P_3 double domination and matching cut

1 Introduction

Our goal is to design a $O(2^{2n/5})$ algorithm for matching cut. If G has a leaf vertex or two adjacent degree-two vertices, then G has a matching cut. Thus, we may assume that our input graph G does not have a leaf vertex or two adjacent degree-two vertices. One implication of this assumption is that $\gamma(G) \leq 3n/8$.

Strategies:

1. Let S be a set of vertices such that every P_3 of $G[V \setminus S]$ has a vertex with at least two neighbors in S. Then we have a $O(2^{|S|})$ algorithm to check if G has a matching cut; this is by a reduction to 2-SAT.

2. An approach is to consider the number k of edges in the matching cut. At one end, we have an algorithm for perfect matching (matching cut with $n/2$ matching edges) that runs in $O(2^{n/3})$ time. Thus, testing for a matching cut with $n/2 - k$ edges can be done in time $O\left(\binom{n}{2k} 2^{(n-2k)/3}\right)$ time. For $2k = \alpha n$ with $\alpha \leq 1/2$, this gives us a bound of $2^{n\beta}$ where $\beta = H(\alpha) + (1-\alpha)/3$, which doesn't seem to help except for tiny values of α, beating $2n/5$ at $\alpha \sim 1/11$.

3. Local search: The current best local search algorithm for 3-SAT runs in time $O(2.792^k)$; we have an independent $O(2.83^k)$ algorithm for matching cut, which it may be possible to improve. An improvement to 2.5^k would be interesting, although still not enough to beat the exact algorithm bound.

4. Improving the analyzsis (/algorithm) of PPSZ for matching cut. This needs an understanding of the details of PPSZ's correctness for general 3-SAT.

Another useful idea, which may not help in sparse graphs, is the following. Let $X = A \cup B$ such that the vertices of B can be ordered as $w_1, w_2, \ldots, w_k$, and every w_i has at least 3 neighbors in $A \cup \{w_1, \ldots, w_{i-1}\}$. Then the matching cut coloring for B is uniquely determined, once A is colored.

2 Bound on P_3 2-dominating sets

For a graph $G = (V, E)$, let $p(G)$ denote the size of the smallest subset S such that every P_3 in $G[V \setminus S]$ has a vertex with at least two neighbors in S.

Our goal is to prove the following proposition.

Proposition 1 *Let G be a n-vertex graph without any leaf vertex and with no two adjacent vertices of degree two. Then $p(G) \leq 2n/5$.*

The above proposition is tight for K_5, $K_{2,3}$, and the Petersen graph.

Observation 2 *Let T be a maximal induced forest; then every vertex outside T has at least two neighbors in T.*

Lemma 3 *Let $G = (V, E)$ be a graph such that none of its connected components is an odd cycle or an isolated vertex. Then in polynomial time, we can partition V into two sets $V_0 \cup V_1$ with $V_i = C_i \cup I_i$ for $i \in \{0, 1\}$ such tha the following hold for $i \in \{0, 1\}$.*

(i) Every vertex in C_i has at least two neighbors in V_{1-i};

(ii) I_i is an independent set;

(iii) If $v \in I_i$, then v has exactly one neighbor in V_{1-i}.

Corollary 4 *If G is a connected n-vertex graph and not an odd cycle, then there is a set T of vertices such that $|T| \leq n/2$ and for every edge uv in $G[V \setminus T]$, one of u, v has at least two neighbors in T.*

Proof Consider a partition of G as in Lemma 3, and let T be the smaller of V_0, V_1. ∎

To prove the proposition, we start with a good partition. We will try to find sets $S_0 \subseteq V_0$ and $S_1 \subseteq V_1$ such that the following holds. If $x \in C_i$ has at least two neighbors in I_i, then x has at least two neighbors in S_{1-i}. We know that if u, v are adjacent vertices in C_i, then u or v has at least two neighbors in S_{1-i}. What we want is that $|S_i| \leq 4/5|V_i|$. It is also sufficient that $|S_0| + |S_1| \leq 4n/5$.

Q: Can we obtain a small fraction of X as a P_3-double dominating set?

Let Y be a maximum size subset of $V \setminus X$ that induces a subgraph of minimum degree at least 3, and let $Z = V \setminus (X \cup Y)$. Let Y_1 be a 2-dominating set of $G[Y]$ of size at most $|Y|/2$. Then $Y_1 \cup U$ is a P_3-double-dominating set of size at most $|Y|/2 + |Z| = n - |Y|/2 - |X|$.

Other strategies:

1. Consider a maximum $P - 5$ packing and let S be the set of second and fourth vertices. Does this work? Does picking the middle vertex of the P_5 give a P_3-dominating set?

2. Prove the proposition for graphs of minimum degree at least 3 (or 4 or higher).

3. Prove the proposition for: 2-degenerate graphs; graphs of maximum degree 3; planar graphs; graphs without 4-cycles.

4. Consider a maximal collection of disjoint closed neighborhoods: $N[v_1], \ldots, N[v_k]$.

5. Let R be a maximal set of vertices such that every pair of them are at distance more than two. Three may also be useful.

3 Useful graph theory facts

The following facts should be useful.

1. Bounds on $\gamma(G)$: if $\delta(G) \geq 2$, then $\gamma(G) \leq 2n/5$, if $\delta(G) \geq 3$, then $\gamma(G) \leq 2n/5$. Further, if k denotes the number of vertices of degree two, then $\gamma(G) \leq (3n + k)/8$.

2. If $d(u)+d(v) \geq 5$ for every pair of adjacent vertices, then $\gamma(G) \leq 3n/8$ (Henning, Schiermeyer, Yeo). Also, if G is C_4, C_5-free, then $\gamma(G) \leq 3n/8$;

3. Any connected graph of order $n \geq 6$ has a vertex-edge dominating set of size at most $n/3$. Further, this is obtained as the set of centers of a maximum P_3 matching. However, it is not clear how to efficiently find such a set.

4. A 2-degenerate graph on n vertices has a feedback vertex set of size at most $2n/5$. [Proved by Boroweicki et al]

5. If G is a forest on n vertices, then G contains a subset S on at least $2n/3$ vertices that induces a subgraph of maximum degree at most one.

6. Let $G = (V, E)$ be a graph of degeneracy d. Then V can be partitioned as $V_1 \cup V_2$ such that $G[V_1]$ is $\lfloor d/2 \rfloor$-degenerate and $G[V_2]$ is $\lceil d/2 \rceil$-degenerate.

7. If G is connected and d-regular, then deleting any one vertex of G results in a graph which is $(d-1)$-degenerate.

8. $\gamma_2(G) \leq 0.418n$ if $\delta \geq 9$.

Facts about cycles:

- A connected graph G has no even cycles if and only if every block of G is either an odd cycle or K_2.
- A graph of minimum degree at least 3 must contain a cycle of length divisible by 3.
- A graph of minimum degree at least 3 must contain a cycle of length divisible by 4.
- A graph of minimum degree at least $2k-1$ must contain a cycle of length divisible by k.
- $\chi(G) > k$ implies G has a cycle of length divisible by k.
- If $\delta(G) \geq k+1$, then G has a cycle of length 2 modulo k. If G contains neither K_k nor $K_{k,k}$ as an induced subgraph, then $\delta \geq k$ suffices. If G is 2-connected and is not a complete or complete bipartite graph, then also $\delta \geq k$ suffices.