MINERVA®

Assignment II

Vector Space Structures

Spring 2019

CS– 111B

Taha Bouhoun

I. New frontiers:

a. First sampling sites scheduled

$v_1 = \{3, 1\}$

$v_2 = \{1, 2\}$

We need to find a combination of the coefficients c_1 and c_2 that could lead us to the point (749, 448)

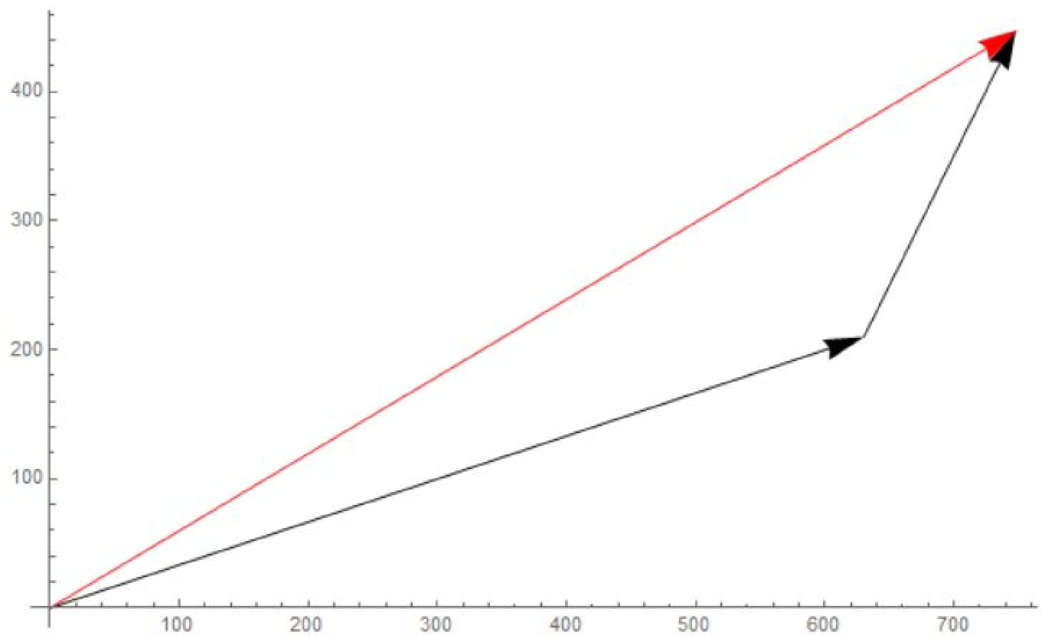
$c_1 \cdot v_1 + c_2 \cdot v_2 = \; < 749,\ 448 >$

```
In[49]:= Solve[c1*v1 + c2*v2 == {749, 448}, {c1, c2}]

Out[49]= {{c1 → 210, c2 → 119}}

In[59]:= Graphics[{Arrow[{{0, 0}, {630, 210}}], Arrow[{{630, 210}, {748, 448}}],
           Red, Arrow[{{0, 0}, {749, 448}}]}, Axes → True]
```

Out[59]=



For a coefficients combination of (210, 119), the drone can reach the cite (749, 448) using the headings {3, 1} and {1, 2}

b. The cites that can be reached by the drone:

Algebraically:

The set S = {$\mathbf{v}_1$, $\mathbf{v}_2$} of vectors in R^2 **spans** R^2 if:

$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = d_1\mathbf{w}_1 + d_2\mathbf{w}_2$ (*)

Where: $\mathbf{w}_1$=[1, 0] and $\mathbf{w}_2$=[0, 1] unit vectors of R^2

(*) Has at least one solution for every set of values of the coefficients d_1, d_2

$$3 \cdot c_1 + 1 \cdot c_2 = 1 \cdot d_1 + 0 \cdot d_2$$

$$1 \cdot c_1 + 2 \cdot c_2 = 0 \cdot d_1 + 1 \cdot d_2$$

$$\begin{matrix} 3 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{matrix}$$

Which can be transformed by a sequence of elementary operations to:

$$1\, c_1 + 0\, c_2 = (2/5)\ d_1 + (-1/5)\, d_2$$

$$0\, c_1 + 1\, c_2 = (-1/5)\, d_1 + (3/5)\ d_2$$

Since the vectors can be transformed into unit vectors of R^2 which are linearly independent, then $\mathbf{v}_1$, $\mathbf{v}_2$ **spans** the space R^2.

Geometrically:

If a combination of the two vectors can construct both unit vectors of R^2 namely $\bar{i}$ and $\bar{j}$ then the two vectors span the space R^2

Using Mathematica, we can solve for the coefficient by which the headings v_1 and v_2 can form the unit vectors of R^2

```
In[60]:= Solve[x*v1 + y*v2 == {1, 0}, {x, y}]

Out[60]= {{x → 2/5, y → -1/5}}

In[62]:= Graphics[{Arrow[{{0, 0}, {6/5, 2/5}}], Arrow[{{6/5, 2/5}, {1, 0}}],
           Red, Arrow[{{0, 0}, {1, 0}}]}, Axes → True]
```

Out[62]=

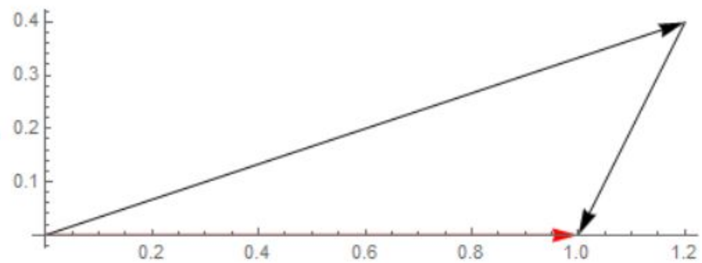


Fig x. Forming the unit vector (1, 0) using the headings v_1 and v_2

```
In[61]:= Solve[x*v1 + y*v2 == {0, 1}, {x, y}]

Out[61]= {{x → -1/5, y → 3/5}}

In[63]:= Graphics[{Arrow[{{0, 0}, {-3/5, -1/5}}], Arrow[{{-3/5, -1/5}, {0, 1}}],
           Red, Arrow[{{0, 0}, {0, 1}}]}, Axes → True]
```

Out[63]=

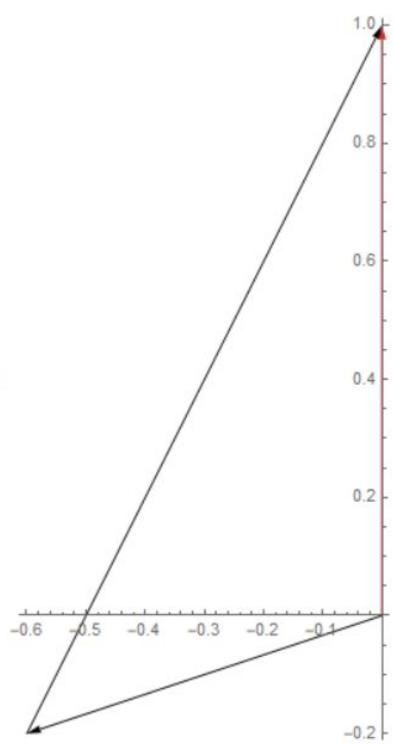


Fig x. Forming the unit vector (0, 1) using the headings v_1 and v_2

c. The storm is coming:

Limited to a single heading $v_1 = \ < 1,\ 3 >$ the closest point from the drone to the cite would be when the vector that intersects with the v_1 and the cite is perpendicular to the heading v_1.

$(-19,\ -5) \rightarrow (3x,\ x)$ and the line of the equation $y = 3x$

We compute x for the scalar product of the two vectors to equal zero:

$(-19 - 3x,\ -5 - x)$ and $(3,\ 1)$

$(-19 - 3x)\,(3) + (-5 - x) = 0$

$-57\ -9x - 5 - x = 0 \rightarrow\ -62 = 10x \rightarrow\ x = \ -62/10\ \rightarrow\ y = \ -186/10$

```
a = line([(-21, -7), (3, 1)],linestyle='--')
b = point((-19, -5), color='black', size=20)
c = circle((-186/10, -62/10), 2, color='red', linestyle='-.')
d = line([(-186/10, -62/10), (-19, -5)], color='green', thickness=0.5)
a+b+c+d
```

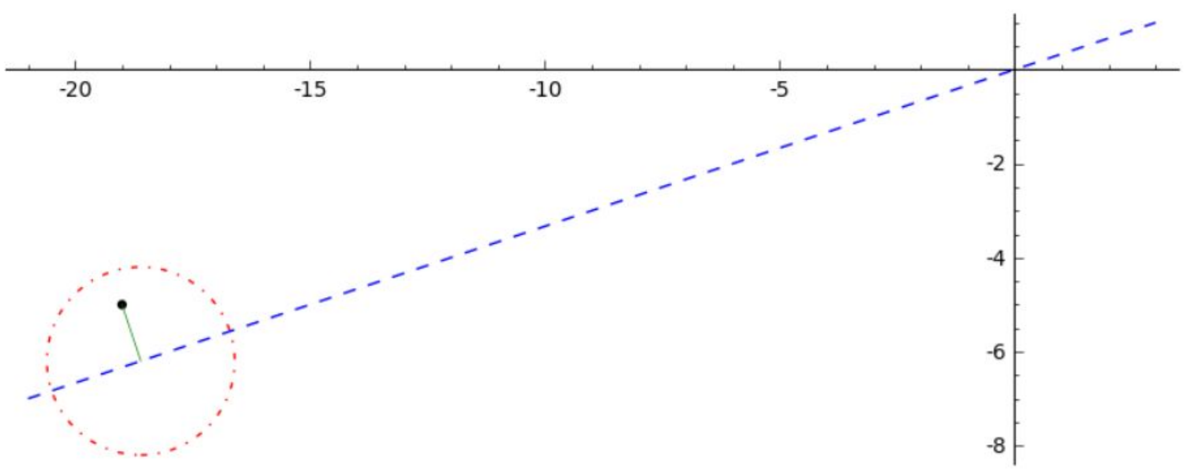


Fig 1. Illustrating the calculations of the nearest point to the cite while using just the v_1 heading.

The arm of the drone can indeed reach the cite (-19, -5) using just the heading (3, 1).

To further confirm our result, we can find the magnitude of the green segment mentioned in the graph: (-62/10, -186/10) → (-19, -5)

$$d = \left[(-18.6-(-19))^2 + (-6.2-(-5))^2\right]^{1/2}$$

$$d = \left[(0.4)^2 + (-1.2)^2\right]^{1/2}$$

$$d = \sqrt{1.6} \approx 1.265 \quad \rightarrow \quad 1.265 < 2$$

Therefore, the drone will be able to scan the cite (-19, -5)

d. New limitations:

```
a = line([(-21, -7), (3, 1)], linestyle='--')
aa = line([(0, 0), (1, -3)], linestyle='--')
b = point((-19, -5), color='black', size=20)
bb = point((2/sqrt(10), -6/sqrt(10)), color='black', size=20)
c = circle((0,0), 2)
d = line([(sqrt(10)/5, -3*sqrt(10)/5), (-20, -2/3*sqrt(10) - 20/3)],
        color='green', thickness=0.3)
e = line([(-sqrt(10)/5, 3*sqrt(10)/5), (-20, 2/3*sqrt(10) - 20/3)],
        color='green', thickness=0.3)
aa+a+b+bb+c+d+e
```

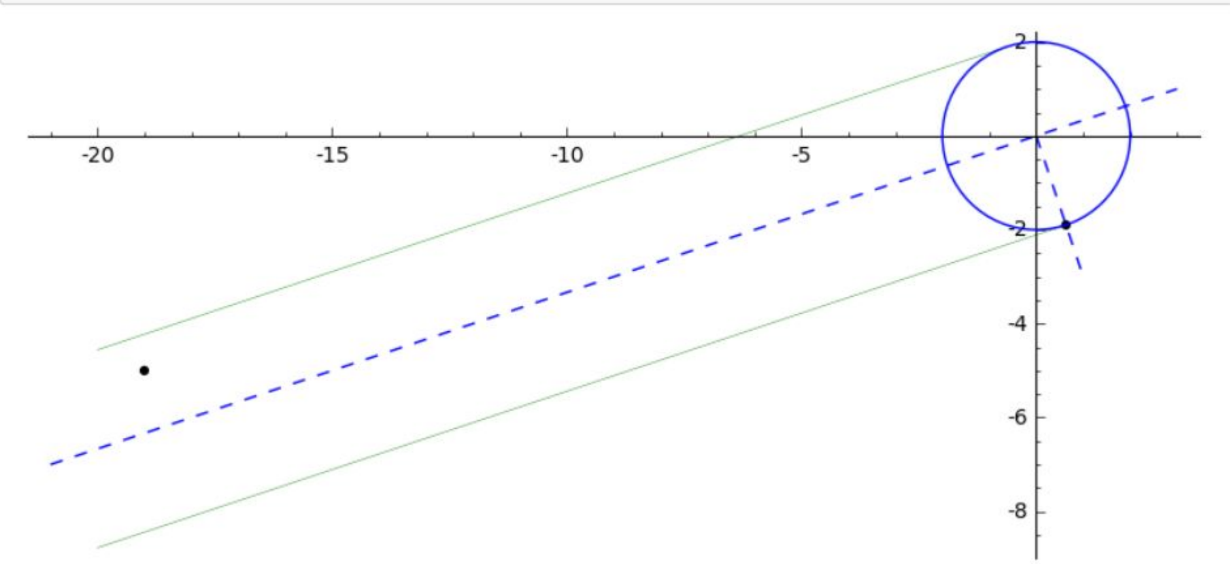


Fig x. The graph showing the potential sampling space based on a single heading v={3, 1}

Algebraically:

First, we know that the slop for the heading v_1 and the lines constructed by the infinite circles are the same. We then exploit the notion that the arm constructs a vector of magnitude 2 that is perpondicular on the heading v_1 As illustrated in the graph, we take the origin as a stating point to compute the point that belongs to the circle and the green line using the notion of the scalar multiplication.

Since $v_1 \perp \text{arm} \rightarrow v_1 \cdot \text{arm} = 0$

This gives us the point $(\frac{\sqrt{10}}{5}, \frac{-3\sqrt{10}}{5})$

With the slop of the line and one point, we can deduce the equation line:

$\frac{-3\sqrt{10}}{5} = \frac{1}{3} \cdot \frac{\sqrt{10}}{5} + b \quad \rightarrow \quad b = -\frac{9\sqrt{10}}{15} - \frac{\sqrt{10}}{15} = \frac{-2\sqrt{10}}{3}$

The line equation would be: $y = \frac{1}{3} \cdot x - \frac{2\sqrt{10}}{3}$

We can use the same strategy to find the equation of the second upper bound for the drone which then would have an equation line of: $y = \frac{1}{3} \cdot x + \frac{2\sqrt{10}}{3}$

Geometrically:

Considering the current location of the drone to be the origin (0, 0) and the length of the arm is 2m, the drone can sample from a space that is constructed by an infinite number of circles which their centers belong to the line of the heading $v_1 = \{3, 1\}$. This would construct two parallel lines to the heading v_1 with a 2m distance apart. We notice that the cite (-19, -5) belongs to that area as proven in part (c).

2. Row space:

Given an $n\times m$ matrix A, the row space of A, denoted Row(A), is the span of the vectors that make up the rows of A.

a. Show that Row(A) is a subspace of R^m

First, the notion of Row(A) means that the rows of the matrix A construct the span, furthermore, we know that each row of A has ***m*** element.

Testing the closure under addition and multiplication:

Let Row(A)$= c_0\mathbf{v}_0 + c_1\mathbf{v}_1 + c_3\mathbf{v}_3 + \; .. + c_m\mathbf{v}_m$

Addition:

Assume that: $U = a_0\mathbf{v}_0 + \; .. + a_m\mathbf{v}_m$ and $W = b_0\mathbf{v}_0 + \; .. + b_m\mathbf{v}_m$

$V+W= a_0\mathbf{v}_0+b_0\mathbf{v}_0+ \; .. + a_m\mathbf{v}_m+ b_m\mathbf{v}_m$

$V+W= (a_0+b_0)\mathbf{v}_0+ \; .. + (a_m+ b_m)\mathbf{v}_m$

Which has the same form of Row(A) and belongs to R^m

Multiplication:

$U = a_0\mathbf{v}_0+a_1\mathbf{v}_1+ \; .. + a_m\mathbf{v}_m$

$k\,U = k\,(a_0\mathbf{v}_0+a_1\mathbf{v}_1+ \; .. + a_m\mathbf{v}_m) \rightarrow k\,U = k\,a_0\mathbf{v}_0+k\,a_1\mathbf{v}_1+ \; .. + k\,a_m\mathbf{v}_m$

$k\,U = (k\,a_1)\,\mathbf{v}_0+(k\,a_1)\,\mathbf{v}_1+ \; .. + (k\,a_m)\mathbf{v}_m$

Therefore, it’s enclosed under addition and multiplication.

The origin being part of the matrix (the zero vector)

One solution for Row(A)$=c_0\mathbf{v}_0 + c_1\mathbf{v}_1+ c_3\mathbf{v}_3+ \; .. +c_m\mathbf{v}_m = 0$

Is when the coefficients $c_0= c_1= .. = c_m = 0$

Which confirms the fact that the vector zero is part of Row(A)

b. Suppose that U is the reduced row echelon form of A. How do we know that Row(A) = Row(U)?

Reasoning:

Row(A) = Row(U) → is equivelant to the fact that row operations do not change the row space of a given matrix.

Any given row produced by a row operation is a linear combination of the original rows and hence in the original row space, this applies for swapping rows, adding rows, and multiplying rows by a scalar. Since row operations are invertible, the reverse is also true. Based on this conclusion, the space spanned by the rows of the matrix A equals the space spanned by the nonzero rows of its Row Reduced Echelon Form (U).

c. Suppose that we have two sets of vectors in R^4

- $v_1 = <1, 2, -1, 3>$, $v_2 = <2, 4, 1, -2>$, $v_3 = <3, 6, 3, -7>$
- $w_1 = <1, 2, -4, 11>$, $w_2 = <2, 4, -5, 14>$

Let V be the span of the v_is and W be the span of the w_js. Use part (b) to show that V = W. Is there another, more tedious way to do this? How?

In[94]:= $V = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

```
Out[94]= {{1, 2, -1, 3}, {2, 4, 1, -2}, {3, 6, 3, -7}}

In[95]:= RowReduce[V] // MatrixForm
```

Out[95]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{8}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
In[96]:= W = (1  2  -4  11
              2  4  -5  14)

Out[96]= {{1, 2, -4, 11}, {2, 4, -5, 14}}

In[97]:= RowReduce[W] // MatrixForm
```

Out[97]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{8}{3} \end{pmatrix}$$

Explanation: According to part (b) we can deduce that

Row(V) = Row(rref(V))

Row(W) = Row(rref(W))

Drawing from the calculation above, we can notice that the span of vectors that constructs the rows of V and W are the same since the last row of V is a zero vector, hence, we conclude that V=W.

A more tedious way is to list all the combinations that the two vectors can form then check for the following relation:

$v_1 <1, 2, -1, 3> + v_2 <2, 4, 1, -2> + v_3 <3, 6, 3, -7> = w_1 <1, 2, -4, 11> + w_2 <2, 4, -5, 14>$

We then construct a matrix by putting all the variables in one side to yield a matrix with a dimension of 4×5 then find its RREF to conclude whether it's consistent and if it actually has a solution that would satisfy the above system of equation. Therefore, we could conclude the equivalence of the spans from thses two sets of vectors.

3. Finding bases:

For each of the following sets of vectors S in a vector space V :

(i) Describe the subspace spanned by the set S.

(ii) Extend the set S to a basis for V.

a. $S = \{<1,\ 2,\ -1,\ 1>,\ <2,\ 1,\ 0,\ 3>\}$ in R^4

(i) The set $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ of vectors in R^4 is **linearly independent** if the only solution of $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = 0$ is $c_1, c_2 = 0$.

In this case, the set S **forms a basis** for span S.

If this is the case, a **subset of S** can be found that forms a basis for span S.

With our vectors $\mathbf{v}_1$, $\mathbf{v}_2$ becomes:

$c_1 \cdot [1,\ 2,\ -1,\ 1] + c_2 \cdot [2,\ 1,\ 0,\ 3] = [0,\ 0,\ 0,\ 0]$

$$c_1 \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Rearranging the left-hand side yields

$[c_1 + 2\cdot c_2,\ 2\cdot c_1 + c_2,\ -c_1 + 0\cdot c_2,\ c_1 + 3\cdot c_2] = [0,\ 0,\ 0,\ 0]$

$$\begin{pmatrix} c_1 + 2 \cdot c_2 \\ 2 \cdot c_1 + c_2 \\ -c_1 + 0 \cdot c_2 \\ 1 \cdot c_1 + 3 \cdot c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We now transform the coefficient matrix of the homogeneous system above to the reduced row echelon form to determine whether the system has a solution

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 0 \\ 1 & 3 \end{pmatrix} => \textit{Echelon form} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The RREF above shows that each column has a pivot which makes them linearly independent. However, in the context of R^4 there would be two free variables, hence, the geometrical form of the span is a plane.

(ii) Forming a basis in R^4:

Along the two vectors in set S, we need two more independent vectors to construct a basis in R4.

$$\begin{matrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

The Row Reduced Echelon Form then would be the identity matrix in R^4

This combination can form any set of vectors in R^4

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

b. $S = \{3x - 1,\ 5x,\ x^2 + 1\}$ in P(3)

(i) We're looking for a set $\forall a_0, a_1, a_2, a_3 \in R \rightarrow \exists \lambda_1, \lambda_2, \lambda_3 \in R$ such that

$$\lambda_1(3x-1) + \lambda_2(5x) + \lambda_3(x^2+1) = a_0 \cdot x^3 + a_1 \cdot x^2 + a_2 \cdot x + a_3$$

$$0\,\lambda_1 + 0\,\lambda_2 + 0\,\lambda_3 = a_0$$

$$\lambda_3 = a_1$$

$$3\,\lambda_1 + 5\,\lambda_2 = a_2$$

$$-\lambda_1 + \lambda_3 = a_3$$

Which translates to the augmented matrix:

$$\begin{matrix} 0 & 0 & 0 & | & a_0 \\ 0 & 0 & 1 & | & a_1 \\ 3 & 5 & 0 & | & a_2 \\ -1 & 0 & 1 & | & a_3 \end{matrix}$$

With Row Reduced Echelon Form:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}$$

Interpretation: for a space of polynomial of degree at most of 3, the three linearily independent column vectors construct a subspace of all polynomials with degree at most 2 since the coefficient for the a_0 is always 0.

(ii) The basis of this set in P(3):

We need to add another column that accounts for the coefficients of the third-degree variables x^3. In this case, it would be the vector (1, 0, 0, 0)

$$\begin{matrix} 0 & 0 & 0 & 1 & | & a_0 \\ 0 & 0 & 1 & 0 & | & a_1 \\ 3 & 5 & 0 & 0 & | & a_2 \\ -1 & 0 & 1 & 0 & | & a_3 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

Corresponding to the RREF expressed on the right side above.

Using this set S, we would be able to construct any polynomial in P(3)

c. S is a set of matrices:

Each matrix has a given coefficient that leads to the form:

$$x_1 \begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix} + x_2 \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3\,x_1 + 7\,x_2 & 0\,x_1 + 0\,x_2 \\ 0\,x_1 + 0\,x_2 & -5\,x_1 + 2\,x_2 \end{pmatrix}$$

Which can be translated into the following matrix where column represents the coefficients x_1 and x_2

$$\begin{pmatrix} 3 & 7 \\ 0 & 0 \\ 0 & 0 \\ -5 & 2 \end{pmatrix} \textit{ to RREF becomes } \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Similar to the set in part (a), the matrix has two pivot which makes its column linearly independent, however, the two free variables would lead the span to take a form of a plane.

(ii) The goal is to have an upper triangle 2×2 matrix:

With the existing set of matrices, we can have the following form

$$x_1 \begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix} + x_2 \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3\,x_1 + 7\,x_2 & 0 \\ 0 & -5\,x_1 + 2\,x_2 \end{pmatrix}$$

We add another matrix that accounts for the upper right values:

$$x_1 \begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix} + x_2 \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix} + x_3 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3\,x_1 + 7\,x_2 & x_3 \\ 0 & -5\,x_1 + 2\,x_2 \end{pmatrix}$$

When taken to the RREF we get a combination that can from any upper right triangle matrix:

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} c_1 & c_3 \\ 0 & c_2 \end{pmatrix}$$

Appendix:

#deduction: throughout the assingment (mainly indirect questions), I used deduction to translate properties of vectors into solutions for the problems like the scanning of the cite (-19, -5), the reasoning was that if the arm can't reach the closest point to the cite (which occurs when v_1 is $\perp$ to the cite) then there's no other way to reach the cite.

#creative_heuristics: For Problem 1 part b, I think that the shortcut to identify whether the two vectors have a combination that lead to the unit vectors of the space R2 was an efficient way to identify the space they span. My reasoning is that at their best, spanning R2 is the biggest space both vectors can cover, hence proving that they're independent and they construct the unit vectors was a quick rule of thumb.