Topics in Computing (CS5160) : Problem Set 1

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- Scan and upload your answer sheet on google classroom.
- Maintain academic honesty. If caught, you will get an F in the course.
- Please write "credit" or "audit" on your answer sheets depending on whether you are crediting or auditing the course.
- Due Date: 20 September (before 11:59pm).

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1. Let $f\colon \{0,1\}^n \to \{0,1\}$ be a Boolean function, and k be the largest natural number such that $|f^{-1}(1)|$ is divisible by 2^k. Show that $\mathsf{D}(f) \geq n-k$, where $\mathsf{D}(f)$ is the decision tree complexity of f. **(10 points)**

2. For every k, define f_k to be the following function taking inputs of length $n = 2^k$:

$$f_k(x_1,\ldots,x_n) = \begin{cases} f_{k-1}(x_1,\ldots,x_{2^{k-1}}) \wedge f_{k-1}(x_{2^{k-1}+1},\ldots,x_{2^k}) & \text{if } k>0 \text{ is even} \\ f_{k-1}(x_1,\ldots,x_{2^{k-1}}) \vee f_{k-1}(x_{2^{k-1}+1},\ldots,x_{2^k}) & \text{if } k \text{ is odd} \\ x_i & \text{if } k=0 \end{cases}$$

   Show that $\mathsf{D}(f_k) = 2^k$. **(10 points)**

3. Show that for a non-constant symmetric Boolean function $f\colon \{0,1\}^n \to \{0,1\}$,

$$\mathsf{s}(f) \geq \left\lceil \frac{n+1}{2} \right\rceil,$$

   where $\mathsf{s}(f)$ is the sensitivity of f. Also, give an example where this bound is tight. Recall, we say a function is *symmetric* if its value depends only on the number of 1s in the input. That is, it depends on the hamming weight of the input. **(10 points)**

4. Show that $\mathsf{s}(f) = \mathsf{bs}(f) = \mathsf{C}(f)$ for every monotone Boolean function f, where $\mathsf{bs}(f)$ and $\mathsf{C}(f)$ respectively denote the block sensitivity and certificate complexity of f. Recall we say a function is monotone if for any x, y such that $x \leq y$, that is, $x_i \leq y_i$ for all $i \in [n]$, $f(x) \leq f(y)$. **(10 points)**

5. Prove that when f is monotone, $\mathsf{s}(f) \leq \mathsf{deg}(f)$. **(10 points)**

6. Define *average sensitivity*, $\mathsf{as}(f)$, of a Boolean function f to be the expected sensitivity of f on a random input: $\mathsf{as}(f) = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} \mathsf{s}(f, x)$.
   Let T be a decision tree and let ℓ_x be the length of the unique path in T consistent with x. Define *average depth* of T to be $\frac{1}{2^n} \sum_{x \in \{0,1\}^n} \ell_x$.
   Show that the average depth of any decision tree for f is at least $\mathsf{as}(f)$. **(10 points)**