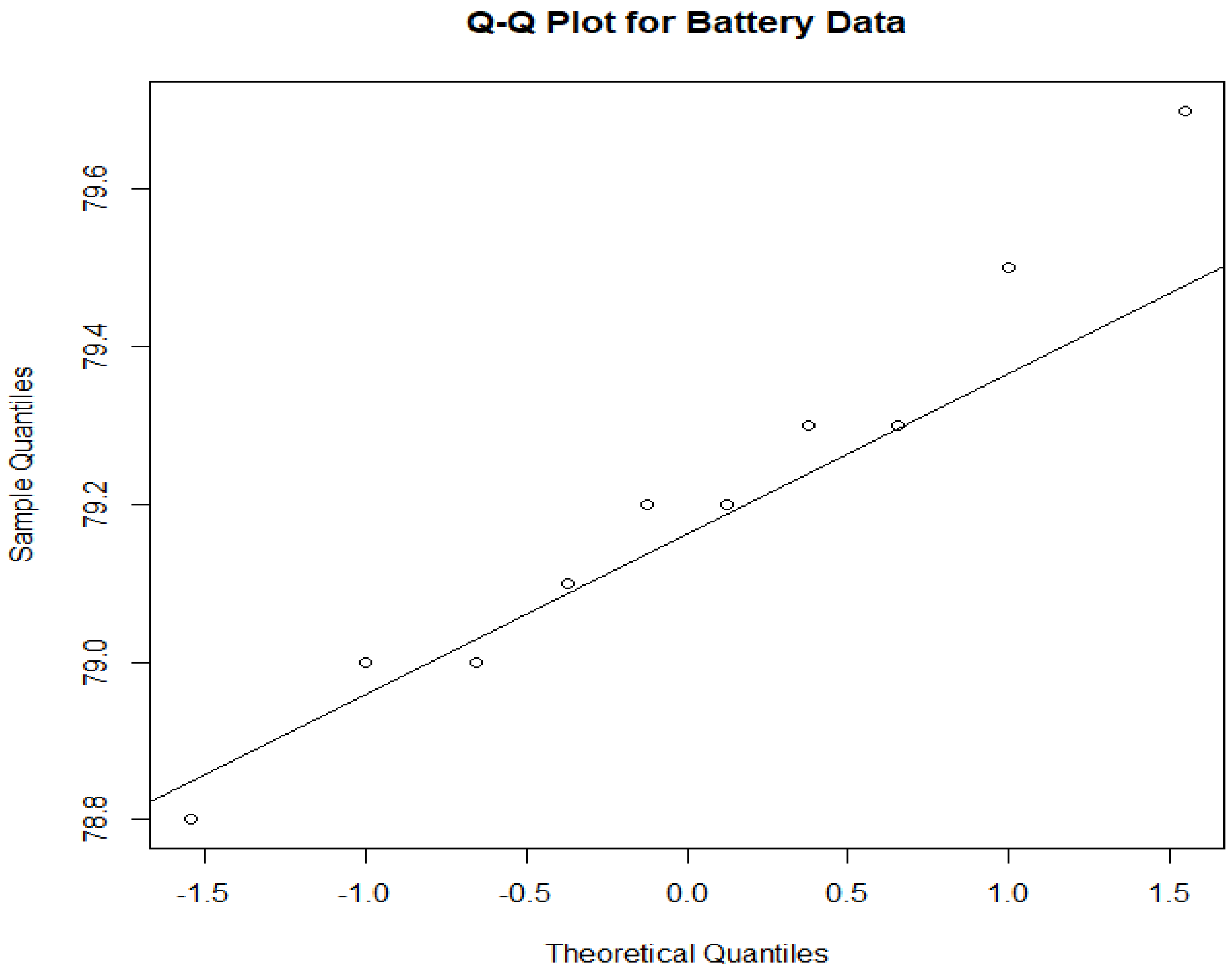
- Use 0.05 significance level to perform your test.

One Sample *t*-test – Exercise

Exercise (Porosity of Battery):

- Sample mean: $\bar{y} = 79.21$
- Sample Size: $n = 10$
- Sample variance: $s^2 = 0.0677$
- Sample variance: $s = 0.2601$
- Significance level: $\alpha = 0.05$
- Check the assumption for the normal approximation.

Porosity of Battery – Q-Q Plot



Degrees of Freedom

- **Degrees of freedom**: It is important to remember that the degrees of freedom of the *t* statistic are always those used to estimate the variance, which may not be sample size minus 1.
  - For example, suppose that we table 100 stones to estimate the average size of stones produced by a gravel crusher. We do not have time to weigh each stone individually. So we weigh the entire 100 in one weighing and choose and weigh 10 stones individually. The test statistic is $t = \frac{\bar{y}_{100} - \mu_0}{s/\sqrt{100}}$ where $s^2 = \sum(y_i - \bar{y}_{10})^2/9$. It has a *t*-distribution with 9 degrees of freedom.

Inference On Proportions

- If $Y \sim Binomial(n, p)$, then $Y = \sum_{i=1}^{n} Y_i$ and $Y_i (i = 1, \cdots, n)$ is a random sample from Bernoulli distribution. So $\frac{Y}{n}$ (the proportion) is considered as a sample mean.
- By Central Limit Theorem (CLT): $\frac{\frac{Y}{n} - p}{\sqrt{p(1-p)/n}} = \frac{Y - np}{\sqrt{np(1-p)}}$ can be approximated by $N(0,1)$.
- Also, $E\left(\frac{Y}{n}\right) = p$; and Var $\left(\frac{Y}{n}\right) = \frac{p(1-p)}{n}$
- This statistic can be used for hypothesis testing.

Test for Proportion

- The null hypothesis: $H_0: p = p_0$.
- Denote $\hat{p} = \frac{y}{n}$. The test statistic is:

$$z = \frac{\frac{Y}{n} - p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{Y - np_0}{\sqrt{np_0(1-p_0)}}$$

- Assumptions for this test are:
  - $Y \sim Binomial(n, p)$.
  - Sample size is large enough: $np_0 \geq 5$ and $n(1 - p_0) \geq 5$.

Test for Proportion – Rejection Region

- For two-sided test ($H_a: p \neq p_0$), the rejection region is:

$$|z| > z_{\alpha/2}$$

- For upper-tailed (right-tailed) test ($H_a: p > p_0$), the rejection region:

$$z > z_\alpha$$

- For lower-tailed (left-tailed) test ($H_a: p < p_0$), the rejection region is:

$$z < -z_\alpha$$

Test for Proportion – Confidence Interval

- For two-sided test ($H_a: p \neq p_0$), construct a $100(1-\alpha)\%$ two-sided confidence interval $\left(\hat{p} - z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$.
- For upper-tailed (right-tailed) test ($H_a: p > p_0$), construct a $100(1-\alpha)\%$ lower confidence interval $\left(\hat{p} - z_{\alpha}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, 1\right)$.
- For lower-tailed (left-tailed) test ($H_a: p < p_0$), construct a $100(1-\alpha)\%$ upper confidence interval $\left(0, \hat{p} + z_{\alpha}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$.

Test for Proportion – *p*-value Approach

- For two-sided test ($H_a: p \neq p_0$):

$$p\text{-value} = 2\Pr(Z > |z|)$$

- For upper-tailed (right-tailed) test ($H_a: p > p_0$):

$$p\text{-value} = \Pr(Z > z)$$

- For lower-tailed (left-tailed) test ($H_a: p < p_0$):

$$p\text{-value} = \Pr(Z < z)$$

Inference on a Proportion – Example 4.4

- **Example 4.4** - An advertisement claims that more than 60% of doctors prefer a particular brand of painkiller. An agency established to monitor truth in advertising conducts a survey consisting of a random sample of 120 doctors. Of the 120 questioned, 82 indicated a preference for the particular brand. Is the advertisement justified?
- **Example 4.4** – Use five steps for hypothesis testing. The significance level is set as 0.05.

Example 4.4 – R Program and Output

1-sample proportions test without continuity correction

data: y out of n, null probability 0.6
X-squared = 3.4722, df = 1, p-value = 0.0312
alternative hypothesis: true p is greater than 0.6
95 percent confidence interval:
 0.6100991 1.0000000
sample estimates:
        p
0.6833333

Test for Proportion – Exercise

- **Exercise (Breaking Strengths Carbon Fiber)** – Padgett and Spurrier (1990) analyzed the breaking strengths of carbon fiber. Suppose that historically, the proportion of nonconforming product is 10%. To test if the true proportion of nonconforming is 10%, 100 fibers are collected and 6 are nonconforming.
- **Question**: Is the true proportion of nonconforming 10%? Use 0.01 significance level to perform your test.

Test for a Proportion - Example 4.5

- **Example 4.5** – A pre-election poll using a random sample of 150 voters indicated that 84 favored candidate Smith, that is, $\hat{p} = 0.56$. We would like to construct a 0.99 confidence interval on the true proportion of voters favoring Smith.
- This will be used as an exercise problem in the class. You can look at the R program.