Assignment 5

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Download all python codes from

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https://github.com/Taha−Adeel/AI1103/tree/main/
    Assignment_5/Codes
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and latex-tikz codes from

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https://github.com/Taha−Adeel/AI1103/tree/main/
    Assignment_5
```

1 Problem (UGC/Math (mathA_Dec 2017) Q.119)

Arrival of customers in a shop is a Poisson process with intensity $\lambda = 2$. Let X be the number of customers entering during the time interval $(1, 2)$, and let Y be the number of customers entering during the time interval $(5, 10)$. Which of the following is true?

(A) X and Y are independent.

(B) $X + Y$ is a Poisson with parameter 6.

(C) $X - Y$ is a Poisson with parameter 8.

(D) $\Pr(X = 0 \mid X + Y = 12) = \left(\frac{5}{6}\right)^{12}$

2 Solution

X and Y are Poisson distributions with parameters $\mu_1 = \lambda\tau_1 = 2\times 1$ and $\mu_2 = \lambda\tau_2 = 2\times 5$ respectively(τ is time-interval). Hence the pmfs (probability mass functions) of random variables X and Y are given by:

$$p_X(x) = \frac{e^{-2}\cdot 2^x}{x!}, \qquad \text{for } x = 0, 1, 2, \ldots \tag{2.0.1}$$

$$p_Y(y) = \frac{e^{-10}\cdot 10^y}{y!}, \qquad \text{for } y = 0, 1, 2, \ldots \tag{2.0.2}$$

(A) In a Poisson process, the occurrences/frequencies of the event do not depend on past or future occurrences. Therefore X and Y are independent, by definition of Poisson process. Hence option (A) is **correct.**

(B)

Lemma 2.1. *If X and Y are two independent Poisson distributions with parameters μ_1 and μ_2 respectively, then the distribution of $X + Y$ is also Poisson with parameter $\mu_1 + \mu_2$.*

Proof. We have for $k \geq 0$, the probability mass function $p_{X+Y}(k)$ is a convolution of pmfs $p_X(x)$ and $p_Y(y)$:

$$p_{X+Y}(k) = \Pr(X + Y = k) = \Pr(Y = k - X) \tag{2.0.3}$$

$$= \sum_i \Pr(Y = k - i | X = i) \times p_X(i) \tag{2.0.4}$$

As X and Y are independent:

$$\Pr(Y = k - i \mid X = i) = \Pr(Y = k - i) = p_Y(k - i) \tag{2.0.5}$$

Simplifying (2.0.4)

$$p_{X+Y}(k) = \sum_{i=0}^{k} p_Y(k - i) \times p_X(i) \tag{2.0.6}$$

$$= \sum_{i=0}^{k} e^{-\mu_2}\frac{\mu_2^{k-i}}{(k-i)!} e^{-\mu_1}\frac{\mu_1^i}{i!} \tag{2.0.7}$$

$$= e^{-(\mu_1+\mu_2)}\frac{1}{k!}\sum_{i=0}^{k}\frac{k!}{i!(k-i)!}\mu_1^i\mu_2^{k-i} \tag{2.0.8}$$

$$= e^{-(\mu_1+\mu_2)}\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}\mu_1^i\mu_2^{k-i} \tag{2.0.9}$$

$$p_{X+Y}(k) = \frac{e^{-(\mu_1+\mu_2)}\cdot(\mu_1+\mu_2)^k}{k!} \tag{2.0.10}$$

□

As X and Y are independent Poisson distributions, using lemma (2.1), $X + Y$ is a Poisson with parameter $\mu_1 + \mu_2 = 12 \neq 6$. Hence option (B) is **incorrect.**

$$p_{X+Y}(k) = \frac{e^{-(12)}\cdot(12)^k}{k!} \tag{2.0.11}$$

(C) The distribution function for $X - Y$ no longer remains Poisson, as $X - Y$ will also attain negative values. Hence option (C) is **incorrect.**

(D)

$$\Pr(X = 0 \mid X + Y = 12) = \frac{\Pr(X = 0, Y = 12)}{\Pr(X + Y = 12)} \tag{2.0.12}$$

As X and Y are independent, and using (2.0.1), (2.0.2), and (2.0.11), we have:

$$\Pr(X=0 \mid X+Y=12) = \frac{\Pr(X=0) \cdot \Pr(Y=12)}{\Pr(X + Y = 12)} \tag{2.0.13}$$

$$= \frac{\dfrac{e^{-2} \times 2^0}{0!} \times \dfrac{e^{-10} \times 10^{12}}{12!}}{\dfrac{e^{-12} \times 12^{12}}{12!}} \tag{2.0.14}$$

$$= \left(\frac{5}{6}\right)^{12} \tag{2.0.15}$$

Hence option (D) is **correct.**

Ans: (A), (D)