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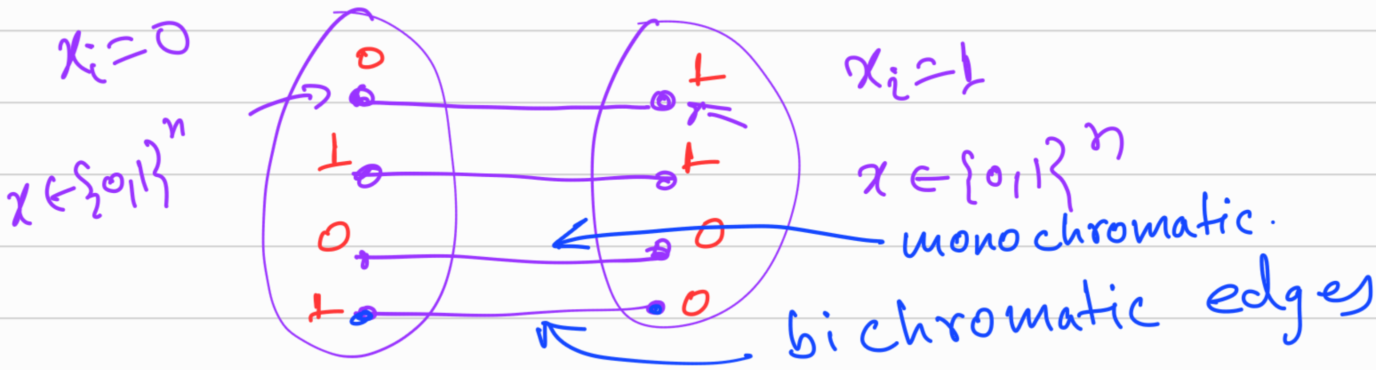
→ if f is computed by a decision tree of size s then $\|\hat{f}\|_1 := \sum_{S \subseteq [n]} |\hat{f}(S)| \leq s$.

→ Defn:- Influence of a variable.

let $f : \{0,1\}^n \to \{0,1\}$

$$\mathrm{Inf}_i(f) := \Pr_{x \sim_u \{0,1\}^n} [f(x^i) \neq f(x)]$$

(x^i is x with i-th bit flipped)



$$\Pr_{x \sim \{0,1\}^n} \left[f(x^i) \neq f(x) \right]$$

$$\boxed{= \left(\frac{1}{2^n} + \frac{1}{2^n} \right)} = \frac{2 \cdot \#\text{ bichromatic edges in } i\text{-th direction}}{2^n}$$

Can $\mathrm{Inf}_i(f) = 0$?

Then function f doesn't depend on variable x_i.

$$\mathrm{Inf}_i(f) \neq 0 \Rightarrow \mathrm{Inf}_i(f) \geq \frac{1}{2^{n-1}}$$

<u>Defn</u>:- Total Influence of a function f

$$\mathrm{Inf}(f) := \sum_{i=1}^{n} \mathrm{Inf}_i(f)$$

Average Sensitivity $= \mathrm{Inf}(f)$.

$$:= \mathop{\mathbb{E}}_{x \sim_u \{0,1\}^n} [s(f,x)]$$

$$as(f) = \mathop{\mathbb{E}}_{x} [s(f,x)]$$

define $s_i(f,x) = \begin{cases} 0 & \text{if } f(x) = f(x^i) \\ 1 & \text{if } f(x) \neq f(x^i) \end{cases}$

$$= \frac{1}{2^n} \sum_{x \in \{0,1\}^n} s(f,x)$$

$$= \frac{1}{2^n} \sum_{x \in \{0,1\}^n} \sum_{i=1}^{n} s_i(f,x)$$

$$= \frac{1}{2^n} \sum_{i=1}^{n} \sum_{x \in \{0,1\}^n} s_i(f,x)$$

$$= \sum_{i=1}^{n} \boxed{\frac{1}{2^n} \sum_{x \in \{0,1\}^n} s_i(f,x)}$$

$$= \sum_{i=1}^{n} \Pr_x \left[f(x) \neq f(x^i) \right]$$

$$= \sum_{i=1}^{n} \mathrm{Inf}_i(f) = \mathrm{Inf}(f).$$

$\Rightarrow$ if $\mathrm{Inf}(f) \neq 0$

$$\Rightarrow \quad \mathrm{Inf}(f) \geq \frac{n}{2^{n-1}}.$$

$$\Rightarrow \mathrm{Inf}(f) \geq \frac{2 \cdot \#\text{bichromatic edges}}{2^n} \geq \frac{2 \cdot n}{2^n}$$

$$\mathrm{Inf}(\text{XOR}/\text{Parity}) = n.$$

$$0 \leq \mathrm{Inf}(f) \leq n$$

Thm :- [KKL '88]

Kahn-Kalai-Linial.

$$f : \{\pm 1\}^n \to \{\pm 1\}$$

$$\mathrm{Inf}_i(f) := \Pr_x[f(x) \neq f(x^i)]$$

Then,
$$\boxed{\mathrm{Inf}_i(f) = \sum_{S:\, i \in S} \hat{f}(S)^2}$$

Recall, $$f(x) = \sum_{S \subseteq [n]} \hat{f}(S) \prod_{i \in S} x_i$$

$$\sum_{S \subseteq [n]} \hat{f}(S)^2 = 1 \quad \text{[Parseval's Thm]}$$

$$\sum_{S \subseteq [n]} \hat{f}(S)^2 = E[f^2]$$

$$f : \{\pm 1\}^n \to \mathbb{R}$$

Devise a function $g_1 : \{0,1\}^{n-1} \to \{0,1,-1\}$

$$|g_1(y)| = \begin{cases} 1 & \text{if } f(1,y) \neq f(0,y) \\ 0 & \text{o/w.} \end{cases}$$

$$g_1(y) := f(1,y) - f(0,y)$$

$$g_1 : \{\pm 1\}^{n-1} \to \{\pm 1, 0\}$$

$$f : \{\pm 1\}^n \to \{\pm 1\}.$$

$$g_1(y) := \underbrace{\frac{f(1,y) - f(-1,y)}{2}}$$

$$\mathrm{Inf}_1[f] := \Pr_{x \sim \{\pm 1\}^n}[f(x) \neq f(x')]$$

$$= \Pr_{y \sim \{\pm 1\}^{n-1}}[g_1(y) \in \{\pm 1\}]$$

$$= \Pr_y[g_1^2(y) = 1]$$

$$= \mathbb{E}_y \left[g_1^2(y) \right]$$

$$\boxed{f(y_1, \dots, y_n) = \sum_{S \subseteq [n]} \hat{f}(S) \prod_{i \in S} y_i}$$

$$= \sum_{S : 1 \notin S} \hat{f}(S) \prod_{i \in S} y_i$$

$$+ \sum_{S : 1 \in S} \hat{f}(S) \prod_{i \in S} y_i$$

$$\frac{f(1, y_2, \dots, y_n) - f(-1, y_2, \dots, y_n)}{2}$$

$$= \sum_{S : 1 \in S} \hat{f}(S) \prod_{i \in S \setminus \{1\}} y_i$$

$$g_1(y) = \sum_{S : 1 \in S} \hat{f}(S) \prod_{i \in S \setminus \{1\}} y_i$$

$$\mathrm{Inf}_1(f) = \mathop{\mathbb{E}}_{y}\left[g_1(y)^2\right]$$

from Parseval's equality

$$= \sum_{S:1\in S} \hat{f}(S)^2$$

$$\boxed{\begin{array}{l} f:\{\pm1\}^n \to \mathbb{R} \\ \mathbb{E}[f^2] = \sum_{S\subseteq[n]} \hat{f}(S)^2 \end{array}}$$

$$\mathrm{Inf}_i(f) = \sum_{S:i\in S} \hat{f}(S)^2$$

$$\mathrm{Inf}(f) = \sum_{i=1}^{n} \mathrm{Inf}_i(f)$$

$$= \sum_{i=1}^{n} \sum_{S:i\in S} \hat{f}(S)^2$$

$$= \sum_{S \subseteq [n]} |S| \hat{f}(S)^2$$

[KKL-Lemma]

Obs 1: $\mathrm{Inf}(f) \leq \deg(f)$

$$\mathrm{Inf}(f) = \sum_{S \subseteq [n]} |S| \hat{f}(S)^2$$

$$\leq \deg(f) \sum_{S \subseteq [n]} \hat{f}(S)^2$$

$$= \deg(f) \cdot 1 .$$

Fourier Entropy-Influence Conjecture. [Friedgut-Kalai '98]

$$\sum_{S \subseteq [n]} \hat{f}(S)^2 = 1$$

$$\mathbb{H}(f) = \sum_{S \subseteq [n]} \hat{f}(S)^2 \log \frac{1}{\hat{f}(S)^2}$$

$$\mathrm{Inf}(f) = \sum_{S \subseteq [n]} |S| \hat{f}(S)^2$$

$\exists$ a universal constant $C > 0$ s.t. for all $f : \{\pm 1\}^n \to \{\pm 1\}$

$$\mathbb{H}(f) \leq C \cdot \mathrm{Inf}(f).$$

$$\sum \hat{f}(S)^2 \log \frac{1}{\hat{f}(S)^2} \leq C \cdot \sum |S| \hat{f}(S)^2$$

Q:- Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean fn over n vars and deg d.

What is an upper bound on $\underline{n}$ in terms of $\underline{d}$?

$$MUX : \{0,1\}^{n+2^n} \to \{0,1\}$$

Address function

$$MUX(x_1, \dots, x_n, y_0, \dots, y_n)$$

$$= y_{bin(x_1, \dots, x_n)}$$

$\deg \leq n+1$

$\# \text{vars} = n + 2^n := M$

$\deg = O(\log M)$

Thm :- Let $f : \{0,1\}^n \to \{0,1\}$ of deg d then

$n \leq d \cdot 2^{d-1}$

$\Rightarrow \quad d = \Omega(\log n)$.

when n is the no. of vars f depends on.

Thm :- $\quad$ → multilinear.

Let $p : \{0,1\}^n \to \mathbb{R}$ of deg d.

Then $\Pr_{x \sim_u \{0,1\}^n} [p(x) \neq 0] \geq \frac{1}{2^d}$

$f : \{0,1\}^n \to \{0,1\}$ ~~of~~ deg d.

$$Inf_i(f) = \sum_{S : i \in S} \hat{f}(S) \prod_{j \in S \setminus \{i\}} x_j$$

$Inf_i(f)(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$

$$\Pr_x [Inf_i(f) \neq 0] \geq \frac{1}{2^{d-1}}$$

=

$$\sum_{i=1}^{n} \mathrm{Inf}_i(f) \leq d$$

$$\Downarrow$$

$$\frac{n}{2^{d-1}}$$

$$\Rightarrow \quad n \leq d \cdot 2^{d-1}.$$