Group, Ring, and Field Theory Lecture Notes

Abstract Group G and Symmetries

Let G be an abstract group. Find an object X where the symmetries of X are isomorphic to G.

Cayley Graph

- A Cayley graph is an ordered, directed graph where points represent elements of group G and the set S on which G acts.
- Define actions of G on S:
  - **Left Action**: A map $G \times S \to S$, defined as $(g, s) \mapsto g \cdot s$.
  - **Right Action**: A map $S \times G \to S$, defined as $(s, g) \mapsto s \cdot g$.
- An action is **faithful** if $g \cdot s = s$ for all $s \in S$ implies $g = e$ (the identity element), indicating that G acts bijectively on S.
- **Problem**: Add extra structure to S to cut down the symmetry group to G.
- Extra structure involves the right action of G on S, satisfying $s \cdot g = sg$. Verify that the left action preserves this structure.
- The Cayley graph of G is constructed by drawing lines from s to gs for each $g \in G$ and $s \in S$, coloring lines differently for each generator of G.

Examples

- For $G =$ Klein 4-group, check that any symmetry of the Cayley graph is of the form $s \cdot g \cdot s$ for some $g \in G$.
- **Exercise**: Classify actions of G on itself. There are 8 actions of G on G:
  - Left actions: $g(s) = gs$.
  - Right actions: $g(s) = sg$.
  - Trivial action: $g(s) = s$.
  - Adjoint action: $g(s) = gsg^{-1}$.

Goals

- Classify all groups.
- Find all representations of a group.
- Define an action on some mathematical object that preserves its structure.

Lagrange's Theorem

- The order of an element $g \in G$ divides the order of G.
- If G has prime order, it is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Action and Stabilizer

- Suppose G acts transitively on a set S. Let H be the stabilizer of an element $s \in S$. Then:
$$|G| = |G : H| \cdot |H|.$$

Matrix Groups

- Example: Find the order of $\mathrm{GL}_2(\mathbb{F}_2)$, the general linear group over the field $\mathbb{F}_2$. This group contains 2×2 matrices with non-zero determinants.

Group Products and Extensions

- Direct product: The direct product of groups $G_1, G_2, \ldots, G_n$ is $G = G_1 \times G_2 \times \ldots \times G_n$.
- Warning: Complex group interactions can lead to unexpected behaviors in some products.

Field Theory

- For G acting on a field, explore field extensions and understand the roles of elements with specific order properties within the group structure.