

Theorem. Let x be the number of maximum queens that share the same row/column in a minimum domination set. We have $x \leq 2\gamma - n + 2$.

First, consider the following theorem from this paper: “Cockayne, E.J., 1990. Chessboard domination problems. *Discrete Mathematics*, 86(1-3), pp.13-20.”

Theorem 3 (Spencer [14]). *For any n , $\gamma(Q_n) \geq (n - 1)/2$.*

Proof. Consider a covering of the $n \times n$ board using $\gamma = \gamma(Q_n)$ queens. Suppose that the rows and columns are sequentially labelled $1, \dots, n$ from top to bottom and left to right respectively. A row or column is said to be *occupied* if it contains a queen.

Let column $a, (b)$ be the left most (right most) unoccupied column and let row $c (d)$ be the unoccupied row closest to the top (bottom). Further we set $\delta_1 = b - a$ and $\delta_2 = d - c$ and assume without loss of generality that $\delta_1 \geq \delta_2$.

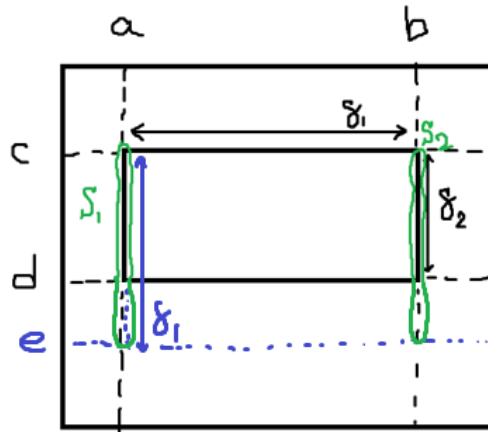
Consider the sets S_1 and S_2 of squares in columns a and b respectively, which lie between rows c and $c + \delta_1 - 1$ inclusive and let $S = S_1 \cup S_2$. Since $\delta_1 \geq \delta_2$, no diagonal intersects both S_1 and S_2 . Hence every queen diagonally dominates at most two squares of S (i.e. at most one per diagonal). Further queens situated above row c or below row $c + \delta_1 - 1$ do not dominate squares of S by row or column.

By definition of c , there are at least $c - 1$ queens above row c . Each row below row d is occupied and $d = c + \delta_2 \leq c + \delta_1$. Therefore all the $n - c - \delta_1$ rows below row $c + \delta_1$ are occupied. Hence there are at least $n - c - \delta_1$ queens below row $c + \delta_1 - 1$.

It follows that at least $(c - 1) + (n - c - \delta_1) = n - \delta_1 - 1$ queens dominate at most 2 squares of S . The remaining queens of which there are at most $\gamma - (n - \delta_1 - 1)$, may cover at most 4 squares of S . Since all the $2\delta_1$ squares of S must be dominated we have

$$2(n - \delta_1 - 1) + 4(\gamma - (n - \delta_1 - 1)) \geq 2\delta_1,$$

which gives $\gamma \geq (n - 1)/2$ as required. \square



Let $x \geq 2$ be the number of maximum queens that share the same row in a minimum domination set. Thus, there is at least one row with x queen.

Case 1. If there is such a row above or below the inner rectangle of the figure, we have:

$$2(n - \delta_1 - 1 + (x - 1)) + 4(\gamma - (n - \delta_1 - 1 + (x - 1))) \geq 2\delta_1 \text{ which gives } x \leq 2\gamma - n + 2$$

In other words, Case 1 represents the scenario in which such a row is one that each queen on it can at most attach 2 squares of S potentially.

Case 2. If there is such a row in the inner rectangle of the figure, we have:

$$2(n - \delta_1 - 1) + 4(\gamma - (n - \delta_1 - 1) - x) + 2 \times 3 + 2(x - 2) \geq 2\delta_1 \text{ which gives } x \leq 2\gamma - n + 2$$

In other words, Case 2 represents the scenario in which such a row is one that each queen on it could at most attach 4 squares of S potentially.

Therefore, $x \leq 2\gamma - n + 2$ in all cases. The board is symmetric, and so, the result is also valid for columns.

Corollary. If $\gamma = n/2$, $x \leq 2$.