Assignment 4

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1 Problem (GATE 2021 (ST) Q.19)

Let $\{X_n\}_{n \geq 1}$ be a sequence of independent and identically distributed random variables each having uniform distribution on (0,2). For $n \geq 1$, let

$$Z_n = -\log_e \left(\prod_{i=1}^{n} (2 - X_i) \right)^{\frac{1}{n}}.$$

Then, as $n \to \infty$, the sequence $\{Z_n\}_{n \geq 1}$ converges almost surely to _____ (Round of to 2 decimal places).

2 Solution (GATE 2021 (ST) Q.19)

Simplifying Z_n, we have

$$Z_n = -\log_e \left(\prod_{i=1}^{n} (2 - X_i) \right)^{\frac{1}{n}} \quad (2.0.1)$$

$$= -\frac{1}{n} \cdot \log_e \left(\prod_{i=1}^{n} (2 - X_i) \right) \quad (2.0.2)$$

$$= \frac{1}{n} \cdot \left(\sum_{i=1}^{n} (-\log_e (2 - X_i)) \right) \quad (2.0.3)$$

Let X and Z be random variables. X follows a uniform distribution from 0 to 2.

$$X \sim \mathcal{U}[0, 2], \quad (2.0.4)$$
$$\text{and let } Z = -\log_e (2 - X) \quad (2.0.5)$$

In our question, we are generating n independent and identically distributed random variables, X_i, where $i \leq n$ from $X \sim \mathcal{U}[0, 2]$ and Z_n finds the average value of $-\log_e (2 - X_i)$. **The Law of Large Numbers** states that for large number of

trials(n here), the average obtained (of $-\log_e (2 - X_i)$ here) should be close to the expected value (of $-\log_e(2 - X) = E(Z)$ here), and will tend to become closer to the expected value as more trials are performed.. Hence as $n \to \infty$, the sequence $\{Z_n\}$ tends to the expectation value of the continuous distribution of Z.

$$\therefore \lim_{n \to \infty} Z_n = E(Z) \quad (2.0.6)$$

The CDF of Z is defined as

$$F_Z(z) = \Pr(Z \leq z) \quad (2.0.7)$$
$$= \Pr(-\log_e(2 - X) \leq z) \quad (2.0.8)$$
$$= \Pr(\log_e(2 - X) \geq -z) \quad (2.0.9)$$
$$= \Pr(2 - X \geq \exp(-z)) \quad (2.0.10)$$
$$= \Pr(X \leq 2 - \exp(-z)) \quad (2.0.11)$$
$$= F_X(2 - \exp(-z)) \quad (2.0.12)$$

The CDF for X ($F_X(x)$), a uniform distribution on $(0, 2)$ is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \leq x \leq 2 \\ 1 & x > 2 \end{cases} \quad (2.0.13)$$

Substituting the above in (2.0.12),

$$F_X(2 - \exp(-z)) =$$
$$\begin{cases} 0 & 2 - \exp(-z) < 0 \\ 1 - \frac{\exp(-z)}{2} & 0 \leq 2 - \exp(-z) \leq 2 \\ 1 & 2 - \exp(-z) > 2 \end{cases} \quad (2.0.14)$$

After some algebra, the above conditions yield

$$F_Z(z) = \begin{cases} 0 & z < -\log_e(2) \\ 1 - \frac{\exp(-z)}{2} & z \geq -\log_e(2) \end{cases} \quad (2.0.15)$$

$$\implies f_Z(z) = \frac{d(F_Z(z))}{dz} = \begin{cases} 0 & z < -\log_e(2) \\ \frac{\exp(-z)}{2} & z \geq -\log_e(2) \end{cases} \quad (2.0.16)$$

Fig. 0: $F_Z(z)$
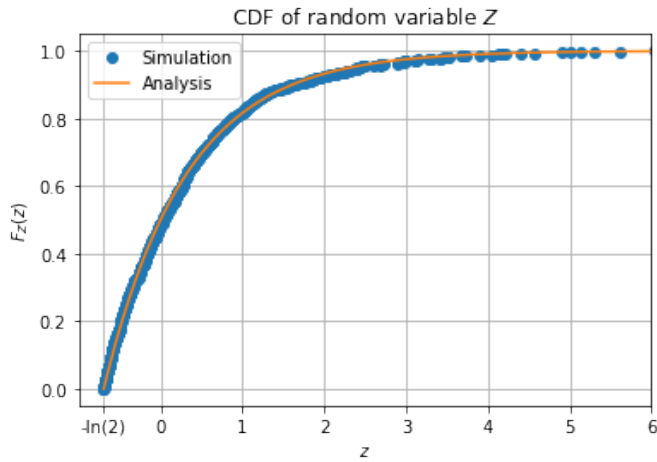
Now calculating the expectation value for Z, we have

$$E(Z) = \int\limits_{-\ln 2}^{\infty} z\, f_Z(z)\, dz \qquad (2.0.17)$$

$$= \int\limits_{-\ln 2}^{\infty} \frac{z\, e^{-z}}{2}\, dz \qquad (2.0.18)$$

$$= \left[\frac{-(z+1)\, e^{-z}}{2} \right]_{-\ln 2}^{\infty} \qquad (2.0.19)$$

$$= 1 - \ln(2) \qquad (2.0.20)$$

$$\approx 0.3068 \qquad (2.0.21)$$

From (2.0.6), we have as $n \to \infty$, $Z_n \to E(Z) = 0.3068 \approx 0.31$(Rounded to 2 decimal places).
Ans: 0.31