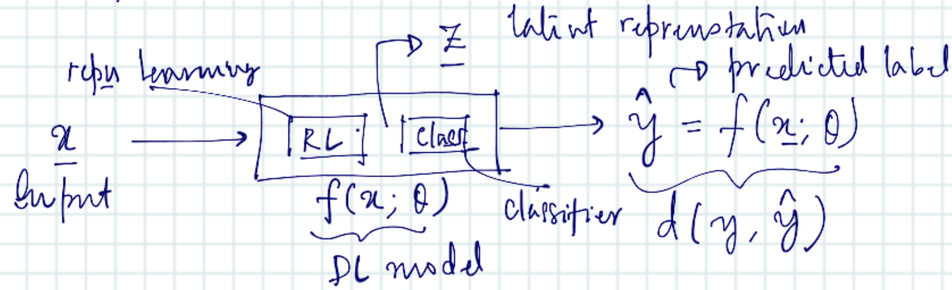
18/8/23

Deep Learning

- ✓ Recap: Metric and Metric spaces
- ✓ Linear space
- ✓ Norm and normed linear spaces
- ✓ Norm ball
- ✓ Convergence
- ✓ Cauchy sequences } Real space
- Extension to metric space
- Inner product



- Since we use iterative algorithms for training models we need notions of convergence.

- Linear space: A non-empty set $\mathcal{X}$ over a field $\mathbb{R}$ such that its elements are closed w.r.t. addition & scalar multiplication. In other words:
  - $x + y = y + x$, $\forall\, x, y \in \mathcal{X}$ (commutativity)
  - $x + (y + z) = (x + y) + z$, $\forall\, x, y, z \in \mathcal{X}$ (associativity)
  - $x + 0 = x$; $x + (-x) = 0$ $\forall\, x \in \mathcal{X}$ (zero element)
  - $\lambda(\mu x) = (\lambda\mu).x$; $\forall\, x \in \mathcal{X}$, $\forall\, \lambda, \mu \in \mathbb{R}$ (scaling)
  - $\lambda(x + y) = \lambda x + \lambda y$; $\forall\, x, y \in \mathcal{X}$, $\forall\, \lambda \in \mathbb{R}$ (scaling sum of elements)
  - $(\lambda + \mu) x = \lambda x + \mu x$; $\forall\, x \in \mathcal{X}$, $\forall\, \lambda, \mu \in \mathbb{R}$

Ex: $\mathcal{X} = \mathbb{R}$ is a lin space

Ex: $\mathcal{X} = \mathbb{Q}$, over rationals.

- Norm: A norm in a linear space $\mathcal{X}$ is a function $\|\cdot\|: \mathcal{X} \to \mathbb{R}$ that satisfies:
  - $\|x\| \geq 0$; $\|x\| = 0 \Leftrightarrow x = 0$; $\forall\, x \in \mathcal{X}$ (non-negativity)
  - $\|\lambda x\| = |\lambda| . \|x\|$; $\forall\, x \in \mathcal{X}$, $\forall\, \lambda \in \mathbb{R}$
  - $\|x + y\| \leq \|x\| + \|y\|$; $\forall\, x, y \in \mathcal{X}$.

- Examples: $\mathcal{X} = \mathbb{R}^d$; let's define the l_p norm $\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p}$ $p \geq 0$

  $\|x\|_2 = \left(\sum_{i=1}^{d} |x_i|^2\right)^{1/2}$; $\|x\|_0 = \{\#$ elements in x that are non-zero$\}$;

  $\|x\|_\infty = \max\{|x_1| \ldots |x_d|\}$

- $\mathcal{X}$ is a normed linear space if it is endowed with a norm $\|\cdot\|$. i.e. $(\mathcal{X}, \|\cdot\|)$ is a normed linear space.

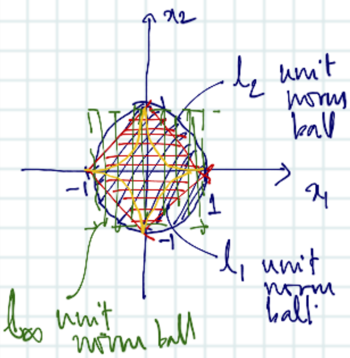
- $d(x, y) = \|x - y\|$ is a valid distance metric. Check!

- Unit Norm ball: B in a normed linear space $(\mathcal{X}, \|\cdot\|)$ is the set of points in $\mathcal{X}$ that satisfy

  $$B = \{x \in \mathcal{X} \mid \|x\| \leq 1\}$$

  - $\mathcal{X} = \mathbb{R}^2$; How does the unit norm ball look like for $\|\cdot\|_2$?

    i.e. find $x \in \mathcal{X}$ such that $\sqrt{x_1^2 + x_2^2} \leq 1$. This is a disc.
    $\hookrightarrow x = (x_1, x_2)$



  - $\mathcal{X} = \mathbb{R}^2$, $\|\cdot\|_1$: $x \in \mathcal{X}$ such that $|x_1| + |x_2| \leq 1$

  - $\mathcal{X} = \mathbb{R}^2$, $\|\cdot\|_\infty$: $x \in \mathcal{X}$ such that $\max\{|x_1|, |x_2|\} \leq 1$

  - Sparse representations of signals make of $\|\cdot\|_0$ norm. $\underset{x \in \mathbb{R}^d}{\operatorname{argmin}} \|Ax - b\|_2 + \lambda \|x\|_0$
    $\to$ Compressed sensing.

  - A linear space $\mathcal{X}$ is said to be convex if for any $x, y \in \mathcal{X}$ and for any $0 \leq \lambda \leq 1$,
    $\lambda x + (1-\lambda) y \in \mathcal{X}$

- A sequence (x_n) is a map from the set of natural numbers $\mathbb{N}$ to $\mathbb{R}$.

- A sequence of real numbers (a_n) is said to converge to x if $\forall \varepsilon > 0$ $\exists N$ such that $|a_n - x| < \varepsilon$ for all $n > N$.

- Ex: $(-1)^n \xrightarrow[n \to \infty]{} ?$

  Ex: $x_n = \left(1 + \frac{1}{n}\right)^n$ $(\mathcal{X} \in \mathbb{R})$

  $$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

Ex: $x_n = \frac{1}{n}$; $\lim_{n \to \infty} \frac{1}{n} = 0$

- Let (x_n) be a sequence of real numbers. This sequence is Cauchy if for any $\epsilon > 0$, $\exists N$
  $|x_n - x_m| < \varepsilon$ for all $n, m > N$.