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# ASSIGNMENT 02

## Q1)

Sol) $V = \left\{ \begin{bmatrix} x \\ x/2 \end{bmatrix} : x \in R \right\}$

Let $u_1 = \begin{bmatrix} x_1 \\ x_1/2 \end{bmatrix}, u_2 = \begin{bmatrix} x_2 \\ x_2/2 \end{bmatrix}$ and $u_3 = \begin{bmatrix} x_3 \\ x_3/2 \end{bmatrix}$

$\forall\ x_1, x_2, x_3 \in V$

(i) $u_1 + u_2 \in V$

$$\begin{bmatrix} x_1 \\ x_1/2 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2/2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ (x_1+x_2)/2 \end{bmatrix} \in R$$

(ii) $u_1 + (u_2 + u_3) = (u_1 + u_2) + u_3$

L.H.S = $u_1 + (u_2 + u_3)$

$$\sim \begin{bmatrix} x_1 \\ x_1/2 \end{bmatrix} + \left( \begin{bmatrix} x_2 \\ x_2/2 \end{bmatrix} + \begin{bmatrix} x_3 \\ x_3/2 \end{bmatrix} \right)$$

$$\sim \begin{bmatrix} x_1 \\ \frac{x_1}{2} \end{bmatrix} + \begin{bmatrix} x_2 + x_3 \\ (x_2 + x_3)/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} x_1 + x_2 + x_3 \\ (x_1 + x_2 + x_3)/2 \end{bmatrix}$$

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$R.H.S = (u_1+u_2)+u_3$

$$\sim \left(\begin{bmatrix} x_1 \\ x_1/2 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2/2 \end{bmatrix}\right) + \begin{bmatrix} x_3 \\ x_3/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} x_1+x_2 \\ (x_1+x_2)/2 \end{bmatrix} + \begin{bmatrix} x_3 \\ x_3/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} x_1+x_2+x_3 \\ (x_1+x_2+x_3)/2 \end{bmatrix}$$

$L.H.S = R.H.S$

<u>proved</u>

(iii) $u_1+O=u_1 \quad \forall u_1 \in V \; \exists O \in V \text{ in } O = \begin{bmatrix} a \\ b \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_1/2 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1/2 \end{bmatrix}$$

$x_1 + a = x_1 \quad \Rightarrow \boxed{a=0}$

$\frac{x_1}{2} + b = \frac{x_1}{2} \quad \Rightarrow \boxed{b=0}$

$\therefore O = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V$

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(iv) $u_1 + (-u_1) = 0$ $\forall u \in V \, \exists \, 0 = \begin{bmatrix} a \\ b \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_1/2 \end{bmatrix} + (-1)\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 - a = 0$ , $\frac{x_1}{2} - b = 0$

$\boxed{a = x_1}$ , $\boxed{b = \frac{x_1}{2}}$

$$\begin{bmatrix} x_1 \\ x_1/2 \end{bmatrix} \in V$$

(v) $u_1 + u_2 = u_2 + u_1$

$$\begin{bmatrix} x_1 \\ x_1/2 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2/2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2/2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_1/2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + x_2 \\ (x_1 + x_2)/2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ (x_1 + x_2)/2 \end{bmatrix}$$

L.H.S = R.H.S

proved

(vi) $ku_1 \in V$

$$k\begin{bmatrix} x_1 \\ x_1/2 \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_1/2 \end{bmatrix} \in R[v]$$

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(vii) $(k_1+k_2)u_1 = k_1u_1 + k_2u_1$

$$(k_1+k_2)\begin{bmatrix} x_1 \\ x_1/2 \end{bmatrix} = k_1\begin{bmatrix} x_1 \\ x_1/2 \end{bmatrix} + k_2\begin{bmatrix} x_1 \\ x_1/2 \end{bmatrix}$$

$$\begin{bmatrix} (k_1+k_2)x_1 \\ (k_1+k_2)x_1/2 \end{bmatrix} = \begin{bmatrix} k_1x_1 \\ k_1x_1/2 \end{bmatrix} + \begin{bmatrix} k_2x_1 \\ k_2x_1/2 \end{bmatrix}$$

$$\begin{bmatrix} k_1x_1 + k_2x_1 \\ (k_1x_1 + k_2x_1)/2 \end{bmatrix} = \begin{bmatrix} k_1x_1 + k_2x_1 \\ (k_1x_1 + k_2x_1)/2 \end{bmatrix}$$

L.H.S = R.H.S

Proved

(viii) $k(u_1+u_2) = ku_1 + ku_2$

$$k\left(\begin{bmatrix} x_1 \\ x_1/2 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2/2 \end{bmatrix}\right) = k\begin{bmatrix} x_1 \\ x_1/2 \end{bmatrix} + k\begin{bmatrix} x_2 \\ x_2/2 \end{bmatrix}$$

$$k\left(\begin{bmatrix} x_1+x_2 \\ (x_1+x_2)/2 \end{bmatrix}\right) = \begin{bmatrix} kx_1 \\ kx_1/2 \end{bmatrix} + \begin{bmatrix} kx_2 \\ kx_2/2 \end{bmatrix}$$

$$\begin{bmatrix} kx_1 + kx_2 \\ (kx_1 + kx_2)/2 \end{bmatrix} = \begin{bmatrix} kx_1 + kx_2 \\ (kx_1 + kx_2)/2 \end{bmatrix}$$

L.H.S = R.H.S

proved

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(ix) $k_1(k_2u) = k_1k_2u$

$$k_1\left(k_2\begin{bmatrix} x \\ x/2 \end{bmatrix}\right) = k_1k_2\begin{bmatrix} x \\ x/2 \end{bmatrix}$$

$$\begin{bmatrix} k_1k_2x \\ (k_1k_2x)/2 \end{bmatrix} = \begin{bmatrix} k_1k_2x \\ (k_1k_2x)/2 \end{bmatrix}$$

L.H.S = R.H.S

Proved

(x) $1 \cdot u_1 = u_1$

L.H.S $= 1 \cdot u_1$

$$= 1 \cdot \begin{bmatrix} x_1 \\ \frac{x_1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ \frac{x_1}{2} \end{bmatrix}$$

$= u_1$

L.H.S = R.H.S

Proved

V is Vector Space as it satisfies all axioms.

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# Q2)

**$P_2$ is subspace of $P_3$:**

- Subset $W$ of a vector is called a subspace of $V$ if $W$ is itself a vector space under the addition and scalar multiplication ~~of~~ on $V$.

$P_2 = \{1, x, x^2\}$
$P_3 = \{1, x, x^2, x^3\}$
as $P_2 \subseteq P_3$
$P_3 \rightarrow$ Vector Space
$P_2 \rightarrow$ Subspace

Every polynomial upto degree 2 is also a polynomial of degree upto 3 $P_2 \subseteq P_3$.
The sum of two polynomials is a polynomial.

**$P_n$ is Subset of $P_{n+1}$:**

$P_n = \{1, x, x^2, x^3, \dots, x^n\}$
$P_{n+1} = \{1, x, x^2, \dots, x^n, x^{n+1}\}$
as $P_n \subseteq P_{n+1}$

Every polynomial up to degree $n$ is also polynomial of degree upto $n+1$. $P_n$ is subset of $P_{n+1}$.

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Q3)

Sol) $S = \{ (\frac{3}{4}, \frac{5}{2}, \frac{3}{2}), (3, 4, \frac{7}{2}), (\frac{-3}{2}, 6, 2) \}$

$k_1 V_1 + k_2 V_2 + k_3 V_3 = 0$

Let $V_1 = (\frac{3}{4}, \frac{5}{2}, \frac{3}{2}), V_2 = (3, 4, \frac{7}{2}),$

$V_3 = (\frac{-3}{2}, 6, 2)$

$k_1 (\frac{3}{4}, \frac{5}{2}, \frac{3}{2}) + k_2 (3, 4, \frac{7}{2}) +$

$k_3 (\frac{-3}{2}, 6, 2) = (0, 0, 0)$

$$A_b = \begin{bmatrix} 3/4 & 3 & -3/2 \\ 5/2 & 4 & 6 \\ 3/2 & 7/2 & 2 \end{bmatrix}$$

$$|A_b| = \begin{vmatrix} 3/4 & 3 & -3/2 \\ 5/2 & 4 & 6 \\ 3/2 & 7/2 & 2 \end{vmatrix}$$

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$$|A_b| = 3\begin{vmatrix} \frac{1}{4} & 1 & -\frac{1}{2} \\ 5/2 & 4 & 6 \\ 3/2 & 7/2 & 2 \end{vmatrix}$$

$$|A_b| = \frac{3}{4}\begin{vmatrix} 1 & 1 & -1/2 \\ 10 & 4 & 6 \\ 6 & 7/2 & 2 \end{vmatrix}$$

$$|A_b| = \frac{3}{8}\begin{vmatrix} 1 & 1 & -1 \\ 10 & 4 & 12 \\ 6 & 7/2 & 4 \end{vmatrix}$$

~~$|A_b| =$~~ $$|A_b| = \frac{3}{8}\begin{vmatrix} 1 & 0 & 0 \\ 10 & -6 & 22 \\ 6 & -5/2 & 10 \end{vmatrix} \quad C_2 - C_1, \; C_3 + C_1$$

Expand by $R_1$

$$|A_b| = \frac{3}{8}\left\{1\begin{vmatrix} -6 & 22 \\ -\frac{5}{2} & 10 \end{vmatrix} - 0 + 0\right\}$$

$$|A_b| = \frac{3}{8}(-60 + 55)$$

$$|A_b| = -\frac{15}{8} \neq 0$$

$\therefore$ S is Linearly independent.

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Q4)

Sol) $A = \begin{bmatrix} -9 & -6 & -22 \\ 1 & 2 & 2 \\ 4 & 2 & 10 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} -9 & -6 & -22 \\ 1 & 2 & 2 \\ 4 & 2 & 10 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} -9-\lambda & -6 & -22 \\ 1 & 2-\lambda & 2 \\ 4 & 2 & 10-\lambda \end{bmatrix}$

Characteristic Eqn:

$\lambda^3 - tr(A)\lambda^2 + (C_{11} + C_{22} + C_{33})\lambda - \det(A) = 0$

$\lambda^3 - (-9+2+10)\lambda^2 + (16-2-12)\lambda - (0) = 0$

R.W

$C_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 2 \\ 2 & 10 \end{vmatrix} = 20-4 = 16$

$C_{22} = (-1)^{2+2} \begin{vmatrix} -9 & -22 \\ 4 & 10 \end{vmatrix} = (-90 + 88) = -2$

$C_{33} = (-1)^{3+3} \begin{vmatrix} -9 & -6 \\ 1 & 2 \end{vmatrix} = -18+6 = -12$

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$\lambda^3 - 3\lambda^2 + 2\lambda = 0$

$\lambda(\lambda^2 - 3\lambda + 2) = 0$

$\boxed{\lambda = 0}$ , $\lambda^2 - 3\lambda + 2 = 0$

$\lambda^2 - 2\lambda - \lambda + 2 = 0$

$\lambda(\lambda - 2) - 1(\lambda - 2) = 0$

$(\lambda - 2)(\lambda - 1) = 0$

$\boxed{\lambda = 2}$ , $\boxed{\lambda = 1}$

Eigen values:

$\lambda = (0, 1, 2)$

$\lambda = 0$ :

$$A - \lambda I = \begin{bmatrix} -9-\lambda & -6 & -22 \\ 1 & 2-\lambda & 2 \\ 4 & 2 & 10-\lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -9 & -6 & -22 \\ 1 & 2 & 2 \\ 4 & 2 & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ -9 & -6 & -22 \\ 4 & 2 & 10 \end{bmatrix} R_{12}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 12 & -4 \\ 0 & -6 & 2 \end{bmatrix} \begin{array}{l} R_2 + 9R_1, \\ R_3 - 4R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 6 & -2 \\ 0 & -6 & 2 \end{bmatrix} \frac{1}{2} R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 6 & -2 \\ 0 & 0 & 0 \end{bmatrix} R_3 + R_2$$

$\Rightarrow x + 2y + 2z = 0$

$6y - 2z = 0$

Let $z = t$

$\cancel{\Rightarrow 6y - 2(t) = 0}$

$\Rightarrow 6y - 2(t) = 0$

$$\boxed{y = \frac{t}{3}}$$

$\Rightarrow x + 2\left(\frac{t}{3}\right) + 2t = 0$

$$\boxed{x = -\frac{8t}{3}}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -8/3 \\ 1/3 \\ 1 \end{bmatrix}$$

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~~Basis = {[-8/3, 1/3, 1]}~~

$\lambda = 1$ :

$$A - \lambda I = \begin{bmatrix} -10 & -6 & -22 \\ 1 & 1 & 2 \\ 4 & 2 & 9 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 1 & 2 \\ -10 & -6 & -22 \\ 4 & 2 & 9 \end{bmatrix} R_{12}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ -5 & -3 & -11 \\ 4 & 2 & 9 \end{bmatrix} \frac{1}{2} R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix} \begin{matrix} R_2 + 5R_1 \\ R_3 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_3 + R_2$$

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$x+y+2z=0$
$2y-z=0$
Let $\boxed{z=t}$

$\Rightarrow 2y-t=0$
$\boxed{y=\frac{t}{2}}$

$\Rightarrow x+\left(\frac{t}{2}\right)+2(t)=0$

$x=-2t-\frac{t}{2}$

$\cancel{x=\frac{-4t-t}{2}}$ $\quad x=\frac{-4t-t}{2}$

$\cancel{\boxed{x=\frac{-5t}{2}}}$ $\quad \boxed{x=\frac{-5t}{2}}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -5/2 \\ 1/2 \\ 1 \end{bmatrix}$$

$\underbrace{\lambda=2}$:

$$A-\lambda I=\begin{bmatrix} -11 & -6 & -22 \\ 1 & 0 & 2 \\ 4 & 2 & 8 \end{bmatrix}$$

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$$\sim \begin{bmatrix} 1 & 0 & 2 \\ -11 & -6 & -22 \\ 4 & 2 & 8 \end{bmatrix} R_{12}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ -11 & -6 & -22 \\ 2 & 1 & 4 \end{bmatrix} \frac{1}{2} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & -6 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} R_2 + 11R_1, \\ R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \frac{1}{6} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_3 + R_2$$

$$x + 2z = 0$$

$$-y = 0$$

$$\Rightarrow \boxed{y = 0}$$

Let $z = t$

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$\Rightarrow x + 2(t) = 0$

$$\boxed{x = -2t}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Basis} = \left\{ \begin{bmatrix} -8/3 \\ 1/3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5/2 \\ 1/2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

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Q5)

Sol) $A = \begin{bmatrix} 4 & -5 \\ -3 & 2 \end{bmatrix}$

$A^6 = PD^6P^{-1}$

$A - \lambda I = \begin{bmatrix} 4 & -5 \\ -3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 4-\lambda & -5 \\ -3 & 2-\lambda \end{bmatrix}$

$\det(A) = \begin{vmatrix} 4-\lambda & -5 \\ -3 & 2-\lambda \end{vmatrix} = 0$

$\{(4-\lambda)(2-\lambda)\} - (-3)(-5) = 0$

$\{(4-\lambda)(2-\lambda)\} - 15 = 0$

$8 - 2\lambda - 4\lambda + \lambda^2 - 15 = 0$

$\lambda^2 - 6\lambda - 7 = 0$

$\lambda^2 - 7\lambda + \lambda - 7 = 0$

$\lambda(\lambda - 7) + 1(\lambda - 7) = 0$

$(\lambda - 7)(\lambda + 1) = 0$

$\boxed{\lambda = 7}, \quad \boxed{\lambda = -1}$

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Eigen Values of $\lambda = (-1, 7)$

$\underline{\lambda = -1}$:

$$A - \lambda I = \begin{bmatrix} 4-\lambda & -5 \\ -3 & 2-\lambda \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & -5 \\ -3 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \frac{1}{5}R_1, \ \frac{1}{3}R_2$$

$$\sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad R_2 + R_1$$

$$x - y = 0$$

let $\boxed{y = t}$

$$\Rightarrow x - t = 0$$

$$\boxed{x = t}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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$\lambda = 7:$

$$A-\lambda I = \begin{bmatrix} 4-7 & -5 \\ -3 & 2-7 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3 & -5 \\ -3 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 5 \\ -3 & -5 \end{bmatrix} (-1)R_1$$

$$\sim \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} R_2 + R_1$$

$\Rightarrow 3x + 5y = 0$

Let $\boxed{y = t}$

$3x + 5t = 0$

$$\boxed{x = \frac{-5t}{3}}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -5/3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

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$Basis = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \end{bmatrix} \right\}$

$\therefore$ dim (eigenspace) = 2

$\therefore$ A is diagonalizable

$D = \begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix} \Rightarrow D^6 = \begin{bmatrix} (-1)^6 & 0 \\ 0 & (7)^6 \end{bmatrix}$

$P = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix}$

$P^{-1}$:

$[P : I]$

$\sim \left[\begin{array}{cc|cc} 1 & -5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & -5 & 1 & 0 \\ 0 & 8 & -1 & 1 \end{array}\right] R_2 - R_1$

$\sim \left[\begin{array}{cc|cc} 1 & -5 & 1 & 0 \\ 0 & 1 & -1/8 & 1/8 \end{array}\right] \frac{1}{8} R_2$

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$$\sim\left[\begin{array}{cc|cc} 1 & 0 & 3/8 & 5/8 \\ 0 & 1 & -1/8 & 1/8 \end{array}\right] R_1+5R_2$$

$$\Rightarrow P^{-1} = \begin{bmatrix} 3/8 & 5/8 \\ -1/8 & 1/8 \end{bmatrix}$$

$$A^6 = PD^6P^{-1}$$

$$A^6 = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} (-1)^6 & 0 \\ 0 & (7)^6 \end{bmatrix} \begin{bmatrix} 3/8 & 5/8 \\ -1/8 & 1/8 \end{bmatrix}$$

Ans