APPENDIX: STRUCTURE OF 2-ISOGENY AND 3-ISOGENY GRAPHS IN SMALL CHARACTERISTIC

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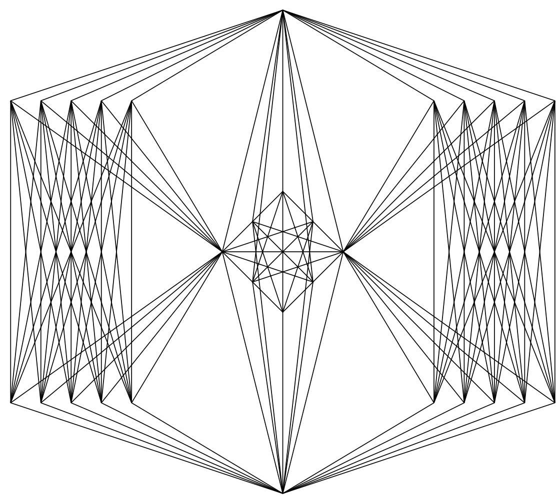

Appendix A. Small prime graph data

We provide descriptions of the spine for the primes $p = 2, 3, 5, 7, 11, 13$. In the cases of $p = 2, 3$, we also provide a description of the 2- and 3-isogeny graphs over $\mathbb{F}_p$. These small primes yield special cases in our structure theorems due to the collision of exceptional j-invariants over $\mathbb{F}_p$.

A.1. $p = 2$**.** There is a unique supersingular j-invariant over $\overline{\mathbb{F}}_2$, namely $j = 0$. Let $E : y^2 + y = x^3$ denote a representative of this isomorphism class. The size of the automorphism group of E is 24.

Since E is supersingular, $E[2] = \mathcal{O}_E$, so there are no degree-2 isogenies of E and the isogeny graph $\mathcal{G}_2(\overline{\mathbb{F}}_2)$ consists of a single vertex and no edges.

The 3-torsion points of E are all defined over $\mathbb{F}_{2^2}$ and since there is a single vertex in the graph, these four edges are all loops.

A.2. $p = 3$**.** There is a unique supersingular j-invariant over $\overline{\mathbb{F}}_3$, namely $j = 0$. Let $E : y^2 = x^3 + x$ denote a representative of this isomorphism class. The size of the automorphism group of E is 12.

The 2-torsion points of E are all defined over $\mathbb{F}_{3^2}$ and since there is a single vertex in the graph, these three edges are all loops.

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Since E is supersingular, $E[3] = \mathcal{O}_E$, so there are no degree-3 isogenies of E and the isogeny graph $\mathcal{G}_3(\overline{\mathbb{F}}_3)$ consists of a single vertex and no edges.

A.3. $p = 5$. The $\mathcal{G}_\ell(\mathbb{F}_5)$ vertices:

- There are 2 isomorphism classes of Elliptic Curves, both of which have j-invariant 0
- Choose $E_0 : y^2 = x^3 + 1$ as the representative of an isomorphism class of elliptic curves with j-invariant 0.
- Choose $E_0^t : y^2 = x^3 + 2$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 0.
- all of the 2-torsion points of the vertices are defined over $\mathbb{F}_{5^2}$.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{5^2}$

The $\mathcal{G}_2(\mathbb{F}_5)$ edges:

- There is one outgoing 2-isogeny of E_0, namely the isogeny $\phi_0 : E_0 \to E_0^t$ generated by the 2-torsion point $(4, 0)$ in E_0.
- There is one outgoing 2-isogeny of E_0^t, namely the isogeny $\phi_0^t : E_0^t \to E_0$ generated by the 2-torsion point $(2, 0)$ in E_0^2.

The $\mathcal{G}_3(\mathbb{F}_5)$ edges:

Let z_2 be a number such that $[\mathbb{F}_5(z_2) : \mathbb{F}_5] = 2$.

- There are two outgoing 3-isogenies of E_0, namely the isogenies
  - $\psi_{0,1} : E_0 \to E_0^t$, generated by the 3-torsion point $(0, 1)$ in E_0
  - $\psi_{0,2} : E_0 \to E_0^t$, generated by the 3-torsion point $(1, z_2 + 2)$ in E_0
- there are two outgoing 3-isogenies of E_0^t, namely the isogenies
  - $\psi_{0,1}^t : E_0^t \to E_0$, generated by the 3-torsion point $(0, z_2 + 2)$ in E_0^t.
  - $\psi_{0,2}^t : E_0^t \to E_0$, generated by the 3-torsion point $(3, 2)$ in E_0^t.

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_5)$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_5)$:

- E_0 and E_0^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 0, denoted [0].

How the edges of $\mathcal{G}_2(\mathbb{F}_5)$ change when going to $\mathcal{G}_2(\overline{\mathbb{F}}_5)$:

- ϕ_0 and ϕ_0^t become equivalent isogenies and hence become one edge of the form $([0], [0])$
- [0] attains two new isogeny equivalence classes of the form $([0], [0])$.

How the edges of $\mathcal{G}_3(\mathbb{F}_5)$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_5)$:

- $\psi_{0,1}$ and $\psi_{0,1}^t$ become equivalent isogenies and hence become one edge of the form $([0], [0])$
- $\psi_{0,2}$ and $\psi_{0,2}^t$ become equivalent isogenies and hence become one edge of the form $([0], [0])$
- [0] attains two new isogeny equivalence classes of the form $([0], [0])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_5)$ vertices:

- There is one supersingular isomorphism class of elliptic curves, namely $j = 0$.
- $E_0 : y^2 = x^3 + 1$ is a representative of the isomorphism class $j = 0$

The $\mathcal{G}_2(\overline{\mathbb{F}}_5)$ edges:

- The isomorphism class [0] has three loops.

The $\mathcal{G}_3(\overline{\mathbb{F}}_5)$ edges:

- The isomorphism class [0] has four loops.

A.4. $p = 7$. The $\mathcal{G}_\ell(\mathbb{F}_7)$ vertices:

- There are 2 isomorphism classes of Elliptic Curves, both of which have j-invariant 6
- Choose $E_6 : y^2 = x^3 + x$ as the representative of an isomorphism class of elliptic curves with j-invariant 6.
- Choose $E_6^t : y^2 = x^3 + 3x$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 6.
- all of the 2-torsion points of the vertices are defined over $\mathbb{F}_{7^2}$.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{7^4}$

The $\mathcal{G}_2(\mathbb{F}_7)$ edges:

- There is one outgoing 2-isogeny of E_6, namely the isogeny $\phi_6 : E_6 \to E_6^t$ generated by the 2-torsion point $(0, 0)$ in E_6.
- There are three outgoing 2-isogeny of E_6^t, namely the isogenies
  - $\phi_{6,1}^t : E_6^t \to E_6$ generated by the 2-torsion point $(0, 0)$ in E_6^2.
  - $\phi_{6,2}^t : E_6^t \to E_6^t$ generated by the 2-torsion point $(2, 0)$ in E_6^2.
  - $\phi_{6,3}^t : E_6^t \to E_6^t$ generated by the 2-torsion point $(5, 0)$ in E_6^2.

The $\mathcal{G}_3(\mathbb{F}_7)$ edges:

- There are no outgoing 3-isogenies of E_6 in this graph.
- There are no outgoing 3-isogenies of E_6^t in this graph.

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_7)$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_7)$:

- E_6 and E_6^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 6, denoted [6].

How the edges of $\mathcal{G}_2(\mathbb{F}_7)$ change when going to $\mathcal{G}_2(\overline{\mathbb{F}}_7)$:

- ϕ_6 and $\phi_{6,1}^t$ become equivalent isogenies and hence become one edge of the form $([6], [6])$.
- $\phi_{6,2}^t$ does not become equivalent to any other isogenies.
- $\phi_{6,3}^t$ does not become equivalent to any other isogenies.
- No new isogenies are added to the graph.

How the edges of $\mathcal{G}_3(\mathbb{F}_7)$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_7)$:

- [6] attains four new isogeny equivalence classes of the form $([6], [6])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_7)$ vertices:

- There is one supersingular isomorphism class of elliptic curves, namely $j = 6$.
- $E_6 : y^2 = x^3 + x$ is a representative of the isomorphism class $j = 6$

The $\mathcal{G}_2(\overline{\mathbb{F}}_7)$ edges:

- The isomorphism class [6] has three loops.

The $\mathcal{G}_3(\overline{\mathbb{F}}_7)$ edges:

- The isomorphism class [6] has four loops.

A.5. $p = 11$. The $\mathcal{G}_\ell(\mathbb{F}_{11})$ vertices:

- There are four isomorphism classes of elliptic curves
- Two of the isomorphism classes of elliptic curves have j-invariant 0.
- The other Two isomorphism classes of elliptic curves have j-invariant 1.

- Choose $E_0 : y^2 = x^3 + 1$ as the representative of an isomorphism class of elliptic curves with j-invariant 0.
- Choose $E_0^t : y^2 = x^3 + 7$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 0.
- Choose $E_1 : y^2 = x^3 + x$ as the representative of an isomorphism class of elliptic curves with j-invariant 1.
- Choose $E_1^t : y^2 = x^3 + 7x$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 1.
- all of the 2-torsion points of the vertices are defined over $\mathbb{F}_{11^2}$.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{11^2}$

The $\mathcal{G}_2(\mathbb{F}_{11})$ edges:

- There is one outgoing 2-isogeny of E_0, namely the isogeny $\phi_0 : E_0 \to E_1^t$ generated by the 2-torsion point $(10, 0)$ in E_0.
- There is one outgoing 2-isogeny of E_0^t, namely the isogeny $\phi_0^t : E_0^t \to E_1^t$ generated by the 2-torsion point $(5, 0)$ in E_0^t.
- There is one outgoing 2-isogeny of E_1, namely the isogeny $\phi_1 : E_1 \to E_1^t$ generated by the 2-torsion point $(0, 0)$ in E_1.
- There are three outgoing 2-isogeny of E_1^t, namely the isogenies
  - $\phi_{1,1}^t : E_1^t \to E_1$ generated by the 2-torsion point $(0, 0)$ in E_1^2.
  - $\phi_{1,2}^t : E_1^t \to E_0$ generated by the 2-torsion point $(2, 0)$ in E_1^2.
  - $\phi_{1,3}^t : E_1^t \to E_1^t$ generated by the 2-torsion point $(8, 0)$ in E_1^2.

The $\mathcal{G}_3(\mathbb{F}_{11})$ edges:

Let z_2 be a number such that $[\mathbb{F}_{11}(z_2) : \mathbb{F}_{11}] = 2$.

- There are two outgoing 3-isogeny of E_0, namely the isogenies
  - $\psi_{0,1} : E_0 \to E_0^t$ generated by the 3-torsion point $(0, 1)$ in E_0.
  - $\psi_{0,2} : E_0 \to E_1$ generated by the 3-torsion point $(6, 2z_2 + 7)$ in E_0.
- There are two outgoing 3-isogeny of E_0^t, namely the isogenies
  - $\psi_{0,1}^t : E_0^t \to E_0$ generated by the 3-torsion point $(0, 5z_2 + 1)$ in E_0^t.
  - $\psi_{0,2}^t : E_0^t \to E_1$ generated by the 3-torsion point $(4, 2)$ in E_0^t.
- There are two outgoing 3-isogeny of E_1, namely the isogenies
  - $\psi_{1,1} : E_1 \to E_0$ generated by the 3-torsion point $(5, 3)$ in E_1.
  - $\psi_{1,2} : E_1 \to E_0^t$ generated by the 3-torsion point $(6, z_2 + 9)$ in E_1.
- There are two outgoing 3-isogeny of E_1^t, namely the isogenies
  - $\psi_{1,1}^t : E_1^t \to E_1^t$ generated by the 3-torsion point $(3, 2)$ in E_1^t.
  - $\psi_{1,2}^t : E_1^t \to E_1^t$ generated by the 3-torsion point $(8, 3z_2 + 5)$ in E_1^t.

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{11})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{11})$:

- E_0 and E_0^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 0, denoted [0].
- E_1 and E_1^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 1, denoted [1].

How the edges of $\mathcal{G}_2(\mathbb{F}_{11})$ change when going to $\mathcal{G}_2(\overline{\mathbb{F}}_{11})$:

- ϕ_0 and ϕ_0^t become equivalent isogenies and hence become one edge of the form $([0], [1])$.
- ϕ_1 and $\phi_{1,1}^t$ become equivalent isogenies and hence become one edge of the form $([1], [1])$.
- $\phi_{1,2}^t$ does not become equivalent to any other isogenies.

- $\phi_{1,3}^t$ does not become equivalent to any other isogenies.
- Two new isogeny equivalence classes are added to this graph of the form $([0],[1])$.

How the edges of $\mathcal{G}_3(\mathbb{F}_{11})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{11})$:

- $\psi_{0,1}$ and $\psi_{0,1}^t$ become equivalent isogenies and hence become one edge of the form $([0],[0])$.
- $\psi_{0,2}$ and $\psi_{0,2}^t$ become equivalent isogenies and hence become one edge of the form $([0],[1])$.
- $\psi_{1,1}$ does not become equivalent to any other isogenies.
- $\psi_{1,2}$ does not become equivalent to any other isogenies.
- $\psi_{1,1}^t$ does not become equivalent to any other isogenies.
- $\psi_{1,2}^t$ does not become equivalent to any other isogenies.
- Two new isogeny equivalence classes are added to this graph of the form $([0],[1])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{11})$ vertices:

- There are 2 supersingular isomorphism class of elliptic curves, namely $j=0$ and $j=1$.
- $E_0 : y^2 = x^3 + 1$ is a representative of the isomorphism class $j = 0$
- $E_1 : y^2 = x^3 + x$ is a representative of the isomorphism class $j = 1$.

The $\mathcal{G}_2(\overline{\mathbb{F}}_{11})$ edges:

- There are three edges of the form $([0],[1])$.
- There are two edges of the form $([1],[0])$.
- There is one edges of the form $([1],[1])$.

The $\mathcal{G}_3(\overline{\mathbb{F}}_{11})$ edges:

- There is one edge of the form $([0],[0])$.
- There are three edges of the form $([0],[1])$.
- There are two edges of the form $([1],[0])$.
- There are two edges of the form $([1],[1])$.

A.6. $p = 13$. The $\mathcal{G}_\ell(\mathbb{F}_{13})$ vertices:

- There are two isomorphism classes of elliptic curves, both of which have j-invariant 5.
- Choose $E_5 : y^2 = x^3 + x + 4$ as the representative of an isomorphism class of elliptic curves with j-invariant 5.
- Choose $E_5^t : y^2 = x^3 + 10x + 7$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 5.
- all of the 2-torsion points of the vertices are defined over $\mathbb{F}_{13^2}$.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{13^4}$

The $\mathcal{G}_2(\mathbb{F}_{13})$ edges:

- There is one outgoing 2-isogeny of E_5, namely the isogeny $\phi_5 : E_5 \to E_5^t$ generated by the 2-torsion point $(10,0)$ in E_5.
- There is one outgoing 2-isogeny of E_5^t, namely the isogeny $\phi_5^t : E_5^t \to E_5$ generated by the 2-torsion point $(7,0)$ in E_5^t.

The $\mathcal{G}_3(\mathbb{F}_{13})$ edges:

- There are no isogenies in this graph

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{13})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{13})$:

- E_5 and E_5^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 5, denoted [5].

How the edges of $\mathcal{G}_2(\mathbb{F}_{13})$ change when going to $\mathcal{G}_2(\overline{\mathbb{F}}_{13})$:

- ϕ_5 and ϕ_5^t become equivalent isogenies and hence become one edge of the form $([5],[5])$.
- Two new isogeny equivalence classes are added to this graph of the form $([5],[5])$.

How the edges of $\mathcal{G}_3(\mathbb{F}_{13})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{13})$:

- Four new isogeny equivalence classes are added to this graph of the form $([5],[5])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{13})$ vertices:

- There is one supersingular isomorphism class of elliptic curves, namely $j = 5$
- $E_5 : y^2 = x^3 + x + 4$ is a representative of the isomorphism class $j = 5$

The $\mathcal{G}_2(\overline{\mathbb{F}}_{13})$ edges:

- There are three edges of the form $([5],[5])$.

The $\mathcal{G}_3(\overline{\mathbb{F}}_{13})$ edges:

- There are four edges of the form $([5],[5])$.

A.7. $p = 17$. The $\mathcal{G}_\ell(\mathbb{F}_{17})$ vertices:

- There are four isomorphism classes of elliptic curves
- Two of the isomorphism classes of elliptic curves have j-invariant 0.
- The other two isomorphism classes of elliptic curves have j-invariant 8.
- Choose $E_0 : y^2 = x^3 + 1$ as the representative of an isomorphism class of elliptic curves with j-invariant 0.
- Choose $E_0^t : y^2 = x^3 + 6$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 0.
- Choose $E_8 : y^2 = x^3 + 4x + 9$ as the representative of an isomorphism class of elliptic curves with j-invariant 8.
- Choose $E_8^t : y^2 = x^3 + 15x + 3$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 8.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{17^2}$

The $\mathcal{G}_3(\mathbb{F}_{17})$ edges:

Let z_2 be a number such that $[\mathbb{F}_{17}(z_2) : \mathbb{F}_{17}] = 2$.

- There are two outgoing 3-isogeny of E_0, namely the isogenies
  - $\phi_{0,1} : E_0 \to E_0^t$ generated by the 3-torsion point $(0, 1)$ in E_0.
  - $\phi_{0,2} : E_0 \to E_8$ generated by the 3-torsion point $(4, 7z_2 + 5)$ in E_0.
- There are two outgoing 3-isogeny of E_0^t, namely the isogenies
  - $\phi_{0,1}^t : E_0^t \to E_0$ generated by the 3-torsion point $(0, 2z_2 + 16)$ in E_0^t.
  - $\phi_{0,2}^t : E_0^t \to E_8^t$ generated by the 3-torsion point $(3, 4)$ in E_0^t.
- There are two outgoing 3-isogeny of E_8, namely the isogenies
  - $\phi_{8,1} : E_8 \to E_8^t$ generated by the 3-torsion point $(1, 7z_2 + 5)$ in E_8.
  - $\phi_{8,2} : E_8 \to E_0$ generated by the 3-torsion point $(14, 2)$ in E_8.
- There are two outgoing 3-isogeny of E_8^t, namely the isogenies
  - $\phi_{8,1}^t : E_8^t \to E_0^t$ generated by the 3-torsion point $(2, 4z_2 + 15)$ in E_8^t.
  - $\phi_{8,2}^t : E_8^t \to E_8$ generated by the 3-torsion point $(5, 4)$ in E_8^t.

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{17})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{17})$:

- E_0 and E_0^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 0, denoted [0].
- E_8 and E_8^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 8, denoted [8].

How the edges of $\mathcal{G}_3(\mathbb{F}_{17})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{17})$:

- $\phi_{0,1}$ and $\phi_{0,1}^t$ become equivalent isogenies and hence become one edge of the form $([0],[0])$.
- $\phi_{0,2}$ and $\phi_{0,2}^t$ become equivalent isogenies and hence become one edge of the form $([0],[8])$.
- $\phi_{8,1}$ and $\phi_{8,2}^t$ become equivalent isogenies and hence become one edge of the form $([8],[8])$.
- $\phi_{8,2}$ and $\phi_{8,1}^t$ become equivalent isogenies and hence become one edge of the form $([8],[0])$.
- Two new isogeny equivalence classes are added to this graph of the form $([0],[8])$.
- Two new isogeny equivalence classes are added to this graph of the form $([8],[8])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{17})$ vertices:

- There are 2 supersingular isomorphism class of elliptic curves, namely $j=0$ and $j=8$.
- $E_0 : y^2 = x^3+1$ is a representative of the isomorphism class $j=0$
- $E_1 : y^2 = x^3+4x+9$ is a representative of the isomorphism class $j=8$.

The $\mathcal{G}_3(\overline{\mathbb{F}}_{17})$ edges:

- There is one edge of the form $([0],[0])$.
- There are three edges of the form $([0],[8])$.
- There is one edge of the form $([8],[0])$.
- There are three edges of the form $([8],[8])$.

A.8. $p=19$. The $\mathcal{G}_\ell(\mathbb{F}_{19})$ vertices:

- There are four isomorphism classes of elliptic curves
- Two of the isomorphism classes of elliptic curves have j-invariant 7.
- The other two isomorphism classes of elliptic curves have j-invariant 18.
- Choose $E_7 : y^2 = x^3+3x+16$ as the representative of an isomorphism class of elliptic curves with j-invariant 7.
- Choose $E_7^t : y^2 = x^3+12x+14$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 7.
- Choose $E_{18} : y^2 = x^3+x$ as the representative of an isomorphism class of elliptic curves with j-invariant 18.
- Choose $E_{18}^t : y^2 = x^3+8x$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 18.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{19^4}$

The $\mathcal{G}_3(\mathbb{F}_{19})$ edges:

- There are no edges in this graph

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{19})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{19})$:

- E_7 and E_7^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 7, denoted [7].
- E_{18} and E_{18}^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 18, denoted [18].

How the edges of $\mathcal{G}_3(\mathbb{F}_{19})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{19})$:

- Four new isogeny equivalence classes are added to this graph of the form $([18],[7])$.
- Two new isogeny equivalence classes are added to this graph of the form $([7],[18])$.
- Two new isogeny equivalence classes are added to this graph of the form $([7],[7])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{19})$ vertices:

- There are 2 supersingular isomorphism class of elliptic curves, namely $j=7$ and $j=18$.
- $E_7 : y^2 = x^3+3x+16$ is a representative of the isomorphism class $j=7$
- $E_{18} : y^2 = x^3+x$ is a representative of the isomorphism class $j=18$.

The $\mathcal{G}_3(\overline{\mathbb{F}}_{19})$ edges:

- There are four edges of the form $([18],[7])$.
- There are two edge of the form $([7],[18])$.
- There are two edges of the form $([7],[7])$.

A.9. $p=23$. The $\mathcal{G}_\ell(\mathbb{F}_{23})$ vertices:

- There are six isomorphism classes of elliptic curves
- Two of the isomorphism classes of elliptic curves have j-invariant 0.
- Two of the isomorphism classes of elliptic curves have j-invariant 3.
- The last two isomorphism classes of elliptic curves have j-invariant 19.
- Choose $E_0 : y^2 = x^3+1$ as the representative of an isomorphism class of elliptic curves with j-invariant 0.
- Choose $E_0^t : y^2 = x^3+20$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 0.
- Choose $E_3 : y^2 = x^3+x$ as the representative of an isomorphism class of elliptic curves with j-invariant 3.
- Choose $E_3^t : y^2 = x^3+17x$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 3.
- Choose $E_{19} : y^2 = x^3+8x+1$ as the representative of an isomorphism class of elliptic curves with j-invariant 19.
- Choose $E_{19}^t : y^2 = x^3+2x+20$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 19.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{23^2}$

The $\mathcal{G}_3(\mathbb{F}_{23})$ edges:

Let z_2 be a number such that $[\mathbb{F}_{23}(z_2) : \mathbb{F}_{23}] = 2$.

- There are two outgoing 3-isogeny of E_0, namely the isogenies
  - $\phi_{0,1} : E_0 \to E_0^t$ generated by the 3-torsion point $(0,1)$ in E_0.
  - $\phi_{0,2} : E_0 \to E_3$ generated by the 3-torsion point $(20, 8z_2+15)$ in E_0.
- There are two outgoing 3-isogeny of E_0^t, namely the isogenies
  - $\phi_{0,1}^t : E_0^t \to E_0$ generated by the 3-torsion point $(0, 8z_2+15)$ in E_0^t.
  - $\phi_{0,2}^t : E_0^t \to E_3$ generated by the 3-torsion point $(13,3)$ in E_0^t.

- There are two outgoing 3-isogeny of E_3, namely the isogenies
  - $\phi_{3,1} : E_3 \to E_0^t$ generated by the 3-torsion point $(5, 5z_2 + 18)$ in E_3.
  - $\phi_{3,2} : E_3 \to E_0$ generated by the 3-torsion point $(18, 10)$ in E_3.
- There are two outgoing 3-isogeny of E_3^t, namely the isogenies
  - $\phi_{3,1}^t : E_3^t \to E_{19}^t$ generated by the 3-torsion point $(1, 8)$ in E_3^t.
  - $\phi_{3,2}^t : E_3^t \to E_{19}$ generated by the 3-torsion point $(22, 4z_2 + 19)$ in E_3^t.
- There are two outgoing 3-isogeny of E_{19}, namely the isogenies
  - $\phi_{19,1} : E_{19} \to E_3^t$ generated by the 3-torsion point $(16, 4)$ in E_{19}.
  - $\phi_{19,2} : E_{19} \to E_{19}^t$ generated by the 3-torsion point $(20, z_2+22)$ in E_{19}.
- There are two outgoing 3-isogeny of E_{19}^t, namely the isogenies
  - $\phi_{19,1}^t : E_{19}^t \to E_{19}$ generated by the 3-torsion point $(13, 9)$ in E_{19}^t.
  - $\phi_{19,2}^t : E_{19}^t \to E_3^t$ generated by the 3-torsion point $(15, 9z_2 + 19)$ in E_{19}^t.

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{23})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{23})$:

- E_0 and E_0^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 0, denoted [0].
- E_3 and E_3^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 3, denoted [3].
- E_{19} and E_{19}^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 19, denoted [19].

How the edges of $\mathcal{G}_3(\mathbb{F}_{23})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{23})$:

- $\phi_{0,1}$ and $\phi_{0,1}^t$ become equivalent isogenies and hence become one edge of the form $([0], [0])$.
- $\phi_{0,2}$ and $\phi_{0,2}^t$ become equivalent isogenies and hence become one edge of the form $([0], [3])$.
- $\phi_{19,1}$ and $\phi_{19,2}^t$ become equivalent isogenies and hence become one edge of the form $([19], [3])$.
- $\phi_{19,2}$ and $\phi_{19,1}^t$ become equivalent isogenies and hence become one edge of the form $([19], [19])$.
- $\phi_{3,1}$ does not become equivalent to any other isogenies.
- $\phi_{3,2}$ does not become equivalent to any other isogenies.
- $\phi_{3,1}^t$ does not become equivalent to any other isogenies.
- $\phi_{3,2}^t$ does not become equivalent to any other isogenies.
- two new isogeny equivalence classes are added to this graph of the form $([0], [3])$.
- two new isogeny equivalence classes are added to this graph of the form $([19], [19])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{23})$ vertices:

- There are 3 supersingular isomorphism class of elliptic curves, namely $j = 0$, $j = 3$, and $j = 19$.
- $E_0 : y^2 = x^3 + 1$ is a representative of the isomorphism class $j = 0$
- $E_3 : y^2 = x^3 + x$ is a representative of the isomorphism class $j = 3$
- $E_{19} : y^2 = x^3 + 8x + 1$ is a representative of the isomorphism class $j = 19$.

The $\mathcal{G}_3(\overline{\mathbb{F}}_{23})$ edges:

- There are three edges of the form $([19], [19])$.
- There is one edge of the form $([19], [3])$.

- There are two edge of the form $([3],[19])$.
- There are two edges of the form $([3],[0])$.
- There are three edges of the form $([0],[3])$.
- There is one edge of the form $([0],[0])$.

A.10. $p = 29$. The $\mathcal{G}_\ell(\mathbb{F}_{29})$ vertices:

- There are six isomorphism classes of elliptic curves
- Two of the isomorphism classes of elliptic curves have j-invariant 0.
- Two of the isomorphism classes of elliptic curves have j-invariant 2.
- The last two isomorphism classes of elliptic curves have j-invariant 25.
- Choose $E_0 : y^2 = x^3 + 1$ as the representative of an isomorphism class of elliptic curves with j-invariant 0.
- Choose $E_0^t : y^2 = x^3 + 11$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 0.
- Choose $E_2 : y^2 = x^3 + 3x + 28$ as the representative of an isomorphism class of elliptic curves with j-invariant 2.
- Choose $E_2^t : y^2 = x^3 + 8x + 11$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 2.
- Choose $E_{25} : y^2 = x^3 + 9x + 19$ as the representative of an isomorphism class of elliptic curves with j-invariant 25.
- Choose $E_{25}^t : y^2 = x^3 + 20x + 4$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 25.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{29^2}$

The $\mathcal{G}_3(\mathbb{F}_{29})$ edges:

Let z_2 be a number such that $[\mathbb{F}_{29}(z_2) : \mathbb{F}_{29}] = 2$.

- There are two outgoing 3-isogeny of E_0, namely the isogenies
  - $\phi_{0,1} : E_0 \to E_0^t$ generated by the 3-torsion point $(0, 1)$ in E_0.
  - $\phi_{0,2} : E_0 \to E_{25}$ generated by the 3-torsion point $(20, z_2 + 12)$ in E_0.
- There are two outgoing 3-isogeny of E_0^t, namely the isogenies
  - $\phi_{0,1}^t : E_0^t \to E_0$ generated by the 3-torsion point $(0, 8z_2 + 9)$ in E_0^t.
  - $\phi_{0,2}^t : E_0^t \to E_{25}^t$ generated by the 3-torsion point $(10, 5)$ in E_0^t.
- There are two outgoing 3-isogeny of E_2, namely the isogenies
  - $\phi_{2,1} : E_2 \to E_{25}^t$ generated by the 3-torsion point $(1, 12z_2 + 28)$ in E_2.
  - $\phi_{2,2} : E_2 \to E_2^t$ generated by the 3-torsion point $(2, 10)$ in E_2.
- There are two outgoing 3-isogeny of E_2^t, namely the isogenies
  - $\phi_{2,1}^t : E_2^t \to E_{25}$ generated by the 3-torsion point $(14, 5)$ in E_2^t.
  - $\phi_{2,2}^t : E_2^t \to E_2$ generated by the 3-torsion point $(28, 3z_2 + 7)$ in E_2^t.
- There are two outgoing 3-isogeny of E_{25}, namely the isogenies
  - $\phi_{25,1} : E_{25} \to E_0$ generated by the 3-torsion point $(8, 9)$ in E_{25}.
  - $\phi_{25,2} : E_{25} \to E_2^t$ generated by the 3-torsion point $(20, 14z_2 + 23)$ in E_{25}.
- There are two outgoing 3-isogeny of E_{25}^t, namely the isogenies
  - $\phi_{25,1}^t : E_{25}^t \to E_0^t$ generated by the 3-torsion point $(8, 3)$ in E_{25}^t.
  - $\phi_{25,2}^t : E_{25}^t \to E_2$ generated by the 3-torsion point $(9, 11z_2 + 16)$ in E_{25}^t.

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{29})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{29})$:

- E_0 and E_0^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 0, denoted [0].
- E_2 and E_2^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 2, denoted [2].
- E_{25} and E_{25}^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 25, denoted [25].

How the edges of $\mathcal{G}_3(\mathbb{F}_{29})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{29})$:

- $\phi_{0,1}$ and $\phi_{0,1}^t$ become equivalent isogenies and hence become one edge of the form $([0],[0])$.
- $\phi_{0,2}$ and $\phi_{0,2}^t$ become equivalent isogenies and hence become one edge of the form $([0],[25])$.
- $\phi_{2,1}$ and $\phi_{2,2}^t$ become equivalent isogenies and hence become one edge of the form $([2],[25])$.
- $\phi_{2,2}$ and $\phi_{2,1}^t$ become equivalent isogenies and hence become one edge of the form $([2],[2])$.
- $\phi_{25,1}$ and $\phi_{25,2}^t$ become equivalent isogenies and hence become one edge of the form $([25],[0])$.
- $\phi_{25,2}$ and $\phi_{25,1}^t$ become equivalent isogenies and hence become one edge of the form $([25],[2])$.
- Two new isogeny equivalence classes are added to this graph of the form $([0],[25])$.
- Two new isogeny equivalence classes are added to this graph of the form $([19],[19])$.
- Two new isogeny equivalence classes are added to this graph of the form $([2],[2])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{29})$ vertices:

- There are 3 supersingular isomorphism class of elliptic curves, namely $j=0$, $j=2$, and $j=125$.
- $E_0 : y^2 = x^3 + 1$ is a representative of the isomorphism class $j = 0$
- $E_3 : y^2 = x^3 + 3x + 28$ is a representative of the isomorphism class $j = 2$
- $E_{19} : y^2 = x^3 + 9x + 19$ is a representative of the isomorphism class $j = 25$.

The $\mathcal{G}_3(\overline{\mathbb{F}}_{29})$ edges:

- There are three edges of the form $([2],[2])$.
- There is one edge of the form $([2],[25])$.
- There are two edge of the form $([25],[25])$.
- There is one edge of the form $([25],[0])$.
- There are three edges of the form $([0],[25])$.
- There is one edge of the form $([0],[0])$.

A.11. $p = 31$. The $\mathcal{G}_\ell(\mathbb{F}_{31})$ vertices:

- There are six isomorphism classes of elliptic curves
- Two of the isomorphism classes of elliptic curves have j-invariant 2.
- Another two isomorphism classes of elliptic curves have j-invariant 4.
- The last two isomorphism classes of elliptic curves have j-invariant 23.
- Choose $E_2 : y^2 = x^3 + 2x + 3$ as the representative of an isomorphism class of elliptic curves with j-invariant 2.

- Choose $E_2^t : y^2 = x^3 + 16x + 19$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 2.
- Choose $E_4 : y^2 = x^3 + 11x + 26$ as the representative of an isomorphism class of elliptic curves with j-invariant 4.
- Choose $E_4^t : y^2 = x^3 + 22x + 18$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 4.
- Choose $E_{23} : y^2 = x^3 + x$ as the representative of an isomorphism class of elliptic curves with j-invariant 23.
- Choose $E_{23}^t : y^2 = x^3 + 13x$ as the representative of the isomorphism class of the twists elliptic curves with j-invariant 23.
- all of the 3-torsion points of the vertices are defined over $\mathbb{F}_{31^4}$

The $\mathcal{G}_3(\mathbb{F}_{31})$ edges:

- There are no edges in this graph

How the vertices of $\mathcal{G}_\ell(\mathbb{F}_{31})$ change when going to $\mathcal{G}_\ell(\overline{\mathbb{F}}_{31})$:

- E_2 and E_2^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 2, denoted [2].
- E_2 and E_2^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 2, denoted [2].
- E_{23} and E_{23}^t become isomorphic to each other to make the equivalence class of isogenies with j-invariant 23, denoted [23].

How the edges of $\mathcal{G}_3(\mathbb{F}_{31})$ change when going to $\mathcal{G}_3(\overline{\mathbb{F}}_{31})$:

- Four new isogeny equivalence classes are added to this graph of the form $([23], [4])$.
- Two new isogeny equivalence classes are added to this graph of the form $([4], [23])$.
- Two new isogeny equivalence classes are added to this graph of the form $([4], [2])$.
- Two new isogeny equivalence classes are added to this graph of the form $([2], [4])$.
- Two new isogeny equivalence classes are added to this graph of the form $([2], [2])$.

The $\mathcal{G}_\ell(\overline{\mathbb{F}}_{31})$ vertices:

- There are 3 supersingular isomorphism class of elliptic curves, namely $j = 2$, $j = 4$, and $j = 23$.
- $E_2 : y^2 = x^3 + 2x + 3$ is a representative of the isomorphism class $j = 2$
- $E_4 : y^2 = x^3 + 11x + 26$ is a representative of the isomorphism class $j = 4$
- $E_{23} : y^2 = x^3 + x$ is a representative of the isomorphism class $j = 23$.

The $\mathcal{G}_3(\overline{\mathbb{F}}_{31})$ edges:

- There are four edges of the form $([23], [4])$.
- There are two edges of the form $([4], [23])$.
- There are two edges of the form $([4], [2])$.
- There are two edges of the form $([2], [4])$.
- There are two edges of the form $([2], [2])$.

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