CS111B Assignment 2: Vector Space Structure

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Question 1

Part a

Since the drone is limited to two headings, v_1 and v_2, a given coordinate can be reached if the linear combination of the two headings can yield the coordinate. We would evaluate the following expression:

$$c_1 v_1 + c_2 v_2 \stackrel{?}{=} <749, 448>$$

$$c_1 <3, 1> + c_2 <1, 2> \stackrel{?}{=} <749, 448>$$

$$3 c_1 + c_2 = 749$$
$$c_1 + 2 c_2 = 448$$

Using matrix to solve the system of linear equation:

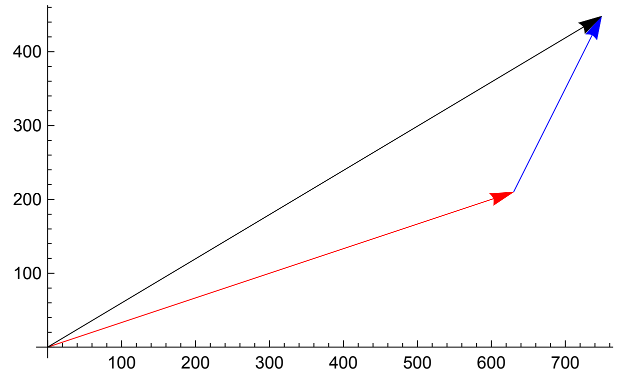
$$\begin{pmatrix} 3 & 1 & 749 \\ 1 & 2 & 448 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 210 \\ 0 & 1 & 119 \end{pmatrix}$$

$$c_1 = 210$$
$$c_2 = 119$$

Since a linear combination of the two headings, where $c_1 = 210$ and $c_2 = 119$, yields the desired coordinates, it is possible for the drone to traverse to the coordinate <749, 448>. The plot below illustrates the path if the drone were to use v_1 210 times (red) and then use v_2 119 times (blue).

In[323]:=
```
V1 = Arrow[{{0, 0}, {630, 210}}];
V2 = Arrow[{{630, 210}, {749, 448}}];
W1 = Arrow[{{0, 0}, {749, 448}}];
D1 = {749, 448};
Graphics[{Red, V1, Blue, V2, Black, W1}, Axes → True]
```

Out[327]=



Part b

As seen above, when reducing the two headings into RREF form, we get an identity matrix. This indicates that the two vectors are linearly independent. In $\mathbb{R}^2$, we need only 2 linearly independent columns to span the space. If we show that the unit vectors $<1, 0>$ and $<0, 1>$ are in the span of v_1 and v_2, then this means the linear combination of v_1 and v_2 can give us any point in the $\mathbb{R}^2$space.

Algebraically, it can be expressed as:

$$c_1 <3, 1> + c_2 <1, 2> = <1, 0>$$

$$3 c_1 + c_2 = 1$$
$$c_1 + 2 c_2 = 0$$

$$c_1 = 2/5$$
$$c_2 = -1/5$$

$$c_1 <3, 1> + c_2 <1, 2> = <0, 1>$$

$$3 c_1 + c_2 = 0$$
$$c_1 + 2 c_2 = 1$$

$$c_1 = -1/5$$
$$c_2 = 3/5$$

Given that the two linearly independent unit vectors are in the span of v_1 and v_2, it means that the linear combination of v_1 and v_2 can span the entire $\mathbb{R}^2$space, because

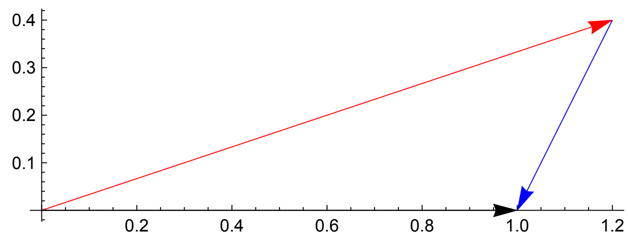
$$c_1 < 1, 0 > + c_2 < 0, 1 > = < a, b >$$

Geometrically, we can also use the unit vectors to explain why the drone can traverse to all points in $\mathbb{R}^2$. I will plot how the linear combination of v_1 and v_2 can give us the two unit vectors, because it is intuitive to see that given two unit vectors in a 2-dimensional space, any point can be reached.

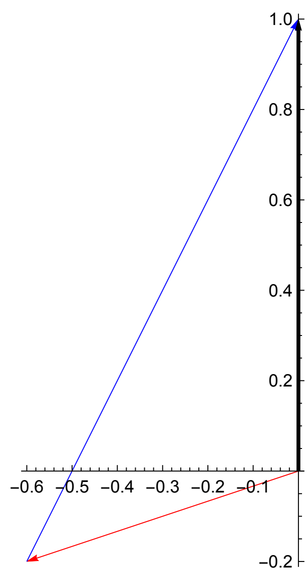
In[343]:=
```
V3 = Arrow[{{0, 0}, {6/5, 2/5}}];
V4 = Arrow[{{6/5, 2/5}, {1, 0}}];
W2 = Arrow[{{0, 0}, {1, 0}}];
Graphics[{Red, V3, Blue, V4, Black, W2}, Axes → True]

V5 = Arrow[{{0, 0}, {-3/5, -1/5}}];
V6 = Arrow[{{-3/5, -1/5}, {0, 1}}];
W3 = Arrow[{{0, 0}, {0, 1}}];
Graphics[{Red, V5, Blue, V6, Black, Thick, W3}, Axes → True]
```

Out[346]=



Out[350]=



Part c

We can solve this problem algebraically and geometrically.

Algebraically, we just have to find a point along the traversed line that is within 2 meters radius to the desired site, which has the coordinates $<-19, -5>$ (negative because we defined East as positive-x and North as positive-y). We will use the distance formula to evaluate distance between two points.

$$c_1 < 3, 1 >$$

An intuitive constant c_1 that would get us close to the site is -6.

$$-6 < 3, 1 > \ = \ < -18, -6 >$$

Then we use the distance formula $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ to evaluate whether it is within 2 meters.

$$D = \sqrt{(-19 - (-18))^2 + (-5 - (-6))^2} = \sqrt{1 + 1} = \sqrt{2}$$

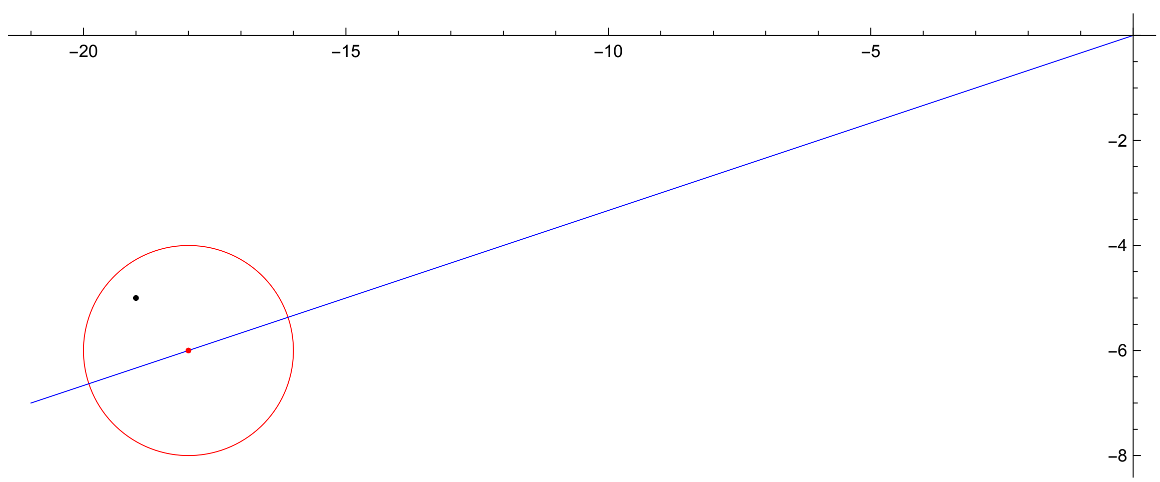
$$D = \sqrt{2} < 2$$

At this point, the distance is smaller than 2 meters. Hence, it is possible for the drone to get close enough for the collection arm to reach the site.

Geometrically, we will have a circle with the radius of 2 around the site plotted along with the single line where the drone can traverse. If the line goes within the 2-meter radius, then it means that the drone can reach the site. The black point represents the site. The red circle represents the 2-meter radius around the point <-18, -6 >. The blue line represents the line where the drone can traverse.

In[357]:=
```
Graphics[{Red, Circle[{-18, -6}, 2], Blue, Line[{{0, 0}, {-21, -7}}],
  Black, Point[{-19, -5}], Red, Point[{-18, -6}]}, Axes → True]
```

Out[357]=



Part d

The closest distance between a point and a line exists when the distance is perpendicular to the line. This can be proven geometrically. Whenever the distance is at an angle that is not 90°, that distance forms the hypotenuse to a right triangle (where the adjacent and opposite sides are the 90° distance line and the side along the line). The hypotenuse is always the longest side of a right triangle. Hence, to prevent having the hypotenuse as the distance, the angle between the distance and the line must be 90°.

Going back to the context of the question, a sample site that can be reached when the distance is perpendicular cannot be reached at any other point along the line. Thus, the set of potential sites that are reachable will be within the range 2 meters perpendicular to the drone’s path. The sets of points will form two lines above and below the original line.

First, we will express the original line as a scalar multiple of the heading.

$$L = c \cdot v_1$$

Then, we will express the lines above and below L as

$$L_a = c \cdot v_1 + 2\,\hat{p}$$

$$L_b = c \cdot v_1 - 2\,\hat{p}$$

where $\hat{p}$ is a unit vector perpendicular to line L.

We have the slope of line L, so the slope of any line perpendicular to it will be the negative reciprocal.

$$L_{\text{slope}} = \frac{1}{3}$$

$$L_{\text{perpendicular}} = -3$$

$$-3\,y = x$$

We can thus express vector $\vec{p}$ as

$$\vec{p} = \langle -3\,y, y \rangle$$

The unit vector $\hat{p}$ in the same direction as $\vec{p}$ will be

$$\hat{p} = \frac{\vec{p}}{\|\vec{p}\|} = \frac{\langle -3\,y, y \rangle}{\sqrt{(-3\,y)^2 + y^2}} = \frac{\langle -3\,y, y \rangle}{y\sqrt{10}} = \left\langle \frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle$$

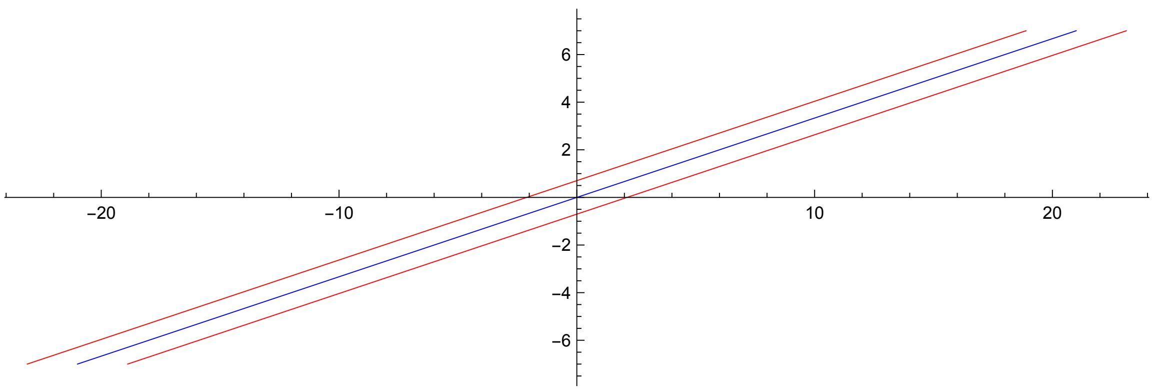
$$L_a = c \cdot v_1 + 2\left\langle \frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle = c\left\langle 3 - \frac{6}{\sqrt{10}}, 1 + \frac{2}{\sqrt{10}} \right\rangle$$

$$L_b = c \cdot v_1 - 2\left\langle \frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle = c\left\langle 3 + \frac{6}{\sqrt{10}}, 1 - \frac{2}{\sqrt{10}} \right\rangle$$

Geometrically, L_a and $L_b$can be plotted along with L. As seen below, the two line envelopes the original line. The area between the L_a and L_b(red lines) represents the range where sample sites are reachable.

In[284]:=
```
Graphics[
 {Blue, Line[{{21, 7}, {-21, -7}}], Red, Line[{{21 + 2.11, 7}, {-21 + 2.11, -7}}],
  Line[{{21 - 2.11, 7}, {-21 - 2.11, -7}}]}, Axes → True]
```

Out[284]=



Question 2

Part a

Given that Row(A) is the span of the vectors that make up the rows of A, we can express Row(A) as

Row(A) = $c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$, where $v's$ are the rows of A.

To show that Row(A) is a subspace of $\mathbb{R}^m$, we have to show that 1) there exists a zero vector and 2) it is closed under addition and scalar multiplication. Given that A is a m×n matrix, we know that there are n rows in A, and each row contains m elements.

Zero Vector:

Row(A) = $c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0$, when c_1, $c_2 \ldots c_n = 0$

Addition Closure:

Let v be $c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$and w be $k_1 w_1 + k_2 w_2 + \ldots + k_n w_n$, where v and w are subsets of $\mathbb{R}^m$.

$v + w = c_1 v_1 + k_1 w_1 + c_2 v_2 + k_2 w_2 \ldots + c_n v_n + k_n w_n$

$v + w$ is closed in the vector space $\mathbb{R}^m$, because it still holds the same dimension of $\mathbb{R}^m$ of 1 row and m elements. This is because $v_n's$ and $w_n's$ have the same dimension as $\mathbb{R}^m$.

Scalar Multiplication Closure:

Let v be $c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$ and k be a scalar, where v is a subset of $\mathbb{R}^m$.

$k v = k[c_1 v_1 + c_2 v_2 + \ldots + c_n v_n] = k c_1 v_1 + k c_2 v_2 + \ldots + k c_n v_n$

$k v$ is closed in the vector, because it still maintains the same dimension of $\mathbb{R}^m$. Multiplying every element in vectors $v_n's$ do not change its dimensions.

Part b

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,m} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n,1} & a_{n,2} & \ldots & a_{n,m} \end{pmatrix} \xrightarrow{\text{rref}} U = \begin{pmatrix} 1 & 0 & \ldots & u_{1,m} \\ 0 & 1 & \ldots & u_{2,m} \\ \ldots & \ldots & \ldots & \ldots \\ u_{n,1} & u_{n,2} & \ldots & u_{n,m} \end{pmatrix}$$

When reducing matrix A to RREF, we are using row operations to cancel out terms and make them 0's and 1's. There are three types of row operations: rows swapping, scalar multiplication, and row addition. Rows swapping involves, as the words suggest, swapping the positions of two rows. Scalar multiplication is when we multiply a scalar to a row. Row addition is when we add one row to another. With a series of these row operations, we can transform matrix A to matrix U.

We are given that Row(matrix) is the span of the row vectors of the matrix. The span of vectors is the set of all linear combination of the vectors. Linear combination involves multiplying each vector by a scalar and adding them together. Sounds familiar? Scalar multiplication and row addition are also row operations. In other words, we can say that when transforming matrices to RREF, we are performing linear combinations of the rows. Knowing this, we can reason that Row(A) = Row(U) because the rows in U are essentially the linear combination of the rows in A. Hence, the linear combination of A, Row(A), and the linear combination of U, Row(U), will be the same.

Part c

Using Part (b) to show that V = W, we must first transform the sets of vectors into RREF.

$$V = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & 1 & 3 \\ 3 & -2 & -7 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$W = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ -4 & -5 \\ 11 & 14 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

As we can see from the RREF, both sets of vectors have two linearly independent vectors. After removing the linearly dependent vector, the two sets form identical bases that spans a space. The linear combination of v_1, $v_{2,}$ v_3 must also span W and the linear combination of w_1, w_2 also must span V. Since both V and W are the span of these linearly independent vectors, they must then be equivalent to each other.

A more tedious way to go about proving V = W is to actually write out the linear combination of all the vectors.

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = c_4 w_1 + c_5 w_2$$

Then, re-writing the linear combination as a system of equations and solving them.

$$c_1 \langle 1, 2, -1, 3\rangle + c_2 \langle 2, 4, 1, -2\rangle + c_3 \langle 3, 6, 3, -7\rangle = c_4 \langle 1, 2, -4, 11\rangle + c_5 \langle 2, 4, -5, 14\rangle$$

$$c_1 + 2c_2 + 3c_3 = c_4 + 2c_5$$

... so on and so forth

Finally, putting all the variables on one side of the equation, placing them into a matrix, and transforming the matrix into RREF. If the RREF is consistent (has at least a solution), then we can conclude that V = W.

In[276]:=
```
V = {{1, 2, 3}, {2, 4, 6}, {-1, 1, 3}, {3, -2, -7}};
RowReduce[V] // MatrixForm
W = {{1, 2}, {2, 4}, {-4, -5}, {11, 14}};
RowReduce[W] // MatrixForm
```

Out[277]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Out[279]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Question 3

Part a

i)

The subspace spanned by set S is the linear combination of the components in S, expressed as

$$\text{span}(S) = c_1 \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} c_1 + 2c_2 \\ 2c_1 + c_2 \\ -c_1 \\ c_1 + 3c_2 \end{pmatrix}$$

To determine what span(S) looks like, we will put these column vectors into the matrix form.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 0 \\ 1 & 3 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The RREF of the matrix shows that each column has a pivot, indicating that the column vectors are linearly independent. In the $\mathbb{R}^4$ space, when there are 2 linearly independent vectors, there will be 2 free variables. Geometrically, the span will thus be a plane.

ii)

To form a basis in $\mathbb{R}^4$, there needs to be 4 elements in the basis. These 4 elements must also be linearly independent from each other. Hence, I will add 2 more column vectors to set S to make it a basis. The 2 vectors are

$$v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The new set S will form a basis for $\mathbb{R}^4$, because 1) the vectors in set S are linearly independent and 2) the vectors span the space $\mathbb{R}^4$.

Linear Independence:

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Spanning $\mathbb{R}^4$:

The linear combination of the column vectors in RREF can form any vector in the $\mathbb{R}^4$ vector space.

$$c_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_4\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

In[260]:=
```
M1 = {{1, 2}, {2, 1}, {-1, 0}, {1, 3}};
RowReduce[M1] // MatrixForm
M2 = {{1, 2, 0, 0}, {2, 1, 0, 0}, {-1, 0, 1, 0}, {1, 3, 0, 1}};
RowReduce[M2] // MatrixForm
```

Out[261]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Out[263]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Part b

i)

The subspace spanned by set S is the linear combination of the components in S, expressed as

$$\text{span}(S) = c_1 (3x - 1) + c_2(5x) + c_3(x^2 + 1)$$

Given that the most general form of $P(3)$ is $ax^3 + bx^2 + cx + d$, we can express span(S) as

$$(0)x^3 + (c_3)x^2 + (3c_1 + 5c_2)x + (-c_1 + c_3)1$$

$a = 0$

$b = c_3$

$c = 3c_1 + 5c_2$

$d = -c_1 + c_3$

Then, with each column vector representing a general form of $P(3)$,

$$c_1\begin{pmatrix}0\\0\\3\\-1\end{pmatrix}+c_2\begin{pmatrix}0\\0\\5\\0\end{pmatrix}+c_3\begin{pmatrix}0\\1\\0\\1\end{pmatrix}=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$$

Finally, putting the column vectors into matrix form and transforming it to RREF.

$$\begin{pmatrix}0&0&0\\0&0&1\\3&5&0\\-1&0&1\end{pmatrix}\xrightarrow{\text{rref}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{pmatrix}$$

With 3 linearly independent vectors spanning the $P(3)$ space, the subspace spanned by set S will have 3 free variables and take form as a hyperplane.

ii)

To extend set S to span the vector space P(3), we will add one that contains x^3. The new set S that forms the basis for P(3) will be

$$S=\{3x-1,\ 5x,\ x^2+1,\ x^3\}$$

Linear Independence:

$$\begin{pmatrix}0&0&0&1\\0&0&1&0\\3&5&0&0\\-1&0&1&0\end{pmatrix}\xrightarrow{\text{rref}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

Spanning P(3):

$$c_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+c_2\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+c_3\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+c_4\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$$

In[351]:=
```
M3 = {{0, 0, 0}, {0, 0, 1}, {3, 5, 0}, {-1, 0, 1}};
RowReduce[M3] // MatrixForm
M4 = {{0, 0, 0, 1}, {0, 0, 1, 0}, {3, 5, 0, 0}, {-1, 0, 1, 0}};
RowReduce[M4] // MatrixForm
```

Out[352]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Out[354]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Part c

i)

The subspace spanned by set S is the linear combination of the components in S, expressed as

$$\text{span}(S) = c_1 \begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix} + c_2 \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 c_1 + 7 c_2 & 0 \\ 0 & -5 c_1 + 2 c_2 \end{pmatrix}$$

To determine what span(S) looks like, we will put these column vectors into the matrix form.

$$\begin{pmatrix} 3 & 7 \\ 0 & 0 \\ 0 & 0 \\ -5 & 2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Again, the RREF of the matrix shows two linearly independent vectors. The subspace spanned by the set S will hence have 2 free variables and take form as a plane.

ii)

The given set S does not form a basis for the vector space of upper-triangular 2×2 matrices, because the two vectors do not span the space. Hence, we need to add a third vector to satisfy this condition. The vector is

$$v_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The new set S will form a basis for upper-triangular 2×2 matrices, because 1) the vectors in set S are linearly independent and 2) the vectors span the space.

Linear Independence:

$$c_1 \begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix} + c_2 \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 c_1 + 7 c_2 & c_3 \\ 0 & -5 c_1 + 2 c_2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 7 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ -5 & 2 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Spanning upper-triangular 2×2 matrices:

The linear combination of the column vectors in RREF can form any vector in the vector space.

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

In[264]:=
```
M5 = {{3, 7}, {0, 0}, {0, 0}, {-5, 2}};
RowReduce[M5] // MatrixForm
M6 = {{3, 7, 0}, {0, 0, 1}, {0, 0, 0}, {-5, 2, 0}};
RowReduce[M6] // MatrixForm
```

Out[265]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Out[267]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Appendix: HC Applications

#dataviz: In the first question, data visualization serves two purposes. One, when it was hard to approach a problem analytically, it is often stimulating/inspiring to visualize the problem. It was especially helpful in Problem 1d, where I only realized that the perpendicular lines from the drone's path can be expressed as vectors after I drew it out. Two, it serves to validate my analytical approach. In Problem 1c, I first identified a point that is 2m within the radius of the drone and used the distance formula to show that. By plotting the radius around the point as well as the drone's path, it serves as another line of supporting evidence to validate my answer.