09/10/2023

LB :- $\exists$ a function $f : \{0,1\}^n \to \{0,1\}$

s.t. $\text{DNF-Size}(f) + \text{CNF-Size}(f)$
$$= N$$

Then the size of any decision tree computing f is at least $2^{\Omega(\log^2 N)}$

$$N^{\Omega(\log N)}.$$

Fourier Analysis of Boolean functions.

Up until now

$$f : \{0,1\}^n \rightarrow \{0,1\}$$

From now onwards,

$$f : \{-1, +1\}^n \rightarrow \{-1, +1\}$$

Or real-valued functions.

$$f : \{\pm 1\}^n \rightarrow \mathbb{R}.$$

$\Rightarrow$ vector space of all functions from $\{\pm 1\}^n$ to $\mathbb{R}$.

→ A particularly nice basis for this space. is given by the following.

$$\{ f \mid f : \{\pm 1\}^n \to \mathbb{R} \}$$

$$\{\pm 1\}^n \quad \underbrace{\begin{matrix} \to \\ \to \\ \to \\ \to \\ \to \\ \to \end{matrix}}_{2^n} \begin{bmatrix} \cdot \in \mathbb{R}. \\ \\ \\ \\ \\ \\ \end{bmatrix}_{2^n \times 1}$$

column vector

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} \qquad \mathbb{R}^{2^n}$$

Fourier basis: $\chi_S : \{\pm 1\}^n \to \{\pm 1\}$

$\forall S \subseteq [n]$, $\chi_S(x) = \prod_{i \in S} x_i$

In particular, $\chi_\emptyset(x) = 1$

constant

1 function.

$\Rightarrow$ χ_S outputs -1

iff there are odd nos.

of -1 in S.

in the input.

$\Rightarrow$ PARITY Functions.

Fact 1 :- They are a basis for the space $f : \{\pm 1\}^n \to \mathbb{R}$.

Pf :- $f^b : \{0,1\}^n \to \{0,1\}$

Consider the polynomial rep. f^b.

$$P_{f^b}(x_1, \dots, x_n) = f^b(x_1, \dots, x_n)$$

$0 \longleftrightarrow 1$

$1 \longleftrightarrow -1$

We want this correspondence.

$P_{f^b} : \{0,1\}^n \to \{0,1\}$

$$p'_{f^b} = 1 - 2 \cdot p_{f^b}$$

$$p' : \{0,1\}^n \to \{\pm 1\}$$

$$F : \{\pm 1\}^n \to \{\pm 1\}$$

$$F(x_1, \dots, x_n) = p'_{f^b}\left(\frac{1-x_1}{2}, \dots, \frac{1-x_n}{2}\right)$$

$$F : \{\pm 1\}^n \to \{\pm 1\}$$

$$F(x) = \sum_{S \subseteq [n]} c_S \prod_{i \in S} x_i$$

$\in \mathbb{R}$

No higher powers

$x_i^2 = 1$

$\wedge$ parity.

For arbitrary functions

$$f : \{\pm 1\}^n \to \mathbb{R}$$

$$f(x) = \sum_{S \subseteq [n]} c_S \prod_{i \in S} x_i$$

"Fourier representation"

$$\text{AND} : \{\pm 1\}^n \to \{\pm 1\}$$

$$\text{AND}(-1, -1, \ldots, -1) = -1$$

$$\text{AND}(1, \ldots) = 1$$

$-1 = \text{True} = 1$

$1 = \text{False} = 0$

$$\text{AND}^b : \{0,1\}^n \to \{0,1\}$$

$$\prod_{i=1}^{n} x_i$$

$$\text{AND}(x_1, \ldots, x_n)$$

$$= 1 - 2 \cdot \prod_{i=1}^{n} \left(\frac{1 - x_i}{2} \right)$$

$$= \left(1 - \frac{2}{2^n}\right) + \sum_{\emptyset \neq S \subseteq [n]} \frac{(-1)^{|S|+1}}{2^{n-1}} \prod_{i \in S} x_i$$

$\nearrow \in \mathbb{R}$ $\qquad\qquad$ $\nearrow \in \mathbb{R}$

$\widehat{AND}(\emptyset)$ $\qquad\qquad$ $\widehat{AND}(S)$

Inner poduct :

$f : \{\pm 1\}^n \to \mathbb{R} \qquad g : \{\pm 1\}^n \to \mathbb{R}$

$$\langle f, g \rangle := \mathbb{E}_x \left[f(x) \cdot g(x) \right]$$

$$= \frac{1}{2^n} \sum_{x \in \{\pm 1\}^n} f(x) \cdot g(x)$$

Over uniform distribution over $\{\pm 1\}^n$

Defn: f and g are orthogonal functions if $\langle f, g \rangle = 0$.

Example:

$$\langle \chi_S, \chi_T \rangle$$

$$= \mathbb{E}_x \left[\prod_{i \in S} x_i \prod_{j \in T} x_j \right]$$

$$= \mathbb{E}_x \left[\prod_{i \in S \Delta T} x_i \right]$$

1

when $S \Delta T = \{S \setminus T\} \cup \{T \setminus S\}$

$$= \prod_{i \in S \Delta T} \mathbb{E}_x [x_i]$$

$$\mathbb{E}_{x \sim_u \{\pm 1\}^n} [x_i]$$

$$= \frac{1}{2} \cdot 1 + \frac{1}{2}(-1)$$

$$= 0$$

$= 0$. if $S \Delta T \neq \emptyset$

if $S \Delta T = \phi$

$\Rightarrow S = T$

from 1,

$$\mathbb{E}_x \left[\prod_{i \in S} x_i \prod_{j \in T} x_j \right]$$

$$= \mathbb{E}_x \left[\prod_{i \in S} x_i^2 \right]$$

$$= \mathbb{E}_x [1] = 1.$$

$$\langle \chi_S, \chi_T \rangle = \begin{cases} 1 & \text{if } S = T \\ 0 & \text{if } S \neq T \end{cases}$$

"Orthonormal" basis.

$$f(x) = \sum_{S \subseteq [n]} c_S \prod_{i \in S} x_i$$

$$= \sum_{S \subseteq [n]} c_S \, \chi_S(x)$$

$$\langle f, \chi_T \rangle = ?$$

$$\langle f, \chi_T \rangle = \mathbb{E}_x\left[f(x) \cdot \chi_T(x) \right]$$

$$= \mathbb{E}_x\left[\left(\sum_{S \subseteq [n]} c_S \, \chi_S(x) \right) \cdot \chi_T(x) \right]$$

$$= \mathbb{E}_x\left[\sum_{S \subseteq [n]} c_S \, \chi_S(x) \cdot \chi_T(x) \right]$$

$$= \sum_{S \subseteq [n]} c_S \, \mathbb{E}_x\left[\chi_S(x) \cdot \chi_T(x) \right]$$

$$= \sum_{S \subseteq [n]} c_S \, \langle \chi_S, \chi_T \rangle$$

$$\langle f, \chi_T \rangle = c_T$$

$$\parallel$$

$$\hat{f}(T)$$

fourier coefficient

$$f(x_1, \ldots, x_n) = \sum_{S \subseteq [n]} \hat{f}(S) \prod_{i \in S} x_i$$

$$= \sum_{S \subseteq [n]} \hat{f}(S)\, \chi_S(x)$$

Consider

$$\langle f, g \rangle = \mathbb{E}_x [f(x) \cdot g(x)]$$

$$= \mathbb{E}_x \left[\sum_{S,T} \hat{f}(S) \cdot \hat{g}(T) \chi_S(x) \chi_T(x) \right]$$

$$= \sum_{S,T \subseteq [n]} \hat{f}(S) \cdot \hat{g}(T) \cdot \underbrace{\mathbb{E}_x [\chi_S(x) \cdot \chi_T(x)]}_{\in \{1, 0\}}$$

$$= \sum_S \hat{f}(S) \cdot \hat{g}(S).$$

$$\langle f, g\rangle = \mathbb{E}_x[f(x)\cdot g(x)]$$

$$= \sum_{S\subseteq[n]} \hat{f}(S)\cdot\hat{g}(S)$$

$$= \langle \hat{f}, \hat{g}\rangle$$

Plancherel's Identity

$f=g$ and $f:\{\pm 1\}^n \to \{\pm 1\}$

$$\langle f, f\rangle = \mathbb{E}_x\left[f(x)^2\right] = 1$$

$$= \sum_{S\subseteq[n]} \hat{f}(S)^2$$

$$\sum_{S \subseteq [n]} \hat{f}(S)^2 = 1.$$

"Parseval's Identity"

Q: $-1 \leq \hat{f}(S) \leq 1$,

$$\hat{f}(S) = \langle f, \chi_S \rangle$$

$$= \mathbb{E}_x \left[f(x) \cdot \chi_S(x) \right]$$

$$f : \{\pm 1\}^n \to \{\pm 1\}$$

$$\chi_S : \{\pm 1\}^n \to \{\pm 1\}$$

$$\boxed{f(x) = \sum_{S \subseteq [n]} \hat{f}(S) \prod_{i \in S} x_i}$$

Fourier degree of f $= \max\{|S| \mid \hat{f}(S) \neq 0\}$

Q: Fourier-degree $=$ real-degree?

$$\{0,1\}^n \qquad\qquad \{\pm 1\}^n$$

$$\prod_{i \in T} x_i = \prod_{i \in T} \left(\frac{\cancel{1} - x_i}{2} \right)$$

$$MAJ : \{\pm 1\}^3 \to \{\pm 1\}$$

$$\left(\frac{1+x_1}{2}\right) \cdot \left(\frac{1+x_2}{2}\right)\left(\frac{1+x_3}{2}\right) \cdot \underline{1}$$

$$+ \left(\frac{1+x_1}{2}\right)\left(\frac{1+x_2}{2}\right)\left(\frac{1-x_3}{2}\right) \quad \underline{1}$$

$$+ \left(\frac{1+x_1}{2}\right)\left(\frac{1-x_2}{2}\right)\left(\frac{1+x_3}{2}\right) \quad \underline{1}$$

$$+ \left(\frac{1+x_1}{2}\right)\left(\frac{1-x_2}{2}\right)\left(\frac{1-x_3}{2}\right) (-1)$$

$$+\left(\frac{1-x_1}{2}\right)\left(\frac{1+x_2}{2}\right)\left(\frac{1+x_3}{2}\right)1$$

$$+\left(\frac{1-x_1}{2}\right)\left(\frac{1+x_2}{2}\right)\left(\frac{1-x_3}{2}\right)(-1)$$

$$+\left(\frac{1-x_1}{2}\right)\left(\frac{1-x_2}{2}\right)\left(\frac{1+x_3}{2}\right)(-1)$$

$$+\left(\frac{1-x_1}{2}\right)\left(\frac{1-x_2}{2}\right)\left(\frac{1-x_3}{2}\right)(-1)$$

$$=\frac{1}{2}x_1+\frac{1}{2}x_2+\frac{1}{2}x_3-\frac{1}{2}x_1x_2x_3$$

$$\left.x_1=x_2\right\} \Rightarrow x_1+\frac{1}{2}x_3-\frac{1}{2}x_3$$

$$x_1$$

$$\widehat{MAJ}(\emptyset) = 0$$

$$\widehat{MAJ}(\{1\}) = \frac{1}{2}$$

$$\widehat{MAJ}(\{1,2,3\}) = -\frac{1}{2}$$

$$\sum_{S \subseteq [3]} \widehat{MAJ}(S)^2 = \frac{1}{4} \times 4$$

$$= 1.$$

Thm :- Let $f : \{\pm 1\}^n \to \{\pm 1\}$.

Then $L(f) \geq \sum_{S \subseteq [n]} |\hat{f}(S)|$

min size of a decision tree computing f

$\|$ L_1-norm $\|\hat{f}\|_1$

(L_1-norm of f) $\|\hat{f}\|_1 := \sum_{S \subseteq [n]} |\hat{f}(S)|$

Prop :-

(1) $\|\widehat{f+g}\|_1 \leq \|\hat{f}\|_1 + \|\hat{g}\|_1$

$$\sum_{S \subseteq [n]} |\widehat{f+g}(S)| = \sum_{S \subseteq [n]} |\hat{f}(S) + \hat{g}(S)|$$

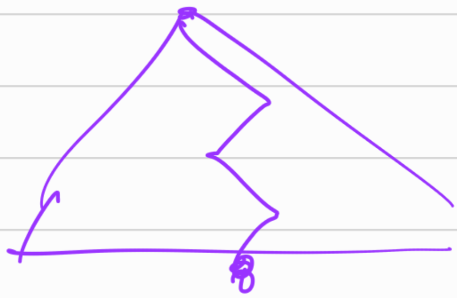
$$\leq \sum_{S \subseteq [n]} |\hat{f}(S)| + |\hat{g}(S)|$$

(2) $\|\widehat{f\cdot g}\|_1 \le \|\hat{f}\|_1 \cdot \|\hat{g}\|_1$

$$\left(\sum_{S,T} \hat{f}(S)\cdot \hat{g}(T) \underbrace{\prod_{i\in S} x_i \prod_{i\in T} x_i}_{S\Delta T}\right)$$

$$\le \left(\sum |\hat{f}(S)|\right)\left(\sum |\hat{g}(T)|\right)$$

$$= \|\hat{f}\|_1 \cdot \|\hat{g}\|_1$$



q

Indicator fn for this.