MINERVA®

Special Bases: LBA

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Assignment IV

Special Bases

1. **EigenLBA** (#vectors, #transformations, #computationaltools, #theoretical-tools)
   In this project, you will analyze the flow of traffic in a system relevant to your global location. For example, go to a local museum and analyze the foot traffic of visitors in a small wing of the museum over a self-selected duration of time. You may also consider other, similar networks. You will construct a model to predict how (foot) traffic evolves over future time-steps.

The model we picked for this LBA is the local Hyderabad supermarket where, according to our estimation, receives 5000 customers throughout the day.
The system we constructed consists of 6 nodes that represent the major sections of the supermarket covering a space of about $1000\ m^2$

To facilitate the way we count the people at each node, Oscar and I had to pick a point in the network where we can see the three nodes (Fig. 1). To cover the nodes 4, 5, and 6, Oscar was standing at the exit of the supermarket. While I was standing in the clothing section to monitor the people on nodes 2, 1, and 3.

We agreed on a time interval of **one hour** to supervise the flow of people within this system and came up with a matrix A that illustrates how many people were at each node on average as we were taking the count every 15 minutes. In other words, the matrix A represents the average people on a given node.

Assumptions:

- Although improbable, we assumed that there's a fixed number of people in the market during a one-hour interval of time.
- We picked major sections from the market to monitor, but the supermarket has different ways to move between the nodes. Therefore, not all people counted at one node are going to transfer to one of the five nodes.

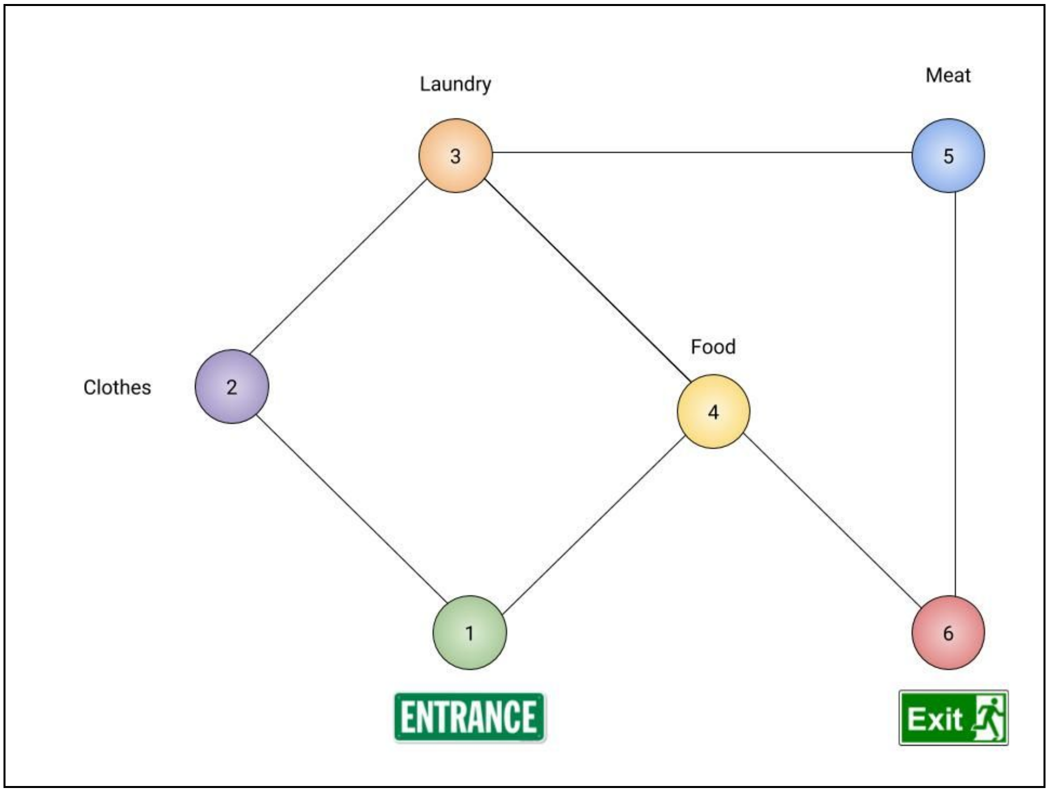

Figure (1). Diagram showing the layout of different sections in the Supermarket

The Markov matrix corresponding to the system

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 4 & 0 & 3 & 7 & 0 & 0 \\ 0 & 0 & 6 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 & 6 & 18 \end{pmatrix}$$

The matrix M corresponding to the normalization of the matrix A

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & \frac{3}{5} & \frac{2}{12} & 0 & 0 & 0 \\ 0 & \frac{2}{5} & \frac{1}{12} & \frac{1}{10} & \frac{1}{10} & 0 \\ \frac{4}{5} & 0 & \frac{3}{12} & \frac{7}{10} & 0 & 0 \\ 0 & 0 & \frac{6}{12} & 0 & \frac{3}{10} & \frac{2}{20} \\ 0 & 0 & 0 & \frac{2}{10} & \frac{6}{10} & \frac{18}{20} \end{pmatrix}$$

Using Mathematica, we found that, except the first vector, the system has imaginary numbers in the other eigenvectors (as a joke, we thought that they might represent imaginary people, maybe the market is haunted, and there are ghosts living in there).

```
In[84]:= M = ( 0      0      0      0      0      0
               2/10   6/10   2/12   0      0      0
               0      4/10   1/12   1/10   1/10   0
               8/10   0      3/12   7/10   0      0
               0      0      6/12   0      3/10   2/20
               0      0      0      2/10   6/10   18/20 )

In[100]:= Eigenvalues[M]
Out[100]=
 {1, Root[-744 - 135 #1 + 20200 #1^2 - 47500 #1^3 + 30000 #1^4 &, 4],
  Root[-744 - 135 #1 + 20200 #1^2 - 47500 #1^3 + 30000 #1^4 &, 3],
  Root[-744 - 135 #1 + 20200 #1^2 - 47500 #1^3 + 30000 #1^4 &, 2],
  Root[-744 - 135 #1 + 20200 #1^2 - 47500 #1^3 + 30000 #1^4 &, 1], 0}
```

In the context of the problem, we can consider imaginary eigenvalues, hence, we picked the first value to proceed our calculation with for the rest of this problem.

Finding the eigenvector of A is the same as getting the nullspace of the following matrix:

$$(A - \lambda \cdot I)$$

Since solving for the eigenvector is written as:

$$(A - \lambda \cdot I) .v = 0$$

The eigenvector corresponding to the eigenvalue of is:

$$v_1 = \begin{pmatrix} 0 \\ \frac{1}{100} \\ \frac{3}{125} \\ \frac{1}{50} \\ \frac{4}{25} \\ 1 \end{pmatrix}$$

Long term behavior ($\lim(n) \rightarrow \infty\ M_n v$)

Distribution 1:

```
In[97]:= M20.(1/18, 3/18, 4/18, 2/18, 1/18, 7/18) // N // MatrixForm
```

$$\text{In[97]:= M20.}\begin{pmatrix} 1/18 \\ 3/18 \\ 4/18 \\ 2/18 \\ 1/18 \\ 7/18 \end{pmatrix} \text{ // N // MatrixForm}$$

Out[97]//MatrixForm=

$$\begin{pmatrix} 0. \\ 0.00894559 \\ 0.0205103 \\ 0.0187838 \\ 0.131789 \\ 0.819971 \end{pmatrix}$$

Distribution 2:

$$\text{In[98]:= M20.}\begin{pmatrix} 4/18 \\ 3/18 \\ 1/18 \\ 7/18 \\ 1/18 \\ 2/18 \end{pmatrix} \text{ // N // MatrixForm}$$

Out[98]//MatrixForm=

$$\begin{pmatrix} 0. \\ 0.00930766 \\ 0.0209097 \\ 0.0201406 \\ 0.131766 \\ 0.817876 \end{pmatrix}$$

Distribution 3:

In[99]:= $M20.\begin{pmatrix} 7/18 \\ 4/18 \\ 2/18 \\ 3/18 \\ 1/18 \\ 1/18 \end{pmatrix}$ // N // MatrixForm

Out[99]//MatrixForm=

$$\begin{pmatrix} 0. \\ 0.00952418 \\ 0.0211291 \\ 0.0207857 \\ 0.13177 \\ 0.816791 \end{pmatrix}$$

As mentioned in the eigenvector corresponding to this system, the people in the supermarket are going to check out in the long term so we can notice that regardless of the distribution that we pick in the start, the fraction of the number of people is concentrated at node 6 (which is the exit).

These results align with our thinking that all people are required to pass by the exit when they enter the market. Furthermore, we predicted that the exit would have the most concentration of people at any time as it takes a considerable amount of time to check out and pay. This is confirmed by our model (having over 80% of the people in the market at the exit regardless of the initial distribution).

The stochastic nature of this system arises when we don't account for people who are just browsing and might not end up buying anything since, in this case, they're not required to pass by the exit and they might just avoid queuing and exit from the entrance of the market.

Pictures of our visit to the supermarket:

2. **QR Factorization** (#vectors, #transformations)

Suppose that we start with a square matrix A with independent columns and then use Gram-Schmidt on the columns to form an orthogonal matrix Q whose columns are the corresponding orthonormal columns. A can be factored as

$$A = QR$$

where R is an upper triangular matrix. Do the following to determine the general form of R and prove that this factorization always holds.

(a) Find Q using the Gram-Schmidt algorithm. Be sure to record your steps carefully, you will need them later.

$$A = \begin{pmatrix} 2 & 3 \\ 0 & -2 \end{pmatrix}$$

Step 1: Normalize the first column of A

$$q_1 = \frac{v_1}{\| v_1 \|}$$

$$q_1 = \frac{v_1}{\| v_1 \|} = \frac{1}{\sqrt{4+0}} \times \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Step 2: Find the intermediate vector by applying the following relation to the second vector of the matrix A:

$$a_2 = v_2 - (v_2 \cdot q_1) \times q_1$$

$$a_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \left(\begin{pmatrix} 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

Step 3: Normalizing the vector found in step 2

$$q_2 = \frac{a_2}{\| a_2 \|}$$

$$q_2 = \frac{a_2}{\| a_2 \|} = \frac{1}{\sqrt{0 + (-2)^2}} \times \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Step 4: [Optional] Verifying whether the two new vectors form an orthonormal basis (we can just check if their dot product is equal to zero)

$$q_1 \cdot q_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0$$

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(b) Now find a matrix *R* such that *A* = *QR*. Hint: *What is an easy way to do this?*
The easy shortcut to finding R is to left multiply both parts of the equation by the inverse of the matrix Q

$$Q^{-1}A = Q^{-1}QR \qquad \rightarrow \qquad Q^{-1}A = I \times R \qquad \rightarrow \qquad R = Q^{-1}A$$

$$R = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

(c) Repeat part (a) and (b) using the matrix:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$q_1 = \frac{v_1}{\| v_1 \|}$$

$$q_1 = \frac{v_1}{\| v_1 \|} = \frac{1}{\sqrt{2}}. \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$a_2 = v_2 - (v_2 \cdot q_1) \times q_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \times \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

$$q_2 = \frac{a_2}{\| a_2 \|}$$

$$q_2 = \sqrt{\frac{3}{2}} \cdot \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \sqrt{\frac{3}{2}} \end{pmatrix}$$

$$a_3 = v_3 - [(v_3 \cdot q_1) \times q_1] - [(v_3 \cdot q_2) \times q_2]$$

$$a_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}\right] - \left[\begin{pmatrix} \frac{1}{6} \\ -\frac{1}{6} \\ \frac{1}{3} \end{pmatrix}\right]$$

$$a_3 = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \quad \rightarrow \quad q_3 = \frac{a_3}{\| a_3 \|} \quad \rightarrow \quad q_3 = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$Q = [\ q_1 \ \ | \ \ q_2 \ \ | \ \ q_3\ ]$$

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$R = Q^{-1}A$$

$$R = \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix}$$

(d) Given a vector b that lies in the same plane as two orthonormal vectors q_1 and q_2, for example, two of the columns of Q, draw a diagram illustrating how we can write b as a linear combination of q_1 and q_2. How are the projections of b onto the vectors q_1 and q_2 related to this question? Be sure to label your diagram accordingly.

The vector b is on the same plane as q_1 and q_2. Since q_1 and q_2 are orthonormal then they span R^2. Furthermore, the vector b can be written as a linear combination of the two vectors since b belongs to the plan R^2

As mentioned in Fig. X, the vector b has coordinates on the basis constructed by the two vectors, thus the projection of the vector b on either q_1 or q_2 is a form of multiplying the vector basis vector by a constant in order to represent the vector b on that basis.

The projection of b on q_1 is expressed by $c_1 \times q_1$

The projection of b on q_2 is expressed by $c_2 \times q_2$

$$b = k_1 \cdot q_1 + k_2 \cdot q_2$$

$k_1 = b_{q1}$ and $k_2 = b_{q2}$

$$b_{q_1} = b - (b \cdot q_1) \times q_1$$

$$b_{q_2} = b - (b \cdot q_2) \times q_2$$

b_{q1} and b_{q2} are the points corresponding to the projection of b on both vectors.
We can use vector addition to show the vector b as a combination of vectors q_1 and q_2

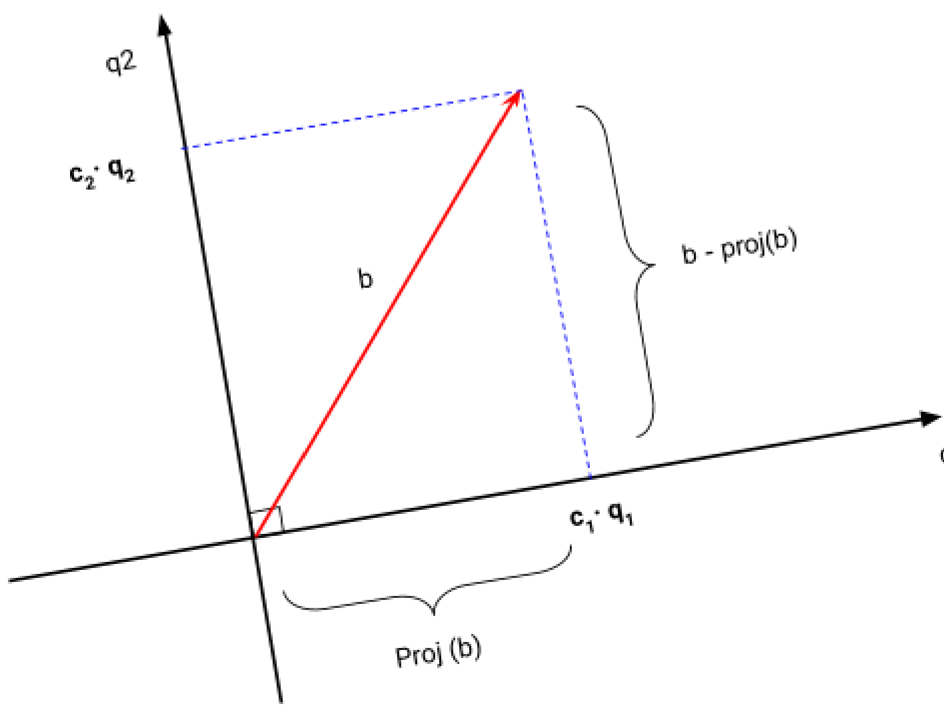

Fig (x). A diagram to show the combination of q_1 and q_2 to find b

(e) In a similar fashion to (c), how can we write a vector c as a linear combination of orthonormal vectors q_1, q_2, and q_3, such as columns of Q, assuming that c lies in the subspace spanned by these 3 vectors? Be sure your answer mentions projections.

The vector c would be a combination of the three vectors as it's in the space R^3 spanned by the three orthonormal vectors.

$$c = k_1 \cdot q_1 + k_2 \cdot q_2 + k_3 \cdot q_3 \qquad (e.1)$$

We make a projection of c on vectors q_1, q_2, and q_3 which is going to result in the coordinates of the vector with respect of the standard basis q_1, q_2, and q_3

$$c_{q1} = c - (c \cdot q_1) \times q_1$$

$$c_{q2} = c - (c \cdot q_2) \times q_2$$

$$c_{q3} = c - (c \cdot q_3) \times q_3$$

In a nutshell, the constant k_1, k_2, and k_3 that were expressed in equation (e.1) are the coordinates of the point c (b_{q1}, b_{q2}, b_{q3}) which are also the projection of c on the vectors in the standard basis.

(f) Examine each of the matrices Q and R that you found in the examples above. How is R related to the steps you performed to find Q? Hint: *What are the values the different projections that you calculated when performing Gram-Schmidt?*

Following the process of the Gram Schmidt algorithm, we realize that the elements constructing the upper triangular matrix R are the dot product of each vector of A by its projection $\mathbf{q}_i$ as follows:

$$R = \begin{pmatrix} a_1 \cdot q_1 & a_2 \cdot q_1 & a_3 \cdot q_1 \\ 0 & a_2 \cdot q_2 & a_3 \cdot q_3 \\ 0 & 0 & a_3 \cdot q_3 \end{pmatrix}$$

Recalling the process of Gram Schmidt when we had to project on the existing orthonormal vectors as we move through the column of the matrix A. Another way to approach this is to think about the scalar product of all the lower triangle of R are equal to zero as we proceed with the algorithm of Gram Schmidt.

(g) Using your observations above, propose what the entries of R must be in general. If you are stuck, just consider the 3x3 case.

The entries of R in general:

$$R = \begin{pmatrix} a_1 \cdot q_1 & a_2 \cdot q_1 & \dots & a_n \cdot q_1 \\ 0 & a_2 \cdot q_2 & \dots & a_n \cdot q_2 \\ 0 & 0 & \dots & \vdots \\ 0 & 0 & 0 & a_n \cdot q_n \end{pmatrix}$$

(h) For each of the examples in (a) and (b), what is the change of basis matrix from the basis for Col(A) to the basis for Col(Q)? How is this related to R?

Example (a)

$$RREF\,(A \mid Q_1) = RREF\begin{pmatrix} 2 & 3 & 1 \\ 0 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

$$RREF\,(A \mid Q_2) = RREF\begin{pmatrix} 2 & 3 & 0 \\ 0 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{3}{4} \\ 0 & 1 & \frac{1}{2} \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{1}{2} & -\frac{3}{4} \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}^{-1} = R^{-1}$$

Example (b)

$$RREF\,(A|Q_1) = \begin{pmatrix} 1 & 1 & 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 1 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$RREF\,(A \mid Q_2) = \begin{pmatrix} 1 & 1 & 0 & \frac{1}{\sqrt{6}} \\ 1 & 0 & 1 & -\frac{1}{\sqrt{6}} \\ 0 & 1 & 1 & \sqrt{\frac{3}{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{\sqrt{6}} \\ 0 & 1 & 0 & \sqrt{\frac{3}{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$RREF\,(A \mid Q_3) = \begin{pmatrix} 1 & 1 & 0 & -\frac{1}{\sqrt{3}} \\ 1 & 0 & 1 & \frac{1}{\sqrt{3}} \\ 0 & 1 & 1 & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2\sqrt{3}} \\ 0 & 1 & 0 & -\frac{1}{2\sqrt{3}} \\ 0 & 0 & 1 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} \\ 0 & \sqrt{\frac{3}{2}} & -\frac{1}{2\sqrt{3}} \\ 0 & 0 & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix}^{-1} = R^{-1}$$

At the last minute, I realized that I might have switched the sides for the change of basis from the basis of col(Q) to col(A) rather than Col(A) to the basis for Col(Q). Therefore, the calculations above represent the former, and to get the transformation from Col(A) to Col(Q) we just have to inverse the matrix C which in this case would be R.

Appendix:

Gram Schmidt implementation in Python:

```
from numpy import *

def proj(u, v):
    return ((sum(array(v)*array(u)) / sum(array(u)**2))*array(u))


def Gram_Schmidt(Matrix):
    Orth = []
    for i in range(len(Matrix)):
        curr_Vec = array(Matrix[i])

        for v in Orth:
            curr_Vec =  curr_Vec - proj(v, curr_Vec)


        curr_Vec = curr_Vec / linalg.norm(curr_Vec)
        Orth.append(curr_Vec)

    return Orth


test = array([[2, 0], [3, -2]])
print(array(Gram_Schmidt(test)))
# Output: [[ 1,  0], [ 0, -1]]
```

HC Applications:

#deduction: In the second problem, I used deduction to figure out the resulting matrix of R and how it relates to the algorithm of Gram Schmidt. Furthermore, when disentangling the change of basis matrix between Q and A.

#algorithms: As mentioned in the appendix, I coded an algortihm that works with any nxn matrix to find it's orthonormal form, it's true that there was an embedded Mathematica function for that, but I thought it would be interesting to pass from the algorithms in part (a) and (b) to a general format to any matrix.

#network: we constructed an organized network based on the local supermarket of Hyderabad, we then translate that to the Markov matrix to simulate the long term behavior and how is that related to the real-life context.

References:

Class 10.1 video Problem 2 question a following the steps of the GS Algorithm. Discrete Mathematical Systems. Retrieved from: https://drive.google.com/file/d/1n5yQrPwDSpJL0LdgPii6to7wFK02sefw/view

Karen E. Smith (2015). Math217: Summary of the change of basis and all that. The University of Michigan. Retrieved from: http://www.math.lsa.umich.edu/~kesmith/CoordinateChange.pdf