$$D(f) \leq C^0(f) \cdot C^1(f).$$

24/08/23

Thm:- For all $f: \{0,1\}^n \to \{0,1\}$

$$D(f) \leq C^0(f) \cdot C^1(f)$$

Proof:-

Claim:- Let (S, α) and (T, β) be a 0-certificate and 1-certificate respectively.

Then $S \cap T \neq \emptyset$.

Furthermore, $\exists\, i \in S \cap T$

s.t. $\alpha(i) \neq \beta(i)$.

<u>Proof of Claim:</u> Exercise!

Consider the following algo.

<u>Step 0:</u> $\mathcal{X} = \{0,1\}^n$

Repeat $C'(f)$ times.

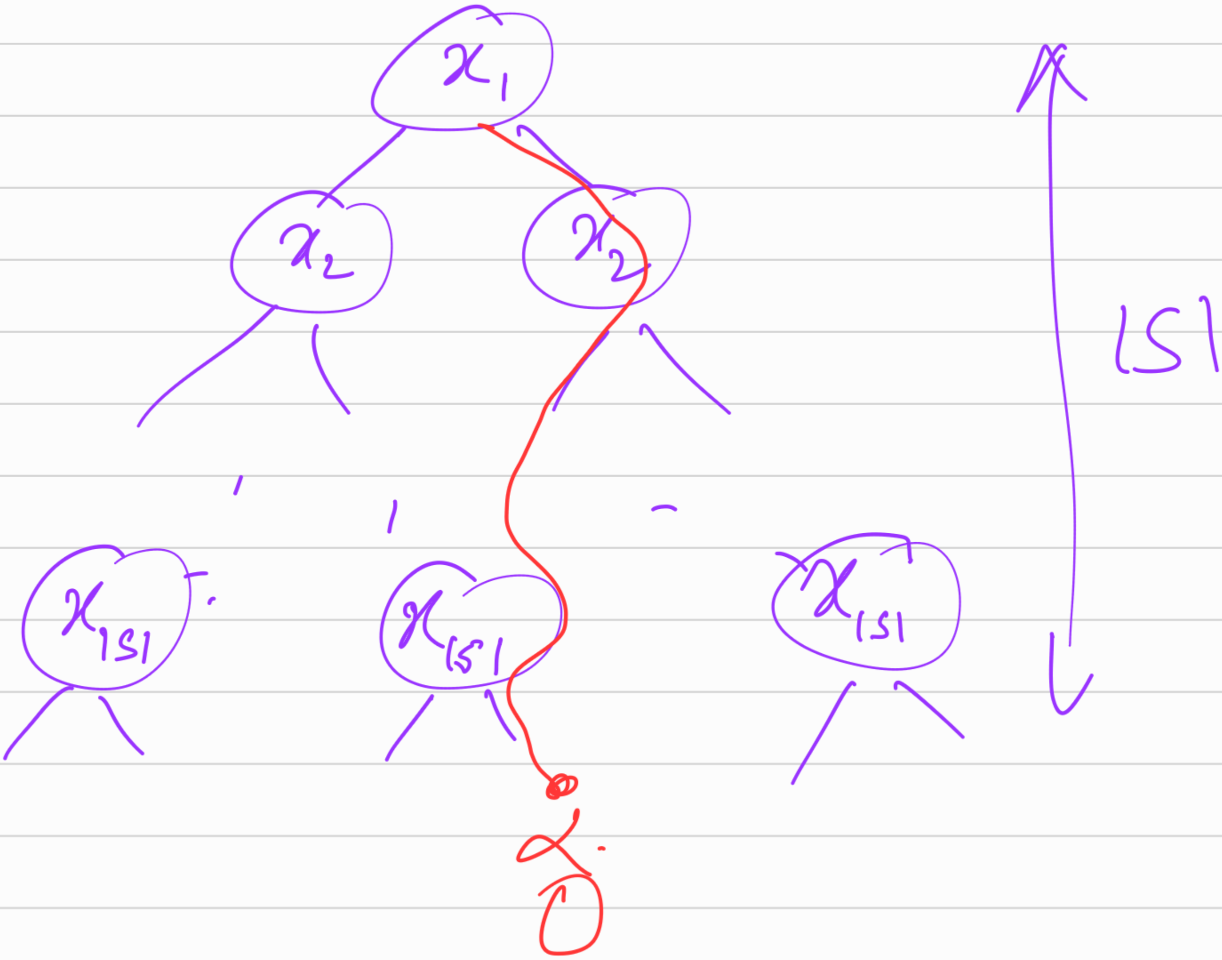
<u>Step 1:</u>- if $f(x)$ is constant for all $x \in \mathcal{X}$, then

Output $f(x)$.

Otherwise, Pick a 0-certificate (S, α)

where $S \subseteq [n]$ and $\alpha : S \to \{0,1\}$

Query the variables in S.



If you match with α then output D and halt.

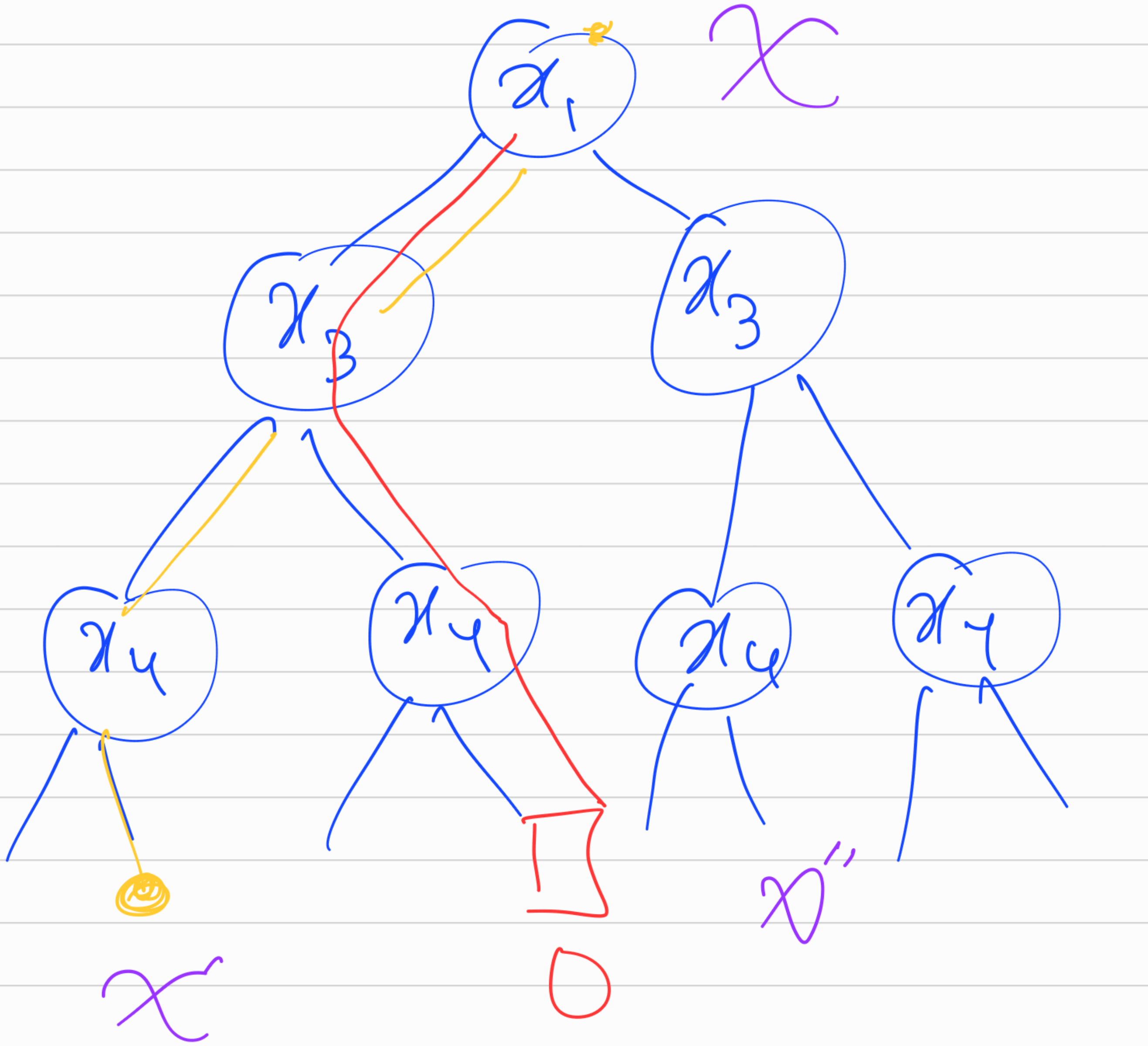
Else. Prune X to be consistent with revealed answers.

Step 2: Pick a $y \in X$
Output $f(y)$. and halt.

Worst case

\# queries made by the algo $\leq C^1(f) \cdot C^0(f)$

$$S = \{x_1, x_3, x_4\}$$



Proof of Correctness:

Claim :- 0-certificate and 1-certificates intersect in a contradictory way.

→ Claim :-

Let f_i be the function after i^{th} iteration of of Step 1.

Then, $C'(f_i) \leq C'(f_{i-1}) - 1$

$f_i = f(\cdot, -, -, -,)$

with revealed bits set accordingly.

Proof:- Because every 0-certificate intersects every 1-certificate

After one iteration of Step 1 you know the value to one variable from every 1-certificate.

Thm :-

$$D(f) \leq C^0(f) \cdot C^1(f)$$

$$\leq C(f)^2$$

P in decision tree

= NP in decision tree

$\cap$ coNP in decision tree.