MATH250A Homework 1

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Problem 1

Order 1: This is the trivial group with only the identity element e, which is commutative under any operation.

Order 2: Let G be a group of order 2 with elements $\{e, x\}$. Define another group G' similarly with elements $\{e', x'\}$. We can construct an isomorphism $f : G \to G'$ defined by:

$$f(e) = e', \quad f(x) = x'.$$

Any other mapping would break the homomorphism condition. Thus, G is unique up to isomorphism. Since $x \cdot x = e$, we conclude that every group of order 2 is abelian.

Order 3: Let G be a group of order 3 with elements $\{e, a, b\}$. To prove G is abelian, consider the multiplication of elements. The only non-trivial case is $ab = ba$. If $ab = a$ or $ab = b$, then a or b must equal e, which is a contradiction. Therefore, $ab = ba$, showing that G is abelian.

Order 4: Let G be a group of order 4 with elements that can have orders 1, 2, or 4. If G has an element of order 4, then G must be cyclic, e.g., $\{e, x, x^2, x^3\}$, and thus abelian. If all elements have order 1 or 2, then G is of the form $\{e, g, h, gh\}$ where $g^2 = h^2 = e$ and $gh = hg$. To show $gh = hg$, we have $(gh)^2 = ghgh = e = gghh = g^2h^2$. Therefore, G is abelian.

Order 5: By Lagrange's theorem, the order of each element must be 1 or 5, making G cyclic, hence abelian.

Problem 2

By Lagrange's Theorem, a group G of order 4 can only have elements of order 1, 2, or 4. If G is a group of order 4 with an element of order 4, then G must be a cyclic group with elements $\{1, x, x^2, x^3\}$. It is trivial to see that this group $G \cong \mathbb{Z}/4\mathbb{Z}$.

Now we shift our discussion to groups of order 4 with elements of order 1 or 2, and we want to show that it is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. As we have seen in the first question, we can write $G = \{1, x, y, xy\}$, where x, y have order 2, and G is abelian with $xy = yx$.

We define a mapping:

$$f : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to G,$$

with

$$f(\overline{a}, \overline{b}) = x^a y^b.$$

The mapping is listed below:

$$f(\overline{0}, \overline{0}) \mapsto 1, \quad f(\overline{1}, \overline{0}) \mapsto x, \quad f(\overline{0}, \overline{1}) \mapsto y, \quad f(\overline{1}, \overline{1}) \mapsto xy.$$

(This shows that f is a bijection.)

We show that f is a homomorphism. Let $\overline{a}, \overline{b}, \overline{c}, \overline{d} \in \mathbb{Z}/2\mathbb{Z}$. Then

$$f(\overline{a}, \overline{b}) \cdot f(\overline{c}, \overline{d}) = x^a y^b \cdot x^c y^d = x^{a+c} y^{b+d} = f(\overline{a+c}, \overline{b+d}).$$

Thus we get an isomorphism between G and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The two groups above are clearly non-isomorphic since one contains an element of order 4, while the other does not.

Probelm 3

(a)

We know that $H \subseteq G$ is a subgroup of finite index. We write $(G : H) = n$. Therefore,

$$G/H = \{xH \mid x \in G\}$$

has exactly n elements since $(G : H) = n$. We define a mapping

$$f : G \to S_n,$$

where S_n is isomorphic to the permutation of G/H. We define an action of G on the set of cosets G/H with $g \cdot (xH) = (gx)H$ for $g, x \in G$. This action induces a homomorphism where $f(g)$ is the permutation of the cosets defined by

$$g \cdot (xH) = (gx)H.$$

We know that

$$N = \ker(f)$$

is a normal subgroup of G by the first isomorphism theorem.

Now, we want to show that $\ker(f) \subseteq H$. For all $g \in \ker(f)$,

$$g(xH) = (gx)H = xH.$$

Thus, $g \in xH$. Therefore, $\ker(f) \subseteq H$. Hence, we find a normal subgroup N of G contained in H and also of finite index.

(b)

Let $(G : H_1) = n$ and $(G : H_2) = m$. We first define a group action on the coset space. For $g \in G$ and $x \in G$, define the action:

$$g \cdot (xH_1, xH_2) = (gxH_1, gxH_2).$$

There we have a homomorphism:

$$\phi : G \to \mathrm{Sym}(H_1) \times \mathrm{Sym}(H_2)$$

with $\phi(g) = (\sigma_{H_1}, \sigma_{H_2})$, where σ_{H_1} and σ_{H_2} are the permutations of the cosets of H_1 and H_2 induced by left multiplication.

For all $g \in \ker(\phi)$, we have:

$$g \cdot (xH_1, xH_2) = (gxH_1, gxH_2) = (xH_1, xH_2),$$

which implies $g \in H_1 \cap H_2$. For all $g \in H_1 \cap H_2$, we have:

$$g \cdot (xH_1, xH_2) = (xH_1, xH_2),$$

so $g \in \ker(\phi)$. Therefore, $\ker(\phi) = H_1 \cap H_2$.

By the first isomorphism theorem, we have:

$$G/(H_1 \cap H_2) \cong \mathrm{Im}(\phi) = \mathrm{Sym}(H_1) \times \mathrm{Sym}(H_2),$$

thus:

$$[G : H_1 \cap H_2] = |\mathrm{Im}(\phi)| = |\mathrm{Sym}(H_1) \times \mathrm{Sym}(H_2)| = m! \cdot n!.$$

Problem 4

We can write $(G : H) = n$. Let $G/H = \{gH \mid g \in G\}$ and $H\backslash G = \{Hg \mid g \in G\}$.

Define the mapping:

$$\phi : G/H \to H\backslash G, \quad \phi(gH) \mapsto Hg^{-1}.$$

We first check that this is well-defined. If $g = g'$, then:

$$\phi(gH) = Hg^{-1} = Hg'^{-1} = \phi(g'H).$$

Injective: Suppose $\phi(gH) = \phi(g'H)$. Then:

$$Hg^{-1} = Hg'^{-1} \implies gH = g'H.$$

Surjective: For any $Hg \in H\backslash G$, there exists $g' \in G$ such that:

$$\phi(g'H) = Hg.$$

We see that ϕ is a bijection, thus:

$$|G/H| = |H\backslash G| = n.$$