Triangle Removal Lemma

Prashanth Sriram, Taha Adeel Mohammed, Ketan Sabne

Indian Institute of Technology Hyderabad

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Outline

Introduction

Introduction

Theorem (Triangle Removal Lemma)

For every $\epsilon > 0$, there is a $\delta > 0$ such that if G is a simple and undirect graph which can be made free of triangles by making atleast ϵn^2 deletions, then G has atleast δn^3 triangles.

Motivation

We can easily conclude that G contains at least ϵn^2 triangles. But the strength of the triangle removal lemma is that, instead of quadratic, the number of triangles is cubic. It was first proved by Ruzsa and Szemeredi, who also observed it implies Roth's theorem.

Preliminaries

Notation

First we fix some notation. For us, a graph $G = (V, E)$ is simple and undirected, with vertex set V and edge set E. For two disjoint subsets A and B of V, we define $e(A, B)$ to be the number of edges across the two sets.

Definition (Edge Density)

For disjoint subsets A and B of V, we define the edge density as follows:

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

Definition (ϵ-regular pair (A,B))

The above pair is called ϵ-regular if for every $X \subset A$ and $Y \subset B$ with $|X| \geq \epsilon|A|$ and $|Y| \geq \epsilon|B|$, we have $|d(X, Y) - d(A, B)| < \epsilon$.

Notation (Contd)

Definition (ϵ regular partition)

The partition $V = V_0 \cup V_1 \cup \cdots \cup V_k$ is called ϵ-regular if

- $|V_0| \leq \epsilon n$
- $|V_1| = |V_2| = \cdots = |V_k|$
- At most ϵk^2 pairs (V_i, V_j) are not ϵ-regular.

Szemeredi's Regularity Lemma

Theorem (Szemeredi's Regularity Lemma)

For every $\epsilon > 0$ and every positive integer t, there exists integers $T(\epsilon, t)$ and $N(\epsilon, t)$ such that every graph G with $n \geq N$ vertices contains a ϵ-regular partition $V = V_0 \cup V_1 \cup \cdots \cup V_k$ with $t \leq k \leq T$.

The crucial aspect of this theorem is the fact the k given here is bounded above. In other words, this theorem could be taken to mean that any large enough graph can "roughly" be decomposed into boundedly many equi-sized clusters, which "roughly" behave "randomly" with each other.

Proof

Theorem

Theorem

(Triangle removal lemma). For any $0 < \epsilon < 1$, there is $\delta = \delta(\epsilon) > 0$ such that, whenever $G = (V, E)$ is ϵ-far from being triangle-free, then it contains at least $\delta|V|^3$ triangles.

Proof.

Let $G = (V, E)$ be an ϵ-far from being triangle-free graph and $|V| = n$. We can assume $n > N(\frac{\epsilon}{4}, \lfloor\frac{4}{\epsilon}\rfloor)$ by just taking δ sufficiently small so that

$$\delta \cdot N(\frac{\epsilon}{4}, \lfloor\frac{4}{\epsilon}\rfloor)^3 < 1$$

Now consider the $\frac{\epsilon}{4}$-regular partition $U = V_0, V_1, \ldots V_k$ given by Szemeredi's regularity lemma. Let $c = |V_1| = \cdots = |V_k|$ and G' be a graph obtained from G by deleting the following edges: □

Proof(contd..)

Proof.

- All edges which are incident in V_0: there are at most $\frac{\epsilon n^2}{4}$ edges
- All edges inside the clusters $V_1, \ldots, V_k$: the number of edges is at most
$$c^2 k < \frac{n^2}{k} < \frac{\epsilon n^2}{4}$$
- All edges that lie in irregular pairs: there are less than
$$(\frac{\epsilon}{4}.k^2).c^2 < \frac{\epsilon n^2}{4} \text{ edges.}$$
- All edges lying in a pair of clusters whose density is less than $\frac{\epsilon}{2}$: their cardinality is at most $\binom{k}{2} \frac{\epsilon c^2}{2} < \frac{\epsilon.n^2}{4}$.

As we can see if we sum up number of deleted edges from above 4 conditions then we can say that number of deleted edges is less than ϵn^2. So, from lemma, we can say that G' contains a triangle.Means some cases are left let's find out. □

Proof(contd..)

Proof.

The three vertices of such triangle belong to three remaining distinct clusters , let's assume it as V_1, V_2, V_3. We'll show that in fact these clusters support many triangles.Assume a vertex $v_1 \in V_1$ typical if it has at least $\frac{\epsilon c}{4}$ adjacent vertices in V_2 and at least $\frac{\epsilon c}{4}$ adjacent vertices in V_3 . So, by hypothesis,

$$d(V_i', V_j') \geq \frac{\epsilon}{4} \tag{1}$$

□

A Lower Bound on number of Typical vertices

- Observe that (the number of vertices in V_1 with at least $\frac{\epsilon c}{4}$ adjacent vertices in V_2) $\geq (1 - \epsilon/4) \cdot c$. Why?
  - Assume that this is not the case $\Rightarrow$ atleast $\frac{\epsilon c}{4}$ vertices in V_1 have less than $\frac{\epsilon c}{4}$ neighbours in V_2.
  - Consider the set of these vertices in V_1 as V_1'. Now, the number of edges between V_1' and V_2 is less than $|V_1'| \cdot \frac{\epsilon c}{4}$.
  - So,

$$d(V_1', V2) < \frac{|V_1'| \cdot \frac{\epsilon c}{4}}{|V_1'| \cdot |V2|} \tag{2}$$

$$= \frac{\epsilon}{4} \tag{3}$$

  - But, since V_1' has $\geq \frac{\epsilon c}{4}$ vertices, it should satisfy (1), contradicting our assumption

Lower Bound on number of Typical vertices (contd)

- Applying similar logic for the number of vertices in V_1 with atleast $\frac{\epsilon c}{4}$ adjacent vertices in V_3, we have:
  - At max, $\frac{\epsilon c}{4}$ vertices in $V_!$ doesn't have more than $\frac{\epsilon c}{4}$ neighbours in V_2
  - At max, $\frac{\epsilon c}{4}$ vertices in $V_!$ doesn't have more than $\frac{\epsilon c}{4}$ neighbours in V_3
- Thus, the number of typical vertices in V_1 is atleast

$$(1 - 2\frac{\epsilon}{4}) \cdot c = (1 - \frac{\epsilon}{2}) \cdot c \tag{4}$$

$$> \frac{c}{2} \quad \text{(Since } \epsilon < 1\text{)} \tag{5}$$

- The number of typical vertices in V_1 is atleast $\frac{c}{2}$

Triangles formed by a typical vertex

Let $v_1 \in V_1$ be a typical vertex. Let $V_2' \subset V_2$ and $V_3' \subset V_3$ be the vertices adjacent to v_1

Figure: Triangles formed by a typical vertex (Src: Yuri Lima's notes on Szemeredi's regularity Lemma)

Triangles formed by a Typical vertex

- As you can observe, every edge between V_2' and V_3' generate a triangle with v_1 as the third point
- Number of such triangles with v_1 as a vertex is equal to the number of edges across V_2' and V_3'

$$
\begin{aligned}
e(V_2', V_3') &= d(V_2', V_3') \cdot |V_2'| \cdot |V_3'| && (6)\\
&\geq \frac{\epsilon}{4} \cdot |V_2'| \cdot |V_3'| && (7)\\
&\geq \frac{\epsilon^3 c^2}{4^3} && (8)
\end{aligned}
$$

Triangles formed by Typical Vertices

- Summing this over all such $v_1 \in V_1$ typical, G' has atleast

$$> \frac{c}{2} \cdot \frac{\epsilon^3 c^2}{4^3} = \frac{\epsilon^3 c^3}{2 \cdot 4^3} \tag{9}$$

$$> (\frac{\epsilon c}{4})^3 \tag{10}$$

  many triangles

- Since $c > \frac{N}{2T(\epsilon/4, \lfloor 4/\epsilon \rfloor)}$, the above quantity is $\geq$

$$(\frac{\epsilon}{4} \cdot \frac{n}{2 \cdot T(\epsilon/4, \lfloor 4/\epsilon \rfloor)})^3 = (\frac{\epsilon}{8 \cdot T(\epsilon/4, \lfloor 4/\epsilon \rfloor)})^3 \cdot n^3 \tag{11}$$

$$= \delta(\epsilon) \cdot n^3 \tag{12}$$

Thank you!

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