Topics in Computing (CS5160) : Problem Set 2

Department of Computer Science and Engineering
IIT Hyderabad

- Scan and upload your answer sheets on google classroom.
- Maintain academic honesty. If caught, you will get an F in the course.
- Please write "credit" or "audit" on your answer sheets depending on whether you are crediting or auditing the course.
- Due Date: 31 October (before 11:59pm).

---

1. Compute the Fourier representation of the following functions:

   (a) the *not-all-equal* function $\mathsf{NAE}_n \colon \{-1,1\}^n \to \{-1,1\}$, defined by $\mathsf{NAE}_n(x) = -1$ if and only if the bits $x_1, \ldots, x_n$ are not all equal; **(3 points)**

   (b) the *minimum* function $\mathsf{min}_n \colon \{-1,1\}^n \to \{-1,1\}$, defined by $\mathsf{min}_n(x) = \min_{i=1}^{n}\{x_i\}$; and **(2 points)**

   (c) the *sortedness* function $\mathsf{sort} \colon \{-1,1\}^4 \to \{-1,1\}$, defined by $\mathsf{sort}(x) = -1$ if and only if $x_1 \leq x_2 \leq x_3 \leq x_4$ or $x_1 \geq x_2 \geq x_3 \geq x_4$. **(3 points)**

2. How many Boolean functions $f \colon \{-1,1\}^n \to \{-1,1\}$ have exactly 1 nonzero Fourier coefficient? Also, prove that there are no Boolean functions $f \colon \{-1,1\}^n \to \{-1,1\}$ with exactly 2 nonzero Fourier coefficients. **(2+10 points)**

3. Given two functions $f \colon \{-1,1\}^n \to \mathbb{R}$ and $g \colon \{-1,1\}^n \to \mathbb{R}$, we define their *convolution* $f * g \colon \{-1,1\}^n \to \mathbb{R}$ to be another function given as follows

$$f * g(x) = \mathbb{E}_{y \sim \{-1,1\}^n}[f(x \circ y) g(y)],$$

   where $x \circ y = (x_1 y_1, x_2 y_2, \ldots, x_n y_n)$ i.e., bit-wise product, and $y \sim \{-1,1\}^n$ denotes y is sampled w.r.t uniform distribution. Show that for all $S \subseteq [n]$,

$$\widehat{f * g}(S) = \widehat{f}(S) \cdot \widehat{g}(S).$$

   (10 points)

4. Let $f\colon \{-1,1\}^n \to \{-1,1\}$ be a Boolean function. Show that

$$\mathsf{as}(f) \geq \mathsf{Var}(f) = 4 \cdot \Pr_x[f(x) = 1] \cdot \Pr_x[f(x) = -1].$$

Recall, $\mathsf{as}(f)$ denotes the average sensitivity of f and $\mathsf{Var}(f)$ denotes the variance of f when x is chosen uniformly at random from $\{-1,1\}^n$. Further, recall variance of a random variable Z is defined as $\mathsf{Var}(Z) = \mathbb{E}[(Z - \mathbb{E}[Z])^2]$. **(20 points)**

5. Let $p\colon \{0,1\}^n \to \mathbb{R}$ and $q\colon \{0,1\}^n \to \mathbb{R}$ be multilinear polynomials of degree at most d. Show that if $p(x) = q(x)$ for all $x \in \{0,1\}^n$ with the Hamming weight of x at most d, i.e., $|x| \leq d$, then $p = q$ as a polynomial.

Recall the hamming weight of x (denoted $|x|$) is the number of 1's in x, i.e., $|x| = \sum_{i=1}^n x_i$. **(10 points)**

6. Show that for any Boolean function $f\colon \{-1,1\}^n \to \{-1,1\}$ of degree d,

   (a) for all $S \subseteq [n]$, either $\widehat{f}(S) = 0$ or $|\widehat{f}(S)| \geq \frac{1}{2^{d-1}}$. **(5 points)**

   (b) the L_1-norm of $\widehat{f}$, $\|\widehat{f}\|_1 := \sum_{S\subseteq[n]} |\widehat{f}(S)| \leq 2^{d-1}$. **(5 points)**

7. Show that for any monotone Boolean function $f\colon \{-1,1\}^n \to \{-1,1\}$, for all $i \in [n]$, $\mathsf{Inf}_i(f) = \widehat{f}(\{i\})$.

Let $x, y \in \{-1,1\}^n$. Define $x \leq y$ if and only if for all $i \in [n]$, $x_i \leq y_i$. We then say that a Boolean function $f\colon \{-1,1\}^n \to \{-1,1\}$ is *monotone* if for any x, y such that $x \leq y$, we have $f(x) \leq f(y)$. **(10 points)**

8. We say that a multilinear polynomial $p\colon \{0,1\}^n \to \mathbb{R}$ approximates a Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ if $|f(x) - p(x)| \leq 1/3$, for all $x \in \{0,1\}^n$. We define the *approximate degree* of f, denoted $\widetilde{\deg}(f)$, as the least degree among all multilinear polynomials that approximate f. Show that $\widetilde{\deg}(f) = \Omega(\sqrt{\mathsf{bs}(f)})$, where $\mathsf{bs}(f)$ is the block-sensitivity of f. **(10 points)**

9. Our goal here is to give a game-theoretic lower bound on the minimum size, $L(f)$, of a decision tree computing a function $f\colon \{0,1\}^n \to \{0,1\}$. There are two players in this game, namely Prover and Delayer. Given an input vector $x \in \{0,1\}^n$, the goal of the Prover is to output $f(x)$. The goal of Delayer is to delay this happening as long as possible. The game proceeds in rounds. In each round, the Prover suggests a variable x_i to be set in this round, and Delayer either chooses a value 0 or 1 for x_i or leaves the choice to the Prover. In this last case, the Delayer scores one point, but the Prover can then choose the value of x_i. The game is over when the Prover outputs $f(x)$. Let $\mathsf{Score}(f)$ denote the maximal number of points the Delayer can earn in this game independent of what strategy the Prover uses. Show that $L(f) \geq 2^{\mathsf{Score}(f)}$. **(10 points)**

Hint: Prove the converse direction: if f can be computed by a decision tree of size S, then the Prover has a strategy under which the Delayer can earn at most $\log S$ points.