MARTINGALES

Conditional Expectation

Given two R.V.s Y and Z

Definition: $E[Y \mid Z=z] = \sum_y y \Pr(Y=y \mid Z=z)$,

where the summation is taken over all y in the range of Y.

Example: We independently roll two standard 6-sided dice. Let X_1 be the num that shows on the 1st die, X_2 be the num that shows on the 2nd die. Let X be the sum of the numbers on the two dice.

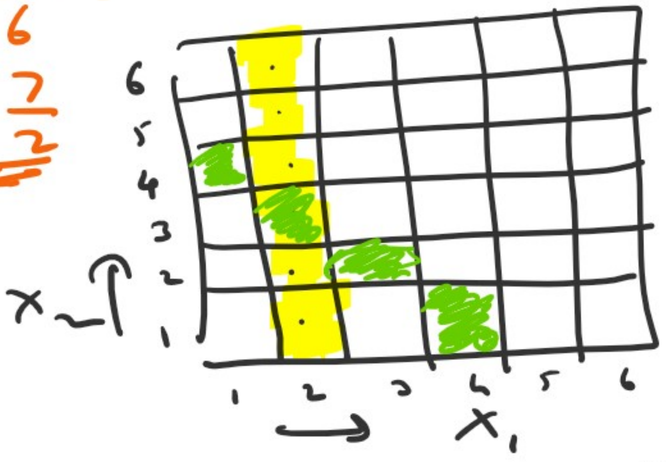
$$\underbrace{E[X \mid X_1]}_{f(X_1)} = \sum_x x \Pr(X=x \mid X_1) = \sum_{x=X_1+1}^{X_1+6} x \cdot \frac{1}{6}$$

$$= \frac{6X_1+21}{6}$$

$$E[X \mid X_1=2] = \sum_x x \Pr(X=x \mid X_1=2)$$

$$= 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6}$$

$$\frac{\;\cdot\;}{6} = X_1 + \frac{7}{2}$$



$$= 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6} + 7\cdot\frac{1}{6} + 8\cdot\frac{1}{6}$$

$$= \frac{33}{6}$$

$$= \frac{11}{2}$$

$$E[X_1 \mid X = 5] = \sum_{x_1=1}^{4} x_1 \Pr[X_1 = x_1 \mid X = 5]$$

$$= \frac{1\cdot 1}{4} + \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 3 + \frac{1}{4}\cdot 4$$

$$= \frac{4\cdot 5}{2\cdot 4}$$

$$= \frac{5}{2}$$

$$E[E[X \mid X_1]]$$
$$= E\left[X_1 + \frac{7}{2}\right]$$
$$= E[X_1] + \frac{7}{2}$$
$$= \frac{7}{2} + \frac{7}{2}$$
$$= 7$$
$$= E[X]$$

Proposition 1 (Linearity of conditional expectation)

Let X, Y, Z be the 3 discrete R.V.s with finite expectations. Then,

$$E[X+Y \mid Z=z] = E[X \mid Z=z] + E[Y \mid Z=z]$$

?.

$$\ldots [Y \mid Z = z]$$

Proof:

$$E[X+Y \mid Z=z] = \sum_i \sum_j (i+j) \Pr(X=i, Y=j \mid Z=z)$$

$$= \sum_i \sum_j i \Pr(X=i, Y=j \mid Z=z) + \sum_i \sum_j j \cdot \Pr(X=i, Y=j \mid Z=z)$$

$$= \sum_i i \sum_j \Pr(X=i, Y=j \mid Z=z) + \sum_j j \sum_i \Pr(X=i, Y=j \mid Z=z)$$

$$= \sum_i i \Pr(X=i \mid Z=z) + \sum_j j \Pr(Y=j \mid Z=z)$$

$$= E[X \mid Z=z] + E[Y \mid Z=z].$$

$\square$

can be generalized to

$$E\left[\sum_{i=1}^{n} X_i \,\Big|\, Z=z\right] = \sum_{i=1}^{n} E[X_i \mid Z=z]$$

$$ = \sum_{i=1} E[X_i \mid Z = z] $$

Proposition 2: For any two R.V.s X and Y

$$ E[X] = \sum_y P_r(Y=y)\, E[X \mid Y=y] $$

Proof:

$$
\begin{aligned}
\sum_y P_r(Y=y)\, E[X \mid Y=y] &= \sum_y P_r(Y=y) \sum_x x P_r(X=x \mid Y=y) \\
&= \sum_x \sum_y x\, \underbrace{P_r(X=x \mid Y=y)\, P_r(Y=y)} \\
&= \sum_x x \sum_y P_r(X=x, Y=y) \\
&= \sum_x x\, P_r(X=x) \\
&= \underline{E[X]}
\end{aligned}
$$

$\square$

Definition: The expression $E[Y \mid Z]$ is

not a real value.

a random variable $f(Z)$ that takes values $E[Y|Z=z]$ when $Z=z$.

In other words, $f(z) = E[Y|Z=z]$.

What is $E\big[E[Y|Z]\big]$?

$E[Y] = E\big[E[Y|X_1, X_2, \ldots, X_k]\big]$

<u>Proposition 3</u>: $E[Y] = E\big(E[Y|Z]\big)$, where Y and Z are two discrete R.V.s.

<u>Proof</u>:

$$E\Big[\underbrace{E[Y|Z]}_{f(Z)}\Big] = \sum_z \overbrace{E[Y|Z=z]}^{f(z)} P_r(Z=z)$$

$$= E[Y] \quad \left(\begin{array}{c}\text{from} \\ \text{Prop 3}\end{array}\right)$$

$\square$

<u>Proposition</u>: Let X, Y, Z be Discrete R.V.s.

$$E[X|Y] = E\big[E[X|Y,Z]\,\big|\,Y\big]$$

$$f(Y) \qquad\qquad g(Y)$$

To prove this, it is enough to show that for all y in the Range of Y

$$f(y) = g(y).$$

In other words, we show that for any y in the range of Y

$$E[X \mid Y=y] = E\big[E[X \mid Y, Z] \mid Y=y\big]$$

Proof:

$$E\big[E[X \mid Y, Z] \mid Y=y\big]$$

$$= \sum_z E[X \mid Y=y, Z=z] \, P_r(Z=z \mid Y=y)$$

$$= \sum_z \sum_x x \cdot P_r(X=x \mid Y=y, Z=z) \, P_r(Z=z \mid Y=y)$$

$$= \sum_{z,x} x \cdot \frac{P_r(X=x, Y=y, Z=z)}{\cancel{P_r(Y=y, Z=z)}} \cdot \frac{\cancel{P_r(Z=z, Y=y)}}{P_r(Y=y)}$$

$$= \sum_{z,x} x . \Pr \frac{(X=x, Y=y, Z=z)}{\Pr(Y=y)}$$

$$= \sum_{z,x} x . \Pr\left(X=x, Z=z \middle| Y=y\right)$$

$$= \sum_{x} x \sum_{z} \Pr\left(X=x, Z=z \middle| Y=y\right)$$

$$= \sum_{x} x \Pr\left(X=x \middle| Y=y\right)$$

$$= E\left[X \middle| Y=y\right]$$

$\square$

General form,

<u>Prop:</u> $E\left[X \middle| Y_1, Y_2, \ldots, Y_i\right] = E\left[E\left[X|Y_1, \ldots, Y_{i+k}\right] \middle| Y_1, Y_2, \ldots Y_i\right]$

$k \geq 1$

<u>Martingale</u>

<u>Definition</u>: A sequence of random variables

Definition: A sequence of random variables $Z_0, Z_1, \ldots$ is a **martingale** with respect to the sequence $X_0, X_1, \ldots$ if for all $n \geq 0$, the following conditions hold:

(i) Z_n is a function of $X_0, X_1, \ldots, X_n$,

(ii) $E[|Z_n|] < \infty$,

(iii) $E[Z_{n+1} | X_0, X_1, \ldots, X_n] = Z_n$

A sequence of R.V. $Z_0, Z_1, \ldots$ is called a **martingale** when it is a martingale with respect to itself. That is, $E[|Z_n|] < \infty$ and

$$E[Z_{n+1} | Z_0, Z_1, \ldots, Z_n] = Z_n.$$

Example.

Gambler players a sequence of fair games

R.V. X_i : the amount the gambler wins on the i^{th} game.

can be $-$ve or $+$ve.

$$E[X_i] = 0, \quad \forall i > 0$$

Take X_0 to be a suitable constant.

Let Z_i be the gambler's total winnings at the end of the i^{th} game.

$Z_0 \rightarrow$ initial capital

$$E[Z_{n+1} \mid X_0, X_1, \ldots, X_n] = Z_n + E[X_{n+1}]$$

$$= Z_n + 0$$

$$= Z_n$$

So $Z_0, Z_1, \ldots, Z_n$ is a martingale w.r.t. $X_0, X_1, \ldots, X_n$.

<u>Doob Martingale</u>

Let $X_0, X_1, \ldots, X_n$ be a sequence of R.V.s and let Y be a R.V.

of R.V.s and let Y be a R.V. with $E[|Y|] < \infty$. Assume Y depends on $X_0, \ldots, X_n$. Then, we define

$$Z_i := E[Y \mid X_0, X_1, \ldots, X_i] \quad , \quad i = 0, 1, \ldots, n$$

$\rightarrow f(X_0, \ldots, X_i)$

This sequence $Z_0, Z_1, \ldots, Z_n$ is a martingale (called <u>Doob Martingale</u>) w.r.t. $X_0, X_1, \ldots, X_n$, since

(i) Z_i is a function of $X_0, X_1, \ldots, X_i$, $i = 0, 1, \ldots, n$

(ii)
$$\begin{aligned} E[|Z_i|] &= E\big[\,|E[Y \mid X_0, \ldots, X_i]|\,\big] \\ &\leq E\big[E[|Y| \mid X_0, \ldots, X_i]\big] \\ &= E[|Y|] \\ &< \underline{\underline{\infty}} \end{aligned}$$

$< \alpha$

(iii) $E[Z_{i+1} | X_0, X_1, \ldots, X_i]$

$$= E\Big[E[Y | X_0, \ldots, X_{i+1}] \Big| X_0, X_1, \ldots, X_i \Big]$$

$$= E[Y | X_0, X_1, \ldots, X_i]$$

$$= Z_i$$

In most application, we start with

$Z_0 = E[Y]$ (this would correspond to X_0 being a trivial R.V. that Y is independent of)

fn. of $X_1, \ldots, X_n$

$Z_0, Z_1, \ldots, Z_n$ will be a sequence of refined estimates of Y, progressively using more info of R.V.s $X_1, X_2, \ldots X_n$.

$$Z_0 = E[Y]$$

$$Z_1 = E[Y | X_1]$$

$$Z_1 = E[Y \mid X_1]$$

$$Z_2 = E[Y \mid X_1, X_2]$$

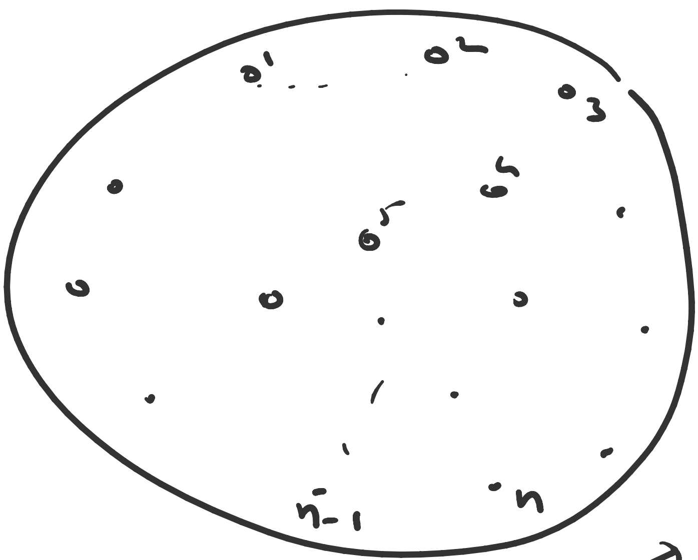
$$\vdots$$

$$Z_n = \overbrace{E[Y \mid X_1, X_2, \dots, X_n]}^{E[A \mid A] = A.} = Y.$$

Examples of Doob Martingale

1. Edge exposure martingale

$G(n,p)$: random graph model. $0 \le p \le 1$.



$$m = \binom{n}{2}$$

Let $G \sim G(n,p)$

no. of vertices

Let $m = \binom{n}{2}$. Label the possible edge slots of G in some order.

There are $m = \binom{n}{2}$ edge slots.

$$X_j = \begin{cases} 1, & \text{if edge in the } j^{th} \text{ slot is present in } G \\ 0, & \text{otherwise.} \end{cases}$$

$X_1, X_2, \ldots, X_m$

size of largest indep set / largest clique / min vertex cover

Let F be any finite-valued function defined over graphs.

Let $Z_0 = E[F(G)]$

$Z_1 = E[F(G) \mid X_1]$

$\vdots$

$Z_i = E[F(G) \mid X_1, \ldots X_i]$

Finally, $Z_m = E[F(G) \mid X_1, X_2, \ldots, X_m]$

$= F(G)$

$$= F(G).$$

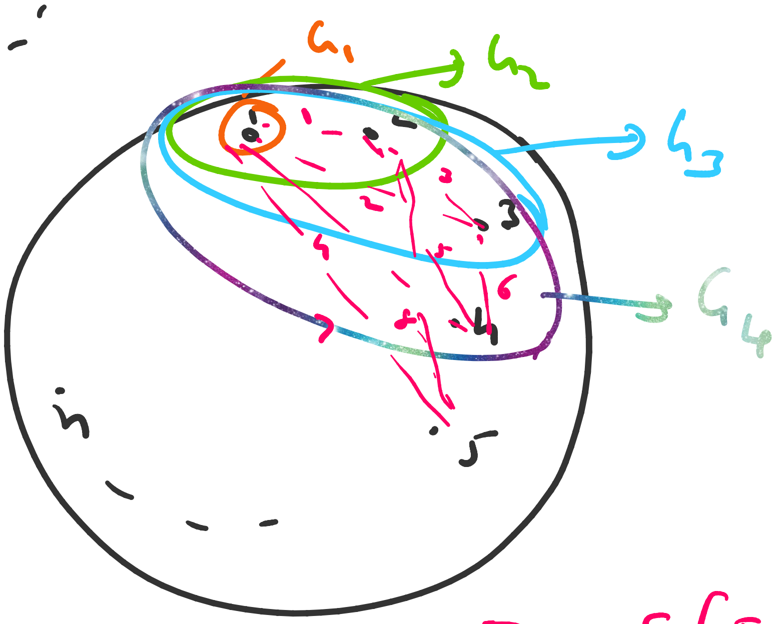
(ii) Vertex exposure Doob Martingale

revealing set of edges connected to a given vertex, one vertex at a time.

Fix an arbitrary numbering of vertices from 1 through n. Let G_i be the subgraph of G induced by the first i vertices.

Let $Z_0 = E[F(G)]$

$$Z_i = E[F(G) \mid G_1, G_2, \ldots, G_i], \quad i = 1, \ldots, n$$



$$Z_n = E[F(G) \mid G_1, \ldots, G_n] = F(G).$$

$$Z_0 = E[F(G)]$$

$$Z_0 = E(F(G))$$

$$Z_1 = E[F(G)]$$

corresponds to edge slot 1

$$Z_2 = E[F(G) \mid X_1]$$

$$Z_3 = E[F(G) \mid X_1, X_2, X_3]$$

$$Z_4 = E(F(G) \mid X_1, X_2, \ldots, X_6]$$

$$\vdots$$

$$Z_n = E[F(G) \mid X_1, \ldots, X_m]$$

Tail Inequalities for Martingales