Topics in Combinatorics

Exam II (out of 30 marks)
(Date: 30 March 2022. Timing: 11:00 to 13:00 hours (Duration: 2 hours))

Instructions

- Questions have *page limits* mentioned in square brackets against them. You are expected to complete your answer within the allowed number of pages. The answer paper you submit should be neat and clean. You may use separate paper(s) for doing rough work which you need not submit.

Problems

1. *Kattappa* and *Baahubali* play a game on board with positions labelled $\{0, 1, \ldots, k\}$. Initially, n stones are at position k. *Kattappa* and *Baahubali* play for r rounds, where each round has the following structure: *Kattappa* names a subset S of the stones on the board, and then, *Baahubali* either moves all stones in S one position to the left, or moves all stones in S^C (the *complement* of S) one position to the left. Any stone that is moved leftward from 0 is discarded. If the number of stones on the board becomes 1 or 0, *Baahubali* loses. Prove that if $n \cdot \sum_{i=0}^{k} \binom{r}{i} 2^{-r} > 1$, then *Baahubali* has a winning strategy. **15 marks** [2 pages]

2. **Definition**: Given a graph G, a collection of permutations (or total orders) of the vertices of G is said to be 3-*mixing* if for every pair of edges, say ab, bc, in G that share a vertex (b in this case), there is a permutation in the collection in which the shared vertex b appears between the other two vertices. For example, if the graph G is a 6-cycle on vertices $\{a, b, c, d, e, f\}$ having edges ab, bc, cd, de, ef and fa, then $\mathcal{C} = \{\sigma_1, \sigma_2\}$ is a 3-mixing family of permutations of the vertices of G where $\sigma_1 : f, e, d, c, b, a$ and $\sigma_2 : e, c, f, d, a, b$. The pairs $(ab, bc), (bc, cd), (cd, de), (de, ef)$ are resolved in σ_1 and the remaining pairs that share a vertex, namely $(ef, fa), (fa, ab)$ are resolved in σ_2.

   Let $\beta(G)$ denote the cardinality of a smallest family of permutations of $V(G)$ that is 3-mixing.

   Problem Let G be a graph where every vertex has at most Δ neighbours (that is, the degree of every vertex is at most Δ). Show that, $\beta(G) \in O(\log \Delta)$. **15 marks** [2 pages]