

Markov Chains

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Prerequisites

State Space

The set of all possible states the system/object can be in. Without loss in generality, the state space can be identified with the set $S = \{1, 2, \dots, \ell\}$, where ℓ is a fixed arbitrary natural number.

Random variables

Let $\{X_0, X_1, X_2 \dots\}$ be a sequence of discrete random variables, where each $X_t \in S$, and X_t represents the state of the system at time t .

Definition

- A process is said to be a Markov Process if the probability of transitioning to any particular state is dependent solely on the current state and does not depend on how the current state was reached.
- Mathematically, $\{X_0, X_1, \dots\}$ is called a Markov chain if

$$\Pr((X_n = i_n \mid X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \Pr(X_n = i_n \mid X_{n-1} = i_{n-1}) \quad (1)$$

- This is known as the Markov property

Transition matrix

Transition matrix

- For each pair $i, j \in S$, consider the (conditional) probability $p_{ij} \in [0, 1]$ for the transition of the object or system from state i to j within one time step.
- The $\ell \times \ell$ matrix $P = (p_{ij})_{i,j=1,\dots,\ell}$ of the transition probabilities p_{ij} where

$$p_{ij} \geq 0, \quad p_{ij} = \Pr(X_{n+1} = j \mid X_n = i) \quad (2)$$

is called one-step *transition matrix* or simply transition matrix of the Markov chain.

Transition matrix(contd.)

Theorem

Let $\{X_0, X_1, X_2, \dots\}$ be a Markov chain with a $\ell \times \ell$ transition matrix P . Then the 2-step transition probabilities are given by the matrix P^2 . That is,

$$\Pr(X_{n+2} = j \mid X_n = i) = (P^2)_{ij} \quad (3)$$

Proof.

- Recall matrix multiplication. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $N \times N$ matrices. The product matrix is $A \times B = AB$, with elements

$$(AB)_{ij} = \sum_{k=0}^N a_{ik} b_{kj} \quad (4)$$



Proof(Contd.)

$$\Pr(X_2 = j | X_0 = i) = \sum_{k=0}^{\ell} \Pr(X_2 = j | X_1 = k, X_0 = i) \Pr(X_1 = k | X_0 = i) \quad (5)$$

$$= \sum_{k=0}^{\ell} \Pr(X_2 = j | X_1 = k) \Pr(X_1 = k | X_0 = i) \quad (6)$$

$$= \sum_{k=0}^{\ell} p_{kj} p_{ik} \quad (7)$$

$$= \sum_{k=0}^{\ell} p_{ik} p_{kj} = (P^2)_{ij} \quad (8)$$

$$\Pr(X_2 = j | X_0 = i) = \Pr(X_{n+2} = j | X_n = i) = (P^2)_{ij} \quad (9)$$

- Similarly it can be proved that the t -step transition matrix is equal to P^t .



Example question

Problem (GATE 2008 (CS), Q.27)

Aishwarya studies either computer science or mathematics everyday. If she studies computer science on a day, then the probability she studies mathematics the next day is 0.6. If she studies mathematics on a day, then the probability she studies computer science the next day is 0.4. Given that Aishwarya studies computer science on Monday, what is the probability she studies computer science on Wednesday?

- 1 0.24
- 2 0.36
- 3 0.4
- 4 0.6

Solution

State space

Consider the state space $S = \{1, 2\}$, where 1 represents Aishwarya studying CS and 2 represents her studying maths on a particular day.

Markov chain

Let $\{X_0, X_1, \dots\}$ be a series of random variables (which form a Markov chain) where $X_i \in S$ represents her studying CS or maths on the i th day ($i=0$ for Monday)

Solution

Transition matrix

The one step state transition matrix $P = (p_{ij})$ for the markov chain (where $p_{ij} = \Pr(X_{n+1} = j | X_n = i)$) is :-

$$P = \begin{matrix} & \overbrace{\begin{matrix} 1 & 2 \end{matrix}}^{X_{n+1}} \\ \begin{matrix} 1 \\ 2 \end{matrix}^{X_n} & \left\{ \begin{matrix} 1 & \begin{bmatrix} x & 0.6 \end{bmatrix} \\ 2 & \begin{bmatrix} 0.4 & y \end{bmatrix} \end{matrix} \right\} = \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix} \end{matrix} \quad (10)$$

As $X_n = 0$ and $X_n = 1$ are mutually exclusive, we can easily calculate x and y .

$$\begin{aligned} x &= \Pr(X_{n+1} = 0 | X_n = 0) = 1 - \Pr(X_{n+1} = 1 | X_n = 0) \\ &= 0.4 \end{aligned} \quad (11)$$

$$\begin{aligned} y &= \Pr(X_{n+1} = 1 | X_n = 1) = 1 - \Pr(X_{n+1} = 0 | X_n = 1) \\ &= 0.6 \end{aligned} \quad (12)$$

Solution

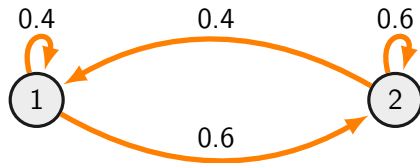


Figure: Markov Diagram

Solution

- Given that her initial state is $X_0 = 1$ (\because she studies CS on Monday($n=0$)).
- The $\Pr(X_{n+2} = j | X_n = i)$ is given by the (i, j) th position of P^2 .
- Therefore $\Pr(X_2 = 1 | X_0 = 1)$ ($\because n=2$ for Wednesday) is the $(1, 1)$ th position of P^2 .

$$P^2 = \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix} \times \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix} \quad (13)$$

- \therefore The probability she studies computer science on Wednesday is $(P^2)_{11} = 0.4$.
(Ans: Option (3))