02/11/23

Noise-Operators

$$f : \{\pm 1\}^n \to \{\pm 1\} / \mathbb{R}$$

Let $0 \leq \rho \leq 1$.

Define. $T_\rho f : \{\pm 1\}^n \to \mathbb{R}$

$$T_\rho f(x) := \mathop{\mathbb{E}}_{y \sim N_\rho(x)} [f(y)]$$

$y \sim N_\rho(x)$ as follows:

$$y_i = \begin{cases} x_i & \text{with prob } \frac{1+\rho}{2} \\ -x_i & \text{with prob } \frac{1-\rho}{2} \end{cases}$$

Fact 1 :- T_ρ is a linear operator.

$$T_\rho(cf+g)(x) = c \cdot T_\rho f + T_\rho g$$

$$\hookrightarrow \mathbb{E}_{y \sim N_\rho(x)}\left[c \cdot f(y) + g(y)\right]$$

$$= c \cdot \mathbb{E}_{y \sim N_\rho(x)}\left[f(y)\right] + \mathbb{E}_{y \sim N_\rho(x)}\left[g(y)\right]$$

$$= c \cdot T_\rho f(x) + T_\rho g(x)$$

Prop :- $T_\rho \chi_S : \{\pm 1\}^n \to \mathbb{R}$

when $\chi_S : \{\pm 1\}^n \to \{\pm 1\}$

$$\chi_S(x) = \prod_{i \in S} x_i$$

$$T_\rho \chi_S(x) = \mathbb{E}_{y \sim N_\rho(x)}\left[\chi_S(y)\right]$$

when $S=\emptyset$ $\quad \mathbb{E}_{y\sim N_\rho}[1] = 1$

$$= \mathbb{E}_{y \sim N_\rho(x)}\left[\prod_{i \in S} y_i\right]$$

$$= \prod_{i \in S} \mathbb{E}_{y_i \sim N_\rho(x_i)}\left[y_i\right]$$

$$= \prod_{i \in S}\left[\left(\frac{1+\rho}{2}\right)\cdot x_i + \left(\frac{1-\rho}{2}\right)(-x_i)\right]$$

$$= \prod_{i \in S} \rho\, x_i = \rho^{|S|}\cdot \chi_S(x)$$

$$T_\rho f(x) = \sum_{S \subseteq [n]} \hat{f}(S) \cdot \rho^{|S|} \cdot \chi_S(x)$$

$$T_\rho f(x) = \mathbb{E}_{y \sim N_\rho(x)} [f(y)]$$

$$T_1 f(x) = f(x)$$

$$T_0 f(x) = \cancel{f(-x)}$$

$$= \hat{f}(\emptyset) = \mathbb{E}_{y \sim N_0(x)} [f(y)]$$

p-norm:- $f : \{\pm 1\}^n \to \mathbb{R}$

$$\|f\|_p = \left(\mathbb{E}_x [|f(x)|^p] \right)^{1/p}$$

$$\|T_\rho f\|_p \le \|f\|_p$$

(Bonami - Beckner)

$\Rightarrow$ Hypercontractivity theorem.

$$\|T_\rho f\|_q \le \|f\|_p$$

when $q \ge p \ge 1$.

and $0 \le \rho \le \sqrt{\frac{p-1}{q-1}}$

Thm :- let $f : \{\pm 1\}^n \to \mathbb{R}$

Then $\|T_{1/2} f\|_2 \le \|f\|_{4/3}$

$\blacksquare$

$$f : \{\pm 1\}^n \to \{\pm 1\}$$

$$D_i f(x) = \frac{f(x_1, \dots, 1, x_{i+1}, \dots, x_n) - f(x_1, \dots, -1, \dots, x_n)}{2}$$

$$= \sum_{S : i \in S} \hat{f}(S) \cdot \prod_{j \in S \setminus \{i\}} x_j$$

$$D_i f : \{\pm 1\}^n \to \{1, 0, -1\}$$

Applying Hypercontractivity to $D_i f$.

$$\|T_{1/2} D_i f\|_2 \le \|D_i f\|_{4/3}$$

$$\|$$

$$\|T_{1/2} D_i f\|_2^2 \le \|D_i f\|_{4/3}^2$$

$$= \mathbb{E}_x \left[|T_{1/2} D_i f(x)|^2 \right]$$

$$= \mathbb{E}_{x \sim \{\pm 1\}^n} \left[\left(T_{1/2} D_i f(x) \right)^2 \right]$$

$$= \sum_{S \subseteq [n]} \left(\widehat{T_{1/2} D_i f}(S) \right)^2$$

$$T_{1/2} D_i f(x) = \sum_{S : i \in S} \left(\frac{1}{2}\right)^{|S|-1} \cdot \hat{f}(S) \prod_{j \in S \setminus \{i\}} x_j$$

$$= \sum_{S : i \in S} \left(\frac{1}{4}\right)^{|S|-1} \cdot \hat{f}(S)^2$$

$$\shortparallel$$

$$\frac{1}{4} \cdot \mathrm{Inf}_i(f)$$

$$\|D_i f\|_{4/3}^2 = \left(\left(\mathbb{E}_x\left[|D_i f(x)|^{4/3}\right]\right)^{3/4}\right)^2$$

$$= \left(\mathbb{E}_x\left[|D_i f(x)|^{4/3}\right]\right)^{3/2}$$

$$= \mathbb{E}_x\left[\,|D_i f(x)|\,\right]^{3/2}$$

$$= \boxed{\mathbb{E}_x\left[(D_i f(x))^2 \right]}^{3/2}$$

$$= \left(\mathrm{Inf}_i(f) \right)^{3/2}$$

$$\frac{1}{4} \cdot \mathrm{Inf}_i(f) \le \mathrm{Inf}_i(f)^{3/2}$$

$$\overset{!!}{}$$

$$\sum_{S : i \in S} \left(\frac{1}{4}\right)^{|S|-1} \cdot \hat{f}(S)^2 \le \left(\mathrm{Inf}_i(f) \right)^{3/2}$$

$$\|$$

$$\| T_{1/2} D_i f \|_2^2$$

Lemma:- Let $f : \{\pm 1\}^n \to \{\pm 1\}$.

Set $\varepsilon > 0$, and $k := \dfrac{2 \cdot \mathrm{Inf}(f)}{\varepsilon}$.

Further define

$$\underline{J} = \left\{ i \in [n] \;\middle|\; \mathrm{Inf}_i(f) \geq 100^{-k} \right\}$$

$$\mathcal{L} = \left\{ S \subseteq [n] \;\middle|\; S \subseteq J \text{ and } |S| \leq k \right\}$$

Then
$$\sum_{S \notin \mathcal{L}} \hat{f}(S)^2 \leq \varepsilon$$

$$\frac{1}{100^k} \cdot |J| \leq \mathrm{Inf}(f)$$

$$\Rightarrow |J| \leq \mathrm{Inf}(f) \cdot 100^{O(\mathrm{Inf}(f))}$$

$\leq \mathrm{Inf}(f) \cdot 2^{O(\mathrm{Inf}(f))}$

KKL:- $\exists\ i$ s.t.

$$\mathrm{Inf}_i(f) \geq \Omega\left(\frac{\log n}{n}\right)$$

Assume $E[f] = 0$ ~~when $0 \leq \delta < 1$~~

Proof:-

Case 1:-

$$\mathrm{Inf}(f) \geq \frac{\log n}{1000}.$$

Then,
$$\sum_{i=1}^{n} \mathrm{Inf}_i(f) = \mathrm{Inf}(f) \geq \frac{\log n}{1000}$$

$$\max_i \{\mathrm{Inf}_i(f)\} \geq \frac{\log n}{1000 \cdot n}$$

Case 2 :- $\mathrm{Inf}(f) \le \frac{\log n}{1000}$

Choose $\varepsilon = \frac{1}{10}$

$$\sum_{i \in J} \mathrm{Inf}_i(f)$$

where J as is previous lemma

$$= \sum_{i \in J} \sum_{S : i \in S} \hat{f}(S)^2$$

$$\ge \sum_{\emptyset \neq S : S \subseteq J} |S| \, \hat{f}(S)^2$$

$$\ge \sum_{\emptyset \neq S : S \subseteq J} \hat{f}(S)^2$$

$$\ge 1 - \varepsilon \ge \frac{9}{10}$$

$$\Rightarrow \underline{\sum_{i \in J} \mathrm{Inf}_i(f) \geq \frac{9}{10}}$$

$$|J| \leq \mathrm{Inf}(f) \cdot 2^{O(\mathrm{Inf}(f))}$$

$$\leq \frac{\log n}{1000} \cdot 2^{O\left(\frac{\log n}{1000}\right)}$$

$$\leq \frac{\log n}{1000} \cdot 2^{\beta \log n}$$

where $\beta < 1$.

$$\leq \frac{n^{\beta} \cdot \log n}{1000}$$

$$\sum_{i \in J} \mathrm{Inf}_i(f) \geq \frac{9}{10}$$

$$\max_{i \in J} \mathrm{Inf}_i(f) \cdot |J| \geq \frac{9}{10}$$

$$\max_{i \in J} \mathrm{Inf}_i(f) \geq \frac{9}{10 \cdot n^{\beta} \cdot \log n}$$

where $\beta < 1$.

$$\geq \frac{\log n}{n}$$