Theory of Neural Dynamics and Application to ML based on RC

Exercise sheet 2 (16.05.2024)

Reduction of the 4D Hodgkin-Huxley neuron into a 2D model and stability analysis of the resting states of two neuron models

Problem 1

A hard-working but sometimes careless neuroscientist said the following: *" Each of the following statements about the Hodgkin-Huxley (HH) neuron model is sufficient to replace the gating variables n and h into a single variable W:*

(A) The differential equations of n and h of the HH neuron model have the same time constants, i.e., $\tau_n(V) = \tau_h(V)$ *.*

(B) Both n and h have the same activation functions, i.e., $n_\infty(V) = h_\infty(V)$*.*

(C) Both n and h have same time constant <u>*and*</u> *the same activation function.*

(D) Both n and h have same time constants <u>*and*</u> *activation functions that are identical after some additive rescaling.*

(E) Both n and h have same time constant <u>*and*</u> *activation functions that are identical after some multiplicative rescaling. "*

Question: Which of the following condition(s) would a hard-working and careful student, like you, say is(are) required to reduce the 4D Hodgkin-Huxley neuron model to a 2D neuron model?

(1) The statements (A), (B), (C), (D), and (E) are all not satisfied, but the gating variable m is faster than voltage variable V but slower than the gating variables n and h.

(2) Statement (D) is satisfied, and m is faster than V but instantaneously slower than n and h.

(3) The statements (A), (B), (C), (D), and (E) are all satisfied, and m is faster than V, n, and h.

(4) Statement (C) is satisfied, and m is faster than V, n, and h.

(5) Statement (E) is satisfied, and the gating variable m is faster than V, n, and h.

(6) Statement (B) is satisfied, and m is slower than V, n, and h.

(7) Statement (A) is satisfied, and m is faster than V and n but slower than h.

(8) Statement (A) is satisfied, and m is slower than V but faster than n and h.

(9) Only statements (3) and (5) could allow this dimensionality reduction.

(10) None of the above.

Problem 2

Another version of the FitzHugh-Nagumo (FHN) neuron model is given below. This version can have only one fixed, no matter the parameter values α and I.

(1) Calculate the coordinates (v_e, w_e) of this fixed point.

(2) Determine the nature of this fixed point (stable or unstable) when $\alpha > 1$ and also when $\alpha < 1$.

$$\begin{cases} dv &= (v - \frac{v^3}{3} - w + I)dt, \\ \dfrac{dw}{dt} &= \varepsilon(\alpha + v), \end{cases} \tag{1}$$

where $v \in \mathbb{R}$ represents the membrane potential, which measures the electric potential difference across the neuron's cell membrane. $w \in \mathbb{R}$ represents the recovery variable and it is related to the neuron's recovery process, and the parameters are given by α and the external current $I \in \mathbb{R}_+$. The timescale separation between v and w is given by $0 < \varepsilon \ll 1$.

Problem 3

The Hindmarsh-Rose (HR) neuron model given below can have up to three fixed points depending on the values of some parameters.

(1) Under what condition on the parameters would the HR neuron model given below admit a unique fixed point?

(2) Calculate the coordinates [i.e., (x_e, y_e, z_e)] of this unique fixed point using the parameter values given below. (*For the parameter $I \geq 0$, you can set it to whatever values you like.*)

(3) Determine (numerically if necessary) for which range of values of the parameter I would the unique fixed point (x_e, y_e, z_e) calculated in (2) be: (a) stable and (b) unstable.

$$\begin{cases} \dfrac{dx}{dt} = y - ax^3 + bx^2 - z + I \\ \dfrac{dy}{dt} = c - dx^2 - y \\ \dfrac{dz}{dt} = r\left(s(x - x_{\text{rest}}) - z\right) \end{cases} \tag{2}$$

where

- $x \in \mathbb{R}$ represents the membrane potential, which measures the electric potential difference across the neuron's cell membrane. It determines the neuron's excitability and plays a crucial role in generating and transmitting electrical signals within the neuron.

- $y \in \mathbb{R}$ represents the recovery variable and it is related to the neuron's recovery process. It is associated with the neuron's ability to recover from depolarization or firing events. It influences the neuron's excitability and plays a role in controlling the firing patterns.

- $z \in \mathbb{R}$ represents another auxiliary variable associated with the neuron's dynamics. It can be related to the neuron's ionic currents or other physiological processes. z captures the slow dynamics of the neuron's behavior.

- The real constant parameters of the model are represented by a, b, c, d, r, s, I, and x_{rest}). Typically the parameters have values: $a = 1.0$; $b = 3.0$; $c = 1.0$; $d = 5.0$; $r = 0.006$, $s = 5.2$, $x_{\text{rest}} = -1.56$ is the resting membrane potential of the neuron. Its value is usually set to a desired baseline potential. It's important to note that these values are not fixed or universally standardized and can be adjusted based on the specific application. The parameter values can be chosen to replicate certain observed behaviors, such as spiking and bursting patterns, or to match experimental data in specific neuronal systems. $I \geq 0$ is the external current input to the neuron, and its value can vary depending on the specific situation or stimulation being applied.

- Use the parameter $(a, b, c, d, r, s, x_{\text{rest}})$ values given above for this exercise.
- *Hints: You can eliminate y and z in the first question by their expressions in terms of x calculated from the second and third equations. So that you only have one cubic polynomial in terms of x and use the depressed cubic and Cardano's formula (https://en.wikipedia.org/wiki/Cubic_equation) to solve for the unique fixed point.*