Lecture-2

Background Information

Abstract: In this lecture we provide background information needed to study the subjects of Analog Communication course.

Review of Complex Numbers

The complex number j is defined as

$$j^2 = -1 \rightarrow j = \sqrt{-1}.$$

A complex number in general can be written as

$$z = x + yj.$$

Example:

$$z_1 = 2 + 3j \qquad z_2 = 4 - 6j \qquad z_3 = -2 - 5j.$$

Complex Conjugate

The complex conjugate of $z = x + yj$ is obtained as

$$z^* = x - yj.$$

Example: What is the complex conjugate of

$$z = 2 - 5j.$$

Answer: $z^* = 2 + 5j.$

Example: If $z = x + yj$, show that $zz^* = x^2 + y^2$.

Solution:

$$zz^* = (x + yj)(x - yj) \rightarrow$$

$$zz^* = x^2 - xyj + yxj + y^2 \rightarrow$$

$$zz^* = x^2 + y^2.$$

Example: If $z = 3 - 4j$, find zz^*.

Solution: $zz^* = 3^2 + 4^2 \rightarrow zz^* = 25$.

Complex Division

The division of two complex numbers $a + bj$ and $c + dj$ is calculated as

$$\frac{a + bj}{c + dj} \rightarrow \frac{a + bj}{c + dj} \times \frac{c - dj}{c - dj} \rightarrow \frac{(a + bj)(c - dj)}{c^2 + d^2}.$$

Example: Calculate the result of the complex division

$$\frac{4 + 7j}{2 + 5j}.$$

Solution: The complex division can be performed as

$$\frac{4 + 7j}{2 + 5j} \rightarrow \frac{4 + 7j}{2 + 5j} \times \frac{2 - 5j}{2 - 5j} \rightarrow$$

$$\frac{(4 + 7j)(2 - 5j)}{2^2 + 5^2} \rightarrow$$

$$\frac{43 - 6j}{29}.$$

Argand Diagram

The complex numbers can be represented in complex planes named as Argand diagrams.

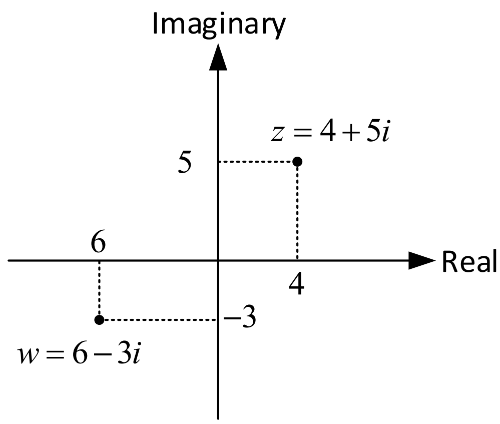


Figure-1 An example for Argand diagram.

Polar Coordinates

The complex number $z = x + yj$ can be represented in polar coordinates as

$$z = |z|e^{j\theta} \tag{1}$$

where

$$|z| = \sqrt{x^2 + y^2} \text{ and } \theta = \arctan\frac{y}{x} + 2n\pi \quad n \in Z.$$

The graphical illustration of the polar coordinates is depicted in Figure-2.

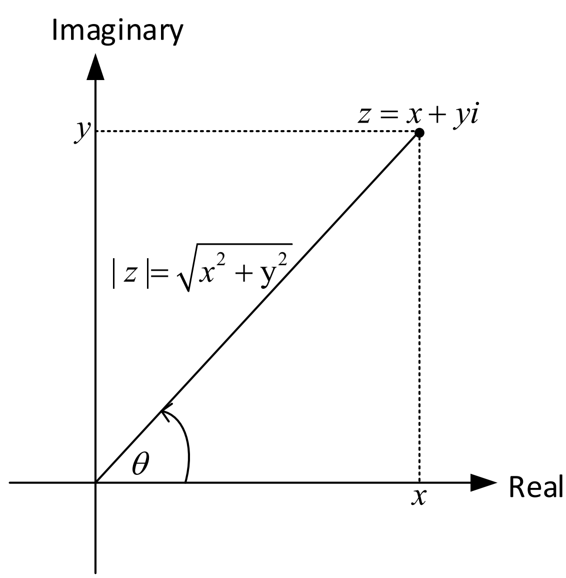


Figure-2 Polar coordinates of a complex number.

Using (1) we get

$$z = |z|e^{j\theta} \rightarrow x + yj = |z|(\cos\theta + j\sin\theta)$$

from which we obtain

$$x = |z| \cos\theta \quad y = |z| \sin\theta.$$

Note that for two complex numbers

$$z = a + bj \quad w = c + dj$$

if $z = w$, then we have

$$a = c \text{ and } b = d.$$

Principal Argument

The argument value θ satisfying

$$-\pi < \theta \leq \pi$$

is called principal argument.

We use principal argument in the polar coordinate representation of complex numbers.

Example: The complex number is given as

$$z = -1 - j.$$

Write z in polar coordinates.

Solution: The complex number can be written as

$$z = -1 - j \rightarrow z = \sqrt{2}e^{\theta} \text{ where } \theta = artan\frac{-1}{-1} + 2n\pi$$

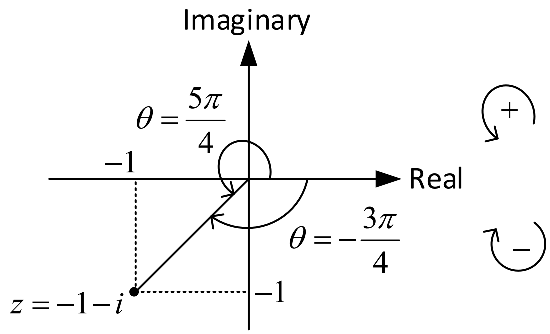
The principal argument can be calculated as

$$\theta = artan\frac{-1}{-1} \rightarrow \theta = -\frac{3\pi}{4}.$$

Hence, the complex number $z = -1 - j$ can be expressed in polar coordinates as

$$z = \sqrt{2}e^{-\frac{3\pi}{4}j}.$$

The calculation of the parameters of the polar coordinates of $z = -1 - j$ is illustrated in Figure-3.



$$-\frac{3\pi}{4} = -2\pi + \frac{5\pi}{4}$$

Figure-3 Polar coordinates of a complex number.

Properties

$$z_1 = r_1 \cos\theta_1 + jr_1 \sin\theta_1 \rightarrow z_1 = r_1 e^{j\theta_1}$$

$$z_2 = r_2 \cos\theta_2 + jr_2 \sin\theta_2 \rightarrow z_2 = r_2 e^{j\theta_2}$$

a)

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + j \sin(\theta_1 + \theta_2)]$$

b)

$$z_1/z_2 = r_1/r_2 [\cos(\theta_1 - \theta_2) + j \sin(\theta_1 - \theta_2)]$$

c)

$$z_1^n = r_1^n \cos n\theta_1 + jr_1^n \sin n\theta_1$$

d)

$$z_1^{1/n} = r_1^{1/n} \cos\frac{\theta_1}{n} + jr_1^{1/n} \sin\frac{\theta_1}{n}$$

Example: If the Fourier transform of $x(t)$ is $X(f)$, then calculate the Fourier transform of

$$y(t) = x(t)e^{j2\pi f_0 t}$$

in terms if $X(f)$.

Solution: Using the property

$$x_1(t)x_2(t) \overset{FT}{\leftrightarrow} X_1(f) * X_2(f)$$

we can calculate the Fourier transform

$$x(t)e^{j2\pi f_0 t}$$

as

$$X(f) * FT\{e^{j2\pi f_0 t}\} \rightarrow X(f) * \delta(f - f_0)$$

in which using the property

$$f(t) * \delta(t - t_0) = f(t - t_0)$$

we obtain

$$X(f - f_0).$$

Thus, we showed that

$$x(t)e^{j2\pi f_0 t} \overset{FT}{\leftrightarrow} X(f - f_0).$$

Example: If the Fourier transform of $x(t)$ is $X(f)$, then calculate the Fourier transform of

$$y(t) = x(t)e^{-j2\pi f_0 t}$$

in terms if $X(f)$.

Solution: Following a similar approach as in the previous example, we get

$$x(t)e^{-j2\pi f_0 t} \overset{FT}{\leftrightarrow} X(f + f_0).$$

Example: If the Fourier transform of $x(t)$ is $X(f)$, then calculate the Fourier transform of

$$y(t) = x(t)e^{j2\pi f_0 t} + x(t)e^{-j2\pi f_0 t}$$

in terms if $X(f)$.

Solution: Using the results of previous two examples, we can calculate the Fourier transform of $y(t)$ as

$$Y(f) = X(f - f_0) + X(f + f_0)$$

Example: If the Fourier transform of $x(t)$ is $X(f)$, then calculate the Fourier transform of

$$y(t) = x(t)\cos 2\pi f_0 t$$

Solution: Substituting

$$\cos 2\pi f_0 t = \frac{1}{2}\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right]$$

into

$$y(t) = x(t)\cos 2\pi f_0 t$$

we get

$$y(t) = \frac{1}{2}x(t)\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right]$$

and using the result of the previous example, we can write

$$Y(f) = \frac{1}{2}[X(f - f_0) + X(f + f_0)]$$

Thus,

$$x(t)\cos 2\pi f_0 t \overset{FT}{\leftrightarrow} \frac{1}{2}[X(f - f_0) + X(f + f_0)]$$

Exercise: If the Fourier transform of $x(t)$ is $X(f)$, then calculate the Fourier transform of

$$y(t) = x(t)\sin 2\pi f_0 t$$

Rectangular Pulse and Fourier Transform of Rectangular Pulse

The rectangular pulse is defined as

$$g(t) = \begin{cases} 1 & -\frac{T}{2} < t < \frac{T}{2} \\ 0 & otherwise. \end{cases}$$

The graphical representation of the rectangular pulse is depicted in Figure-4.

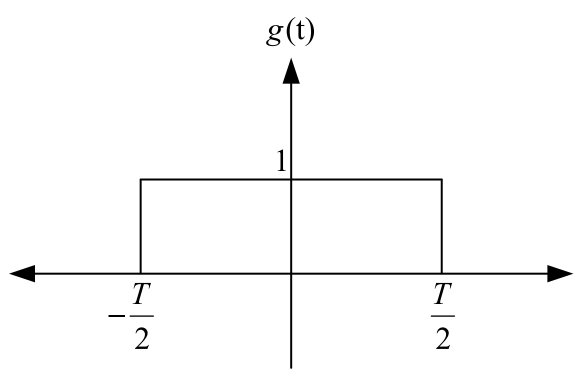


Figure-4 Rectangular pulse in time domain.

The Fourier transform of the rectangular pulse can be calculated as

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt \rightarrow G(f) = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi ft} dt \rightarrow$$

where changing parameters as

$$u = -j2\pi ft \;\rightarrow\; du = -j2\pi f dt \rightarrow \; dt = -\frac{du}{j2\pi f}$$

we get

$$G(f) = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi ft} dt \rightarrow G(f) = -\frac{1}{j2\pi f} \int_{j\pi fT}^{-j\pi fT} e^{u} du$$

weading to

$$G(f) = -\frac{1}{j2\pi f} \left[e^{-j\pi fT} - e^{j\pi fT} \right] \rightarrow$$

$$G(f) = \frac{1}{\pi f} \underbrace{\frac{1}{2j} \left[e^{j\pi fT} - e^{-j\pi fT} \right]}_{\sin \pi fT} \rightarrow$$

which results in

$$G(f) = \frac{\sin \pi fT}{\pi f} \tag{2}$$

The $sinc(\cdot)$ function is defined as

$$sinc(x) = \frac{\sin(\pi x)}{\pi x} \tag{3}$$

The Fourier transform expression in (2) can be expressed in terms of $sinc(\cdot)$ function as

$$G(f) = \frac{\sin \pi fT}{\pi f} \rightarrow G(f) = \frac{T}{T} \frac{\sin \pi fT}{\pi f} \rightarrow$$

$$G(f) = T \frac{\sin \pi fT}{\pi fT} \rightarrow G(f) = T\, sinc(fT).$$

If the amplitude of the rectangular pulse is A other than 1, then the Fourier transform of the rectangular pulse happens to be as

$$G(f) = AT\, sinc(fT) \tag{4}$$

The graph of the $G(f) = T\ sinc(fT)$ is depicted in Figure-5.

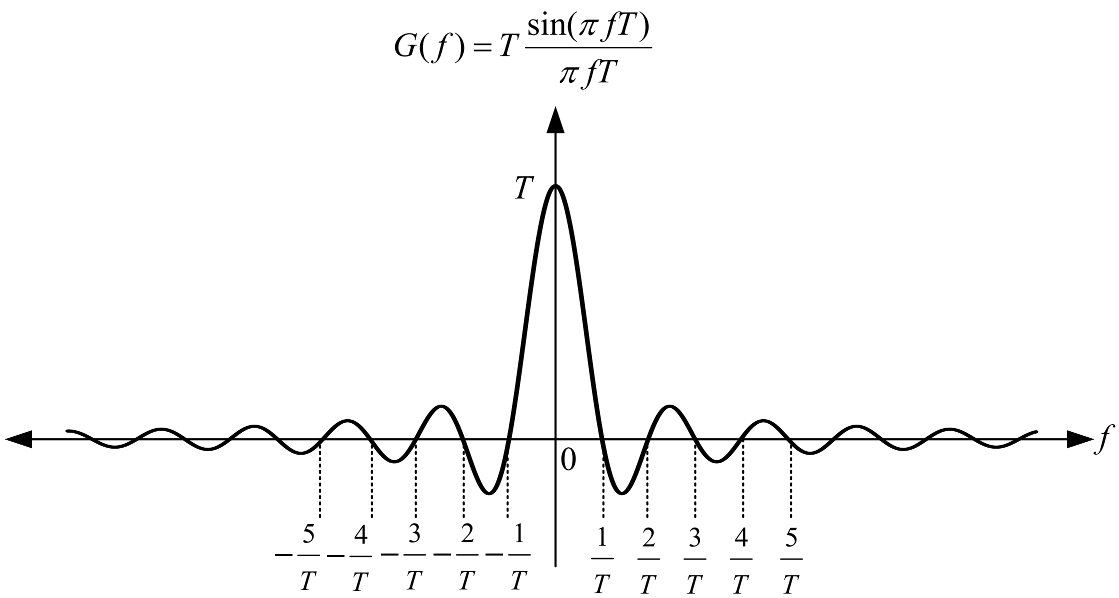


Figure-5 Graph of the $G(f) = AT\ sinc(fT)$.

Thus, we have

$$g(t) = \begin{cases} A & -\frac{T}{2} < t < \frac{T}{2} \\ 0 & otherwise. \end{cases} \quad \overset{FT}{\leftrightarrow} \quad AT\ sinc(fT)$$

and this result is graphically illustrated in Figure-5.

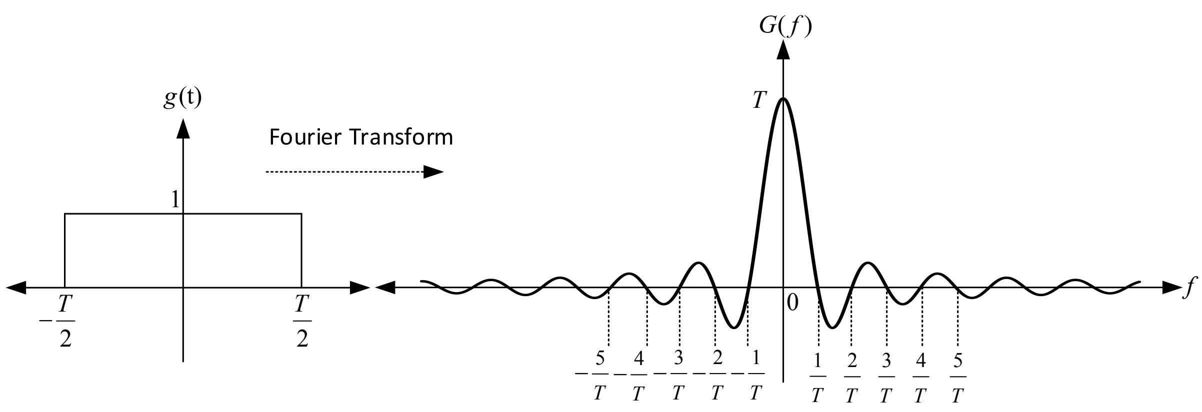


Figure-5 Rectangular pulse and its Fourier transform.

Example: If $g_R(t)$ is the rectangular pulse function draw the graph of $g_R(-t)$.

Solution: The graph of $g(at)$ is obtained dividing the horizontal axis of $g(t)$ by a. Accordingly, it is not difficult to comprehend that $g_R(t) = g_R(-t)$, i.e., the graph of $g_R(-t)$ is the same as that of the graph of $g_R(t)$.

Using the duality property

$$\text{if } g(t) \overset{FT}{\leftrightarrow} G(f) \text{ then } G(t) \overset{FT}{\leftrightarrow} g(-f)$$

for rectangular pulse $g_R(t)$, we can write the property

$$g_R(t) \overset{FT}{\leftrightarrow} Tsinc(fT)$$

$$Tsinc(tT) \overset{FT}{\leftrightarrow} g_R(f)$$

which is graphically illustrated in Figure-6.

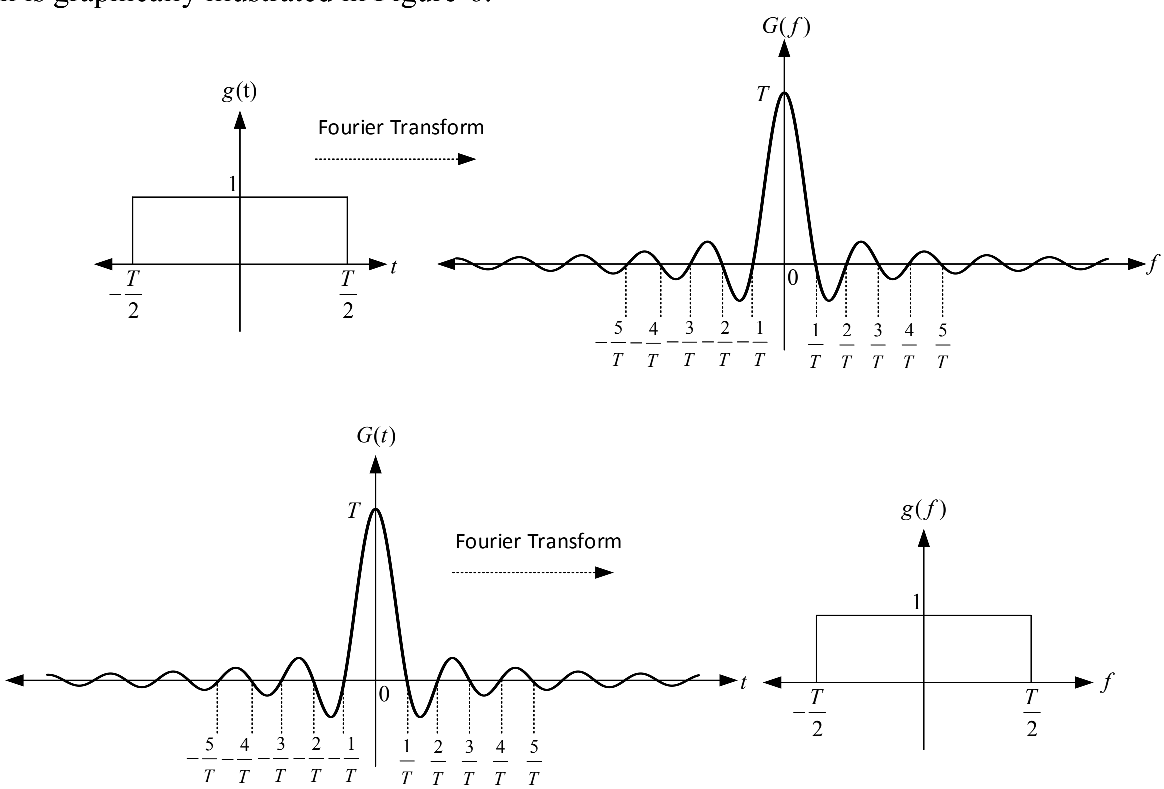


Figure-6 Duality Property for the Rectangular pulse and its Fourier transform.

Signum function and its Fourier transform

The signum function is defined as

$$sgn(t) = \begin{cases} +1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0. \end{cases} \tag{5}$$

The signum function is depicted in Figure-7.

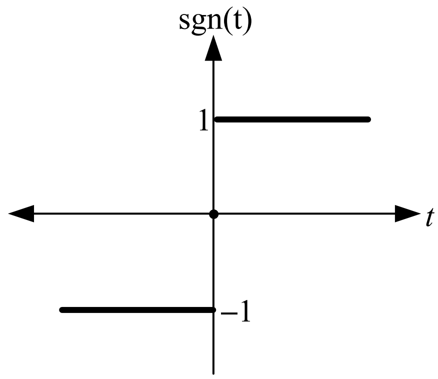


Figure-7 Signum function.

The signum function can be written in terms of the unit step functions $u(t)$ as

$$sgn(t) = u(t) - u(-t) \tag{6}$$

Fourier Transform of the Signum Function

We will use the differentiation property for the calculation of the Fourier transform of the signum function. First, let's recall the differentiation property.

If

$$g(t) \overset{FT}{\leftrightarrow} G(f)$$

then we have

$$\frac{dg(t)}{dt} \overset{FT}{\leftrightarrow} j2\pi f G(f).$$

Taking the derivative of the signum function in (6), we get

$$sgn'(t) = \delta(t) - \delta(-t)\frac{d(-t)}{dt} \rightarrow sgn'(t) = \delta(t) + \delta(-t) \tag{7}$$

Since $\delta(t) = \delta(-t)$, (7) reduces to

$$sgn'(t) = 2\delta(t) \tag{8}$$

Taking the Fourier transform of both sides of (8), we obtain

$$j2\pi f\ Sgn(f) = 2$$

from which we get

$$Sgn(f) = \frac{1}{j\pi f}.$$

Thus, we have

$$sgn(t) \overset{FT}{\leftrightarrow} \frac{1}{j\pi f}.$$