CS5120: Assignment One

Deadline: 23/01/2023

1. Let A be a randomized Monte Carlo algorithm that solves a decision problem in polynomial time, and which is correct with probability at least $\frac{1}{n^3}$, where n is the size of the input, and the probability is over the random choices made by the algorithm. Describe a polynomial time algorithm to solve the same problem whose probability of correctness is at least $\frac{99}{100}$.

2. Let $G = (V, E)$ where $V - \{v_0, v_1, \ldots, v_{n-1}\}$ and $E = \{\{v_i, v_{i+1} | 0 \leq i \leq n\}$ with indices taken modulo n. That is, G is a cycle graph on n vertices. Find the probability that Karger's algorithm finds the cut $A = \{v_1, v_2\}, B = \{v_3, \ldots, v_n\}$.

3. Let X be picked u.a.r from $\{1, 2, \ldots, 3000\}$. In each of the following two cases, decide with justification whether the pair A, B of events are independent.

   (i) A: X is divisible by 2. B: X is divisible by 3.

   (ii) A: X is divisible by 200. B: X is divisible by 300.

4. Suppose that you see a stream of distinct numbers. When you see the first number, you store its value, and subsequently, when you see the ith number, you do the following: with probability p_i, you replace the current number stored with the ith number. The goal is to maintain numbers from the stream with uniform distribution, that is: after seeing i numbers, the probability of maintaining each of the numbers seen so far must be $\frac{1}{i}$. What should be the value of p_i?

5. Consider a set of m clauses over n variables, where each clause is an OR of three literals. An example is $\{(x \vee \bar{y} \vee \bar{z}), (\bar{x} \vee z \vee \bar{w})\}$ with $m = 2$ clauses, over $n = 4$ variables. For a random assignment of True/False to each variable, calculate the expected number of clauses that are made true by the assignment.

6. Since $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$, we have $P(A \cup B) \leq P(A) + P(B)$. More generally, we have $Pr(A_1 \cup \ldots \cup A_n) \leq Pr(A_1) + \ldots + Pr(A_n)$. This inequality is often called the "union bound". Suppose that a fair coin is tossed n times.

   (i) Using the union bound, find an upper bound on the probability that there is a sequence of $\log_2 n + k$ consecutive heads.

   (ii) What is the expected number of sequences of r consecutive heads?