MINERVA®

Assignment V

CS111A - Fall 2018

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Part A

A.1 Handle the Miami heat

(a) Generate a plot that illustrate the constraint on the input variables

$V(x,\ y,\ z) = x\,y\,z = 4000\ m^3$

$x,\ y\ \geq 30 \quad and \quad z \geq 4$

$z\ =\ 4000\ /\ xy$

$4000\ /\ xy\ \geq\ 4 \qquad \rightarrow \qquad 1000\ /\ xy\ \geq\ 1 \qquad \rightarrow \qquad y \leq 1000\ /\ x$

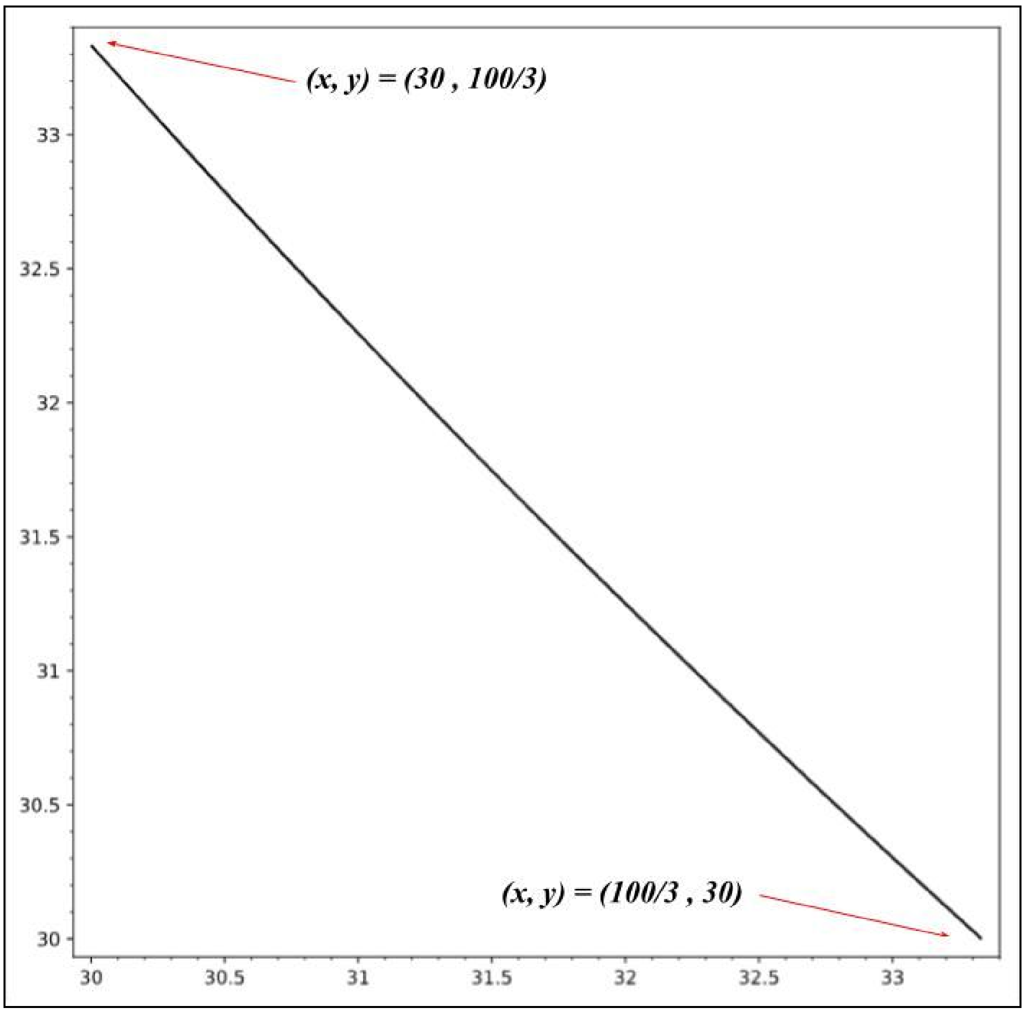

Fig 1. The contour graph shows the constraint posed in the heat loss problem, along with the possible values that width and length of the building can take. (*Plotted in Sagemath*)

Notes about the graph:

We constructed the graph assuming that $z = 4$, in that case, the ***x*** and ***y*** values have to belong to the curve plotted. However, if we want to extrapolate to the general case of $z \geq 4$, then we can highlight the whole area under the curve as long as ***x*** and ***y*** are equal or more than 30. In other words, picking numbers that doesn't belong to the curve means that **z** is going to be bigger than 4

(b) Find the dimensions that minimize heat loss

$$H(x,\ y,\ z) = \ 6xy\ + 16yz\ + 20xz$$

$$V(x,\ y,\ z) = x\,y\,z = 4000\ m^3 \qquad \rightarrow \qquad z\ =\ 4000\,/\,xy$$

Defining the maximum and minimum values that the variables can take:

$$30 \geq x \geq \tfrac{100}{3}$$

$$30 \geq y \geq \tfrac{100}{3}$$

$$4 \geq z \geq \tfrac{40}{9}$$

We use the Lagrange multiplier:

$$dH(x,\ y,\ z) = \ <\ 6y + 20z\ ,\ 6x + 16z,\ 16y + 20x >$$

$$dV(x,\ y,\ z) = \ < yz,\ xz,\ xy >$$

The system of equations is:

$$6y + 20z = \lambda\, yz\ \ \ldots\ldots\ldots\ \ (1)$$

$$6x + 16z = \lambda\, xz\ \ \ldots\ldots\ldots\ \ (2)$$

$$16y + 20x = \lambda\, xy \ldots\ldots\ldots\ \ (3)$$

$$x\,y\,z = 4000 \ldots\ldots\ldots\ \ (4)$$

Since ***x***, ***y***, and ***z*** are not equal to 0, we multiply (1) by ***x***, (2) by ***y***, and (3) by ***z***

$$6xy + 20xz = \lambda\, xyz\ \ \ldots\ldots\ldots\ \ (1)$$

$$6xy + 16yz = \lambda\, xyz\ \ \ldots\ldots\ldots\ \ (2)$$

$16yz + 20xz = \lambda\, xyz$ ……… (3)

We get that:

$6xy + 20xz = 6xy + 16yz \quad \rightarrow \quad 20xz = 16yz$

$6xy + 20xz = 16yz + 20xz \quad \rightarrow \quad 6xy = 16yz$

$20x = 16y \quad \rightarrow \quad y = \frac{5}{4}\, x$

$6x = 16z \quad \rightarrow \quad z = \frac{3}{8}\, x$

We plug in the results into the constraint function:

$x\,(\frac{5}{4}\, x)\,(\frac{3}{8}\, x) = 4000 \quad \rightarrow \quad \frac{15}{32}\, x^3 = 4000$

$x = (128000\,/\,15)^{1/3} = 20.434918 \approx 20.435$

$y \approx \frac{5}{4}(20.435) \approx 25.537$ and $z \approx \frac{3}{8}(20.435) \approx 7.6631$

All the values suggested above for the variables are out of the domain constrained in the problem

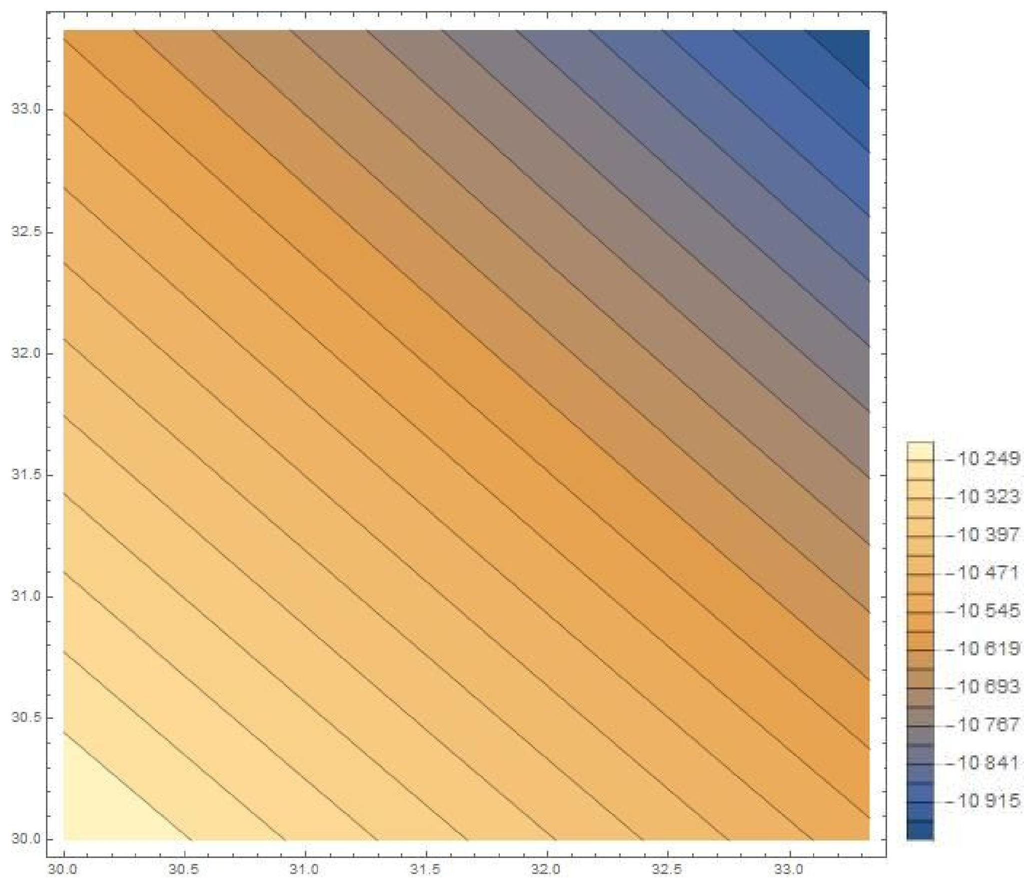

Fig 3. The contour plot of the heat function along with the domain that the constraints impose on the problem, in this case, ***z*** is substituted by conditioning on ***x*** and ***y***.

Drawing from the graph, we notice that within the interval of x and y, the heat loss function is always on the rise and we can read using the legend on the right side of the graph. This means that in light of those constraints, the lowest heat loss value corresponds to the lowest values of x and y, hence, the highest value of z.

Therefore, we can conclude that within the interval drawn above, the minimum for the heat loss function corresponds to **(x, y, z) = (30, 30, 40/9)**

After plugging this values into *H(x, y, z)* we get:

$$H(30,\ 30,\ 40/9) = \ 6(30)(30)\ + 16(30)(40/9)\ + 20(30)(40/9)$$

$$H(30,\ 30,\ 40/9) = 5400 + 19200/9 + 24000/9$$

$$H(30,\ 30,\ 40/9) = 10200\ units$$

(c) Would you design a building with even less heat loss when removing the variables

Indeed, we can design a building with less heat loss if we removed the constraints, supposedly, it's the same solution we had **(x, y, z) = (20.34, 25.53 , 7.66)** if we want to keep the volume fixed at 4000 m^3 otherwise, the change in volume would lead us to different scenarios that might work best in minimizing the heat loss.

A.2 Breaking Lagrange

(a) Try to use Lagrange to solve the problem

The objective function $f(x,\ y) =\ 2x + 3y$

The constraint $\sqrt{x} + \sqrt{y}\ = 5 \qquad \rightarrow \qquad g(x,\ y) = \sqrt{x} + \sqrt{y}$

Computing the partial derivatives of both functions:

$$df(x,\ y) = \ < 2\ ,\ 3 >$$

$$dg(x,\ y) = \ < \tfrac{1}{2\sqrt{x}}\ ,\ \tfrac{1}{2\sqrt{y}} >$$

We construct the system of equations to solve using Lagrange Multipliers:

$$df(x,\ y) = \lambda\ dg(x,\ y)$$

$$2 = \lambda\ \tfrac{1}{2\sqrt{x}} \qquad \rightarrow \qquad 4 = \lambda\ \tfrac{1}{\sqrt{x}} \qquad \rightarrow \qquad \sqrt{x} = \ \tfrac{\lambda}{4}$$

$$3 = \lambda\ \tfrac{1}{2\sqrt{y}} \qquad \rightarrow \qquad 6 = \lambda\ \tfrac{1}{\sqrt{y}} \qquad \rightarrow \qquad \sqrt{y} = \ \tfrac{\lambda}{6}$$

$$\sqrt{x} + \sqrt{y}\ = 5$$

We solve for the value of Lambda, then we plug it into the first 2 equations to get ***x*** and ***y***

$$\tfrac{\lambda}{4} + \tfrac{\lambda}{6} = 5 \qquad \rightarrow \qquad \lambda = 12 \qquad \rightarrow \qquad (x,\ y) = (9,\ 4)$$

(b) Testing the outcome of the values computed above:

$$f(9,\ 4) = 18 + 12 = 30$$

$$f(25,\ 0) = 50 + 0 = 50$$

Indeed, the outcome of $f(25,\ 0)$ gives a larger value than $f(9,\ 4)$ computed in part (a)

(c) Graphing the constraint equation and level curves:

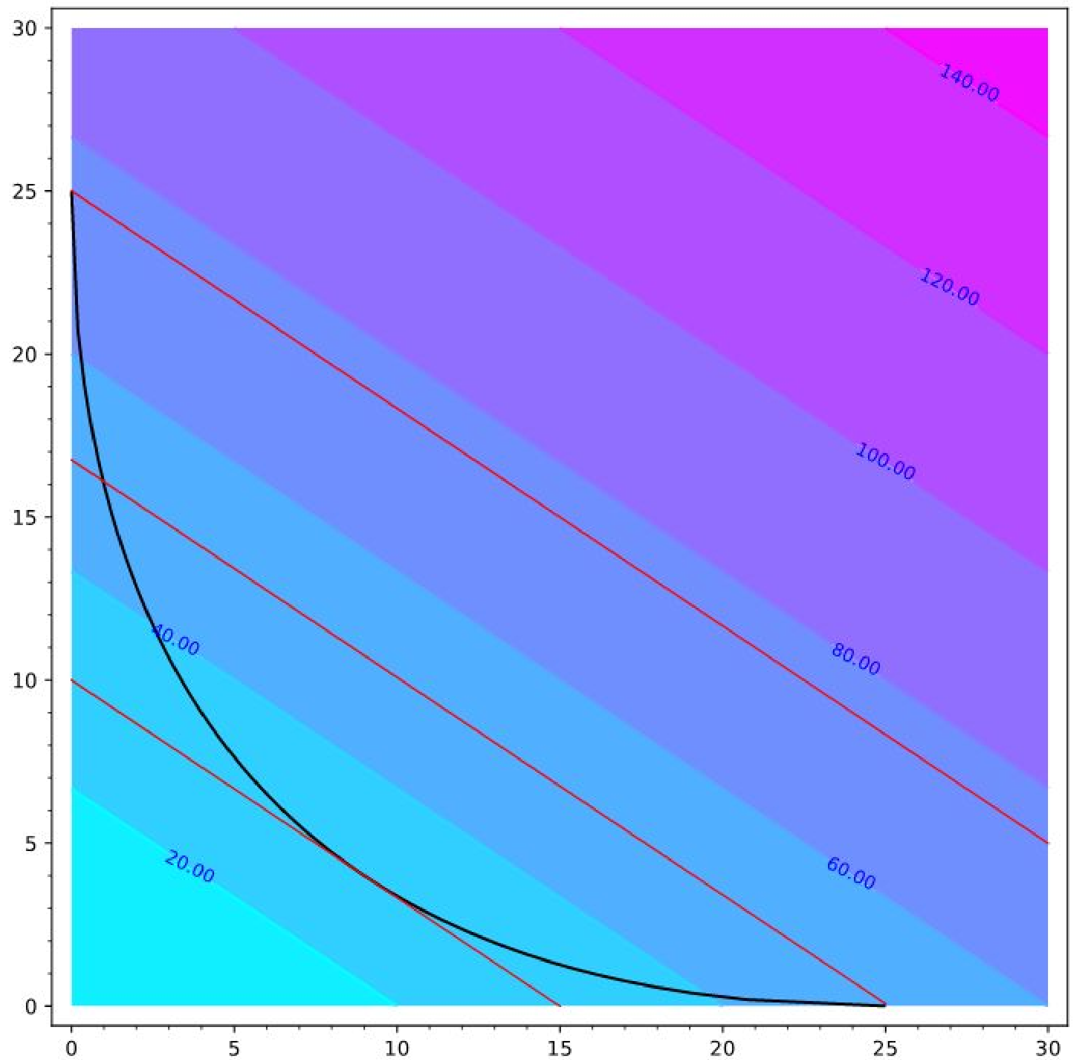

Fig 2. Plot of the contours of the maximizing problem along with the initial solution of Lagrange Multiplier for *(9, 4)=**30**, (25, 0)=**50**, and (0, 25)=**75***

According to the graph above, the maximum value for the function *f(x)* is ***(0, 25)*** as it crosses the level curve at **75**

$f(0,\ 25) =\ 2(0) + 3(25) = 75$ *and* $\sqrt{0} + \sqrt{25} = 5$

(d) The failure of the lagrange method, and what's the significance of the point (9, 4)

Lagrange in this problem didn't work because both (25, 0) and (0, 25) don't satisfy the assumption that the tangent line of the constraint belongs to the same segment as the partial derivatives of the objective function.

The point (9, 4) therefore represent a minimum and not the maximum of the objective function.

Part B

B.1 Cheaper by the dozen

The quantity of the widgets produced by the plant:

$$Q(x, y, z) = xyz$$

Cost of aluminium is 6$ per ton

Cost of iron is 4$ per ton

Cost of magnesium is 8$ per ton

How many tons needed of raw materials to produce 1000 widgets at the lowest cost.

$$C(x, y, z) = 6x + 4y + 8z$$

$$xyz = 1000 \quad \rightarrow \quad g(x, y, z) = xyz$$

$$dC(x, y, z) = \ < 6,\ 4,\ 8 >$$

$$dg(x, y, z) = \ < yz,\ xz,\ xy >$$

$$dC(x, y, z) = \lambda\, dg(x, y, z)$$

$$6 = \lambda\, yz \quad \rightarrow \quad z = 6 / \lambda\, y \quad \ldots\ldots\ldots \ (1)$$

$$4 = \lambda\, xz \quad \rightarrow \quad z = 4 / \lambda\, x \quad \ldots\ldots\ldots \ (2)$$

$$8 = \lambda\, xy \quad \ldots\ldots\ldots \ (3)$$

$$xyz = 1000 \quad \ldots\ldots\ldots \ (4)$$

From (1) and (2) : $y = \frac{3}{2}x$

From (1) and (3) : $z = \frac{3}{4}x$

We plug in the values of ***y*** and **z** in terms of ***x*** into the constraint equation and solve for ***x***

$$x\left(\tfrac{3}{2}x\right)\left(\tfrac{3}{4}x\right) = 1000$$

$$x^3 = 8000 / 9$$

$$x = (8000 / 9)^{1/3} = 9.61499 \simeq 9.615$$

$$y \simeq \tfrac{3}{2}(9.615) \simeq 14.4225$$

$$z \simeq \tfrac{3}{4}(9.615) \simeq 7.21125$$

The lowest cost to produce 1000 product is:

$$C(9.615,\ 14.42,\ 7.21) \simeq 173.05\ \$$$

$$(9.615)\,(14.4225)\,(7.21125) \simeq 1000$$

Appendix:

#constraints: effectively apply the constraints to solve the problems in A.1 and B.1 which are implemented in the Lagrange method.
#optimization: Using optimization techniques (derivatives) to maximize/minimize the problems in hand, and interpret the results in the context of the prompt.