Topics in Computing (CS5160) : Problem Set 3

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- Scan and upload your answer sheets on google classroom.
- Maintain academic honesty. If caught, you will get an F in the course.
- Please write "credit" or "audit" on your answer sheets depending on whether you are crediting or auditing the course.
- Due Date: 30 November (before 11:59pm).

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1. Given a monotone Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ and an input $x \in \{0,1\}^n$, say that the i-th bit x_i of x is "correct" for f if $f(x) = x_i$. Let $c(f)$ denote the expected number of "correct" bits in a uniformly random string x. Show that $c(f) = (n + \mathsf{Inf}(f))/2$. **(10 points)**

2. Let $f\colon \{-1,1\}^n \to \{-1,1\}$. Give a Fourier formula for the expression

$$\mathbb{E}_{x,y,z,w\sim\{-1,1\}^n}[f(x)f(y)f(z)f(w)],$$

   where x, y, z are chosen uniformly at random from $\{-1,1\}^n$ and $w = x \oplus y \oplus z$, i.e., $w_i = x_i y_i z_i$ for all $i \in [n]$. **(10 points)**

3. Let $\rho \in [-1,1]$ and $x \in \{-1,1\}^n$. Recall we say $y \sim N_\rho(x)$ to denote that the random string y is sampled as follows: $y_i = x_i$ with probability $(1+\rho)/2$ and $y_i = -x_i$ with probability $(1-\rho)/2$. For a Boolean function $f\colon \{-1,1\}^n \to \{-1,1\}$, we define *noise stability* of f at ρ as follows

$$\mathsf{Stab}_\rho(f) = \mathbb{E}_{x\sim\{-1,1\}^n, y\sim N_\rho(x)}[f(x)f(y)]$$

   Give a Fourier formula for $\mathsf{Stab}_\rho(f)$. **(10 points)**

4. Let $\varepsilon > 0$. Prove that for every Boolean function $f\colon \{-1,1\}^n \to \{-1,1\}$, there exists a Boolean function $g\colon \{-1,1\}^n \to \{-1,1\}$ depending on at most $2^{O(\mathsf{as}(f)/\varepsilon)}$ variables such that g differs from f on at most an ε fraction of inputs. Recall $\mathsf{as}(f)$ denotes the average sensitivity of f. **(15 points)**

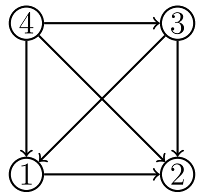

Figure 1: A tournament on 4 vertices

5. A *tournament* is a directed graph obtained by assigning a direction to each edge in an undirected complete graph. (See Figure 1.) We say that a tournament is *acyclic* if it contains no directed cycles. Note that a tournament can be represented by a string in $\{0,1\}^{\binom{n}{2}}$, where every edge is represented by a bit and its value represents the orientation of the edge. Thus, we can define the following Boolean function $T_{\text{acyclic}}\colon \{0,1\}^{\binom{n}{2}} \to \{0,1\}$ such that $T_{\text{acyclic}}(x) = 1$ if and only if x defines an acyclic tournament.

   Prove that $\mathsf{D}(T_{\text{acyclic}}) \geq \binom{n}{2} - \frac{n}{2}$. Recall $\mathsf{D}(f)$ is the deterministic decision tree complexity of f. **(15 points)**

   You will get partial credit even if you can only prove $\Omega(n^2)$ lower bound. On the other hand you will get extra credit if you can prove the tight lower bound of $\binom{n}{2}$.

6. Let T be a tournament and v be a vertex of T. We say that v is a *source* if all edges incident on v are directed *away* from it. For example, the vertex labelled 4 is the source in the tournament shown in Figure 1. Not every tournament has a source. For example, the tournament obtained by flipping the direction of edge $(4,2)$ in Figure 1. Therefore we can consider the following Boolean function $\mathsf{SRC}\colon \{0,1\}^{\binom{n}{2}} \to \{0,1\}$ defined as $\mathsf{SRC}(x) = 1$ if and only if the tournament given by x has a source.

   Show that $\mathsf{D}(\mathsf{SRC}) = O(n)$. **(15 points)**

7. For $1 \leq t \leq n$, let $\mathsf{Th}_t\colon \{0,1\}^n \to \{0,1\}$ be the threshold function defined as follow:

$$\mathsf{Th}_t(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^n x_i \geq t, \\ 0 & \text{otherwise.} \end{cases}$$

   Prove that $\deg(\mathsf{Th}_t) = n$, i.e., any polynomial representing Th_t must have full degree n. **(15 points)**