

Assignment 5

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Download all python codes from

https://github.com/Taha-Adeel/AI1103/tree/main/Assignment_5/Codes

and latex-tikz codes from

https://github.com/Taha-Adeel/AI1103/tree/main/Assignment_5

1 PROBLEM (UGC/MATH (MATHA_Dec 2017) Q.119)

Arrival of customers in a shop is a Poisson process with intensity $\lambda = 2$. Let X be the number of customers entering during the time interval $(1, 2)$, and let Y be the number of customers entering during the time interval $(5, 10)$. Which of the following is true?

(A) X and Y are independent.

(B) $X + Y$ is a Poisson with parameter 6.

(C) $X - Y$ is a Poisson with parameter 8.

(D) $\Pr(X = 0 | X + Y = 12) = \left(\frac{5}{6}\right)^{12}$

2 SOLUTION

X and Y are Poisson distributions with parameters $\mu_1 = \lambda \tau_1 = 2 \times 1$ and $\mu_2 = \lambda \tau_2 = 2 \times 5$ respectively (τ is time-interval). Hence the pmfs (probability mass functions) of random variables X and Y are given by:

$$p_X(x) = \frac{e^{-2} \cdot 2^x}{x!}, \quad \text{for } x = 0, 1, 2, \dots \quad (2.0.1)$$

$$p_Y(y) = \frac{e^{-10} \cdot 10^y}{y!}, \quad \text{for } y = 0, 1, 2, \dots \quad (2.0.2)$$

Lemma 2.1. *If X and Y are two independent Poisson distributions with parameters μ_1 and μ_2 respectively, then the distribution of $X + Y$ is also Poisson with parameter $\mu_1 + \mu_2$.*

Proof. We have for $k \geq 0$, the probability mass function $p_{X+Y}(k)$ is a convolution of pmfs $p_X(x)$ and $p_Y(y)$:

$$p_{X+Y}(k) = \Pr(X + Y = k) = \Pr(Y = k - X) \quad (2.0.3)$$

$$= \sum_i \Pr(Y = k - i | X = i) \times p_X(i) \quad (2.0.4)$$

As X and Y are independent:

$$\Pr(Y = k - i | X = i) = \Pr(Y = k - i) = p_Y(k - i) \quad (2.0.5)$$

Simplifying (2.0.4)

$$p_{X+Y}(k) = \sum_{i=0}^k p_Y(k - i) \times p_X(i) \quad (2.0.6)$$

$$= \sum_{i=0}^k e^{-\mu_2} \frac{\mu_2^{k-i}}{(k-i)!} e^{-\mu_1} \frac{\mu_1^i}{i!} \quad (2.0.7)$$

$$= e^{-(\mu_1 + \mu_2)} \frac{1}{k!} \sum_{i=0}^k \frac{k!}{i!(k-i)!} \mu_1^i \mu_2^{k-i} \quad (2.0.8)$$

$$= e^{-(\mu_1 + \mu_2)} \frac{1}{k!} \sum_{i=0}^k \binom{k}{i} \mu_1^i \mu_2^{k-i} \quad (2.0.9)$$

$$p_{X+Y}(k) = \frac{e^{-(\mu_1 + \mu_2)} \cdot (\mu_1 + \mu_2)^k}{k!} \quad (2.0.10)$$

□

(A) In a Poisson process, the occurrences/frequencies of the event do not depend on past or future occurrences. Therefore X and Y are independent, by definition of Poisson process. Hence option (A) is **correct**.

(B) As X and Y are independent Poisson distributions, using lemma (2.1), $X + Y$ is a Poisson with parameter $\mu_1 + \mu_2 = 12 \neq 6$. Hence option (B) is **incorrect**.

$$p_{X+Y}(k) = \frac{e^{-(12)} \cdot (12)^k}{k!} \quad (2.0.11)$$

(C) The distribution function for $X - Y$ no longer remains Poisson, as $X - Y$ will also attain

negative values. Hence option (C) is **incorrect**.

(D)

$$\Pr(X = 0 \mid X + Y = 12) = \frac{\Pr(X = 0, Y = 12)}{\Pr(X + Y = 12)} \quad (2.0.12)$$

As X and Y are independent, and using (2.0.1), (2.0.2), and (2.0.11), we have:

$$\Pr(X = 0 \mid X + Y = 12) = \frac{\Pr(X = 0) \cdot \Pr(Y = 12)}{\Pr(X + Y = 12)} \quad (2.0.13)$$

$$= \frac{\frac{e^{-2} \times 2^0}{0!} \times \frac{e^{-10} \times 10^{12}}{12!}}{\frac{e^{-12} \times 12^{12}}{12!}} \quad (2.0.14)$$

$$= \left(\frac{5}{6}\right)^{12} \quad (2.0.15)$$

Hence option (D) is **correct**.

Ans: (A), (D)