APPENDIX: STRUCTURE OF 2-ISOGENY AND 3-ISOGENY GRAPHS IN SMALL CHARACTERISTIC

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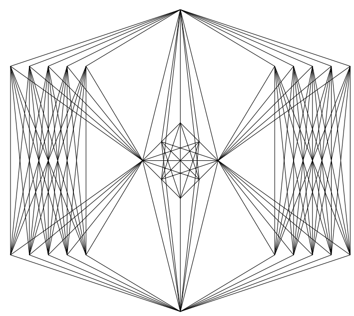

We provide descriptions of the spine for the primes $p = 2, 3, 5, 7, 11, 13$, which are exceptions to our structure theorems. We also provide a description of the 2-isogeny graphs over $\mathbb{F}_p$. These small primes yield special cases in our structure theorems due to the collision of exceptional j-invariants over $\mathbb{F}_p$.

Furthermore, we provide descriptions for a subset of the special-case primes for the 3-isogeny graph structure theorems: namely $\mathcal{P}_3 = \{5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 61, 71, 79, 89, 101, 13$ For missing entries in this file, we refer the reader to `Graph_Viz.ipynb`.

0.1. $p = 2$**.** There is a unique supersingular j-invariant over $\overline{\mathbb{F}}_2$, namely $j = 0$. Let $E : y^2 + y = x^3$ denote a representative of this isomorphism class. The size of the automorphism group of E is 24.

Since E is supersingular, $E[2] = \mathcal{O}_E$, so there are no degree-2 isogenies of E and the isogeny graph $\mathcal{G}_2(\overline{\mathbb{F}}_2)$ consists of a single vertex and no edges.

The 3-torsion points of E are all defined over $\mathbb{F}_{2^2}$ and since there is a single vertex in the graph, these four edges are all loops.

0.2. $p = 3$**.** There is a unique supersingular j-invariant over $\overline{\mathbb{F}}_3$, namely $j = 0$. Let $E : y^2 = x^3 + x$ denote a representative of this isomorphism class. The size of the automorphism group of E is 12.

The 2-torsion points of E are all defined over $\mathbb{F}_{3^2}$ and since there is a single vertex in the graph, these three edges are all loops.

Since E is supersingular, $E[3] = \mathcal{O}_E$, so there are no degree-3 isogenies of E and the isogeny graph $\mathcal{G}_3(\overline{\mathbb{F}}_3)$ consists of a single vertex and no edges.

0.3. $p = 5$**.** The $\mathcal{G}_\ell(\mathbb{F}_5)$ vertices:

- There are 2 isomorphism classes of Elliptic Curves, both of which have j-invariant 0
- Choose $E_0 : y^2 = x^3 + 1$ as the representative of an isomorphism class of elliptic curves with j-invariant 0.

- Choose $E_0^t : y^2 = x^3 + 2$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 0.
- all of the 2-torsion points of the vertices are defined over $\mathbb{F}_{5^2}$.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{5^2}$

The $\mathcal{G}_2(\mathbb{F}_5)$ edges:

- There is one outgoing 2-isogeny of E_0, namely the isogeny $\phi_0 : E_0 \to E_0^t$ generated by the 2-torsion point $(4, 0)$ in E_0.
- There is one outgoing 2-isogeny of E_0^t, namely the isogeny $\phi_0^t : E_0^t \to E_0$ generated by the 2-torsion point $(2, 0)$ in E_0^2.

The $\mathcal{G}_3(\mathbb{F}_5)$ edges:

Let z_2 be a number such that $[\mathbb{F}_5(z_2) : \mathbb{F}_5] = 2$.

- There are two outgoing 3-isogenies of E_0, namely the isogenies
  - $\psi_{0,1} : E_0 \to E_0^t$, generated by the 3-torsion point $(0, 1)$ in E_0
  - $\psi_{0,2} : E_0 \to E_0^t$, generated by the 3-torsion point $(1, z_2 + 2)$ in E_0
- there are two outgoing 3-isogenies of E_0^t, namely the isogenies
  - $\psi_{0,1}^t : E_0^t \to E_0$, generated by the 3-torsion point $(0, z_2 + 2)$ in E_0^t.
  - $\psi_{0,2}^t : E_0^t \to E_0$, generated by the 3-torsion point $(3, 2)$ in E_0^t.

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_5)$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_5)$:

- E_0 and E_0^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 0, denoted [0].

How the edges of $\mathcal{G}_2(\mathbb{F}_5)$ change when going to $\mathcal{G}_2(\overline{\mathbb{F}}_5)$:

- ϕ_0 and ϕ_0^t become equivalent isogenies and hence become one edge of the form $([0], [0])$
- [0] attains two new isogeny equivalence classes of the form $([0], [0])$.

How the edges of $\mathcal{G}_3(\mathbb{F}_5)$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_5)$:

- $\psi_{0,1}$ and $\psi_{0,1}^t$ become equivalent isogenies and hence become one edge of the form $([0], [0])$
- $\psi_{0,2}$ and $\psi_{0,2}^t$ become equivalent isogenies and hence become one edge of the form $([0], [0])$
- [0] attains two new isogeny equivalence classes of the form $([0], [0])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_5)$ vertices:

- There is one supersingular isomorphism class of elliptic curves, namely $j = 0$.
- $E_0 : y^2 = x^3 + 1$ is a representative of the isomorphism class $j = 0$

The $\mathcal{G}_2(\overline{\mathbb{F}}_5)$ edges:

- The isomorphism class [0] has three loops.

The $\mathcal{G}_3(\overline{\mathbb{F}}_5)$ edges:

- The isomorphism class [0] has four loops.

0.4. $p = 7$. The $\mathcal{G}_\ell(\mathbb{F}_7)$ vertices:

- There are 2 isomorphism classes of Elliptic Curves, both of which have j-invariant 6
- Choose $E_6 : y^2 = x^3 + x$ as the representative of an isomorphism class of elliptic curves with j-invariant 6.

- Choose $E_6^t : y^2 = x^3 + 3x$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 6.
- all of the 2-torsion points of the vertices are defined over $\mathbb{F}_{7^2}$.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{7^4}$

The $\mathcal{G}_2(\mathbb{F}_7)$ edges:

- There is one outgoing 2-isogeny of E_6, namely the isogeny $\phi_6 : E_6 \to E_6^t$ generated by the 2-torsion point $(0, 0)$ in E_6.
- There are three outgoing 2-isogeny of E_6^t, namely the isogenies
  - $\phi_{6,1}^t : E_6^t \to E_6$ generated by the 2-torsion point $(0, 0)$ in E_6^2.
  - $\phi_{6,2}^t : E_6^t \to E_6^t$ generated by the 2-torsion point $(2, 0)$ in E_6^2.
  - $\phi_{6,3}^t : E_6^t \to E_6^t$ generated by the 2-torsion point $(5, 0)$ in E_6^2.

The $\mathcal{G}_3(\mathbb{F}_7)$ edges:

- There are no outgoing 3-isogenies of E_6 in this graph.
- There are no outgoing 3-isogenies of E_6^t in this graph.

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_7)$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_7)$:

- E_6 and E_6^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 6, denoted [6].

How the edges of $\mathcal{G}_2(\mathbb{F}_7)$ change when going to $\mathcal{G}_2(\overline{\mathbb{F}}_7)$:

- ϕ_6 and $\phi_{6,1}^t$ become equivalent isogenies and hence become one edge of the form $([6], [6])$.
- $\phi_{6,2}^t$ does not become equivalent to any other isogenies.
- $\phi_{6,3}^t$ does not become equivalent to any other isogenies.
- No new isogenies are added to the graph.

How the edges of $\mathcal{G}_3(\mathbb{F}_7)$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_7)$:

- [6] attains four new isogeny equivalence classes of the form $([6], [6])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_7)$ vertices:

- There is one supersingular isomorphism class of elliptic curves, namely $j = 6$.
- $E_6 : y^2 = x^3 + x$ is a representative of the isomorphism class $j = 6$

The $\mathcal{G}_2(\overline{\mathbb{F}}_7)$ edges:

- The isomorphism class [6] has three loops.

The $\mathcal{G}_3(\overline{\mathbb{F}}_7)$ edges:

- The isomorphism class [6] has four loops.

0.5. $p = 11$. The $\mathcal{G}_\ell(\mathbb{F}_{11})$ vertices:

- There are four isomorphism classes of elliptic curves
- Two of the isomorphism classes of elliptic curves have j-invariant 0.
- The other Two isomorphism classes of elliptic curves have j-invariant 1.
- Choose $E_0 : y^2 = x^3 + 1$ as the representative of an isomorphism class of elliptic curves with j-invariant 0.
- Choose $E_0^t : y^2 = x^3 + 7$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 0.
- Choose $E_1 : y^2 = x^3 + x$ as the representative of an isomorphism class of elliptic curves with j-invariant 1.

- Choose $E_1^t : y^2 = x^3 + 7x$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 1.
- all of the 2-torsion points of the vertices are defined over $\mathbb{F}_{11^2}$.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{11^2}$

The $\mathcal{G}_2(\mathbb{F}_{11})$ edges:

- There is one outgoing 2-isogeny of E_0, namely the isogeny $\phi_0 : E_0 \to E_1^t$ generated by the 2-torsion point $(10, 0)$ in E_0.
- There is one outgoing 2-isogeny of E_0^t, namely the isogeny $\phi_0^t : E_0^t \to E_1^t$ generated by the 2-torsion point $(5, 0)$ in E_0^t.
- There is one outgoing 2-isogeny of E_1, namely the isogeny $\phi_1 : E_1 \to E_1^t$ generated by the 2-torsion point $(0, 0)$ in E_1.
- There are three outgoing 2-isogeny of E_1^t, namely the isogenies
  - $\phi_{1,1}^t : E_1^t \to E_1$ generated by the 2-torsion point $(0, 0)$ in E_1^2.
  - $\phi_{1,2}^t : E_1^t \to E_0$ generated by the 2-torsion point $(2, 0)$ in E_1^2.
  - $\phi_{1,3}^t : E_1^t \to E_1^t$ generated by the 2-torsion point $(8, 0)$ in E_1^2.

The $\mathcal{G}_3(\mathbb{F}_{11})$ edges:

Let z_2 be a number such that $[\mathbb{F}_{11}(z_2) : \mathbb{F}_{11}] = 2$.

- There are two outgoing 3-isogeny of E_0, namely the isogenies
  - $\psi_{0,1} : E_0 \to E_0^t$ generated by the 3-torsion point $(0, 1)$ in E_0.
  - $\psi_{0,2} : E_0 \to E_1$ generated by the 3-torsion point $(6, 2z_2 + 7)$ in E_0.
- There are two outgoing 3-isogeny of E_0^t, namely the isogenies
  - $\psi_{0,1}^t : E_0^t \to E_0$ generated by the 3-torsion point $(0, 5z_2 + 1)$ in E_0^t.
  - $\psi_{0,2}^t : E_0^t \to E_1$ generated by the 3-torsion point $(4, 2)$ in E_0^t.
- There are two outgoing 3-isogeny of E_1, namely the isogenies
  - $\psi_{1,1} : E_1 \to E_0$ generated by the 3-torsion point $(5, 3)$ in E_1.
  - $\psi_{1,2} : E_1 \to E_0^t$ generated by the 3-torsion point $(6, z_2 + 9)$ in E_1.
- There are two outgoing 3-isogeny of E_1^t, namely the isogenies
  - $\psi_{1,1}^t : E_1^t \to E_1^t$ generated by the 3-torsion point $(3, 2)$ in E_1^t.
  - $\psi_{1,2}^t : E_1^t \to E_1^t$ generated by the 3-torsion point $(8, 3z_2 + 5)$ in E_1^t.

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{11})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{11})$:

- E_0 and E_0^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 0, denoted [0].
- E_1 and E_1^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 1, denoted [1].

How the edges of $\mathcal{G}_2(\mathbb{F}_{11})$ change when going to $\mathcal{G}_2(\overline{\mathbb{F}}_{11})$:

- ϕ_0 and ϕ_0^t become equivalent isogenies and hence become one edge of the form $([0], [1])$.
- ϕ_1 and $\phi_{1,1}^t$ become equivalent isogenies and hence become one edge of the form $([1], [1])$.
- $\phi_{1,2}^t$ does not become equivalent to any other isogenies.
- $\phi_{1,3}^t$ does not become equivalent to any other isogenies.
- Two new isogeny equivalence classes are added to this graph of the form $([0], [1])$.

How the edges of $\mathcal{G}_3(\mathbb{F}_{11})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{11})$:

- $\psi_{0,1}$ and $\psi_{0,1}^t$ become equivalent isogenies and hence become one edge of the form $([0],[0])$.
- $\psi_{0,2}$ and $\psi_{0,2}^t$ become equivalent isogenies and hence become one edge of the form $([0],[1])$.
- $\psi_{1,1}$ does not become equivalent to any other isogenies.
- $\psi_{1,2}$ does not become equivalent to any other isogenies.
- $\psi_{1,1}^t$ does not become equivalent to any other isogenies.
- $\psi_{1,2}^t$ does not become equivalent to any other isogenies.
- Two new isogeny equivalence classes are added to this graph of the form $([0],[1])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{11})$ vertices:

- There are 2 supersingular isomorphism class of elliptic curves, namely $j = 0$ and $j = 1$.
- $E_0 : y^2 = x^3 + 1$ is a representative of the isomorphism class $j = 0$
- $E_1 : y^2 = x^3 + x$ is a representative of the isomorphism class $j = 1$.

The $\mathcal{G}_2(\overline{\mathbb{F}}_{11})$ edges:

- There are three edges of the form $([0],[1])$.
- There are two edges of the form $([1],[0])$.
- There is one edges of the form $([1],[1])$.

The $\mathcal{G}_3(\overline{\mathbb{F}}_{11})$ edges:

- There is one edge of the form $([0],[0])$.
- There are three edges of the form $([0],[1])$.
- There are two edges of the form $([1],[0])$.
- There are two edges of the form $([1],[1])$.

0.6. $p = 13$. The $\mathcal{G}_\ell(\mathbb{F}_{13})$ vertices:

- There are two isomorphism classes of elliptic curves, both of which have j-invariant 5.
- Choose $E_5 : y^2 = x^3 + x + 4$ as the representative of an isomorphism class of elliptic curves with j-invariant 5.
- Choose $E_5^t : y^2 = x^3 + 10x + 7$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 5.
- all of the 2-torsion points of the vertices are defined over $\mathbb{F}_{13^2}$.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{13^4}$

The $\mathcal{G}_2(\mathbb{F}_{13})$ edges:

- There is one outgoing 2-isogeny of E_5, namely the isogeny $\phi_5 : E_5 \to E_5^t$ generated by the 2-torsion point $(10,0)$ in E_5.
- There is one outgoing 2-isogeny of E_5^t, namely the isogeny $\phi_5^t : E_5^t \to E_5$ generated by the 2-torsion point $(7,0)$ in E_5^t.

The $\mathcal{G}_3(\mathbb{F}_{13})$ edges:

- There are no isogenies in this graph

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{13})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{13})$:

- E_5 and E_5^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 5, denoted [5].

How the edges of $\mathcal{G}_2(\mathbb{F}_{13})$ change when going to $\mathcal{G}_2(\overline{\mathbb{F}}_{13})$:

- ϕ_5 and ϕ_5^t become equivalent isogenies and hence become one edge of the form $([5],[5])$.
- Two new isogeny equivalence classes are added to this graph of the form $([5],[5])$.

How the edges of $\mathcal{G}_3(\mathbb{F}_{13})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{13})$:

- Four new isogeny equivalence classes are added to this graph of the form $([5],[5])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{13})$ vertices:

- There is one supersingular isomorphism class of elliptic curves, namely $j=5$
- $E_5 : y^2 = x^3 + x + 4$ is a representative of the isomorphism class $j=5$

The $\mathcal{G}_2(\overline{\mathbb{F}}_{13})$ edges:

- There are three edges of the form $([5],[5])$.

The $\mathcal{G}_3(\overline{\mathbb{F}}_{13})$ edges:

- There are four edges of the form $([5],[5])$.

0.7. $p = 17$. The $\mathcal{G}_\ell(\mathbb{F}_{17})$ vertices:

- There are four isomorphism classes of elliptic curves
- Two of the isomorphism classes of elliptic curves have j-invariant 0.
- The other two isomorphism classes of elliptic curves have j-invariant 8.
- Choose $E_0 : y^2 = x^3 + 1$ as the representative of an isomorphism class of elliptic curves with j-invariant 0.
- Choose $E_0^t : y^2 = x^3 + 6$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 0.
- Choose $E_8 : y^2 = x^3 + 4x + 9$ as the representative of an isomorphism class of elliptic curves with j-invariant 8.
- Choose $E_8^t : y^2 = x^3 + 15x + 3$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 8.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{17^2}$

The $\mathcal{G}_3(\mathbb{F}_{17})$ edges:

Let z_2 be a number such that $[\mathbb{F}_{17}(z_2) : \mathbb{F}_{17}] = 2$.

- There are two outgoing 3-isogeny of E_0, namely the isogenies
  - $\phi_{0,1} : E_0 \to E_0^t$ generated by the 3-torsion point $(0,1)$ in E_0.
  - $\phi_{0,2} : E_0 \to E_8$ generated by the 3-torsion point $(4, 7z_2+5)$ in E_0.
- There are two outgoing 3-isogeny of E_0^t, namely the isogenies
  - $\phi_{0,1}^t : E_0^t \to E_0$ generated by the 3-torsion point $(0, 2z_2+16)$ in E_0^t.
  - $\phi_{0,2}^t : E_0^t \to E_8^t$ generated by the 3-torsion point $(3,4)$ in E_0^t.
- There are two outgoing 3-isogeny of E_8, namely the isogenies
  - $\phi_{8,1} : E_8 \to E_8^t$ generated by the 3-torsion point $(1, 7z_2+5)$ in E_8.
  - $\phi_{8,2} : E_8 \to E_0$ generated by the 3-torsion point $(14,2)$ in E_8.
- There are two outgoing 3-isogeny of E_8^t, namely the isogenies
  - $\phi_{8,1}^t : E_8^t \to E_0^t$ generated by the 3-torsion point $(2, 4z_2+15)$ in E_8^t.
  - $\phi_{8,2}^t : E_8^t \to E_8$ generated by the 3-torsion point $(5,4)$ in E_8^t.

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{17})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{17})$:

- E_0 and E_0^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 0, denoted $[0]$.

- E_8 and E_8^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 8, denoted [8].

How the edges of $\mathcal{G}_3(\mathbb{F}_{17})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{17})$:

- $\phi_{0,1}$ and $\phi_{0,1}^t$ become equivalent isogenies and hence become one edge of the form $([0],[0])$.
- $\phi_{0,2}$ and $\phi_{0,2}^t$ become equivalent isogenies and hence become one edge of the form $([0],[8])$.
- $\phi_{8,1}$ and $\phi_{8,2}^t$ become equivalent isogenies and hence become one edge of the form $([8],[8])$.
- $\phi_{8,2}$ and $\phi_{8,1}^t$ become equivalent isogenies and hence become one edge of the form $([8],[0])$.
- Two new isogeny equivalence classes are added to this graph of the form $([0],[8])$.
- Two new isogeny equivalence classes are added to this graph of the form $([8],[8])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{17})$ vertices:

- There are 2 supersingular isomorphism class of elliptic curves, namely $j=0$ and $j=8$.
- $E_0 : y^2 = x^3+1$ is a representative of the isomorphism class $j=0$
- $E_1 : y^2 = x^3+4x+9$ is a representative of the isomorphism class $j=8$.

The $\mathcal{G}_3(\overline{\mathbb{F}}_{17})$ edges:

- There is one edge of the form $([0],[0])$.
- There are three edges of the form $([0],[8])$.
- There is one edge of the form $([8],[0])$.
- There are three edges of the form $([8],[8])$.

0.8. $p=19$. The $\mathcal{G}_\ell(\mathbb{F}_{19})$ vertices:

- There are four isomorphism classes of elliptic curves
- Two of the isomorphism classes of elliptic curves have j-invariant 7.
- The other two isomorphism classes of elliptic curves have j-invariant 18.
- Choose $E_7 : y^2 = x^3+3x+16$ as the representative of an isomorphism class of elliptic curves with j-invariant 7.
- Choose $E_7^t : y^2 = x^3+12x+14$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 7.
- Choose $E_{18} : y^2 = x^3+x$ as the representative of an isomorphism class of elliptic curves with j-invariant 18.
- Choose $E_{18}^t : y^2 = x^3+8x$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 18.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{19^4}$

The $\mathcal{G}_3(\mathbb{F}_{19})$ edges:

- There are no edges in this graph

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{19})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{19})$:

- E_7 and E_7^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 7, denoted [7].
- E_{18} and E_{18}^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 18, denoted [18].

How the edges of $\mathcal{G}_3(\mathbb{F}_{19})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{19})$:

- Four new isogeny equivalence classes are added to this graph of the form $([18],[7])$.
- Two new isogeny equivalence classes are added to this graph of the form $([7],[18])$.
- Two new isogeny equivalence classes are added to this graph of the form $([7],[7])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{19})$ vertices:

- There are 2 supersingular isomorphism class of elliptic curves, namely $j = 7$ and $j = 18$.
- $E_7 : y^2 = x^3 + 3x + 16$ is a representative of the isomorphism class $j = 7$
- $E_{18} : y^2 = x^3 + x$ is a representative of the isomorphism class $j = 18$.

The $\mathcal{G}_3(\overline{\mathbb{F}}_{19})$ edges:

- There are four edges of the form $([18],[7])$.
- There are two edge of the form $([7],[18])$.
- There are two edges of the form $([7],[7])$.

0.9. $p = 23$. The $\mathcal{G}_\ell(\mathbb{F}_{23})$ vertices:

- There are six isomorphism classes of elliptic curves
- Two of the isomorphism classes of elliptic curves have j-invariant 0.
- Two of the isomorphism classes of elliptic curves have j-invariant 3.
- The last two isomorphism classes of elliptic curves have j-invariant 19.
- Choose $E_0 : y^2 = x^3 + 1$ as the representative of an isomorphism class of elliptic curves with j-invariant 0.
- Choose $E_0^t : y^2 = x^3 + 20$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 0.
- Choose $E_3 : y^2 = x^3 + x$ as the representative of an isomorphism class of elliptic curves with j-invariant 3.
- Choose $E_3^t : y^2 = x^3 + 17x$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 3.
- Choose $E_{19} : y^2 = x^3 + 8x + 1$ as the representative of an isomorphism class of elliptic curves with j-invariant 19.
- Choose $E_{19}^t : y^2 = x^3 + 2x + 20$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 19.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{23^2}$

The $\mathcal{G}_3(\mathbb{F}_{23})$ edges:

Let z_2 be a number such that $[\mathbb{F}_{23}(z_2) : \mathbb{F}_{23}] = 2$.

- There are two outgoing 3-isogeny of E_0, namely the isogenies
  - $\phi_{0,1} : E_0 \to E_0^t$ generated by the 3-torsion point $(0, 1)$ in E_0.
  - $\phi_{0,2} : E_0 \to E_3$ generated by the 3-torsion point $(20, 8z_2 + 15)$ in E_0.
- There are two outgoing 3-isogeny of E_0^t, namely the isogenies
  - $\phi_{0,1}^t : E_0^t \to E_0$ generated by the 3-torsion point $(0, 8z_2 + 15)$ in E_0^t.
  - $\phi_{0,2}^t : E_0^t \to E_3$ generated by the 3-torsion point $(13, 3)$ in E_0^t.
- There are two outgoing 3-isogeny of E_3, namely the isogenies
  - $\phi_{3,1} : E_3 \to E_0^t$ generated by the 3-torsion point $(5, 5z_2 + 18)$ in E_3.
  - $\phi_{3,2} : E_3 \to E_0$ generated by the 3-torsion point $(18, 10)$ in E_3.
- There are two outgoing 3-isogeny of E_3^t, namely the isogenies

- $\phi^t_{3,1} : E^t_3 \to E^t_{19}$ generated by the 3-torsion point $(1, 8)$ in E^t_3.
- $\phi^t_{3,2} : E^t_3 \to E_{19}$ generated by the 3-torsion point $(22, 4z_2 + 19)$ in E^t_3.

- There are two outgoing 3-isogeny of E_{19}, namely the isogenies
  - $\phi_{19,1} : E_{19} \to E^t_3$ generated by the 3-torsion point $(16, 4)$ in E_{19}.
  - $\phi_{19,2} : E_{19} \to E^t_{19}$ generated by the 3-torsion point $(20, z_2+22)$ in E_{19}.
- There are two outgoing 3-isogeny of E^t_{19}, namely the isogenies
  - $\phi^t_{19,1} : E^t_{19} \to E_{19}$ generated by the 3-torsion point $(13, 9)$ in E^t_{19}.
  - $\phi^t_{19,2} : E^t_{19} \to E^t_3$ generated by the 3-torsion point $(15, 9z_2 + 19)$ in E^t_{19}.

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{23})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{23})$:

- E_0 and E^t_0 become isomorphic to each other to make the equivalence class of isogenies with j-invariant 0, denoted [0].
- E_3 and E^t_3 become isomorphic to each other to make the equivalence class of isogenies with j-invariant 3, denoted [3].
- E_{19} and E^t_{19} become isomorphic to each other to make the equivalence class of isogenies with j-invariant 19, denoted [19].

How the edges of $\mathcal{G}_3(\mathbb{F}_{23})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{23})$:

- $\phi_{0,1}$ and $\phi^t_{0,1}$ become equivalent isogenies and hence become one edge of the form $([0], [0])$.
- $\phi_{0,2}$ and $\phi^t_{0,2}$ become equivalent isogenies and hence become one edge of the form $([0], [3])$.
- $\phi_{19,1}$ and $\phi^t_{19,2}$ become equivalent isogenies and hence become one edge of the form $([19], [3])$.
- $\phi_{19,2}$ and $\phi^t_{19,1}$ become equivalent isogenies and hence become one edge of the form $([19], [19])$.
- $\phi_{3,1}$ does not become equivalent to any other isogenies.
- $\phi_{3,2}$ does not become equivalent to any other isogenies.
- $\phi^t_{3,1}$ does not become equivalent to any other isogenies.
- $\phi^t_{3,2}$ does not become equivalent to any other isogenies.
- two new isogeny equivalence classes are added to this graph of the form $([0], [3])$.
- two new isogeny equivalence classes are added to this graph of the form $([19], [19])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{23})$ vertices:

- There are 3 supersingular isomorphism class of elliptic curves, namely $j = 0$, $j = 3$, and $j = 19$.
- $E_0 : y^2 = x^3 + 1$ is a representative of the isomorphism class $j = 0$
- $E_3 : y^2 = x^3 + x$ is a representative of the isomorphism class $j = 3$
- $E_{19} : y^2 = x^3 + 8x + 1$ is a representative of the isomorphism class $j = 19$.

The $\mathcal{G}_3(\overline{\mathbb{F}}_{23})$ edges:

- There are three edges of the form $([19], [19])$.
- There is one edge of the form $([19], [3])$.
- There are two edge of the form $([3], [19])$.
- There are two edges of the form $([3], [0])$.
- There are three edges of the form $([0], [3])$.
- There is one edge of the form $([0], [0])$.

0.10. $p = 29$. The $\mathcal{G}_\ell(\mathbb{F}_{29})$ vertices:

- There are six isomorphism classes of elliptic curves
- Two of the isomorphism classes of elliptic curves have j-invariant 0.
- Two of the isomorphism classes of elliptic curves have j-invariant 2.
- The last two isomorphism classes of elliptic curves have j-invariant 25.
- Choose $E_0 : y^2 = x^3 + 1$ as the representative of an isomorphism class of elliptic curves with j-invariant 0.
- Choose $E_0^t : y^2 = x^3 + 11$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 0.
- Choose $E_2 : y^2 = x^3 + 3x + 28$ as the representative of an isomorphism class of elliptic curves with j-invariant 2.
- Choose $E_2^t : y^2 = x^3 + 8x + 11$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 2.
- Choose $E_{25} : y^2 = x^3 + 9x + 19$ as the representative of an isomorphism class of elliptic curves with j-invariant 25.
- Choose $E_{25}^t : y^2 = x^3 + 20x + 4$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 25.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{29^2}$

The $\mathcal{G}_3(\mathbb{F}_{29})$ edges:

Let z_2 be a number such that $[\mathbb{F}_{29}(z_2) : \mathbb{F}_{29}] = 2$.

- There are two outgoing 3-isogeny of E_0, namely the isogenies
  - $\phi_{0,1} : E_0 \to E_0^t$ generated by the 3-torsion point $(0, 1)$ in E_0.
  - $\phi_{0,2} : E_0 \to E_{25}$ generated by the 3-torsion point $(20, z_2 + 12)$ in E_0.
- There are two outgoing 3-isogeny of E_0^t, namely the isogenies
  - $\phi_{0,1}^t : E_0^t \to E_0$ generated by the 3-torsion point $(0, 8z_2 + 9)$ in E_0^t.
  - $\phi_{0,2}^t : E_0^t \to E_{25}^t$ generated by the 3-torsion point $(10, 5)$ in E_0^t.
- There are two outgoing 3-isogeny of E_2, namely the isogenies
  - $\phi_{2,1} : E_2 \to E_{25}^t$ generated by the 3-torsion point $(1, 12z_2 + 28)$ in E_2.
  - $\phi_{2,2} : E_2 \to E_2^t$ generated by the 3-torsion point $(2, 10)$ in E_2.
- There are two outgoing 3-isogeny of E_2^t, namely the isogenies
  - $\phi_{2,1}^t : E_2^t \to E_{25}$ generated by the 3-torsion point $(14, 5)$ in E_2^t.
  - $\phi_{2,2}^t : E_2^t \to E_2$ generated by the 3-torsion point $(28, 3z_2 + 7)$ in E_2^t.
- There are two outgoing 3-isogeny of E_{25}, namely the isogenies
  - $\phi_{25,1} : E_{25} \to E_0$ generated by the 3-torsion point $(8, 9)$ in E_{25}.
  - $\phi_{25,2} : E_{25} \to E_2^t$ generated by the 3-torsion point $(20, 14z_2 + 23)$ in E_{25}.
- There are two outgoing 3-isogeny of E_{25}^t, namely the isogenies
  - $\phi_{25,1}^t : E_{25}^t \to E_0^t$ generated by the 3-torsion point $(8, 3)$ in E_{25}^t.
  - $\phi_{25,2}^t : E_{25}^t \to E_2$ generated by the 3-torsion point $(9, 11z_2 + 16)$ in E_{25}^t.

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{29})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{29})$:

- E_0 and E_0^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 0, denoted [0].
- E_2 and E_2^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 2, denoted [2].
- E_{25} and E_{25}^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 25, denoted [25].

How the edges of $\mathcal{G}_3(\mathbb{F}_{29})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{29})$:

- $\phi_{0,1}$ and $\phi_{0,1}^t$ become equivalent isogenies and hence become one edge of the form $([0],[0])$.
- $\phi_{0,2}$ and $\phi_{0,2}^t$ become equivalent isogenies and hence become one edge of the form $([0],[25])$.
- $\phi_{2,1}$ and $\phi_{2,2}^t$ become equivalent isogenies and hence become one edge of the form $([2],[25])$.
- $\phi_{2,2}$ and $\phi_{2,1}^t$ become equivalent isogenies and hence become one edge of the form $([2],[2])$.
- $\phi_{25,1}$ and $\phi_{25,2}^t$ become equivalent isogenies and hence become one edge of the form $([25],[0])$.
- $\phi_{25,2}$ and $\phi_{25,1}^t$ become equivalent isogenies and hence become one edge of the form $([25],[2])$.
- Two new isogeny equivalence classes are added to this graph of the form $([0],[25])$.
- Two new isogeny equivalence classes are added to this graph of the form $([19],[19])$.
- Two new isogeny equivalence classes are added to this graph of the form $([2],[2])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{29})$ vertices:

- There are 3 supersingular isomorphism class of elliptic curves, namely $j=0$, $j=2$, and $j=125$.
- $E_0 : y^2 = x^3 + 1$ is a representative of the isomorphism class $j = 0$
- $E_3 : y^2 = x^3 + 3x + 28$ is a representative of the isomorphism class $j = 2$
- $E_{19} : y^2 = x^3 + 9x + 19$ is a representative of the isomorphism class $j = 25$.

The $\mathcal{G}_3(\overline{\mathbb{F}}_{29})$ edges:

- There are three edges of the form $([2],[2])$.
- There is one edge of the form $([2],[25])$.
- There are two edge of the form $([25],[25])$.
- There is one edge of the form $([25],[0])$.
- There are three edges of the form $([0],[25])$.
- There is one edge of the form $([0],[0])$.

0.11. $p = 31$. The $\mathcal{G}_\ell(\mathbb{F}_{31})$ vertices:

- There are six isomorphism classes of elliptic curves
- Two of the isomorphism classes of elliptic curves have j-invariant 2.
- Another two isomorphism classes of elliptic curves have j-invariant 4.
- The last two isomorphism classes of elliptic curves have j-invariant 23.
- Choose $E_2 : y^2 = x^3 + 2x + 3$ as the representative of an isomorphism class of elliptic curves with j-invariant 2.
- Choose $E_2^t : y^2 = x^3 + 16x + 19$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 2.
- Choose $E_4 : y^2 = x^3 + 11x + 26$ as the representative of an isomorphism class of elliptic curves with j-invariant 4.
- Choose $E_4^t : y^2 = x^3 + 22x + 18$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 4.

- Choose $E_{23} : y^2 = x^3 + x$ as the representative of an isomorphism class of elliptic curves with j-invariant 23.
- Choose $E_{23}^t : y^2 = x^3 + 13x$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 23.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{31^4}$

The $\mathcal{G}_3(\mathbb{F}_{31})$ edges:

- There are no edges in this graph

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{31})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{31})$:

- E_2 and E_2^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 2, denoted [2].
- E_2 and E_2^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 2, denoted [2].
- E_{23} and E_{23}^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 23, denoted [23].

How the edges of $\mathcal{G}_3(\mathbb{F}_{31})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{31})$:

- Four new isogeny equivalence classes are added to this graph of the form $([23], [4])$.
- Two new isogeny equivalence classes are added to this graph of the form $([4], [23])$.
- Two new isogeny equivalence classes are added to this graph of the form $([4], [2])$.
- Two new isogeny equivalence classes are added to this graph of the form $([2], [4])$.
- Two new isogeny equivalence classes are added to this graph of the form $([2], [2])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{31})$ vertices:

- There are 3 supersingular isomorphism class of elliptic curves, namely $j = 2$, $j = 4$, and $j = 23$.
- $E_2 : y^2 = x^3 + 2x + 3$ is a representative of the isomorphism class $j = 2$
- $E_4 : y^2 = x^3 + 11x + 26$ is a representative of the isomorphism class $j = 4$
- $E_{23} : y^2 = x^3 + x$ is a representative of the isomorphism class $j = 23$.

The $\mathcal{G}_3(\overline{\mathbb{F}}_{31})$ edges:

- There are four edges of the form $([23], [4])$.
- There are two edges of the form $([4], [23])$.
- There are two edges of the form $([4], [2])$.
- There are two edges of the form $([2], [4])$.
- There are two edges of the form $([2], [2])$.

0.12. $p = 41$. The $\mathcal{G}_\ell(\mathbb{F}_{41})$ vertices:

- There are 8 isomorphism classes of Elliptic Curves
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 0
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 3
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 28
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 32

Namely we have the following Elliptic Curves representing isomorphism classes:

- Choose $E_{(0,1)} : y^2 = x^3 + 1$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 0
- Choose $E_{(0,2)} : y^2 = x^3 + 19$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 0
- Choose $E_{(3,1)} : y^2 = x^3 + 27x + 28$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 3
- Choose $E_{(3,2)} : y^2 = x^3 + 28x + 40$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 3
- Choose $E_{(28,1)} : y^2 = x^3 + 38x + 38$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 28
- Choose $E_{(28,2)} : y^2 = x^3 + 17x + 37$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 28
- Choose $E_{(32,1)} : y^2 = x^3 + 5x + 32$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 32
- Choose $E_{(32,2)} : y^2 = x^3 + 37x + 27$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 32
- all the 3-torsion points of the elliptic curves are defined over F_{p^2}.

The $\mathcal{G}_3(\mathbb{F}_{41})$ edges:

- $E_{0,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,0,1}$ generated by the 3-torsion point(s) [(0 : 1 : 1), (0 : 40 : 1)], and has codomain $E_{0,2}$
  $\phi_{1,0,1}$ generated by the 3-torsion point(s) [(16 : 3*z + 16 : 1), (16 : 38*z + 25 : 1)], and has codomain $E_{28,1}$
- $E_{0,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,0,2}$ generated by the 3-torsion point(s) [(0 : 4*z + 35 : 1), (0 : 37*z + 6 : 1)], and has codomain $E_{0,1}$
  $\phi_{1,0,2}$ generated by the 3-torsion point(s) [(29 : 4 : 1), (29 : 37 : 1)], and has codomain $E_{28,2}$
- $E_{3,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,3,1}$ generated by the 3-torsion point(s) [(10 : z + 19 : 1), (10 : 40*z + 22 : 1)], and has codomain $E_{32,2}$
  $\phi_{1,3,1}$ generated by the 3-torsion point(s) [(19 : 15 : 1), (19 : 26 : 1)], and has codomain $E_{3,2}$
- $E_{3,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,3,2}$ generated by the 3-torsion point(s) [(19 : 16 : 1), (19 : 25 : 1)], and has codomain $E_{32,1}$
  $\phi_{1,3,2}$ generated by the 3-torsion point(s) [(32 : 18*z + 14 : 1), (32 : 23*z + 27 : 1)], and has codomain $E_{3,1}$
- $E_{28,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,28,1}$ generated by the 3-torsion point(s) [(13 : 8 : 1), (13 : 33 : 1)], and has codomain $E_{0,1}$
  $\phi_{1,28,1}$ generated by the 3-torsion point(s) [(37 : z + 19 : 1), (37 : 40*z + 22 : 1)], and has codomain $E_{32,1}$

- $E_{28,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,28,2}$ generated by the 3-torsion point(s) [(28 : 13 : 1), (28 : 28 : 1)], and has codomain $E_{32,2}$
  $\phi_{1,28,2}$ generated by the 3-torsion point(s) [(32 : 16*z + 17 : 1), (32 : 25*z + 24 : 1)], and has codomain $E_{0,2}$
- $E_{32,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,32,1}$ generated by the 3-torsion point(s) [(3 : 19 : 1), (3 : 22 : 1)], and has codomain $E_{28,1}$
  $\phi_{1,32,1}$ generated by the 3-torsion point(s) [(33 : 17*z + 36 : 1), (33 : 24*z + 5 : 1)], and has codomain $E_{3,2}$
- $E_{32,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,32,2}$ generated by the 3-torsion point(s) [(13 : 3 : 1), (13 : 38 : 1)], and has codomain $E_{3,1}$
  $\phi_{1,32,2}$ generated by the 3-torsion point(s) [(31 : 2*z + 38 : 1), (31 : 39*z + 3 : 1)], and has codomain $E_{28,2}$

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{41})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{41})$:

All twists $E_{j,1}$, $E_{j,2}$ become isomorphic to each other to make the equivalence class denoted by their common j-invariant.

How the edges of $\mathcal{G}_3(\mathbb{F}_{41})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{41})$:

- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([0], [0]): $\phi_{0,0,1}$ $\phi_{0,0,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([0], [28]): $\phi_{1,0,1}$ $\phi_{1,0,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([3], [32]): $\phi_{0,3,1}$ $\phi_{0,3,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([3], [3]): $\phi_{1,3,1}$ $\phi_{1,3,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([28], [0]): $\phi_{0,28,1}$ $\phi_{1,28,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([28], [32]): $\phi_{1,28,1}$ $\phi_{0,28,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([32], [28]): $\phi_{0,32,1}$ $\phi_{1,32,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([32], [3]): $\phi_{1,32,1}$ $\phi_{0,32,2}$

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{41})$ vertices:

The $\mathcal{G}_3(\overline{\mathbb{F}}_{41})$ edges:

0.13. $p = 47$. Let $\mathbb{F}_{47^2}$ be generated by z with minimal polynomial $x^2 + 45x + 5$.

The $\mathcal{G}_\ell(\mathbb{F}_{47})$ vertices:

- There are 10 isomorphism classes of Elliptic Curves
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 0
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 9
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 10

- 2 of the isomorphism classes of Elliptic Curves have j-invariant 36
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 44

Namely we have the following Elliptic Curves representing isomorphism classes:

- Choose $E_{0,1} : y^2 = x^3 + 1$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 0
- Choose $E_{0,2} : y^2 = x^3 + 39$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 0
- Choose $E_{9,1} : y^2 = x^3 + 24x + 38$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 9
- Choose $E_{9,2} : y^2 = x^3+12x+4$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 9
- Choose $E_{10,1} : y^2 = x^3 + 28x + 16$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 10
- Choose $E_{10,2} : y^2 = x^3+2x+44$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 10
- Choose $E_{36,1} : y^2 = x^3 + x$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 36
- Choose $E_{36,2} : y^2 = x^3 + 41x$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 36
- Choose $E_{44,1} : y^2 = x^3+25x+8$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 44
- Choose $E_{44,2} : y^2 = x^3 + 16x + 15$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 44
- all the 3-torsion points of the elliptic curves are defined over F_{p^2}.

The $\mathcal{G}_3(\mathbb{F}_{47})$ edges:

- $E_{0,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,0,1}$ generated by the 3-torsion point(s) [(0 : 1 : 1), (0 : 46 : 1)], and has codomain $E_{0,2}$
  $\phi_{1,0,1}$ generated by the 3-torsion point(s) [(29 : 6*z + 41 : 1), (29 : 41*z + 6 : 1)], and has codomain $E_{9,2}$
- $E_{0,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,0,2}$ generated by the 3-torsion point(s) [(0 : 8*z + 39 : 1), (0 : 39*z + 8 : 1)], and has codomain $E_{0,1}$
  $\phi_{1,0,2}$ generated by the 3-torsion point(s) [(27 : 4 : 1), (27 : 43 : 1)], and has codomain $E_{9,1}$
- $E_{9,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,9,1}$ generated by the 3-torsion point(s) [(27 : 21 : 1), (27 : 26 : 1)], and has codomain $E_{36,1}$
  $\phi_{1,9,1}$ generated by the 3-torsion point(s) [(31 : 10*z + 37 : 1), (31 : 37*z + 10 : 1)], and has codomain $E_{0,2}$
- $E_{9,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,9,2}$ generated by the 3-torsion point(s) [(17 : 19 : 1), (17 : 28 : 1)], and has codomain $E_{0,1}$

$\phi_{1,9,2}$ generated by the 3-torsion point(s) [(33 : 10*z + 37 : 1), (33 : 37*z + 10 : 1)], and has codomain $E_{36,1}$

- $E_{10,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,10,1}$ generated by the 3-torsion point(s) [(5 : 23*z + 24 : 1), (5 : 24*z + 23 : 1)], and has codomain $E_{44,1}$
  $\phi_{1,10,1}$ generated by the 3-torsion point(s) [(25 : 19 : 1), (25 : 28 : 1)], and has codomain $E_{36,2}$
- $E_{10,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,10,2}$ generated by the 3-torsion point(s) [(17 : 13 : 1), (17 : 34 : 1)], and has codomain $E_{44,2}$
  $\phi_{1,10,2}$ generated by the 3-torsion point(s) [(38 : 6*z + 41 : 1), (38 : 41*z + 6 : 1)], and has codomain $E_{36,2}$
- $E_{36,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,36,1}$ generated by the 3-torsion point(s) [(17 : 18 : 1), (17 : 29 : 1)], and has codomain $E_{9,2}$
  $\phi_{1,36,1}$ generated by the 3-torsion point(s) [(30 : 9*z + 38 : 1), (30 : 38*z + 9 : 1)], and has codomain $E_{9,1}$
- $E_{36,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,36,2}$ generated by the 3-torsion point(s) [(17 : 8 : 1), (17 : 39 : 1)], and has codomain $E_{10,2}$
  $\phi_{1,36,2}$ generated by the 3-torsion point(s) [(30 : 4*z + 43 : 1), (30 : 43*z + 4 : 1)], and has codomain $E_{10,1}$
- $E_{44,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,44,1}$ generated by the 3-torsion point(s) [(1 : 9 : 1), (1 : 38 : 1)], and has codomain $E_{10,1}$
  $\phi_{1,44,1}$ generated by the 3-torsion point(s) [(19 : 16*z + 31 : 1), (19 : 31*z + 16 : 1)], and has codomain $E_{44,2}$
- $E_{44,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,44,2}$ generated by the 3-torsion point(s) [(28 : 15 : 1), (28 : 32 : 1)], and has codomain $E_{44,1}$
  $\phi_{1,44,2}$ generated by the 3-torsion point(s) [(46 : 19*z + 28 : 1), (46 : 28*z + 19 : 1)], and has codomain $E_{10,2}$

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{47})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{47})$:

All twists $E_{j,1}$, $E_{j,2}$ become isomorphic to each other to make the equivalence class denoted by their common j-invariant.

How the edges of $\mathcal{G}_3(\mathbb{F}_{47})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{47})$:

- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([0], [0]): $\phi_{0,0,1}$ $\phi_{0,0,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([0], [9]): $\phi_{1,0,1}$ $\phi_{1,0,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([9], [36]): $\phi_{0,9,1}$ $\phi_{1,9,2}$

- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([9], [0]): $\phi_{1,9,1}$ $\phi_{0,9,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([10], [44]): $\phi_{0,10,1}$ $\phi_{0,10,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([10], [36]): $\phi_{1,10,1}$ $\phi_{1,10,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([36], [9]): $\phi_{0,36,1}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([36], [9]): $\phi_{1,36,1}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([36], [10]): $\phi_{0,36,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([36], [10]): $\phi_{1,36,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([44], [10]): $\phi_{0,44,1}$ $\phi_{1,44,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([44], [44]): $\phi_{1,44,1}$ $\phi_{0,44,2}$

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{47})$ vertices:

The $\mathcal{G}_3(\overline{\mathbb{F}}_{47})$ edges:

0.14. $p = 59$. The $\mathcal{G}_\ell(\mathbb{F}_{59})$ vertices:

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- There are 12 isomorphism classes of Elliptic Curves
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 0
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 15
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 17
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 28
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 47
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 48

Namely we have the following Elliptic Curves representing isomorphism classes:

- Choose $E_{0,1} : y^2 = x^3 + 1$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 0
- Choose $E_{0,2} : y^2 = x^3 + 24$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 0
- Choose $E_{15,1} : y^2 = x^3 + 31x + 57$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 15
- Choose $E_{15,2} : y^2 = x^3 + 58x + 58$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 15
- Choose $E_{17,1} : y^2 = x^3 + x$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 17
- Choose $E_{17,2} : y^2 = x^3 + 55x$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 17
- Choose $E_{28,1} : y^2 = x^3+20x+9$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 28
- Choose $E_{28,2} : y^2 = x^3 + 21x + 13$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 28

- Choose $E_{47,1} : y^2 = x^3+18x+6$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 47
- Choose $E_{47,2} : y^2 = x^3+47x+3$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 47
- Choose $E_{48,1} : y^2 = x^3 + 20x + 20$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 48
- Choose $E_{48,2} : y^2 = x^3 + 19x + 24$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 48
- all the 3-torsion points of the elliptic curves are defined over F_{p^2}.

The $\mathcal{G}_3(\mathbb{F}_{59})$ edges:

- $E_{0,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,0,1}$ generated by the 3-torsion point(s) [(0 : 1 : 1), (0 : 58 : 1)], and has codomain $E_{0,2}$;
  $\phi_{1,0,1}$ generated by the 3-torsion point(s) [(31 : 26*z + 46 : 1), (31 : 33*z + 13 : 1)], and has codomain $E_{48,1}$
- $E_{0,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,0,2}$ generated by the 3-torsion point(s) [(0 : 11*z + 24 : 1), (0 : 48*z + 35 : 1)], and has codomain $E_{0,1}$;
  $\phi_{1,0,2}$ generated by the 3-torsion point(s) [(9 : 1 : 1), (9 : 58 : 1)], and has codomain $E_{48,2}$
- $E_{15,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,15,1}$ generated by the 3-torsion point(s) [(40 : 12*z + 53 : 1), (40 : 47*z + 6 : 1)], and has codomain $E_{17,2}$;
  $\phi_{1,15,1}$ generated by the 3-torsion point(s) [(51 : 8 : 1), (51 : 51 : 1)], and has codomain $E_{15,2}$
- $E_{15,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,15,2}$ generated by the 3-torsion point(s) [(37 : 23*z + 18 : 1), (37 : 36*z + 41 : 1)], and has codomain $E_{15,1}$;
  $\phi_{1,15,2}$ generated by the 3-torsion point(s) [(51 : 6 : 1), (51 : 53 : 1)], and has codomain $E_{17,2}$
- $E_{17,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,17,1}$ generated by the 3-torsion point(s) [(12 : 18 : 1), (12 : 41 : 1)], and has codomain $E_{47,2}$;
  $\phi_{1,17,1}$ generated by the 3-torsion point(s) [(47 : 5*z + 27 : 1), (47 : 54*z + 32 : 1)], and has codomain $E_{47,1}$
- $E_{17,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,17,2}$ generated by the 3-torsion point(s) [(17 : 19 : 1), (17 : 40 : 1)], and has codomain $E_{15,1}$;
  $\phi_{1,17,2}$ generated by the 3-torsion point(s) [(42 : 2*z + 58 : 1), (42 : 57*z + 1 : 1)], and has codomain $E_{15,2}$
- $E_{28,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:

$\phi_{0,28,1}$ generated by the 3-torsion point(s) [(5 : 23 : 1), (5 : 36 : 1)], and has codomain $E_{47,1}$;

$\phi_{1,28,1}$ generated by the 3-torsion point(s) [(13 : 7*z + 26 : 1), (13 : 52*z + 33 : 1)], and has codomain $E_{48,2}$

- $E_{28,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:

  $\phi_{0,28,2}$ generated by the 3-torsion point(s) [(25 : 28 : 1), (25 : 31 : 1)], and has codomain $E_{48,1}$;

  $\phi_{1,28,2}$ generated by the 3-torsion point(s) [(55 : 2*z + 58 : 1), (55 : 57*z + 1 : 1)], and has codomain $E_{47,2}$
- $E_{47,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:

  $\phi_{0,47,1}$ generated by the 3-torsion point(s) [(19 : 3 : 1), (19 : 56 : 1)], and has codomain $E_{17,1}$;

  $\phi_{1,47,1}$ generated by the 3-torsion point(s) [(38 : 12*z + 53 : 1), (38 : 47*z + 6 : 1)], and has codomain $E_{28,1}$
- $E_{47,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:

  $\phi_{0,47,2}$ generated by the 3-torsion point(s) [(15 : 9 : 1), (15 : 50 : 1)], and has codomain $E_{28,2}$;

  $\phi_{1,47,2}$ generated by the 3-torsion point(s) [(37 : 24*z + 47 : 1), (37 : 35*z + 12 : 1)], and has codomain $E_{17,1}$
- $E_{48,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:

  $\phi_{0,48,1}$ generated by the 3-torsion point(s) [(17 : 9 : 1), (17 : 50 : 1)], and has codomain $E_{0,1}$;

  $\phi_{1,48,1}$ generated by the 3-torsion point(s) [(34 : 10*z + 54 : 1), (34 : 49*z + 5 : 1)], and has codomain $E_{28,2}$
- $E_{48,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:

  $\phi_{0,48,2}$ generated by the 3-torsion point(s) [(15 : 12 : 1), (15 : 47 : 1)], and has codomain $E_{28,1}$;

  $\phi_{1,48,2}$ generated by the 3-torsion point(s) [(37 : 9*z + 25 : 1), (37 : 50*z + 34 : 1)], and has codomain $E_{0,2}$

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{59})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{59})$:

All twists $E_{j,1}$, $E_{j,2}$ become isomorphic to each other to make the equivalence class denoted by their common j-invariant.

How the edges of $\mathcal{G}_3(\mathbb{F}_{59})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{59})$:

- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([0], [0]): $\phi_{0,0,1}$ $\phi_{0,0,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([0], [48]): $\phi_{1,0,1}$ $\phi_{1,0,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([15], [17]): $\phi_{0,15,1}$ $\phi_{1,15,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([15], [15]): $\phi_{1,15,1}$ $\phi_{0,15,2}$

- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([17], [47]): $\phi_{0,17,1}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([17], [47]): $\phi_{1,17,1}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([17], [15]): $\phi_{0,17,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([17], [15]): $\phi_{1,17,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([28], [47]): $\phi_{0,28,1}$ $\phi_{1,28,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([28], [48]): $\phi_{1,28,1}$ $\phi_{0,28,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([47], [17]): $\phi_{0,47,1}$ $\phi_{1,47,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([47], [28]): $\phi_{1,47,1}$ $\phi_{0,47,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([48], [0]): $\phi_{0,48,1}$ $\phi_{1,48,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([48], [28]): $\phi_{1,48,1}$ $\phi_{0,48,2}$

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{59})$ vertices:

The $\mathcal{G}_3(\overline{\mathbb{F}}_{59})$ edges:

0.15. $p = 61$**.** Let $\mathbb{F}_{61^4}$ be generatex by z with minimal polynomial $x^4 + 3 * x^2 + 40 * x + 2$.

The $\mathcal{G}_\ell(\mathbb{F}_{61})$ vertices:

- There are 6 isomorphism classes of Elliptic Curves
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 9
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 41
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 50

Namely we have the following Elliptic Curves representing isomorphism classes:

- Choose $E_{9,1} : y^2 = x^3 + 53x + 18$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 9
- Choose $E_{9,2} : y^2 = x^3+8x+46$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 9
- Choose $E_{41,1} : y^2 = x^3 + 40x + 11$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 41
- Choose $E_{41,2} : y^2 = x^3 + 17x + 27$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 41
- Choose $E_{50,1} : y^2 = x^3 + 14x + 36$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 50
- Choose $E_{50,2} : y^2 = x^3+9x+17$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 50
- all the 3-torsion points of the elliptic curves are defined over F_{p^4}.

The $\mathcal{G}_3(\mathbb{F}_{61})$ edges:

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How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{61})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{61})$:

All twists $E_{j,1}$, $E_{j,2}$ become isomorphic to each other to make the equivalence class denoted by their common j-invariant.

How the edges of $\mathcal{G}_3(\mathbb{F}_{61})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{61})$:

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The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{61})$ vertices:

The $\mathcal{G}_3(\overline{\mathbb{F}}_{61})$ edges:

0.16. $p = 71$. Let $\mathbb{F}_{71^2}$ be generated by z with minimal polynomial $x^2 + 69 * x + 7$.

The $\mathcal{G}_\ell(\mathbb{F}_{71})$ vertices:

- There are 14 isomorphism classes of Elliptic Curves
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 0
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 17
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 24
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 40
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 41
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 48
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 66

Namely we have the following Elliptic Curves representing isomorphism classes:

- Choose $E_{0,1} : y^2 = x^3 + 1$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 0
- Choose $E_{0,2} : y^2 = x^3 + 17$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 0
- Choose $E_{17,1} : y^2 = x^3+2x+38$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 17
- Choose $E_{17,2} : y^2 = x^3 + 38x + 31$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 17
- Choose $E_{24,1} : y^2 = x^3 + x$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 24
- Choose $E_{24,2} : y^2 = x^3 + 55x$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 24
- Choose $E_{40,1} : y^2 = x^3 + 68x + 39$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 40
- Choose $E_{40,2} : y^2 = x^3 + 59x + 43$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 40
- Choose $E_{41,1} : y^2 = x^3 + 39x + 16$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 41
- Choose $E_{41,2} : y^2 = x^3 + 31x + 28$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 41
- Choose $E_{48,1} : y^2 = x^3 + 23x + 13$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 48
- Choose $E_{48,2} : y^2 = x^3 + 28x + 64$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 48
- Choose $E_{66,1} : y^2 = x^3 + 62x + 32$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 66
- Choose $E_{66,2} : y^2 = x^3 + 56x + 42$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 66
- all the 3-torsion points of the elliptic curves are defined over F_{p^2}.

The $\mathcal{G}_3(\mathbb{F}_{71})$ edges:

- $E_{0,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,0,1}$ generated by the 3-torsion point(s) [(0 : 1 : 1), (0 : 70 : 1)], and has codomain $E_{0,2}$
  $\phi_{1,0,1}$ generated by the 3-torsion point(s) [(13 : 6*z + 65 : 1), (13 : 65*z + 6 : 1)], and has codomain $E_{41,1}$
- $E_{0,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,0,2}$ generated by the 3-torsion point(s) [(0 : 28*z + 43 : 1), (0 : 43*z + 28 : 1)], and has codomain $E_{0,1}$,
  $\phi_{1,0,2}$ generated by the 3-torsion point(s) [(1 : 14 : 1), (1 : 57 : 1)], and has codomain $E_{41,2}$
- $E_{17,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,17,1}$ generated by the 3-torsion point(s) [(43 : 18 : 1), (43 : 53 : 1)], and has codomain $E_{48,1}$,
  $\phi_{1,17,1}$ generated by the 3-torsion point(s) [(44 : 10*z + 61 : 1), (44 : 61*z + 10 : 1)], and has codomain $E_{40,2}$
- $E_{17,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,17,2}$ generated by the 3-torsion point(s) [(16 : 13 : 1), (16 : 58 : 1)], and has codomain $E_{40,1}$,
  $\phi_{1,17,2}$ generated by the 3-torsion point(s) [(35 : 11*z + 60 : 1), (35 : 60*z + 11 : 1)], and has codomain $E_{48,2}$
- $E_{24,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,24,1}$ generated by the 3-torsion point(s) [(2 : 9 : 1), (2 : 62 : 1)], and has codomain $E_{66,2}$,
  $\phi_{1,24,1}$ generated by the 3-torsion point(s) [(69 : 7*z + 64 : 1), (69 : 64*z + 7 : 1)], and has codomain $E_{66,1}$
- $E_{24,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,24,2}$ generated by the 3-torsion point(s) [(5 : 20 : 1), (5 : 51 : 1)], and has codomain $E_{48,2}$,
  $\phi_{1,24,2}$ generated by the 3-torsion point(s) [(66 : 16*z + 55 : 1), (66 : 55*z + 16 : 1)], and has codomain $E_{48,1}$
- $E_{40,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,40,1}$ generated by the 3-torsion point(s) [(4 : 34 : 1), (4 : 37 : 1)], and has codomain $E_{40,2}$,
  $\phi_{1,40,1}$ generated by the 3-torsion point(s) [(21 : 22*z + 49 : 1), (21 : 49*z + 22 : 1)], and has codomain $E_{17,2}$
- $E_{40,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,40,2}$ generated by the 3-torsion point(s) [(8 : 33 : 1), (8 : 38 : 1)], and has codomain $E_{17,1}$,

$\phi_{1,40,2}$ generated by the 3-torsion point(s) [(59 : 27*z + 44 : 1), (59 : 44*z + 27 : 1)], and has codomain $E_{40,1}$

- $E_{41,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,41,1}$ generated by the 3-torsion point(s) [(40 : 16 : 1), (40 : 55 : 1)], and has codomain $E_{0,1}$,
  $\phi_{1,41,1}$ generated by the 3-torsion point(s) [(41 : 32*z + 39 : 1), (41 : 39*z + 32 : 1)], and has codomain $E_{66,2}$
- $E_{41,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,41,2}$ generated by the 3-torsion point(s) [(11 : 12*z + 59 : 1), (11 : 59*z + 12 : 1)], and has codomain $E_{0,2}$,
  $\phi_{1,41,2}$ generated by the 3-torsion point(s) [(45 : 2 : 1), (45 : 69 : 1)], and has codomain $E_{66,1}$
- $E_{48,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,48,1}$ generated by the 3-torsion point(s) [(29 : 19 : 1), (29 : 52 : 1)], and has codomain $E_{24,2}$,
  $\phi_{1,48,1}$ generated by the 3-torsion point(s) [(56 : 31*z + 40 : 1), (56 : 40*z + 31 : 1)], and has codomain $E_{17,1}$
- $E_{48,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,48,2}$ generated by the 3-torsion point(s) [(21 : 3*z + 68 : 1), (21 : 68*z + 3 : 1)], and has codomain $E_{24,2}$,
  $\phi_{1,48,2}$ generated by the 3-torsion point(s) [(43 : 8 : 1), (43 : 63 : 1)], and has codomain $E_{17,2}$
- $E_{66,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,66,1}$ generated by the 3-torsion point(s) [(25 : 5 : 1), (25 : 66 : 1)], and has codomain $E_{24,1}$,
  $\phi_{1,66,1}$ generated by the 3-torsion point(s) [(39 : 15*z + 56 : 1), (39 : 56*z + 15 : 1)], and has codomain $E_{41,2}$
- $E_{66,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representatives:
  $\phi_{0,66,2}$ generated by the 3-torsion point(s) [(48 : 32 : 1), (48 : 39 : 1)], and has codomain $E_{41,1}$,
  $\phi_{1,66,2}$ generated by the 3-torsion point(s) [(69 : 14*z + 57 : 1), (69 : 57*z + 14 : 1)], and has codomain $E_{24,1}$

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{71})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{71})$:

All twists $E_{j,1}$, $E_{j,2}$ become isomorphic to each other to make the equivalence class denoted by their common j-invariant.

How the edges of $\mathcal{G}_3(\mathbb{F}_{71})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{71})$:

- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([0], [0]): $\phi_{0,0,1}$ $\phi_{0,0,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([0], [41]): $\phi_{1,0,1}$ $\phi_{1,0,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([17], [48]): $\phi_{0,17,1}$ $\phi_{1,17,2}$

- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([17], [40]): $\phi_{1,17,1}$ $\phi_{0,17,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([24], [66]): $\phi_{0,24,1}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([24], [66]): $\phi_{1,24,1}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([24], [48]): $\phi_{0,24,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([24], [48]): $\phi_{1,24,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([40], [40]): $\phi_{0,40,1}$ $\phi_{1,40,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([40], [17]): $\phi_{1,40,1}$ $\phi_{0,40,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([41], [0]): $\phi_{0,41,1}$ $\phi_{0,41,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([41], [66]): $\phi_{1,41,1}$ $\phi_{1,41,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([48], [24]): $\phi_{0,48,1}$ $\phi_{0,48,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([48], [17]): $\phi_{1,48,1}$ $\phi_{1,48,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([66], [24]): $\phi_{0,66,1}$ $\phi_{1,66,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([66], [41]): $\phi_{1,66,1}$ $\phi_{0,66,2}$

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{71})$ vertices:

The $\mathcal{G}_3(\overline{\mathbb{F}}_{71})$ edges:

0.17. $p = 89$. The $\mathcal{G}_\ell(\mathbb{F}_{89})$ vertices:

- There are 12 isomorphism classes of Elliptic Curves
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 0
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 6
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 7
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 13
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 52
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 66

Namely we have the following Elliptic Curves representing isomorphism classes:

- Choose $E_{0,1} : y^2 = x^3 + 1$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 0
- Choose $E_{0,2} : y^2 = x^3 + 56$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 0
- Choose $E_{6,1} : y^2 = x^3 + 24x + 38$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 6
- Choose $E_{6,2} : y^2 = x^3 + 51x + 85$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 6

- Choose $E_{7,1} : y^2 = x^3+7x+38$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 7
- Choose $E_{7,2} : y^2 = x^3 + 63x + 47$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 7
- Choose $E_{13,1} : y^2 = x^3 + 46x + 65$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 13
- Choose $E_{13,2} : y^2 = x^3 + 38x + 87$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 13
- Choose $E_{52,1} : y^2 = x^3+63x+7$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 52
- Choose $E_{52,2} : y^2 = x^3 + 33x + 11$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 52
- Choose $E_{66,1} : y^2 = x^3 + 43x + 60$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 66
- Choose $E_{66,2} : y^2 = x^3 + 13x + 72$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 66
- all the 3-torsion points of the elliptic curves are defined over F_{p^2}.

The $\mathcal{G}_3(\mathbb{F}_{89})$ edges:

- $E_{0,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,0,1}$ generated by the 3-torsion point(s) [(0 : 1 : 1), (0 : 88 : 1)], and has codomain $E_{0,2}$
  $\phi_{1,0,1}$ generated by the 3-torsion point(s) [(11 : 12*z + 47 : 1), (11 : 77*z + 42 : 1)], and has codomain $E_{52,1}$
- $E_{0,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,0,2}$ generated by the 3-torsion point(s) [(0 : 31*z + 25 : 1), (0 : 58*z + 64 : 1)], and has codomain $E_{0,1}$
  $\phi_{1,0,2}$ generated by the 3-torsion point(s) [(26 : 30 : 1), (26 : 59 : 1)], and has codomain $E_{52,2}$
- $E_{6,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,6,1}$ generated by the 3-torsion point(s) [(68 : 25*z + 46 : 1), (68 : 64*z + 43 : 1)], and has codomain $E_{52,2}$
  $\phi_{1,6,1}$ generated by the 3-torsion point(s) [(77 : 43 : 1), (77 : 46 : 1)], and has codomain $E_{13,1}$
- $E_{6,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,6,2}$ generated by the 3-torsion point(s) [(6 : 42 : 1), (6 : 47 : 1)], and has codomain $E_{52,1}$
  $\phi_{1,6,2}$ generated by the 3-torsion point(s) [(67 : 25*z + 46 : 1), (67 : 64*z + 43 : 1)], and has codomain $E_{13,2}$
- $E_{7,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,7,1}$ generated by the 3-torsion point(s) [(19 : 34 : 1), (19 : 55 : 1)], and has codomain $E_{66,2}$
  $\phi_{1,7,1}$ generated by the 3-torsion point(s) [(69 : 12*z + 47 : 1), (69 : 77*z + 42 : 1)], and has codomain $E_{13,1}$

- $E_{7,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,7,2}$ generated by the 3-torsion point(s) [(29 : 39 : 1), (29 : 50 : 1)], and has codomain $E_{13,2}$
  $\phi_{1,7,2}$ generated by the 3-torsion point(s) [(57 : 36*z + 52 : 1), (57 : 53*z + 37 : 1)], and has codomain $E_{66,1}$
- $E_{13,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,13,1}$ generated by the 3-torsion point(s) [(56 : 41 : 1), (56 : 48 : 1)], and has codomain $E_{7,1}$
  $\phi_{1,13,1}$ generated by the 3-torsion point(s) [(84 : 6*z + 68 : 1), (84 : 83*z + 21 : 1)], and has codomain $E_{6,1}$
- $E_{13,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,13,2}$ generated by the 3-torsion point(s) [(40 : 29*z + 32 : 1), (40 : 60*z + 57 : 1)], and has codomain $E_{7,2}$
  $\phi_{1,13,2}$ generated by the 3-torsion point(s) [(60 : 36 : 1), (60 : 53 : 1)], and has codomain $E_{6,2}$
- $E_{52,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,52,1}$ generated by the 3-torsion point(s) [(18 : 5*z + 27 : 1), (18 : 84*z + 62 : 1)], and has codomain $E_{6,2}$
  $\phi_{1,52,1}$ generated by the 3-torsion point(s) [(41 : 20 : 1), (41 : 69 : 1)], and has codomain $E_{0,1}$
- $E_{52,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,52,2}$ generated by the 3-torsion point(s) [(34 : 5*z + 27 : 1), (34 : 84*z + 62 : 1)], and has codomain $E_{0,2}$
  $\phi_{1,52,2}$ generated by the 3-torsion point(s) [(54 : 6 : 1), (54 : 83 : 1)], and has codomain $E_{6,1}$
- $E_{66,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,66,1}$ generated by the 3-torsion point(s) [(9 : 44*z + 24 : 1), (9 : 45*z + 65 : 1)], and has codomain $E_{66,2}$
  $\phi_{1,66,1}$ generated by the 3-torsion point(s) [(56 : 23 : 1), (56 : 66 : 1)], and has codomain $E_{7,2}$
- $E_{66,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,66,2}$ generated by the 3-torsion point(s) [(21 : 33*z + 18 : 1), (21 : 56*z + 71 : 1)], and has codomain $E_{7,1}$
  $\phi_{1,66,2}$ generated by the 3-torsion point(s) [(59 : 19 : 1), (59 : 70 : 1)], and has codomain $E_{66,1}$

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{89})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{89})$:

All twists $E_{j,1}$, $E_{j,2}$ become isomorphic to each other to make the equivalence class denoted by their common j-invariant.

How the edges of $\mathcal{G}_3(\mathbb{F}_{89})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{89})$:

-

- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([0], [0]): $\phi_{0,0,1}$ $\phi_{0,0,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([0], [52]): $\phi_{1,0,1}$ $\phi_{1,0,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([6], [52]): $\phi_{0,6,1}$ $\phi_{0,6,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([6], [13]): $\phi_{1,6,1}$ $\phi_{1,6,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([7], [66]): $\phi_{0,7,1}$ $\phi_{1,7,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([7], [13]): $\phi_{1,7,1}$ $\phi_{0,7,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([13], [7]): $\phi_{0,13,1}$ $\phi_{0,13,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([13], [6]): $\phi_{1,13,1}$ $\phi_{1,13,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([52], [6]): $\phi_{0,52,1}$ $\phi_{1,52,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([52], [0]): $\phi_{1,52,1}$ $\phi_{0,52,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([66], [66]): $\phi_{0,66,1}$ $\phi_{1,66,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([66], [7]): $\phi_{1,66,1}$ $\phi_{0,66,2}$

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{89})$ vertices:

The $\mathcal{G}_3(\overline{\mathbb{F}}_{89})$ edges:

0.18. $p = 101$. Let $\mathbb{F}_{101^2}$ be generatex by z with minimal polynomial $x^2+97*x+2$.

The $\mathcal{G}_\ell(\mathbb{F}_{101})$ vertices:

- There are 14 isomorphism classes of Elliptic Curves
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 0
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 3
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 21
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 57
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 59
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 64
- 2 of the isomorphism classes of Elliptic Curves have j-invariant 66

Namely we have the following Elliptic Curves representing isomorphism classes:

- Choose $E_{0,1} : y^2 = x^3 + 1$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 0
- Choose $E_{0,2} : y^2 = x^3 + 93$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 0
- Choose $E_{3,1} : y^2 = x^3 + 72x + 20$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 3
- Choose $E_{3,2} : y^2 = x^3 + 66x + 83$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 3

- Choose $E_{21,1} : y^2 = x^3 + 77x + 42$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 21
- Choose $E_{21,2} : y^2 = x^3 + 58x + 70$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 21
- Choose $E_{57,1} : y^2 = x^3 + 12x + 65$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 57
- Choose $E_{57,2} : y^2 = x^3+3x+46$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 57
- Choose $E_{59,1} : y^2 = x^3 + 89x + 20$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 59
- Choose $E_{59,2} : y^2 = x^3+66x+3$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 59
- Choose $E_{64,1} : y^2 = x^3+25x+8$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 64
- Choose $E_{64,2} : y^2 = x^3+4x+31$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 64
- Choose $E_{66,1} : y^2 = x^3 + 18x + 54$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 66
- Choose $E_{66,2} : y^2 = x^3 + 10x + 55$ is a representative of one of the isomorphism class of elliptic curves that have j-invariant 66
- all the 3-torsion points of the elliptic curves are defined over F_{p^2}.

The $\mathcal{G}_3(\mathbb{F}_{101})$ edges:

- $E_{0,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,0,1}$ generated by the 3-torsion point(s) [(0 : 1 : 1), (0 : 100 : 1)], and has codomain $E_{0,2}$
  $\phi_{1,0,1}$ generated by the 3-torsion point(s) [(31 : 7*z + 87 : 1), (31 : 94*z + 14 : 1)], and has codomain $E_{64,1}$
- $E_{0,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,0,2}$ generated by the 3-torsion point(s) [(0 : 20*z + 61 : 1), (0 : 81*z + 40 : 1)], and has codomain $E_{0,1}$
  $\phi_{1,0,2}$ generated by the 3-torsion point(s) [(39 : 23 : 1), (39 : 78 : 1)], and has codomain $E_{64,2}$
- $E_{3,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,3,1}$ generated by the 3-torsion point(s) [(65 : 38*z + 25 : 1), (65 : 63*z + 76 : 1)], and has codomain $E_{64,2}$
  $\phi_{1,3,1}$ generated by the 3-torsion point(s) [(89 : 16 : 1), (89 : 85 : 1)], and has codomain $E_{21,1}$
- $E_{3,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative
  $\phi_{0,3,2}$ generated by the 3-torsion point(s) [(28 : 42 : 1), (28 : 59 : 1)], and has codomain $E_{64,1}$
  $\phi_{1,3,2}$ generated by the 3-torsion point(s) [(43 : 39*z + 23 : 1), (43 : 62*z + 78 : 1)], and has codomain $E_{21,2}$
- $E_{21,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative

$\phi_{0,21,1}$ generated by the 3-torsion point(s) [(21 : 39*z + 23 : 1), (21 : 62*z + 78 : 1)], and has codomain $E_{3,1}$

$\phi_{1,21,1}$ generated by the 3-torsion point(s) [(51 : 13 : 1), (51 : 88 : 1)], and has codomain $E_{59,1}$

- $E_{21,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative

  $\phi_{0,21,2}$ generated by the 3-torsion point(s) [(31 : 15*z + 71 : 1), (31 : 86*z + 30 : 1)], and has codomain $E_{59,2}$

  $\phi_{1,21,2}$ generated by the 3-torsion point(s) [(90 : 11 : 1), (90 : 90 : 1)], and has codomain $E_{3,2}$
- $E_{57,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative

  $\phi_{0,57,1}$ generated by the 3-torsion point(s) [(46 : 36 : 1), (46 : 65 : 1)], and has codomain $E_{66,1}$

  $\phi_{1,57,1}$ generated by the 3-torsion point(s) [(88 : 13*z + 75 : 1), (88 : 88*z + 26 : 1)], and has codomain $E_{59,1}$
- $E_{57,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative

  $\phi_{0,57,2}$ generated by the 3-torsion point(s) [(23 : 9*z + 83 : 1), (23 : 92*z + 18 : 1)], and has codomain $E_{66,2}$

  $\phi_{1,57,2}$ generated by the 3-torsion point(s) [(44 : 44 : 1), (44 : 57 : 1)], and has codomain $E_{59,2}$
- $E_{59,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative

  $\phi_{0,59,1}$ generated by the 3-torsion point(s) [(26 : 35 : 1), (26 : 66 : 1)], and has codomain $E_{57,1}$

  $\phi_{1,59,1}$ generated by the 3-torsion point(s) [(82 : 49*z + 3 : 1), (82 : 52*z + 98 : 1)], and has codomain $E_{21,1}$
- $E_{59,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative

  $\phi_{0,59,2}$ generated by the 3-torsion point(s) [(25 : 16*z + 69 : 1), (25 : 85*z + 32 : 1)], and has codomain $E_{57,2}$

  $\phi_{1,59,2}$ generated by the 3-torsion point(s) [(40 : 36 : 1), (40 : 65 : 1)], and has codomain $E_{21,2}$
- $E_{64,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative

  $\phi_{0,64,1}$ generated by the 3-torsion point(s) [(12 : 4 : 1), (12 : 97 : 1)], and has codomain $E_{0,1}$

  $\phi_{1,64,1}$ generated by the 3-torsion point(s) [(17 : 42*z + 17 : 1), (17 : 59*z + 84 : 1)], and has codomain $E_{3,2}$
- $E_{64,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative

  $\phi_{0,64,2}$ generated by the 3-torsion point(s) [(74 : 6 : 1), (74 : 95 : 1)], and has codomain $E_{3,1}$

  $\phi_{1,64,2}$ generated by the 3-torsion point(s) [(76 : 43*z + 15 : 1), (76 : 58*z + 86 : 1)], and has codomain $E_{0,2}$
- $E_{66,1}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative

$\phi_{0,66,1}$ generated by the 3-torsion point(s) [(26 : 25 : 1), (26 : 76 : 1)], and has codomain $E_{66,2}$

$\phi_{1,66,1}$ generated by the 3-torsion point(s) [(78 : 7*z + 87 : 1), (78 : 94*z + 14 : 1)], and has codomain $E_{57,1}$

- $E_{66,2}$ has 2 outgoing equivalence class of isogenies, namely the isogeny equivalence class with representative

  $\phi_{0,66,2}$ generated by the 3-torsion point(s) [(14 : 45*z + 11 : 1), (14 : 56*z + 90 : 1)], and has codomain $E_{66,1}$

  $\phi_{1,66,2}$ generated by the 3-torsion point(s) [(42 : 5 : 1), (42 : 96 : 1)], and has codomain $E_{57,2}$

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{101})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{101})$:

All twists $E_{j,1}$, $E_{j,2}$ become isomorphic to each other to make the equivalence class denoted by their common j-invariant.

How the edges of $\mathcal{G}_3(\mathbb{F}_{101})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{101})$:

- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([0], [0]): $\phi_{0,0,1}$ $\phi_{0,0,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([0], [64]): $\phi_{1,0,1}$ $\phi_{1,0,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([3], [64]): $\phi_{0,3,1}$ $\phi_{0,3,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([3], [21]): $\phi_{1,3,1}$ $\phi_{1,3,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([21], [3]): $\phi_{0,21,1}$ $\phi_{1,21,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([21], [59]): $\phi_{1,21,1}$ $\phi_{0,21,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([57], [66]): $\phi_{0,57,1}$ $\phi_{0,57,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([57], [59]): $\phi_{1,57,1}$ $\phi_{1,57,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([59], [57]): $\phi_{0,59,1}$ $\phi_{0,59,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([59], [21]): $\phi_{1,59,1}$ $\phi_{1,59,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([64], [0]): $\phi_{0,64,1}$ $\phi_{1,64,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([64], [3]): $\phi_{1,64,1}$ $\phi_{0,64,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([66], [66]): $\phi_{0,66,1}$ $\phi_{0,66,2}$
- The following $\mathbb{F}_p$ isogenies become equivalent to the $\overline{\mathbb{F}}_p$ isogeny ([66], [57]): $\phi_{1,66,1}$ $\phi_{1,66,2}$

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{101})$ vertices:

The $\mathcal{G}_3(\overline{\mathbb{F}}_{101})$ edges:

0.19. $p = 139$. The $\mathcal{G}_\ell(\mathbb{F}_{139})$ vertices:

The $\mathcal{G}_3(\mathbb{F}_{139})$ edges:

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{139})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{139})$:

How the edges of $\mathcal{G}_3(\mathbb{F}_{139})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{139})$:

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{139})$ vertices:

The $\mathcal{G}_3(\overline{\mathbb{F}}_{139})$ edges:

0.20. $p = 151$. The $\mathcal{G}_\ell(\mathbb{F}_{151})$ vertices:

The $\mathcal{G}_3(\mathbb{F}_{151})$ edges:

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{151})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{151})$:

All twists $E_{j,1}$, $E_{j,2}$ become isomorphic to each other to make the equivalence class denoted by their common j-invariant.

How the edges of $\mathcal{G}_3(\mathbb{F}_{151})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{151})$:

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{151})$ vertices:

The $\mathcal{G}_3(\overline{\mathbb{F}}_{151})$ edges:

0.21. $p = 199$. The $\mathcal{G}_\ell(\mathbb{F}_{199})$ vertices:

The $\mathcal{G}_3(\mathbb{F}_{199})$ edges:

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{199})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{199})$:

All twists $E_{j,1}$, $E_{j,2}$ become isomorphic to each other to make the equivalence class denoted by their common j-invariant.

How the edges of $\mathcal{G}_3(\mathbb{F}_{199})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{199})$:

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{199})$ vertices:

The $\mathcal{G}_3(\overline{\mathbb{F}}_{199})$ edges:

0.22. $p = 271$. The $\mathcal{G}_\ell(\mathbb{F}_{271})$ vertices:

The $\mathcal{G}_3(\mathbb{F}_{271})$ edges:

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{271})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{271})$:

All twists $E_{j,1}$, $E_{j,2}$ become isomorphic to each other to make the equivalence class denoted by their common j-invariant.

How the edges of $\mathcal{G}_3(\mathbb{F}_{271})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{271})$:

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{271})$ vertices:

The $\mathcal{G}_3(\overline{\mathbb{F}}_{271})$ edges:

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