11/09/23

Thm :- [Nisan–Szegedy]

For all $f:\{0,1\}^n \to \{0,1\}$,

$$\deg(f) = \Omega\left(\sqrt{bs(f)}\right).$$

Proof :-

Let $a \in \{0,1\}^n$ be an input s.t. $bs(f,a) = bs(f) = b$.

Furthermore, assume WLOG,

$$f(a) = 0.$$

Let $B_1, \ldots, B_b$ be the disjoint sensitive blocks at a which witnesses the block sensitivity at a equals $b = bs(f)$.

Let $p(x_1, \ldots, x_n)$ be the polynomial that represents f.

$\deg(p) = \deg(f)$.

Obtain $q(y_1, \ldots, y_b)$ from $p(x_1, \ldots, x_n)$ by the following substitution.

(i) if $x_i \notin \bigcup_{j=1}^{b} B_j$ then

set $x_i = a_i$ in ϕ.

(ii) if $x_i \in B_j$ and

$a_i = 0$

then set $x_i = y_j$ in ϕ.

(iii) if $x_i \in B_j$ and

$a_i = 1$

then set $x_i = 1 - y_j$ in ϕ.

After these substitution

we have obtained a

polynomial $q(y_1, \dots, y_b)$.

Properties of q

(i) q is multilinear polynomial of $\deg(q) \le \deg(p) \le b$

$$q : \{0,1\}^b \to \{0,1\}$$

(ii) $q(0^b) = p(a) = f(a) = 0$

(iii) $q(e_i) = p(a^{B_i}) = f(a^{B_i}) = 1$

$\forall\, i \in \{1, 2, \dots, b\}$

$q(0^b) = 0$

Claim :- $\deg(q) \geq \sqrt{\frac{b}{2}}$

define

$$q_{sym}(y_1, \dots, y_b) = \frac{\sum_{\sigma \in S_b} q(\sigma(y))}{b!}$$

where S_b is the set of permutation over $\{1, \dots, b\}$

and $\sigma(y) = \sigma(y_1, \dots, y_b) = (y_{\sigma(1)}, y_{\sigma(2)}, \dots, y_{\sigma(b)})$

(Symmetrization of a polynomial)

(Minsky & Papert '63).

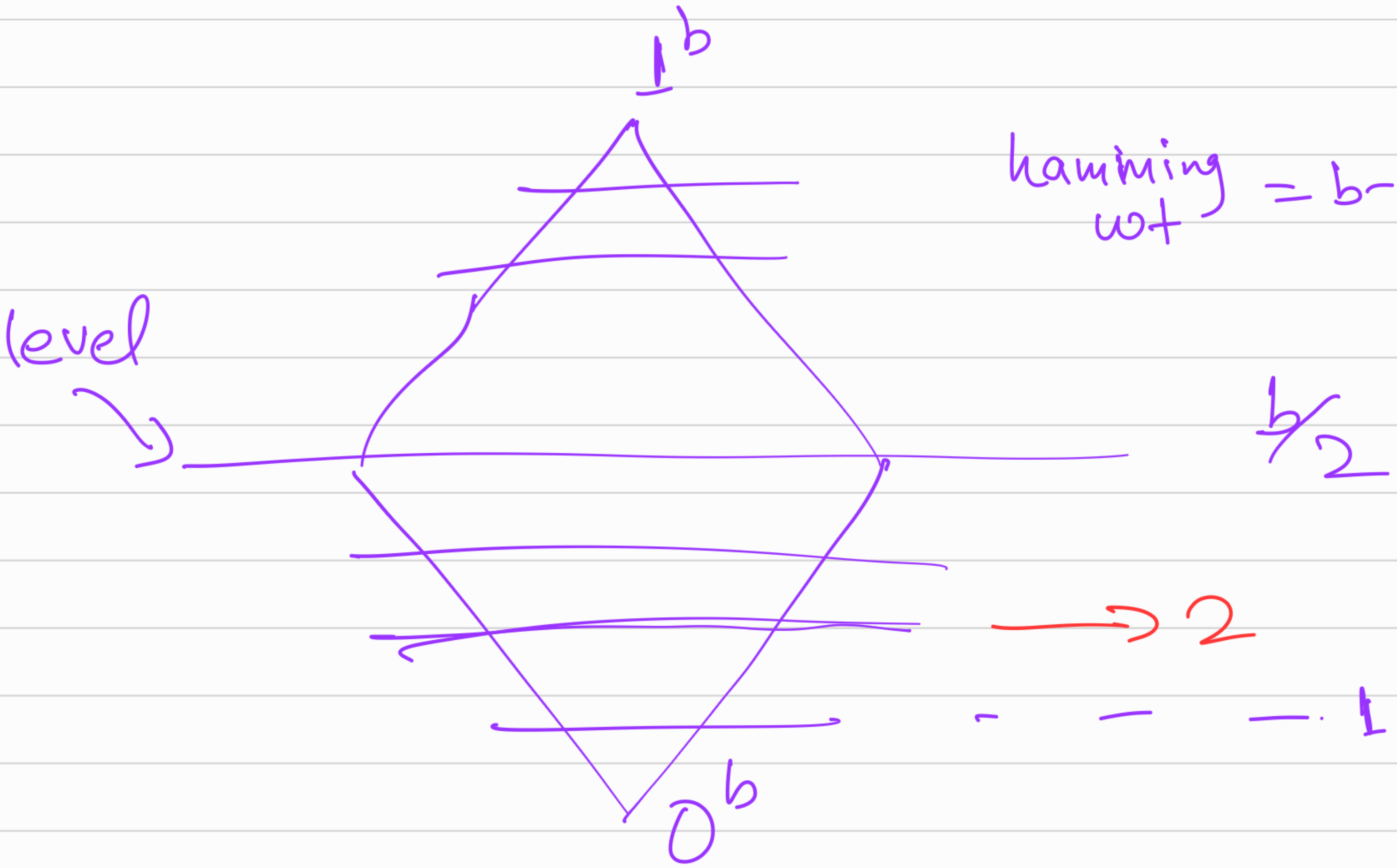
properties of q_{sym}

① $q_{sym}(0^b) = q(0^b) = 0$

② $q_{sym}(e_i) = 1.$

③ $\deg(q_{sym}) \leq \deg(q),$

④ q_{sym} is a univariate polynomial.



$110^{b-2} \longrightarrow$ all inputs in level 2.

$$q_{sym}(110^{b-2}) = \frac{1}{b!} \sum_{\sigma \in S_b} q(\sigma(110^{b-2}))$$

$$= \frac{1}{b!} \sum_{\substack{y \in \{0,1\}^b \\ |y|=2}} q(y)$$

$$\lambda := \left| \left\{ \sigma \mid \sigma(110^{b-2}) = 0110^{b-3} \right\} \right|$$

$$\deg(q_{sym}) \leq \deg(q) \leq \deg(p)$$

Lemma :- Let $h : \mathbb{R} \to \mathbb{R}$ be a univariate polynomial s.t. for every integer k, where $0 \leq k \leq n$,

$$b_1 \leq h(k) \leq b_2 .$$

And $\exists$ a $z \in [0, n]$

s.t. $|h'(z)| \geq c$.

Then $\deg(h) \geq \sqrt{\dfrac{c \cdot n}{c + b_2 - b_1}}$

(Back to proof).

$\Rightarrow \forall\ 0 \leq k \leq b$

$0 \leq q_{sym}(k) \leq 1$

$\Rightarrow q_{sym}(0) = 0$

$\Rightarrow q_{sym}(1) = 1$

$\Rightarrow$ By intermediate mean value Theorem

$\exists$ a point $z \in [0,1]$

s.t. $q'_{sym}(z) \geq \dfrac{q_{sym}(1) - q_{sym}(0)}{1 - 0}$

$= \dfrac{1-0}{1-0} = 1.$

From the Lemma, we have

$\deg(q_{sym}) \geq \sqrt{\dfrac{1 \cdot b}{1+1-0}}$

$= \sqrt{\dfrac{b}{2}}.$

$\blacksquare$

$$\boxed{\begin{array}{l}\text{OPEN :- Show } \exists \text{ a function } f : \{0,1\}^n \to \{0,1\} \\ \text{s.t. } \deg(f) = O\left(\sqrt{bs(f)}\right)\end{array}}$$

$\#$ Boolean function on n variables $= 2^{2^n}$

Q :- How many Boolean function over n variables have $\deg = n$?

$1 - o(1)$

$$\frac{\#\text{ B.f. with } dg = n}{2^{2^n}}$$

$g(m) \in o(1)$ iff $\frac{g(m)}{2} \to 0$ as $m \to \infty$

e.g. $\frac{1}{m}, \frac{1}{2^m}$

$\frac{1}{\sqrt{m}}, \frac{1}{\log m}$

Q': How to count # B.f. with $\deg = n$?

Consider $f : \{0,1\}^n \to \{0,1\}$

It is representable by

a polynomial $p(x_1, \ldots, x_n)$.

$$p(x_1, \ldots, x_n) = \sum_{S \subseteq [n]} c_S \prod_{i \in S} x_i$$

where c_S is the coefficient of monomial $\prod_{i \in S} x_i$

<u>Q'':-</u> How to compute c_S ?

<u>Thm</u>:- $c_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} \cdot f(T)$

where $f(T) = f(\alpha)$ where $\alpha_i = 1$ iff $i \in T$.

Define $X_1^{even} = \{x \mid f(x) = 1, |x| \text{ is even}\}$

$X_1^{odd} = \{x \mid f(x) = 1, |x| \text{ is odd}\}$.

Thm:- $\deg(f) = n$ iff

$|X_1^{even}| \neq |X_1^{odd}|$.

Proof:- $c_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} \cdot f(T)$

$\deg(f) = n$ iff $c_{[n]} \neq 0$

$$C_{[n]} = \sum_{T \subseteq [n]} (-1)^{n-|T|} \cdot f(T)$$

$$= (-1)^n \sum_{T \subseteq [n]} (-1)^{|T|} \cdot f(T)$$

$$= (-1)^n \sum_{T \subseteq [n]} (-1)^{|T|}$$

$$f(T) = 1$$

$$= (-1)^n \left(|X_1^{even}| - |X_1^{odd}| \right)$$

$\boxtimes$

Count # B.f. with deg < n

$\because$ For such functions

$$|X_1^{even}| = |X_1^{odd}|$$

$$\sum_{i=0}^{2^{n-1}} \binom{2^{n-1}}{i} \cdot \binom{2^{n-1}}{i}$$

$$= \binom{2^n}{2^{n-1}} \approx \frac{2^{2^n}}{\sqrt{2^n}}$$

$$\left[\binom{k}{k/2} \simeq \frac{2^k}{\sqrt{k}} \right]$$