Markov Chains

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Prerequisites

State Space

The set of all possible states the system/object can be in. Without loss in generality, the state space can be identified with the set $S = \{1, 2, \ldots, \ell\}$, where ℓ is a fixed arbitrary natural number.

Random variables

Let $\{X_0, X_1, X_2 \ldots\}$ be a sequence of discrete random variables, where each $X_t \in S$, and X_t represents the state of the system at time t.

Definition

- A process is said to be a Markov Process if the probability of transitioning to any particular state is dependent solely on the current state and does not depend on how the current state was reached.
- Mathematically, $\{X_0, X_1, \dots\}$ is called a Markov chain if

$$\Pr\left((X_n = i_n \mid X_{n-1} = i_{n-1}, \dots, X_0 = i_0\right) = \Pr\left(X_n = i_n \mid X_{n-1} = i_{n-1}\right) \tag{1}$$

- This is known as the Markov property

Transition matrix

Transition matrix

- For each pair $i, j \in S$, consider the (conditional) probability $p_{ij} \in [0, 1]$ for the transition of the object or system from state i to j within one time step.
- The $\ell \times \ell$ matrix $\mathrm{P} = (p_{ij})_{i,j=1,\ldots,\ell}$ of the transition probabilities p_{ij} where

$$p_{ij} \geq 0\,, \qquad p_{ij} = \Pr\left(X_{n+1} = j \mid X_n = i\right) \tag{2}$$

is called one-step *transition matrix* or simply transition matrix of the Markov chain.

Transition matrix(contd.)

Theorem

Let $\{X_0, X_1, X_2, ...\}$ *be a Markov chain with a* $\ell \times \ell$ *transition matrix* P*. Then the 2-step transition probabilities are given by the matrix* P^2*. That is,*

$$\Pr(X_{n+2} = j \mid X_n = i) = (P^2)_{ij} \tag{3}$$

Proof.

- Recall matrix multiplication. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $N \times N$ matrices.The product matrix is $A \times B = AB$, with elements

$$(AB)_{ij} = \sum_{k=0}^{N} a_{ik} b_{kj} \tag{4}$$

□

Proof(Contd.)

$$\Pr(X_2 = j | X_0 = i) = \sum_{k=0}^{\ell} \Pr(X_2 = j | X_1 = k, X_0 = i) \Pr(X_1 = k | X_0 = i) \tag{5}$$

$$= \sum_{k=0}^{\ell} \Pr(X_2 = j | X_1 = k) \Pr(X_1 = k | X_0 = i) \tag{6}$$

$$= \sum_{k=0}^{\ell} p_{kj} p_{ik} \tag{7}$$

$$= \sum_{k=0}^{\ell} p_{ik} p_{kj} = (P^2)_{ij} \tag{8}$$

$$\Pr(X_2 = j | X_0 = i) = \Pr(X_{n+2} = j | X_n = i) = (P^2)_{ij} \tag{9}$$

- Similarly it can be proved that the t-step transition matrix is equal to P^t.

□

Example question

Problem (GATE 2008 (CS), Q.27)

Aishwarya studies either computer science or mathematics everyday. If she studies computer science on a day, then the probability she studies mathematics the next day is 0.6. If she studies mathematics on a day, then the probability she studies computer science the next day is 0.4. Given that Aishwarya studies computer science on Monday, what is the probablity she studies computer science on Wednesday?

1. 0.24
2. 0.36
3. 0.4
4. 0.6

Solution

State space

Consider the state space $S = \{1, 2\}$, where 1 represents Aishwarya studying CS and 2 represents her studying maths on a particular day.

Markov chain

Let $\{X_0, X_1, \dots\}$ be a series of random variables(which form a Markov chain) where $X_i \in S$ represents her studying CS or maths on the ith day(i=0 for Monday)

Solution

Transition matrix

The one step state transition matrix $P = (p_{ij})$ for the markov chain (where $p_{ij} = \Pr(X_{n+1} = j \mid X_n = i)$) is :-

$$P = \begin{array}{cc} & \overbrace{\begin{array}{cc} 1 & \quad 2 \end{array}}^{X_{n+1}} \\ X_n \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right. & \begin{bmatrix} x & 0.6 \\ 0.4 & y \end{bmatrix} \end{array} = \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix} \tag{10}$$

As $X_n = 1$ and $X_n = 2$ are mutually exclusive, we can easily calculate x and y.

$$x = \Pr(X_{n+1} = 1 | X_n = 2) = 1 - \Pr(X_{n+1} = 2 | X_n = 1)$$
$$= 0.4 \tag{11}$$
$$y = \Pr(X_{n+1} = 2 | X_n = 2) = 1 - \Pr(X_{n+1} = 1 | X_n = 2)$$
$$= 0.6 \tag{12}$$

Solution

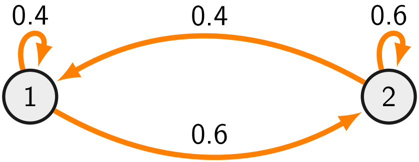

Figure: Markov Diagram

Solution

- Given that her initial state is $X_0 = 1$ ($\because$ she studies CS on Monday(n=0)).
- The $\Pr(X_{n+2} = j \mid X_n = i)$ is given by the (i, j)th position of P^2.
- Therefore $\Pr(X_2 = 1|X_0 = 1)$ ($\because$ n=2 for Wednesday) is the $(1, 1)$th position of P^2.

$$P^2 = \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix} \times \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix} \tag{13}$$

- $\therefore$ The probability she studies computer science on Wednesday is $(P^2)_{1\,1} = 0.4$.
  (**Ans: Option (3)**)