Lecture-1

Background Information

Abstract: In this lecture we provide background information needed to study the subjects of Analog Communication course.

Impulse or Dirac delta function

Continuous time impulse function is defined as

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \qquad \text{and we have} \qquad \int_{-\infty}^{\infty} \delta(t) dt = 1.$$

The impulse function satisfies the properties

a)

$$g(t)\delta(t - t_0) = g(t_0)\delta(t - t_0)$$

$$g(t - t_1)\delta(t - t_0) = g(t_0 - t_1)\delta(t - t_0)$$

b)

$$\int_{-\infty}^{\infty} g(t)\delta(t - t_0)dt = g(t_0)$$

c)

$$\int_{-\infty}^{\infty} g(t - t_1)\delta(t - t_0)dt = g(t_0 - t_1)$$

d)

$$\delta\big(a(t - t_0)\big) = \frac{1}{|a|}\delta(t - t_0)$$

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

e)

$$g(t) * \delta(t - t_0) = g(t - t_0)$$

$$g(t - t_1) * \delta(t - t_0) = g(t - t_0 - t_1)$$

where $*$ denotes the convolution operation.

Complex Exponential Function

The complex exponential function $e^{j\theta}$ is defined as

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

The function $e^{j\theta}$ satisfies the property

$$\left|e^{j\theta}\right| = 1$$

since we have

$$\left|e^{j\theta}\right| = |\cos\theta + j\sin\theta| \rightarrow \left|e^{j\theta}\right| = \sqrt{(\cos\theta)^2 + (\sin\theta)^2}$$

Note that the complex number $a = b + jc$ can be expressed as

$$a = |a|e^{j\angle a}$$

where

$$|a| = \sqrt{b^2 + c^2} \quad \text{and} \quad \angle a = \arctan\frac{c}{b}.$$

Example: Calculate $\angle e^{j\theta}$.

Solution: The expression $\angle e^{j\theta}$ can be written as

$$e^{j\theta} = \cos\theta + j\sin\theta$$

whose phase can be calculated as

$$\angle e^{j\theta} = \tan^{-1}\frac{\sin\theta}{\cos\theta} \rightarrow \angle e^{j\theta} = \tan^{-1}\frac{\sin\theta}{\cos\theta}$$

leading to

$$\angle e^{j\theta} = \tan^{-1}\tan\theta \rightarrow \angle e^{j\theta} = \theta.$$

Note that if $f(x)$ is a function and $f^{-1}(x)$ is the inverse function, then we have the property,

$$f^{-1}(f(x)) = x \quad \text{or} \quad f(f^{-1}(x)) = x.$$

Example: Write $\cos(\theta)$ in terms of the complex exponential functions.

Solution: We can write $e^{j\theta}$ and $e^{-j\theta}$ as in

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{A}$$

$$e^{-j\theta} = \cos\theta - j\sin\theta \tag{B}$$

Summing (A) and (B), we obtain

$$e^{j\theta} + e^{-j\theta} = \cos\theta + j\sin\theta + \cos\theta - j\sin\theta$$

from which we get

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right).$$

Exercise: Show that $\sin\theta$ can be written as

$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

Unit Step Function

The unit step function can be defined either as

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & otherwise \end{cases}$$

or as

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & otherwise. \end{cases}$$

The relation between impulse function and unit step function can be expressed as

$$\delta(t) = \frac{du(t)}{dt}$$

$$u(t) = \int_{-\infty}^{t} \delta(t)dt.$$

Continuous Time Fourier Transform

The Fourier transform of the continuous time signal $g(t)$ is calculated as

$$G(w) = \int_{-\infty}^{\infty} g(t) e^{-jwt} dt \tag{1}$$

where w is the angular frequency and j is the complex number such that $j = \sqrt{-1}$.

Employing $w = 2\pi f$ in (1), we can express the Fourier transform of $g(t)$ as in

$$G(2\pi f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt \rightarrow G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt$$

Note that

$$w = 2\pi f$$

is the angular frequency, and its unit is radian/sec. On the other hand the unit of the f is $Hertz$, i.e., Hz, and we have

$$1\ Hz = \frac{1}{1\ sec}.$$

Inverse Fourier Transform

The inverse Fourier transform of $G(w)$ is defined as

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(w) e^{jwt} dw \tag{2}$$

where w is the angular frequency and j is the complex number such that $j = \sqrt{-1}$.

If we use $w = 2\pi f$ in (2), we obtain

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(2\pi f) e^{j2\pi ft} 2\pi df \rightarrow g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df.$$

In brief, Fourier transform and inverse Fourier transform formulas for the signal $g(t)$ can be written as

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt$$

and

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df$$

respectively.

The Fourier transform function $G(f)$ is a complex function, and it can be expressed as

$$G(f) = |G(f)| e^{j\theta(f)} \tag{3}$$

where $G(f)$ is called magnitude spectrum of $g(t)$ and $\theta(f)$ is called phase spectrum of $g(t)$.

$G(f)$ can also be written as

$$G(f) = X(f) + jY(f) \tag{4}$$

Using (3) and (4) we can write

$$|G(f)| = \sqrt{X^2(f) + Y^2(f)} \qquad \theta(f) = \arctan \frac{Y(f)}{X(f)}.$$

Example: The Fourier transform of a continuous function is given as

$$G(f) = \frac{1}{1 - jf}$$

Find $|G(f)|$ and $\theta(f)$.

Solution: The Fourier transform expression can be written as

$$G(f) = \frac{1}{1 + j2f} \rightarrow G(f) = \frac{1}{1 + 4f^2} - j\frac{2f}{1 + 4f^2}$$

We calculate the magnitude and phase spectrums as

$$|G(f)| = \sqrt{\frac{1}{(1 + 4f^2)^2} + \frac{4f^2}{(1 + 4f^2)^2}} \rightarrow$$

$$|G(f)| = \sqrt{\frac{1}{1 + 4f^2}}$$

$$\theta(f) = \arctan\frac{Im[G(f)]}{Real[G(f)]} \rightarrow \theta(f) = \arctan\frac{-\frac{2f}{1+4f^2}}{\frac{1}{1+4f^2}} \rightarrow \theta(f) = \arctan(-2f)$$

Thus

$$\theta(f) = -\arctan 2f$$

Example: Show that

$$\arctan(-x) = -\arctan x$$

.

Proof: We know that $\tan(-x) = -\tan x$.

If

$$y = \arctan(-x)$$

then we have

$$-x = \tan y$$

from which we get

$$x = \tan(-y)$$

leading to

$$-y = \arctan x$$

which can be written as

$$y = -\arctan x$$

Example: Find

$$\angle \cos(\pi f)$$

Solution:

Example: Given

$$G(f) = 1 + e^{-j2\pi f}$$

find $|G(f)|$

Solution: The complex function

$$G(f) = 1 + e^{-j2\pi f}$$

can be written as

$$G(f) = 1 + e^{-j2\pi f} = 1 + \cos 2\pi f - j\sin 2\pi f$$

and $|G(f)|$ can be calculated as

$$|G(f)| = \sqrt{(1+\cos 2\pi f)^2 + (\sin 2\pi f)^2}$$

leading to

$$|G(f)| = \sqrt{1 + 2\cos 2\pi f + (\cos 2\pi f)^2 + (\sin 2\pi f)^2}$$

where using

$$(\cos 2\pi f)^2 + (\sin 2\pi f)^2 = 1$$

we get

$$|G(f)| = \sqrt{2 + 2\cos 2\pi f}$$

which can be written as

$$|G(f)| = \sqrt{2}\sqrt{1 + \cos 2\pi f}$$

where using

$$\cos 2\pi f = 2\cos^2 \pi f - 1$$

we get

$$|G(f)| = 2\,|\cos(\pi f)|$$

Example: For the given analog signal $g(t) = \delta(t) + \delta(t-1)$ find the magnitude and phase spectrum functions.

Solution: Using the Fourier transform formula

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt$$

for the given function, we obtain

$$G(f) = \int_{-\infty}^{\infty} \big(\delta(t) + \delta(t-1)\big) e^{-j2\pi f t} dt$$

in which using the property

$$\int \delta(t - t_0) f(t) dt = f(t_0)$$

we get

$$G(f) = e^{-j2\pi f 0} + e^{-j2\pi f 1} \rightarrow G(f) = 1 + e^{-j2\pi f} \tag{6}$$

The spectrum expression in (6) can be manipulated as

$$G(f) = 2e^{-j\pi f} \cos(\pi f) \tag{7}$$

From (7), the magnitude and phase spectrums can be calculated as

$$|G(f)| = |2| \underbrace{|e^{-j\pi f}|}_{=1} |\cos(\pi f)| \rightarrow |G(f)| = 2\,|\cos(\pi f)|$$

$$\angle G(f) = \angle 2 + \angle e^{-j\pi f} + \angle \cos(\pi f) \rightarrow \angle G(f) = 0 - \pi f + \angle \cos(\pi f)$$

The period of $\cos(\pi f)$ is 2, and in one period $\angle\cos(\pi f)$ can be written as

$$\theta_c(f) = \angle\cos(\pi f) = \begin{cases} 0 & f \leq 0.5 \text{ or } 1.5 \leq f < 2 \\ \pi & 0.5 < f < 1.5 \end{cases}$$

$\angle\cos(\pi f)$ can be written as

$$\angle\cos(\pi f) = \sum_k \theta_c(f - k2).$$

Fourier transform of the complex exponential signal

The Fourier transform of $e^{jw_0 t}$ is

$$2\pi\delta(w - w_0)$$

i.e., we have the pair

$$e^{jw_0 t} \overset{FT}{\leftrightarrow} 2\pi\delta(w - w_0) \tag{8}$$

If we use $w = 2\pi f$ and $w_0 = 2\pi f_0$ in (8), we get

$$e^{j2\pi f_0 t} \overset{FT}{\leftrightarrow} 2\pi\delta(2\pi f - 2\pi f_0)$$

where the right hand side can be written as

$$2\pi\delta(2\pi f - 2\pi f_0) \rightarrow 2\pi\delta\big(2\pi(f - f_0)\big) \tag{9}$$

in which using the property

$$\delta\big(a(f - f_0)\big) = \frac{1}{|a|}\delta(f - f_0)$$

we simplify the right hand side of (9) as

$$2\pi\delta\big(2\pi(f - f_0)\big) \rightarrow \frac{2\pi}{2\pi}\delta\big((f - f_0)\big) \rightarrow \delta(f - f_0).$$

Thus, we have

$$e^{j2\pi f_0 t} \overset{FT}{\leftrightarrow} \delta(f - f_0) \tag{10}$$

Example: If $G(f) = \delta(f - f_0)$, find $g(t)$ using inverse Fourier transform formula.

Solution: Employing the inverse Fourier transform formula

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} dt$$

for the given Fourier transform we get

$$g(t) = \int_{-\infty}^{\infty} \delta(f - f_0) e^{j2\pi f t} dt$$

in which using the property

$$\int \delta(f - f_0) G(f) df = G(f_0)$$

we obtain

$$g(t) = e^{j2\pi f_0 t}$$

which is the complex exponential function.

Fourier Transforms of Sine and Cosine Functions

The sine and cosine functions $\sin(2\pi f_0 t)$ and $\cos(2\pi f_o t)$ can be written in terms of the complex exponential functions as

$$\sin(2\pi f_0 t) = \frac{1}{2j}\left(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right)$$

$$\cos(2\pi f_0 t) = \frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)$$

Using (10) we can evaluate the Fourier transforms of the sine and cosine signals as

$$\sin(2\pi f_0 t) \overset{FT}{\leftrightarrow} \frac{1}{2j}\left(\delta(f - f_0) - \delta(f + f_0)\right)$$

$$\cos(2\pi f_0 t) \overset{FT}{\leftrightarrow} \frac{1}{2}\left(\delta(f - f_0) + \delta(f + f_0)\right)$$

Properties of Fourier Transform

Linearity:

Let

$$g_1(t) \overset{FT}{\leftrightarrow} G_1(f)$$

$$g_2(t) \overset{FT}{\leftrightarrow} G_2(f)$$

then we have

$$ag_1(t) + bg_2(t) \overset{FT}{\leftrightarrow} aG_1(f) + bG_2(f).$$

Time Scaling:

If

$$g(t) \overset{FT}{\leftrightarrow} G(f)$$

then we have

$$g(at) \overset{FT}{\leftrightarrow} \frac{1}{|a|} G\left(\frac{f}{a}\right).$$

Example: Using

$$\cos(2\pi f_0 t) \overset{FT}{\leftrightarrow} \frac{1}{2}\big(\delta(f - f_0) + \delta(f + f_0)\big)$$

and time scaling property, we can find the Fourier transform of

$$\cos(\pi f_0 t)$$

as

$$\frac{1}{1/2}\,\frac{1}{2}\left(\delta\left(\frac{f}{1/2} - f_0\right) + \delta\left(\frac{f}{1/2} + f_0\right)\right)$$

which can be written as

$$\delta(2f - f_0) + \delta(2f + f_0)$$

which is written as

$$\delta\left(2\left(f - \frac{f_0}{2}\right)\right) + \delta\left(2\left(f + \frac{f_0}{2}\right)\right)$$

leading to

$$\frac{1}{2}\left[\delta\left(f - \frac{f_0}{2}\right) + \delta\left(f + \frac{f_0}{2}\right)\right]$$

In fact, writing $\cos(\pi f_0 t)$ as

$$\cos\left(2\pi\left(\frac{f_0}{2}\right)t\right)$$

and using

$$\cos(2\pi f' t) \overset{FT}{\leftrightarrow} \frac{1}{2}\big(\delta(f - f') + \delta(f + f')\big)$$

we can get the same result.

Duality:

If

$$g(t) \overset{FT}{\leftrightarrow} G(f)$$

then we have

$$G(t) \overset{FT}{\leftrightarrow} g(-f).$$

Example: Using

$$\cos(2\pi f_0 t) \overset{FT}{\leftrightarrow} \frac{1}{2}\big(\delta(f - f_0) + \delta(f + f_0)\big)$$

and duality property we can write

$$\frac{1}{2}\big(\delta(t - t_0) + \delta(t + t_0)\big) \overset{FT}{\leftrightarrow} \cos(2\pi f t_0)$$

Time Shifting:

If

$$g(t) \overset{FT}{\leftrightarrow} G(f)$$

then we have

$$g(t - t_0) \overset{FT}{\leftrightarrow} e^{-j2\pi f t_0} G(f).$$

Frequency Shifting:

If

$$g(t) \overset{FT}{\leftrightarrow} G(f)$$

then we have

$$e^{j2\pi f_0 t} g(t) \overset{FT}{\leftrightarrow} G(f - f_0).$$

Differentiation:

If

$$g(t) \overset{FT}{\leftrightarrow} G(f)$$

then we have

$$\frac{dg(t)}{dt} \overset{FT}{\leftrightarrow} j2\pi f G(f).$$

In general, we have

$$\frac{d^n g(t)}{dt^n} \overset{FT}{\leftrightarrow} (j2\pi f)^n G(f).$$

Convolution in Time Domain

Let

$$g_1(t) \overset{FT}{\leftrightarrow} G_1(f)$$

$$g_2(t) \overset{FT}{\leftrightarrow} G_2(f)$$

then we have

$$g_1(t) * g_2(t) \overset{FT}{\leftrightarrow} G_1(f)G_2(f)$$

where $*$ denotes the convolution operation.

Note:

$$g_1(t) * g_2(t) = \int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau$$

Convolution in Frequency Domain

Let

$$g_1(t) \overset{FT}{\leftrightarrow} G_1(f)$$

$$g_2(t) \overset{FT}{\leftrightarrow} G_2(f)$$

then we have

$$g_1(t)g_2(t) \overset{FT}{\leftrightarrow} G_1(f) * G_2(f)$$

where $*$ denotes the convolution operation.

Note:

$$G_1(f) * G_2(f) = \int_{-\infty}^{\infty} G_1(\tau)G_2(f-\tau)d\tau$$