31/08/23

~~OPEN~~ :- $\exists$? f s.t.

(NOT OPEN) $C(f) = \Omega(bs(f)^2)$

This is known.

But a new proof will still be interesting.

① $s(f,x) \leq bs(f,x) \leq C(f,x) \leq D(f)$

② $C(f) \leq bs(f)^2$.

③ $s(f) \leq bs(f) \underset{?}{\leq} O(s(f)^d)$

$\leq O(s(f)^4)$

<u>known</u>.

<u>Example</u>:- (Rubinstein's function)

$$f_{Rub} : \{0,1\}^n \to \{0,1\}$$

where $x_1, x_2, \ldots, x_{\sqrt{n}}$

$x_{\sqrt{n}+1}, x_{\sqrt{n}+2}, \ldots, x_{2\sqrt{n}}$

$\vdots$

$x_{n-\sqrt{n}+1}, x_{n-\sqrt{n}+2}, \ldots, x_n$

So basically we have.

$\sqrt{n}$ blocks of $\sqrt{n}$ vars.

$f_{RUB}(x) = 1$ iff $\exists$ a block s.t.

it contains two consecutive ones and rest 0's.

$$\sqrt{n}-1 \left\{ \begin{array}{l} 110^{\sqrt{n}-2} \\ 0110^{\sqrt{n}-3} \\ 00110^{\sqrt{n}-4} \end{array} \right.$$

$$bs(f_{RUB}, 0^n) \geq \frac{n}{2}.$$

$\sqrt{n}$ = even

$$\{\{1,2\}, \{3,4\}, \{5,6\}, \dots\}$$

$$s(f_{RUB}, 0^n) = 0$$

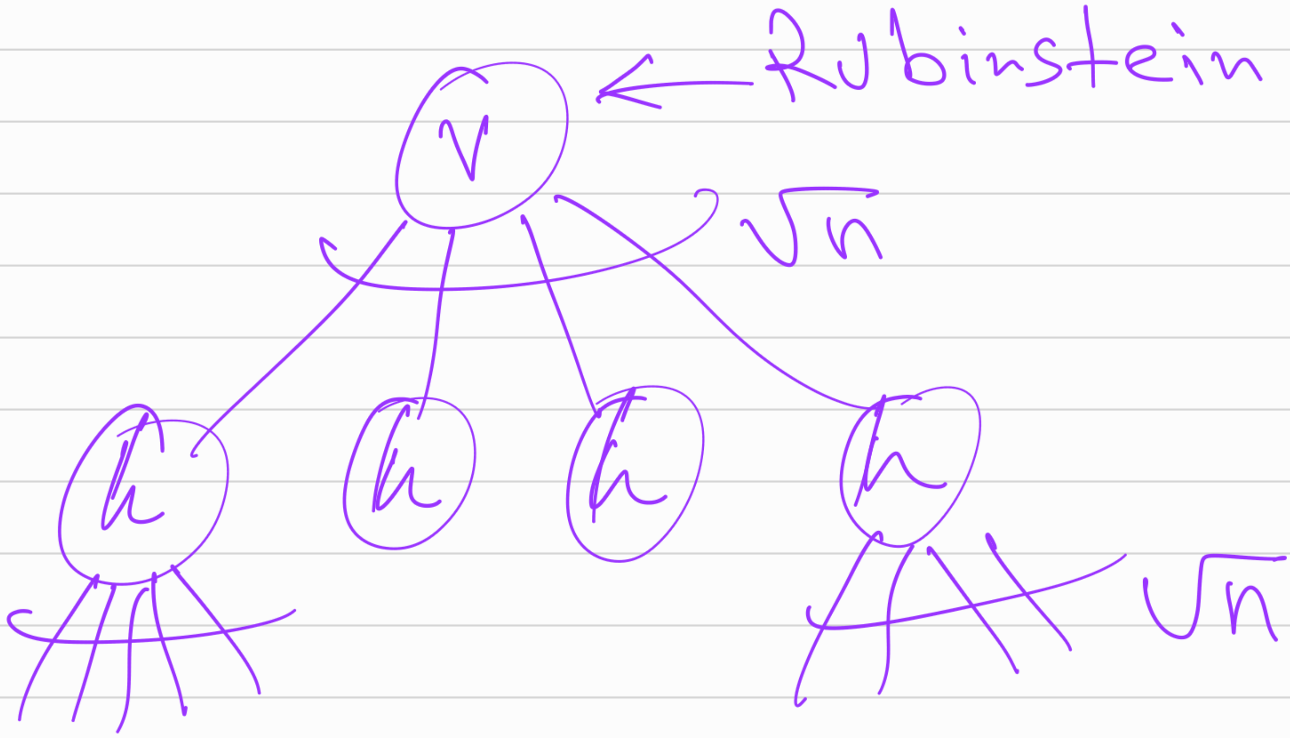
$$s(f_{RUB}, y) \leq 2\sqrt{n}$$

$y = \underbrace{0\cdots010\cdots0}$

$\underbrace{01010101}$ X

$$s(f_{RUB}, z) =$$

where $f_{RUB}(z) = 1$.



<u>Case 1:-</u> Suppose z contains two blocks which matches the required pattern.

$$s(f_{RUB}, z) \le O$$

Case 2 :- Otherwise.

$$s(f_{RUB}, z) \leq \sqrt{n}.$$

Let z be s.t.

$$f_{RUB}(z) = 0.$$

$$s(f_{RUB}, z) \leq 2\sqrt{n}.$$

$$s(f_{RUB}) = O(\sqrt{n})$$

$$bs(f_{RUB}) \geq bs(f_{RUB}, 0^n) \geq \frac{n}{2}$$

Prop (Rubinstein)

$\exists f$: s.t.

$$bs(f) = \Omega(s^2(f)).$$

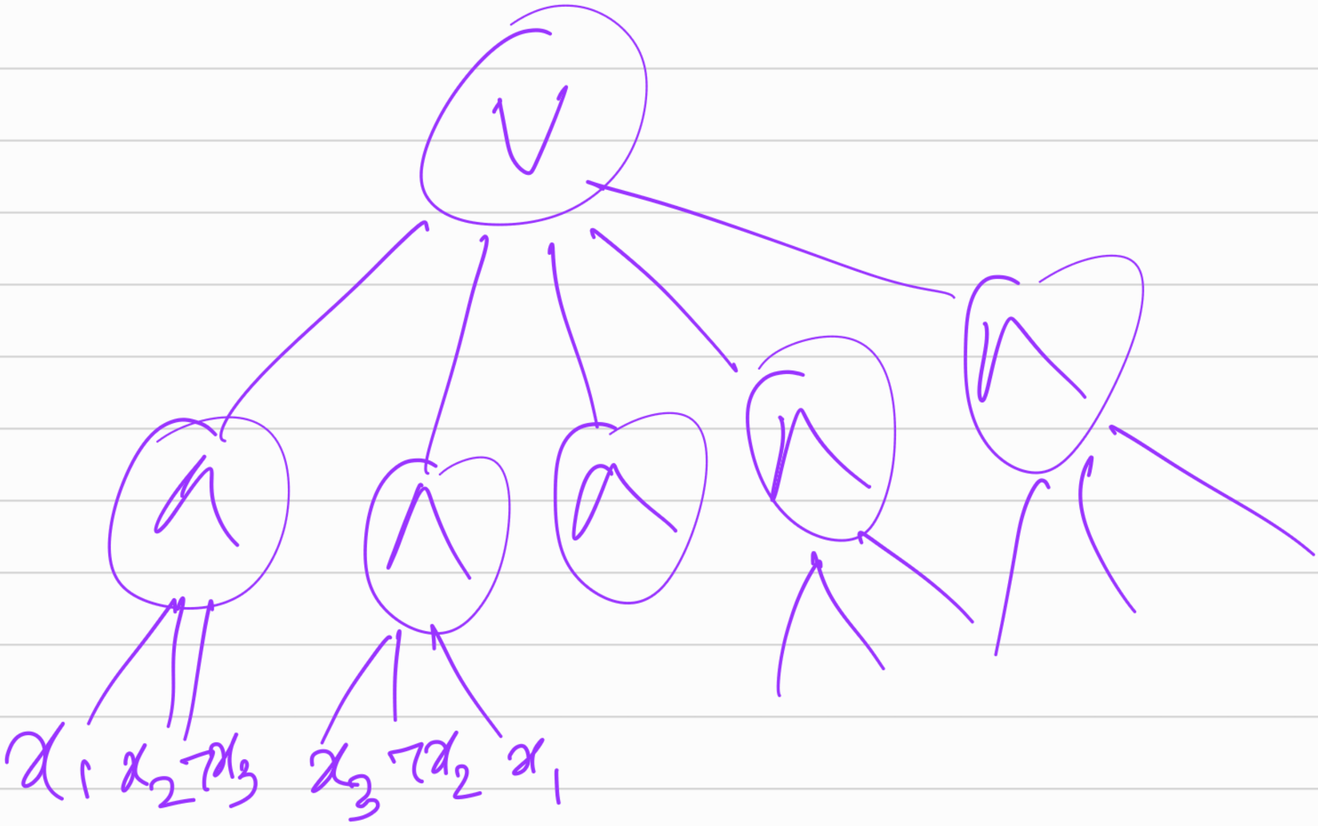
— Relationship D vs bs.

Look at certificates again.

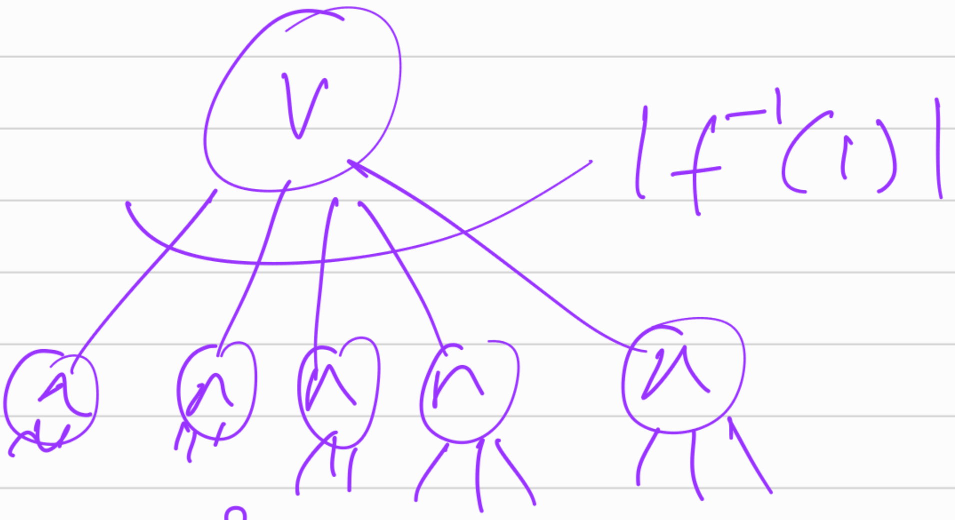
$\Rightarrow$ DNF / CNF

DNF = Disjunctive Normal Form

CNF = Conjunctive Normal Form



Any $f : \{0,1\}^n \to \{0,1\}$



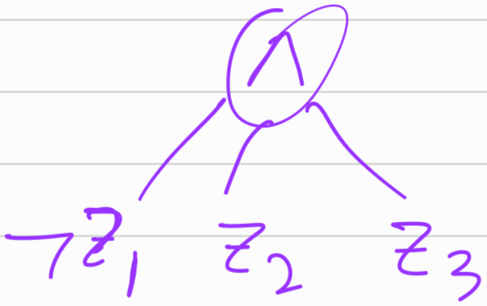
$g : \{0,1\}^3 \to \{0,1\}$

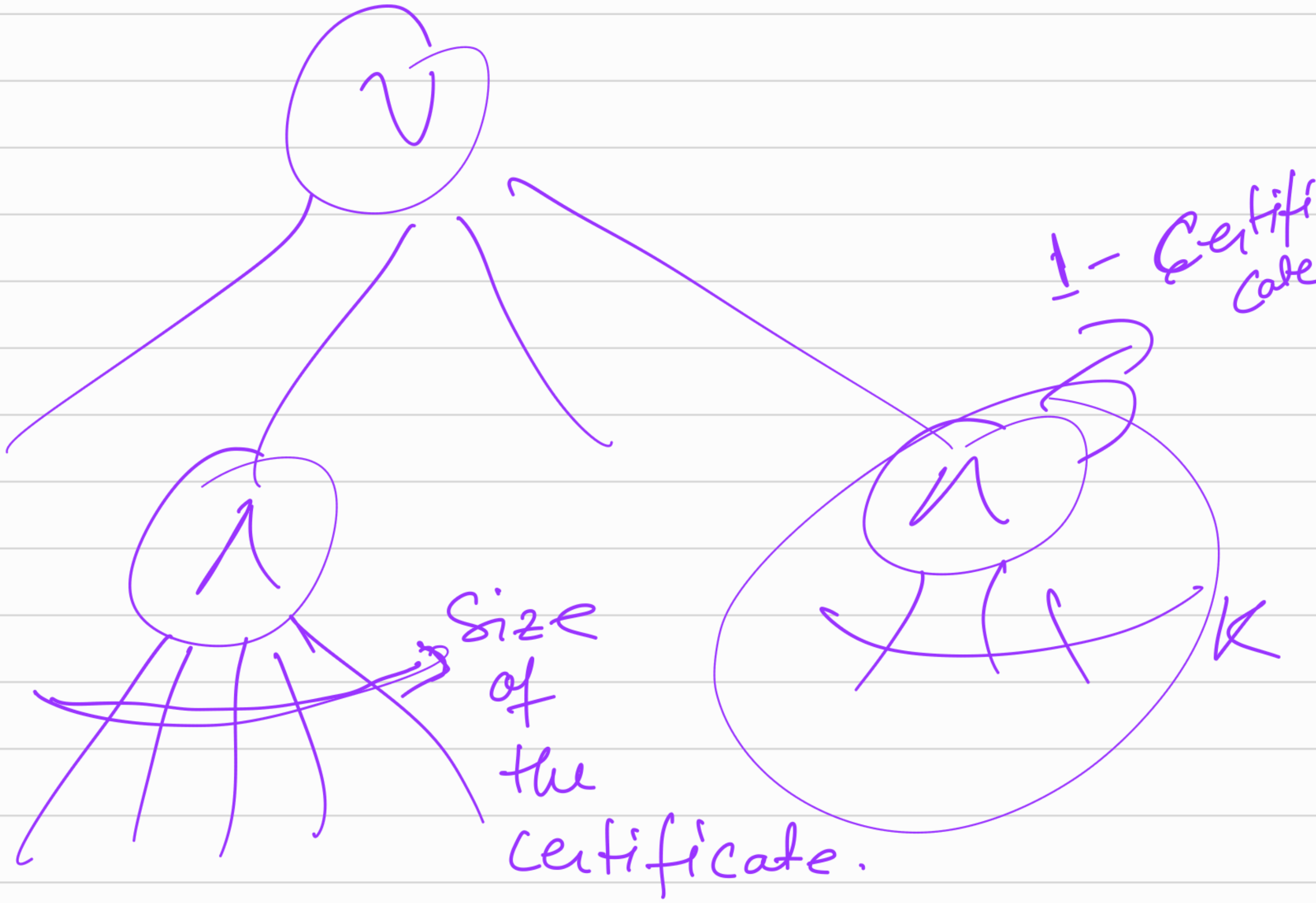
$g(011) = 1.$

$g(100) = 1$

g' s.t.

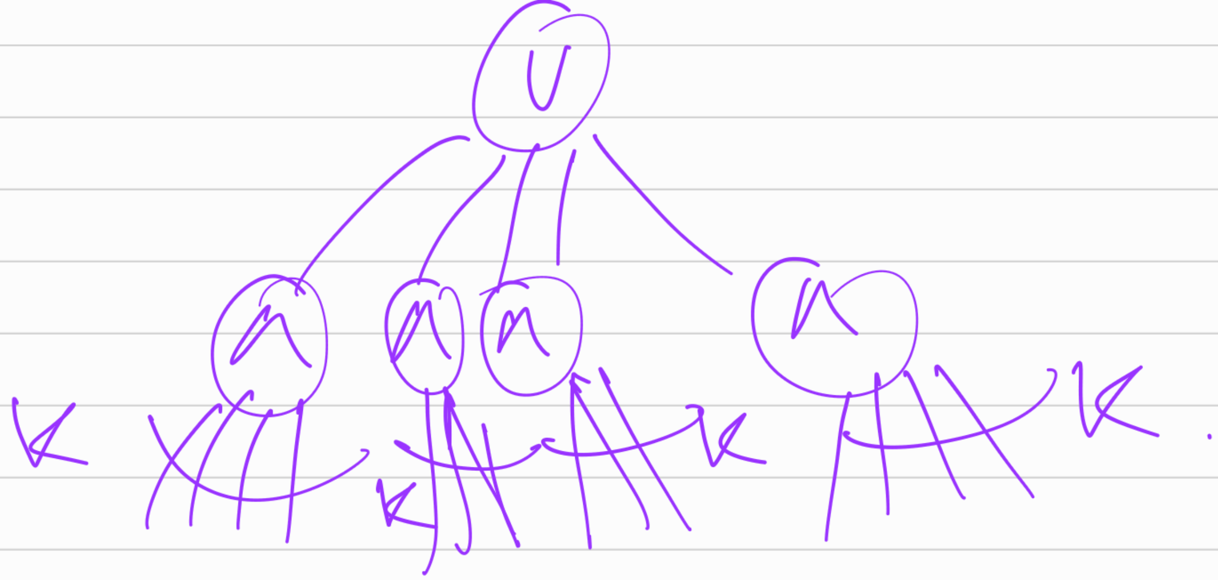
$g'(z) = 1$ if $z = 011.$

$g' =$ 

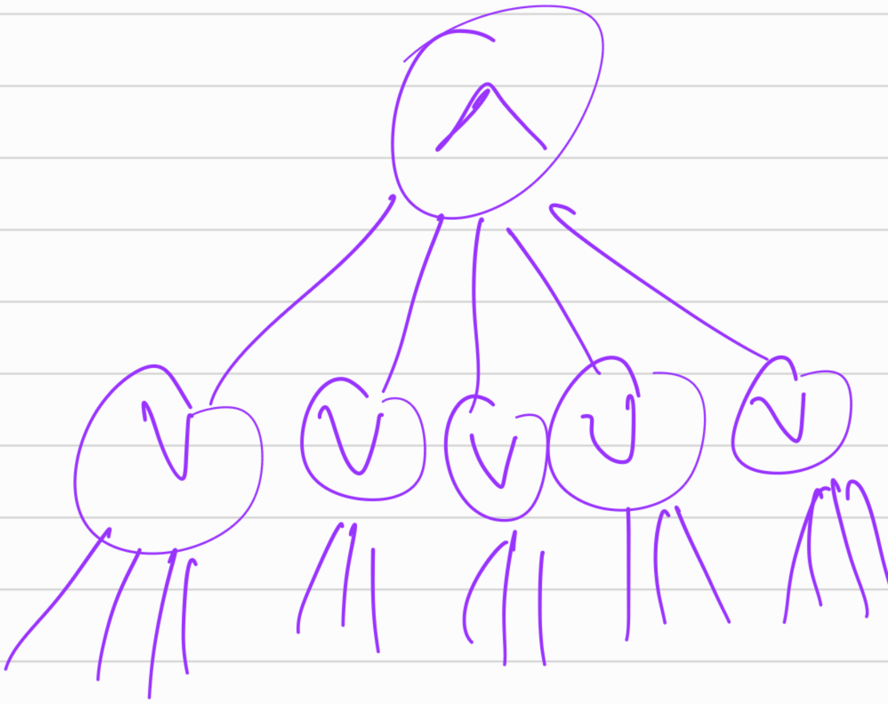


bottom-fan-in $\leq C^1(f)$

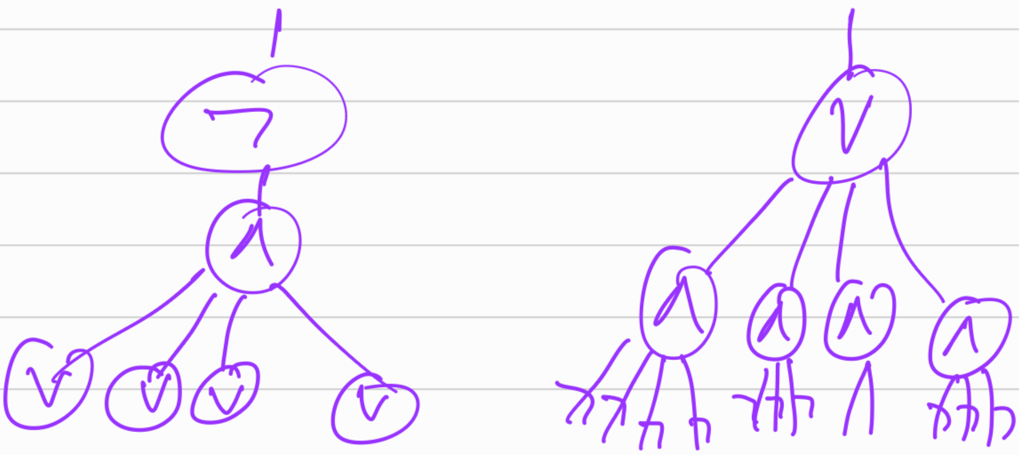
K- DNF

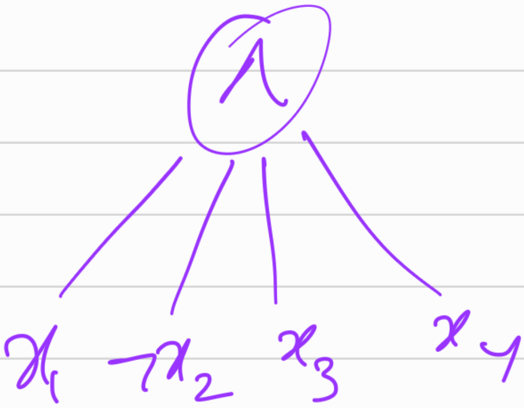


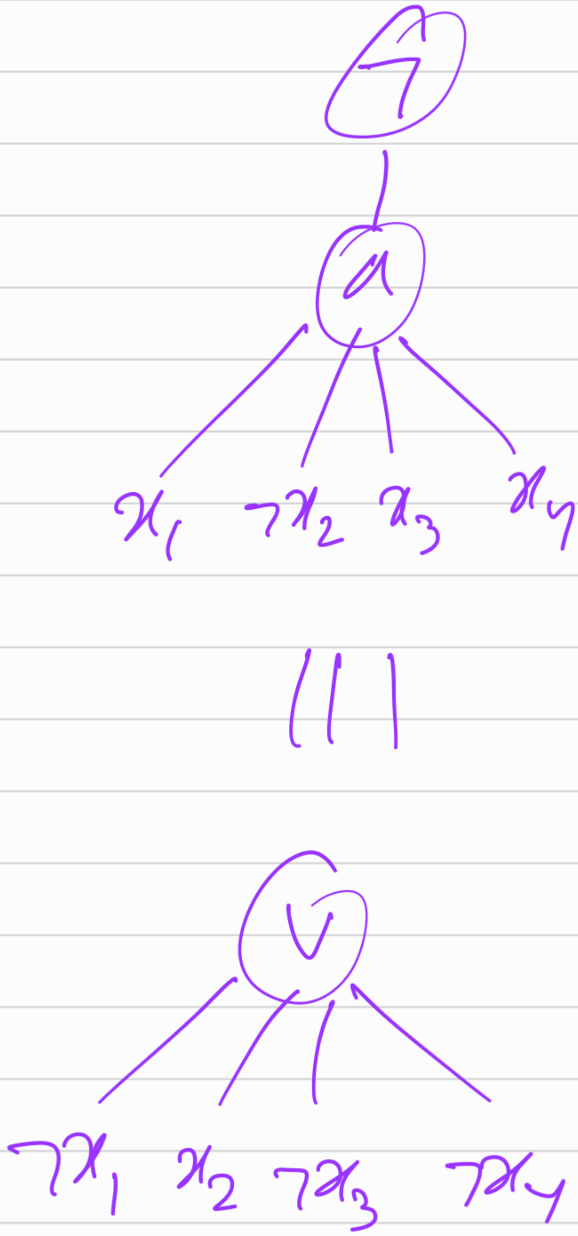
CNF :-



$$\neg(a \wedge b) \equiv \neg a \vee \neg b.$$







Recall:- $D(f) \le C^0(f) \cdot C^1(f)$

Question: Can you improve

on the above inequality?

Thm :- $D(f) \leq C^1(f) \cdot bs(f)$.

Reworded,

if f can be written as a k-DNF

then $D(f) \leq k \cdot bs(f)$.

Algorithm :-

Step 1 :- Repeat the following $bs(f)$ times.

1a :- If no 1- certificate exists, return 0 and stop

1b :- Pick a consistent 1-certificate, (S, α).

1c :- query the variables in S, if match then output 1.

1d :- O/w. discard certificates not consistent with revealed bits and continue.

Step 2: Pick a consistent input y and output $f(y)$.