14/09/2023

Defn:- A monomial in a multilinear polynomial is called a "maxonomial" if it has the highest degree.

Fix a boolean function f.

Q:- Consider a certificate S and a maxonomial M.

Can $S \cap M \neq \emptyset$?

YES!

Thm:- $D(f) \le C^1(f) \cdot \deg(f)$

$\le C^0(f) \cdot \deg(f)$

Proof:-

1. Pick a consistent 1-certificate.

2. Query all variables in it. If it is satisfied answer 1 else prune and repeat.

Claim:- After querying

certificate, the degree of the polynomials decreases by at least 1

$\Rightarrow \quad D(f) \leq C'(f) \cdot \deg(f)$

$\deg(f) \geq \Omega\left(\sqrt{bs(f)}\right)$

$\Rightarrow \quad D(f) \leq \deg(f)$

$\Rightarrow \quad C(f) \leq bs(f)^2$

$\Rightarrow \quad D(f) \leq C'(f) \cdot \deg(f) \leq bs(f)^2 \cdot \deg(f$

$\leq \deg(f)^5.$

<u>Lemma</u>:- Let M be a maxonomial of a Boolean function f.

Then $\exists$ a block $B \subseteq M$ s.t. $f(0) \neq f(0^B)$

<u>Proof</u>:-

Let $p(x_1, \ldots, x_n)$ be the polynomial representing f.

Let q be obtained from p by setting every variable in $[n] \setminus M$ to 0.

$\Rightarrow$ q is a non-zero polynomial.

$$f(0, *, 0, *, \ldots) = q$$

$$q(0^{|M|}) = f(0)$$

$\Rightarrow$ $\exists$ two inputs say a and $b \in \{0,1\}^{|M|}$ s.t. $q(a) \neq q(b)$

$$f(0) = q(a) \quad \underline{\text{WLOG}}$$

$$f(0) \neq q(b)$$

$\blacksquare$

<u>Lemma</u>: $\exists$ a set of $\deg(f) \cdot bs(f)$ many variables

that intersects every maxonomial in the polynomial representing f.

set of Maxonomials.

Proof:- $\{ \quad , \quad \}$

Set of max nomoials

$$\mathcal{T} = \{M_1, \quad , \quad - , \}$$

→ Pick M_1, remove everything from $\mathcal{T}$ that intersects with M_1.

$\rightarrow$ Repeat until $\mathcal{T} = \emptyset$.

<u>Observe</u> monomials picked in this way are disjoint from each other.

$\therefore$ if we pick t monomials in the process then we have found t

disjoint sensitive blocks at 0.

$$\Rightarrow \; bs(f,0) \geq t.$$

But, $bs(f,0) \leq bs(f)$

<u>Thm</u>:- $D(f) \leq \deg(f) \cdot bs(f) \cdot \deg(f)$

$$\leq \deg(f)^4$$

<u>Proof</u>:- From previous lemma we obtain a set of $\deg(f) \cdot bs(f)$

many vars that intersect each maxnomial.

→ query every variable in this set.

→ repeat if the polynomial is not constant.

We know $D(f) \leq \deg^3(f)$

We also know $\exists f$:

$$D(f) \geq \deg^2(f).$$

$\boxed{\text{OPEN: bridge the gap.}}$

→ **Defn:-**

Size of a decision tree.

(1) # nodes in the tree

(2) # leaves

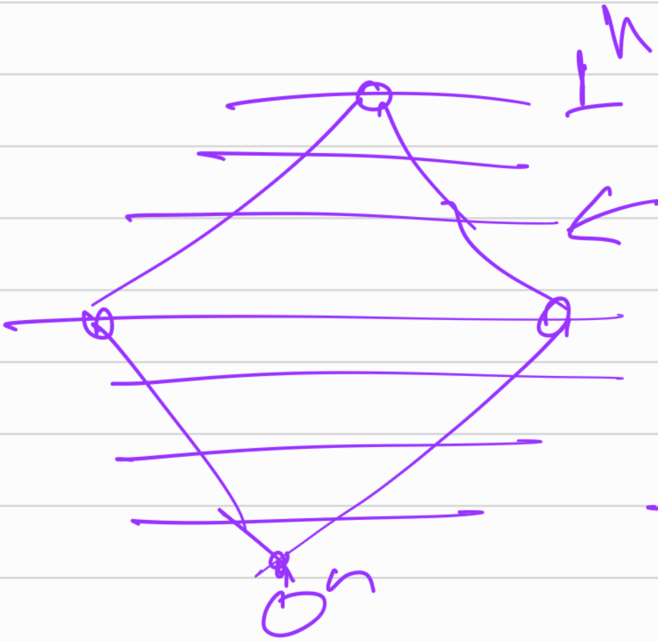
(3) # root-to-leaf paths.

$$(2) = (3) \leq (1) \leq 2 \cdot (2) - 1$$

$L(f) :=$ min size of a decision tree computing f.

e.g.: $L(AND_n) = n+1$

$L(MAJ_n) =$



constant on every level.

Symmetric function.

$$L(MAS_n) \geq 2^{n/2}$$

$n+1$

$\frac{n}{2}+1$