Lecture-3

Hilbert Transform

Abstract: Hilbert Transform

Signum function and its Fourier transform

The signum function is defined as

$$sgn(t) = \begin{cases} +1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0. \end{cases} \tag{1}$$

The signum function is depicted in Figure-1

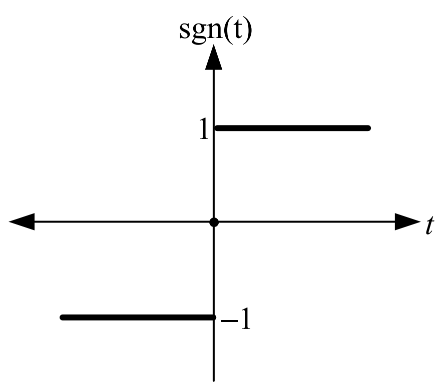

Figure-1 Signum function.

$$sgn(t) \overset{FT}{\leftrightarrow} \frac{1}{j\pi f}.$$

Odd and Even Function:

The function $g(t)$ is an odd function if it satisfies

$$g(-t) = -g(t).$$

It is even function if it satisfies

$$g(-t) = g(t).$$

Odd functions are symmetric w.r.t. origin, i.e., the same distance from the central point for the same x values.

Even functions are symmetric w.r.t. vertical axis, i.e., the same distance from the y-axis for the same x values.

Example: The signum function is an odd function, since it satisfies

$$sgn(-t) = -sgn(t).$$

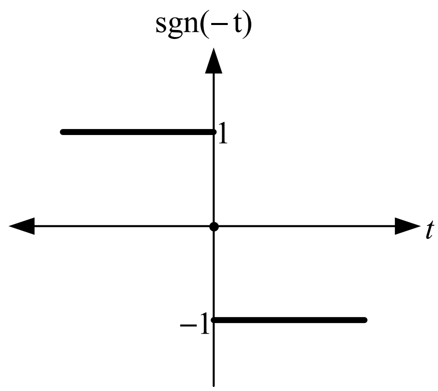

Figure-1 $sgn(-t)$ function.

Note that $sgn(at)$ is obtained by dividing the time axis of $sgn(t)$ by a.

Using the duality property

$$\text{if } g(t) \overset{FT}{\leftrightarrow} G(f) \text{ then } G(t) \overset{FT}{\leftrightarrow} g(-f)$$

we can write that

$$\frac{1}{j\pi t} \overset{FT}{\leftrightarrow} sgn(-f)$$

leading to

$$\frac{1}{\pi t} \overset{FT}{\leftrightarrow} -jsgn(f) \tag{2}$$

Hilbert transform

The Hilbert transform of $g(t)$ is defined as

$$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(\tau)}{t-\tau}d\tau$$

which can also be written as

$$\hat{g}(t) = \frac{1}{\pi t} * g(t).$$

That is Hilbert transform of $g(t)$ is obtained by taking the convolution of $g(t)$ by $\frac{1}{\pi t}$.

The inverse Hilbert transform is calculated as

$$g(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\hat{g}(\tau)}{t-\tau}d\tau$$

which can also be written as

$$g(t) = -\frac{1}{\pi t} * \hat{g}(t).$$

Fourier transform of Hilbert transform

If we take the Fourier transform of both sides of

$$\hat{g}(t) = \frac{1}{\pi t} * g(t)$$

we get

$$FT\{\hat{g}(t)\} = FT\left\{\frac{1}{\pi t}\right\}FT\{g(t)\}$$

$$FT\{\hat{g}(t)\} = FT\left\{\frac{1}{\pi t}\right\}FT\{g(t)\}$$

which can be written as

$$\hat{G}(f) = -jsgn(f)G(f) \tag{3}$$

in which using

$$sgn(f) = \begin{cases} 1 & f > 0 \\ 0 & f = 0 \\ -1 & f < 0 \end{cases}$$

we get

$$\hat{G}(f) = \begin{cases} -jG(f) & f > 0 \\ 0 & f = 0 \\ jG(f) & f < 0 \end{cases}$$

where employing $j = e^{j\frac{\pi}{2}}$ and $-j = e^{-j\frac{\pi}{2}}$, we obtain

$$\hat{G}(f) = \begin{cases} e^{-j\frac{\pi}{2}}G(f) & f > 0 \\ 0 & f = 0 \\ e^{j\frac{\pi}{2}}G(f) & f < 0 \end{cases} \tag{4}$$

If we write $G(f) = |G(f)|e^{j\theta(f)}$, then (4) happens to be

$$\hat{G}(f) = \begin{cases} |G(f)|e^{j\left(\theta(f)-\frac{\pi}{2}\right)} & f > 0 \\ 0 & f = 0 \\ |G(f)|e^{j\left(\theta(f)+\frac{\pi}{2}\right)} & f < 0 \end{cases} \tag{5}$$

When (5) is inspected, we see that Hilbert transform produces a phase shift of $-\pi/2$ for positive frequencies and it produces a phase shift of $+\pi/2$ for negative frequencies.

Besides, it is clear from (5) that

$$\left|\hat{G}(f)\right| = |G(f)|.$$

Example: Calculate $sgn(f)\delta(f - f_0) \quad f_0 > 0$.

Solution: $sgn(f)\delta(f - f_0) = \delta(f - f_0)$.

Example: Calculate $sgn(f)\delta(f + f_0) \quad f_0 > 0$.

Solution: $sgn(f)\delta(f + f_0) = -\delta(f + f_0)$.

Example: Calculate the Hilbert transform of the cosine function

$$g(t) = \cos(2\pi f_0 t).$$

Solution: If we use the time domain definition of the Hilbert transform

$$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(\tau)}{t-\tau} d\tau$$

for the given cosine function, we get

$$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos(2\pi f_0 \tau)}{t-\tau} d\tau$$

which cannot be simplified more in time domain.

Let's try to solve our question using frequency domain.

If we use (3) which is given below for the reminder

$$\hat{G}(f) = -jsgn(f)G(f)$$

for the cosine signal we get

$$\hat{G}(f) = -jsgn(f)\frac{1}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right)$$

from which we get

$$\hat{G}(f) = -j\frac{1}{2}\left(\delta(f-f_0) - \delta(f+f_0)\right)$$

$$\hat{G}(f) = -j\frac{1}{2}\left(\delta(f-f_0) - \delta(f+f_0)\right)$$

leading to

$$\hat{G}(f) = \frac{1}{2j}\left(\delta(f-f_0) - \delta(f+f_0)\right)$$

which is Fourier transform of $\sin(2\pi f_0 t)$.

Thus, we showed that

$$\cos(2\pi f_0 t) \rightarrow HT \rightarrow \sin(2\pi f_0 t)$$

where HT means Hilbert transform.

Exercise: Show that

$$HT\{\sin(2\pi f_0 t)\} = -\cos(2\pi f_0 t).$$

Example: Show that

$$HT\{m(t)\cos(2\pi f_0 t)\} = m(t)\sin(2\pi f_0 t).$$

Solution: Using

$$\hat{G}(f) = -jsgn(f)G(f)$$

for

$$g(t) = m(t)\cos(2\pi f_0 t)$$

we get

$$\hat{G}(f) = -jsgn(f)\ FT\{m(t)\cos(2\pi f_0 t)\}$$

where

$$FT\{m(t)\cos(2\pi f_0 t)\}$$

can be calculated as

$$FT\{m(t)\cos(2\pi f_0 t)\} = FT\{m(t)\} * FT\{\cos(2\pi f_0 t)\}$$

where using

$$FT\{\cos(2\pi f_0 t)\} = \frac{1}{2}\big(\delta(f - f_0) + \delta(f + f_0)\big)$$

we get

$$FT\{m(t)\cos(2\pi f_0 t)\} = M(f) * \frac{1}{2}\big(\delta(f - f_0) + \delta(f + f_0)\big)$$

which results in

$$FT\{m(t)\cos(2\pi f_0 t)\} = \frac{1}{2}\big(M(f - f_0) + M(f + f_0)\big)$$

And we can write,

$$\hat{G}(f) = -jsgn(f)\ \frac{1}{2}\big(M(f - f_0) + M(f + f_0)\big)$$

which equals

$$\hat{G}(f) = \frac{1}{2}\big(-jM(f - f_0) + jM(f + f_0)\big)$$

$$\hat{G}(f) = \frac{1}{2j}\big(M(f - f_0) - M(f + f_0)\big)$$

whose inverse Fourier transform equals

$$\hat{g}(t) = m(t)\sin(2\pi f_0 t)$$

Exercise: Show that

$$HT\{m(t)\sin(2\pi f_0 t)\} = -m(t)\cos(2\pi f_0 t).$$

Properties of Hilbert Transform

1) if $\hat{g}(t) = H\{g(t)\}$ then we have $-g(t) = H\{\hat{g}(t)\}$

2) $g(t)$ and its Hilbert transform $\hat{g}(t)$ are orthogonal functions, i.e., we have

$$\int_{-\infty}^{\infty} g(t)\hat{g}(t)dt = 0.$$