Resilient Reachability for Linear Systems

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Motivation

We consider systems enduring loss of control authority over their actuators. We investigate whether the desired target set for a linear system remains reachable even after such a malfunction, characterized by undesirable inputs of magnitude similar as the control inputs.

Our Contributions

- We describe the problem of resilient reachability and derive an analytical condition for linear systems.
- We establish necessary and sufficient conditions for driftless systems to be resilient to the loss of actuator authority.

Notation

- The unit sphere in $\mathbb{R}^n$ is $\mathbb{U} = \big\{x \in \mathbb{R}^n : \|x\| = 1\big\}$.
- The space of square integrable functions is $\mathcal{L}_2$.
- The dual space of X is $X^* = \mathcal{L}\big(X, \mathbb{R}\big)$. For $S \in \mathcal{L}\big(X, Y\big)$, $S^* \in \mathcal{L}\big(Y^*, X^*\big)$ is the dual linear map. The norm of $f \in X^*$ is $\|f^*\|_{X^*} = \sup\limits_{\|x\|=1} \big\{|f^*(x)|\big\}$.

Problem Statement

A system loses control authority over some of its actuators. The controlled inputs are u, the undesirable ones are w while the state x follows the LTI dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t) + Cw(t), \qquad x(0) = x_0 \in \mathbb{R}^n. \quad (1)$$

The goal set is $G = \big\{x \in \mathbb{R}^n : \|x - x_{goal}\|_{\mathbb{R}^n} \leq \varepsilon\big\}$.

Definition

G is *resiliently reachable* from x_0 at time T if, for any undesirable inputs $w \in W$, there exists a control law $u_w \in U$ driving the system from x_0 to G in time T, i.e. $x_0 \in X_0^T = \big\{x_0 \in \mathbb{R}^n : \forall\, w \in W,\ \exists\, u_w \in U : x(T) \in G\big\}$.

Preliminaries

The solution of (1) can be written in integral form and becomes $x(T) = s + Su + Rw$, state equation in [1].

Proposition 1 (Delfour)

G is resiliently reachable from x_0 at time T iff

$$\sup_{\|x^*\|=1} \Big\{x^*(s - x_{goal}) - \|S^*x^*\| + \|R^*x^*\| - \varepsilon\Big\} \leq 0.$$

General Result

Proposition 1 is too abstract and impractical to use because of the dual terms.

So, we use the Riesz representation theorem to simplify x^* into a scalar product with a certain $h \in \mathbb{U}$, representing a direction. We also express the dual map as $S^*x^* = x^* \circ S$.

Theorem 1

G is resiliently reachable from x_0 at time T iff

$$\max_{h\,\in\,\mathbb{U}} \Bigg\{ \langle h, e^{AT}x_0 - x_{goal}\rangle - \sup_{\|u\|_{\mathcal{L}_2}=1} \Big\{\Big|\Big\langle h, \textstyle\int_0^T e^{A(T-\tau)}Bu(\tau)d\tau\Big\rangle\Big|\Big\} + \sup_{\|w\|_{\mathcal{L}_2}=1} \Big\{\Big|\Big\langle h, \textstyle\int_0^T e^{A(T-\tau)}Cw(\tau)d\tau\Big\rangle\Big|\Big\} - \varepsilon \Bigg\} \leq 0.$$

Theorem 1 is more direct than Proposition 1, as it does not use any dual element. Yet, computing the two supremums is complicated because of the infinite dimension of $\mathcal{L}_2$.

Driftless Systems

Consider a system without drift term, so that (1) becomes:

$$\dot{x}(t) = Bu(t) + Cw(t). \quad (2)$$

Using the equality case of the Cauchy-Schwarz inequality we calculate the value of the supremums of Theorem 1.

Theorem 2

G is resiliently reachable at time T from x_0 iff

$$\max_{h\,\in\,\mathbb{U}} \Big\{\langle h, x_0 - x_{goal}\rangle - \sqrt{T}\,\big\|B^T h\big\| + \sqrt{T}\,\big\|C^T h\big\|\Big\} \leq \varepsilon.$$

Time Evolution of Reachability

To simplify let $d = x_0 - x_{goal}$ and define

$$J : (h,\ t) \mapsto \langle h, d\rangle + \sqrt{t}\Big(\big\|C^T h\big\| - \big\|B^T h\big\|\Big)\ ,$$

so G is resiliently reachable from x_0 by time T if and only if

$$\min_{t\,\in\,[0,T]} \Big\{\max_{h\,\in\,\mathbb{U}} \big\{J(h,t)\big\}\Big\} \leq \varepsilon\ .$$

Hence the reachability by time T can be described as a minimax problem with a DC (Difference of Convex) cost function. As time grows, $\sqrt{t}$ becomes the leading term in J, with its sign determined by

$$g : h \mapsto \|C^T h\| - \|B^T h\|.$$

Intuitions

Call h^* the argument of the max in Theorems 1 and 2.

- h^* is driving the system away from x_{goal}.
- h^* is the travel direction giving the most strength to the undesirable inputs over the controls.
- So h^* is the worst direction for resilient reachability.
- Along direction h, $g(h)$ quantifies the difference of strength between undesirable inputs and controls.
- $\mathrm{sign}\Big(\max g(h)\Big)$ tells which input is the strongest.

Theorem 3

- If $\max\big\{g(h)\big\} > 0$, G is only resiliently reachable up to a certain time,
- If $\max\big\{g(h)\big\} = 0$, G can be either always resiliently reachable, never resiliently reachable, or its resilient reachability depends on time,
- If $\max\big\{g(h)\big\} < 0$, G is resiliently reachable from some time onwards.

Underwater Robot

An underwater robot is propelled by three engines in a 2D plane. The main thruster u_1 has a small bias in the y direction. The system dynamics are:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}10 & 1 & 1\\ 0.2 & -1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}$$

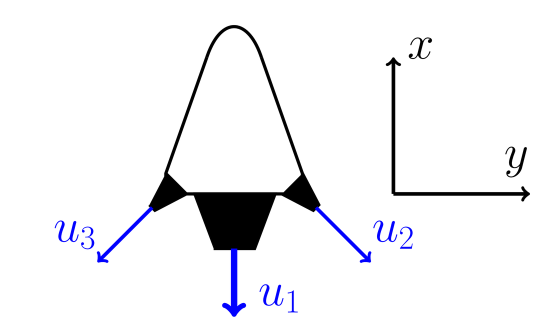

Figure: A model of an underwater robot with three actuators.

When losing control of u_3, $B = \begin{bmatrix}10 & 1\\ 0.2 & -1\end{bmatrix}$, while $C = \begin{bmatrix}1\\ 1\end{bmatrix}$.

We take $h = [h_1, h_2]^T \in \mathbb{U}$, and compute $\max g(h) = -0.02$ for $h_1 \approx 0$. Thus, according to Theorem 3, any target set can be reached in finite time.

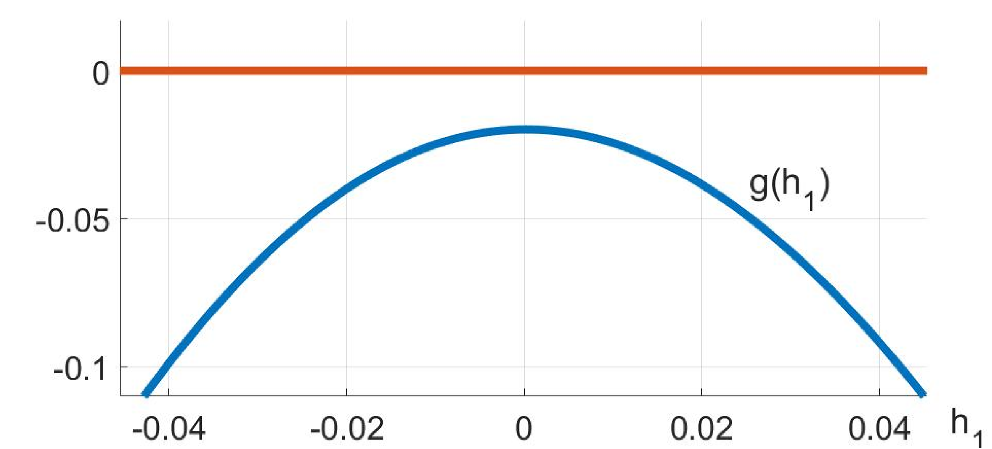

Figure: Maximum of $g(h_1) < 0$

Uncontrolled actuator	$\max\big\{g(h)\big\}$	Resilient Reachability
u_2 or u_3	$-0.02 < 0$	Yes
u_2 and u_3	$1.4 > 0$	No
u_1	$8.6 > 0$	No

Table: Resilient reachability for loss of control of different actuators

Theorem 4

Let $F = BB^T - CC^T$.

- If F is positive definite, then G is resiliently reachable.
- If F is not positive semi-definite, then G is not resiliently reachable.

We compute F for the different cases. When losing control of u_3, $F = \begin{bmatrix}100 & 0\\ 0 & 0.04\end{bmatrix} \succ 0$. But, $F = \pm\begin{bmatrix}98 & 2\\ 2 & -1.96\end{bmatrix}$ for the two other cases, which is not positive semi-definite. So, the robot is only resilient to the loss of one of its side engines.

Conclusions and Future Work

- We described the problem of resilient reachability for a system losing control authority over some of its actuators.
- We derived resilient reachability conditions for linear and driftless systems.
- Our next focus is to design control matrices resilient to the loss of any actuator.

Main References

[1] M. C. Delfour and S. K. Mitter, "Reachability of perturbed systems and min sup problems," *SIAM Journal on Control and Optimization*, vol. 7, pp. 521 – 533, November 1969.

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