Linear Transformation

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Assignment 3 - Linear Transformation

1. Hip to be square (#transformations, #theoreticaltools**)**

Suppose all vectors v in the unit square $[0, 1]\times[0, 1]$ are transformed to Av where A is a 2×2 matrix.

(a) What is the shape of the transformed region when

$$A = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$$

$\forall\ x, y \in R:\ 0 \le x \le 1 \quad and \quad 0 \le y \le 1$

We can table the for corners (extremes) which the values of x and y can take:

(x, y)	$A.v$
(0, 0)	(0, 0)
(1, 0)	(1, 3)
(0, 1)	(-2, 4)
(1, 1)	(-1, 7)

The graph below illustrates the region at which the multiplication of any point that has coordinates $0 \le x \le 1 \quad and \quad 0 \le y \le 1$ by the matrix A will be included within that region.

The table above takes the four corners of the unit square $[0, 1]\times[0, 1]$ and transforms it using the matrix A to the shape of a **parallelogram.**

Once we map the four corners of the unit square, all other points belonging to the unit square is going to fall inside the region constructed by the transformation.

```
In[64]:= points = {{0, 0}, {-1, 7}, {-2, 4}, {1, 3}};

reg = ConvexHullMesh@points
```

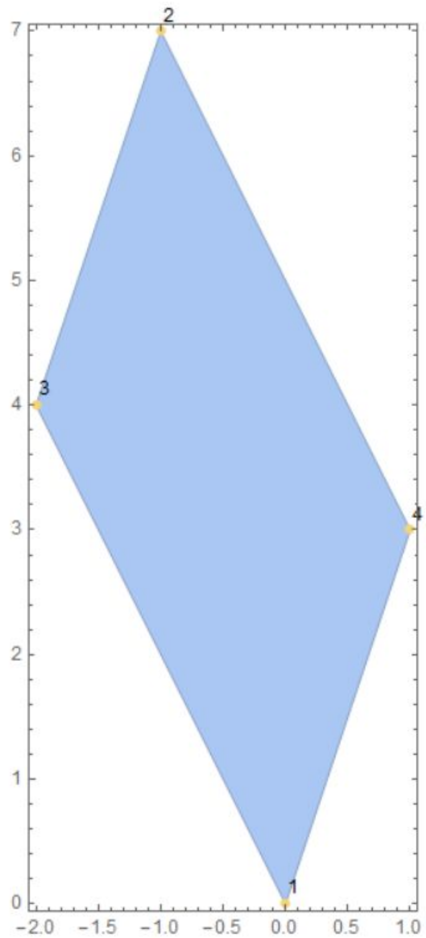


Fig 1. The transformation of the unit square by matrix A

(b) In general, what is the shape of the transformed region?
For a 2×2 matrix, the general

In[157]:= $$\text{point} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \{0, 0\},\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \{1, 0\},\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \{0, 1\},\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \{1, 1\} \right\}$$

Out[157]= $\{\{0, 0\}, \{a, c\}, \{b, d\}, \{a + b, c + d\}\}$

Computing the general form of the transformation of the unit square by a matrix A

For any 2×2 matrix of the form: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Mapping the four edges of the unit square would yield the following points:
$(0\ ,\ 0)\ ,\ (a\ ,\ c)\ ,\ (b\ ,\ d)\ ,\ (a+b\ ,\ c+d)$

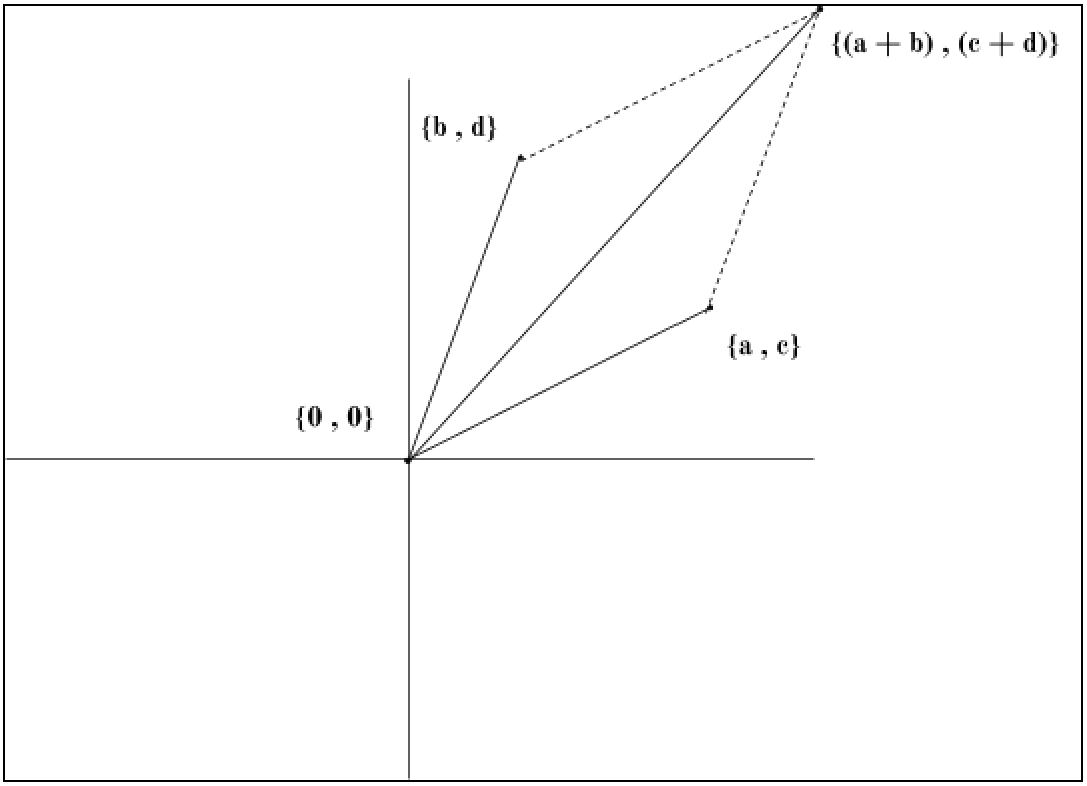


Fig 2. The transformed region of the unit square by an arbitrary matrix A

```
In[117]:= f[a_, b_, c_, d_] :=
   ConvexHullMesh@{{{a, b}, {c, d}}.{0, 0}, {{a, b}, {c, d}}.{1, 0},
     {{a, b}, {c, d}}.{0, 1}, {{a, b}, {c, d}}.{1, 1}}
```

A function that computes the region constructed by the transformed four extremes of the unit square.

We notice that for any Matrix A, the resulting shape has the origin as one of their edges. This observation would lead us to focus on the three floating points that depend on a, b, c, and d

(c) For which matrices A is that region a square?

Since the origin is a fixed point and based on Fig 2. we can check for which values the matrix A would give a square-shaped region. Starting by defining the vectors:

$$\overrightarrow{i} = (a,\ c) - (0,\ 0) = (a,\ c)$$

$$\overrightarrow{j} = (b,\ d) - (0,\ 0) = (b,\ d)$$

$$\overrightarrow{k} = (a+b,\ c+d) - (0,\ 0) = (a+b,\ c+d)$$

We need to satisfy the two conditions:

1. The adjacent vectors are equal in magnitude

$$\| \overrightarrow{i} \| = \| \overrightarrow{j} \|$$

$$\sqrt{a^2+b^2} = \sqrt{c^2+d^2} \quad \rightarrow \quad a^2+b^2 = c^2+d^2$$

2. The scalar product of the adjacent vectors is equal to 0:

This would mean that the two vectors are perpendicular to each other.

$$\overrightarrow{i} \cdot \overrightarrow{j} = 0 \quad \rightarrow \quad a \cdot b + c \cdot d = 0$$

```
In[44]:= Solve[{b, d}.{a, c} == 0 &&
           Norm[{b, d}] == Norm[{a, c}], {a, b, c, d}]

Out[44]= {{b → c, d → -a}, {b → -c, d → a},
          {b → 0, c → 0, d → -a}, {b → 0, c → 0, d → a}}
```

Therefore, the forms of Matrices A that construct a square region are:

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \quad \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

(d) For which A is that region a line segment?

If the region in a line segment, then we can express each of the three vectors in the function of each other.

$$\begin{pmatrix} a \\ c \end{pmatrix} = k_1 \cdot \begin{pmatrix} b \\ d \end{pmatrix}$$

$$\begin{pmatrix} a \\ c \end{pmatrix} = k_2 \cdot \begin{pmatrix} a+b \\ c+d \end{pmatrix}$$

$$\begin{cases} a = k_1 \dot{} b \\ c = k_1 \dot{} d \end{cases}$$

$$\begin{cases} a = k_2 \dot{} (a+b) \\ c = k_2 \dot{} (c+d) \end{cases}$$

$$k_1 = \frac{a}{b} = \frac{c}{d}$$

$$a \cdot d = c \cdot b \rightarrow \textit{Equation (1)}$$

$$k_2 = \frac{a}{a+b} = \frac{c}{c+d}$$

$$a \cdot (c+d) = c \cdot (a+b)$$

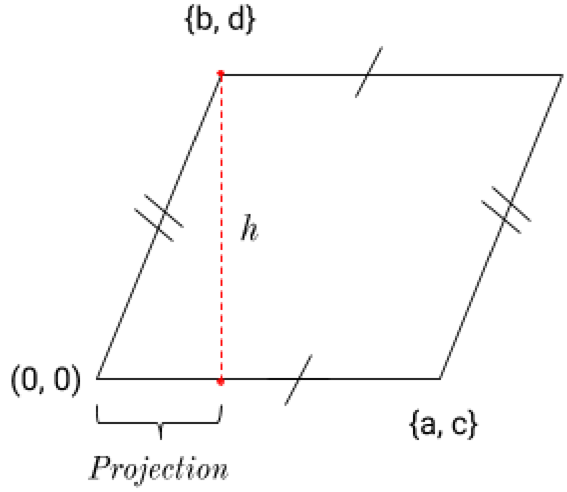
$$a \cdot c + a \cdot d = c \cdot a + c \cdot b \quad \rightarrow \quad a \cdot d = c \cdot b \rightarrow \textit{Equation (2)}$$

We notice that both equations (1) and (2) are the same, thus for any given matrix A we deduce that:

$$\forall \ a, \ b, \ k \ \in R \ \wedge \ a \cdot d = c \cdot b : \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$This \ can \ be \ exressed \ by \ \forall \ k \ \in \ R : \ A = \begin{pmatrix} a & b \\ k \cdot a & k \cdot b \end{pmatrix}$$

(e) For which A is the area of the transformed region still 1?



$$\text{proj}_{\{a,\ c\}}\,\{b,\ d\} = \frac{\{b,\ d\}\cdot\{a,\ c\}}{\|\ \{a,\ c\}\ \|^2}\cdot\{a,\ c\} = \frac{a\cdot b + c\cdot d}{a^2+c^2}\cdot\{a,\ c\}$$

For the area to be equal to 1:

$\|\ \{a,\ c\}\ \|\cdot h = 1$

$$h = \frac{1}{\sqrt{a^2+c^2}}$$

Using Pythagoras formula:

$h^2 = \|\ \{b,\ d\}\ \|^2 - \|\ \text{proj}_{\{a,\ c\}}\,\{b,\ d\}\ \|^2$

$$\frac{1}{a^2+c^2} = \left(b^2+d^2\right) - \left[\frac{(a\cdot b + c\cdot d)^2}{\left(a^2+c^2\right)^2}\right]\left(a^2+c^2\right)$$

$$\frac{1}{a^2+c^2} + \frac{(a\cdot b + c\cdot d)^2}{a^2+c^2} = \left(b^2+d^2\right)$$

$$1 + (a\cdot b + c\cdot d)^2 = \left(b^2+d^2\right)\left(a^2+c^2\right)$$

$$1 + a^2b^2 + c^2d^2 + 2\cdot abcd = a^2b^2 + c^2b^2 + a^2d^2 + c^2d^2$$

$$1 = c^2 b^2 + a^2 d^2 - 2 \cdot abcd \quad \rightarrow \quad 1 = (ad - cb)^2$$

```
In[16]:= Solve[(a*d - b*c)^2 == 1, {a, b, c, d}]

Out[16]= {{d -> (-1 + b c)/a}, {d -> (1 + b c)/a},
          {a -> 0, c -> -1/b}, {a -> 0, c -> 1/b}}
```

2. All about the basis (#transformations, #computationaltools**)**

Let T: $R^3 \rightarrow R^3$ be the map given by

$$T(x, y, z) = (x + 2y + 3z, 3x - 2y + z, 2x - 4y - 2z)$$

(a) Show that T is a linear transformation:

Taking any two vectors in R^3 in the form (x_i, y_i, z_i), we need to prove that:

- $T(v + u) = T(v) + T(u)$

$$T((x_1,\ y_1,\ z_1) + (x_2,\ y_2,\ z_2)) = T((x_1 + x_2), (y_1 + y_2), (z_1 + z_2))$$

$$= [(x_1 + x_2) + 2 \cdot (y_1 + y_2) + 3 \cdot (z_1 + z_2),\ 3 \cdot (x_1 + x_2) - 2 \cdot (y_1 + y_2) + (z_1 + z_2)$$
$$2 \cdot (x_1 + x_2) - 4 \cdot (y_1 + y_2) - 2 \cdot (z_1 + z_2)$$

$$= [(x_1 + 2 \cdot y_1 + 3 \cdot z_1) + (x_2 + 2 \cdot y_2 + 3 \cdot z_2),\ (3 \cdot x_1 - 2 \cdot y_1 + z_1) + (3 \cdot x_2 - 2 \cdot y_2 + z_2)$$
$$(2 \cdot x_1 - 4 \cdot y_1 - 2 \cdot z_1) + (2 \cdot x_2 - 4 \cdot y_2 - 2 \cdot z_2)$$

$$= T(v) + T(u)$$

- $k \cdot T(v) = T(k \cdot v)$

$$k{\cdot}T(x_1,\ y_1,\ z_1) = k{\cdot}(x_1 + 2 \cdot y_1 + 3 \cdot\ z_1,\ 3 \cdot x_1 - 2 \cdot y_1 + z_1,\ 2 \cdot x_1 - 4 \cdot y_1 - 2 \cdot z_1)$$

$$= [k \cdot (x_1 + 2 \cdot y_1 + 3 \cdot\ z_1),\ k \cdot (3 \cdot x_1 - 2 \cdot y_1 + z_1),\ k \cdot (2 \cdot x_1 - 4 \cdot y_1 - 2 \cdot z_1)]$$

$$= T(k \cdot v)$$

(b) Find the matrix A of T with respect to the standard unit basis E of R^n.

$$T = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 2 & -4 & -2 \end{pmatrix}$$

$$A = [[T_1]_E\,|[T_2]_E|\,[T_2]_E]$$

Since we are mapping the vectors in the Standard basis of R^3, then the transformation T is the same as the matrix A that represents it in the standard basis of R^3

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 2 & -4 & -2 \end{pmatrix}$$

(c) Find bases for the fundamental subspaces of A

The row space of A

In[35]:= $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 2 & -4 & -2 \end{pmatrix}$ `// RowReduce // MatrixForm`

Out[35]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The nonzero rows in the reduced row-echelon form are a basis for the row space:

{[[1,0,1],

[0,1, 1]]}

The column space of A

Based on the RREF computed before, the column space for A would be the columns that have their pivot equals to 1:

```
In[35]:= A = (1  2  3
              3 -2  1
              2 -4 -2) // RowReduce // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix} \right\}$$

The null space of A and A^T

```
In[327]:= NullSpace[A]

Out[327]=
          {{-1, -1, 1}}

In[328]:= NullSpace[Transpose[A]]

Out[328]=
          {{1, -1, 1}}
```

(d) Let $S = \{v_1, v_2, v_3\} = \{< 1, 0, -1 >, < 0, 2, 3 >, < -1, 3, 0 >\}$.

i. Show that S is a basis for R^3

We need to prove that the three unit vectors of R^3 are a combination of the vectors in S:

$$\text{In[329]:= Solve}\left[c1 * \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c2 * \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + c3 * \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} == \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \{c1, c2, c3\}\right]$$

$$\text{Out[329]=} \left\{\left\{c1 \rightarrow \frac{9}{11}, c2 \rightarrow \frac{3}{11}, c3 \rightarrow -\frac{2}{11}\right\}\right\}$$

In[330]:= $\text{Solve}\left[c1 * \begin{pmatrix}1\\0\\-1\end{pmatrix} + c2 * \begin{pmatrix}0\\2\\3\end{pmatrix} + c3 * \begin{pmatrix}-1\\3\\0\end{pmatrix} == \begin{pmatrix}0\\1\\0\end{pmatrix}, \{c1, c2, c3\}\right]$

Out[330]= $\left\{\left\{c1 \to \frac{3}{11}, c2 \to \frac{1}{11}, c3 \to \frac{3}{11}\right\}\right\}$

In[331]:= $\text{Solve}\left[c1 * \begin{pmatrix}1\\0\\-1\end{pmatrix} + c2 * \begin{pmatrix}0\\2\\3\end{pmatrix} + c3 * \begin{pmatrix}-1\\3\\0\end{pmatrix} == \begin{pmatrix}0\\0\\1\end{pmatrix}, \{c1, c2, c3\}\right]$

Out[331]= $\left\{\left\{c1 \to -\frac{2}{11}, c2 \to \frac{3}{11}, c3 \to -\frac{2}{11}\right\}\right\}$

Alternative method:

If S is a basis for R^3 then there would be a matrix M that represents the row operation from the standard basis of R^3 to S, therefore, the Row Reduced Echelon form of S should be the standard basis.

In[332]:= $S = \begin{pmatrix}1 & 0 & -1\\0 & 2 & 3\\-1 & 3 & 0\end{pmatrix}$

Out[332]= {{1, 0, -1}, {0, 2, 3}, {-1, 3, 0}}

In[334]:= `RowReduce[S] // MatrixForm`

Out[334]//MatrixForm=

$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$

ii. Find the matrix B of T with respect to S:

Explaining the process:

In[17]:= $T = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 2 & -4 & -2 \end{pmatrix}$

Out[17]= {{1, 2, 3}, {3, -2, 1}, {2, -4, -2}}

In[299]:= T.{1, 0, -1}

Out[299]= {-2, 2, 4}

In[300]:= $\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 2 & 3 & 2 \\ -1 & 3 & 0 & 4 \end{pmatrix}$ // RowReduce // MatrixForm

Out[300]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{20}{11} \\ 0 & 1 & 0 & \frac{8}{11} \\ 0 & 0 & 1 & \frac{2}{11} \end{pmatrix}$$

In[336]:= T.{0, 2, 3}

Out[336]= {13, -1, -14}

In[337]:= $\begin{pmatrix} 1 & 0 & -1 & 13 \\ 0 & 2 & 3 & -1 \\ -1 & 3 & 0 & -14 \end{pmatrix}$ // RowReduce // MatrixForm

Out[337]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & \frac{142}{11} \\ 0 & 1 & 0 & -\frac{4}{11} \\ 0 & 0 & 1 & -\frac{1}{11} \end{pmatrix}$$

In[338]:= T.{-1, 3, 0}

Out[338]= {5, -9, -14}

In[339]:= $\begin{pmatrix} 1 & 0 & -1 & 5 \\ 0 & 2 & 3 & -9 \\ -1 & 3 & 0 & -14 \end{pmatrix}$ // RowReduce // MatrixForm

Out[339]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & \frac{46}{11} \\ 0 & 1 & 0 & -\frac{36}{11} \\ 0 & 0 & 1 & -\frac{9}{11} \end{pmatrix}$$

In[341]:= B // MatrixForm

Out[341]//MatrixForm=

$$\begin{pmatrix} -\frac{20}{11} & \frac{142}{11} & \frac{46}{11} \\ \frac{8}{11} & -\frac{4}{11} & -\frac{36}{11} \\ \frac{2}{11} & -\frac{1}{11} & -\frac{9}{11} \end{pmatrix}$$

iii. Find the change-of-basis matrix P from S to E

$$P = [\ [S_1]_E \ |\ [S_2]_E \ |\ [S_3]_E\ ]$$

$$P = S = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ -1 & 3 & 0 \end{pmatrix}$$

iv. Show that $B = P^{-1}AP$. Explain the mechanics behind this relationship

In[348]:= Inverse[P].A.P // MatrixForm

Out[348]//MatrixForm=

$$\begin{pmatrix} -\frac{20}{11} & \frac{142}{11} & \frac{46}{11} \\ \frac{8}{11} & -\frac{4}{11} & -\frac{36}{11} \\ \frac{2}{11} & -\frac{1}{11} & -\frac{9}{11} \end{pmatrix}$$

In[350]:= B == Inverse[P].A.P

Out[350]= True

Explain the mechanism:

We first need to differentiate between a transformation matrix and a basis. We know that a transformation from E to S requires us to multiply by the Inverse of S but when going backward (S to E) then we need to multiply by S.

The transformation T is expressed in E by the matrix A (which happens to be T because we're in the standard basis). When looking for T in the basis of T, then we need to transform the columns of S using T[E], then find the representation of $S[T(v_i)]_S$ which means the representation of the vectors of T in S.

3. Circuit training (#transformations, #vectors)

Consider the following circuit with two components: a 9V battery and a $3\,\Omega$ resistor
connected by wires. We are interested in modeling this circuit as a graph, with the corresponding incidence matrix. Consider the four corners of the square below to be the nodes (see Figure 1). We need to decide on an orientation for this circuit, so let's take it to be connected clockwise: the top-left corner has an edge towards the top-right corner, which has an edge toward the bottom-right corner and so on.

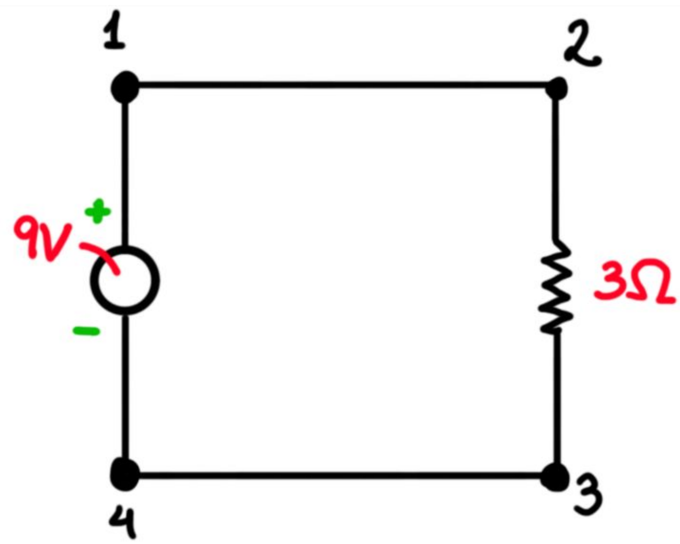


Figure 1: The circuit for Parts (a) to (e).

(a) Write the incidence matrix A that represents the graph in Figure 1. What is the condition for a vector b to be in the column space of A? If we take the potential at the *i-th* node to be x_i, what is the meaning of this condition in terms of potential differences?

The network has 4 nodes and 4 segments → the 4×4 matrix

Column space is the possible potential drops between two nodes.

The incidence matrix A is:

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

For a given vector v to be in the column space of A, v needs to be a combination of the vectors that construct the basis of A:

In the following example, the 4th column of A is dependent on the three first columns; However, we can still express v in terms of the four vectors of A to conserve the structure of the output:

```
In[359]:= V = (k1 * {-1, 0, 0, 1} + k2 * {1, -1, 0, 0} + k3 * {0, 1, -1, 0} + k4 * {0, 0, 1, -1}) //
    MatrixForm
```

$$V = \begin{pmatrix} -k1 + k2 \\ -k2 + k3 \\ -k3 + k4 \\ k1 - k4 \end{pmatrix}$$

We notice that the vector v has its component in terms of the constants and if we add them up, the result is going to be 0.

$-k_1 + k_2 - k_2 + k_3 - k_3 + k_4 + k_1 - k_4 = 0$

And since the basis of the matrix A is the standard basis:

```
In[353]:= A // RowReduce // MatrixForm
```

$$\text{In[361]:=} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ // RowReduce // MatrixForm}$$

Out[361]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then the condition for any given vector to be in the column space of A is to have their component add up to zero.

The meaning of the condition is that the potential difference between two given nodes have to be 0 given that there's no resistance R between the nodes.

(b) We need to model what happens as we pass through each component of the circuit: as we pass through a battery from 'minus' to 'plus' the potential increases by its voltage, in this case, $+9$V. As we pass through wires (straight lines above), there is no voltage change. As we pass through a resistor (on the right), its potential drops. Based on your answers from part (a), what can we say about the potential drop across the resistor?

In[363]:= $$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} . \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} \text{ // MatrixForm}$$

$$b = \begin{pmatrix} -x1 + x2 \\ -x2 + x3 \\ -x3 + x4 \\ x1 - x4 \end{pmatrix}$$

The potential drop occurs when we cross a resistor, otherwise, the potential difference is 0. Hence, according to the matrix A:

$-x_1 + x_2 = 0$

$-x_2 + x_3 = \; -9$

$-x_3 + x_4 = 0$

$x_1 - x_4 = 9$

Therefore, after passing the resistor the potential drops to 0 from 9

(c) Consider the system $A^T \, y \; = \; u$, where y_j is the current through the *j-th* edge. In a regular electrical circuit, there is no current accumulating at any of the nodes, meaning that there are no sources/sinks of current at the nodes. What does this say about the right-hand side u that best models this circuit? Which fundamental subspace represents this?

```
In[365]:= Transpose[A] . (y1, y2, y3, y4) // MatrixForm
```

$$u \begin{pmatrix} -y1 + y4 \\ y1 - y2 \\ y2 - y3 \\ y3 - y4 \end{pmatrix}$$

If $y_1 = y_2 = y_3 = y_4 = I$ $\rightarrow$ then the vector u is the zero vector $\{0, 0, 0, 0\}$

Assuming that the current flows constantly through the matrix, then the vector y would be the **left-nullspace of A**.

(d) Find a basis for the fundamental subspace you found in part (d). What are the resulting constraints on the currents y_j?

```
In[367]:= Transpose[A] // RowReduce // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

According to the results, we have a free variable in the fourth column that we can denote f. The vector y would have the component $y_4 = f$ and so is the case for the other y_i as we have a standard basis.

$$\begin{pmatrix} y1 \\ y2 \\ y3 \\ y4 \end{pmatrix} = \begin{pmatrix} f \\ f \\ f \\ f \end{pmatrix} = f \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

This suggests that the current has to be constant throughout the circuit which aligns with the theory since we have a single enclosed loop without any leaks.

(e) The last piece of the puzzle: Ohm's law gives us a relation between the voltage drop V, current I and resistance R, stating that: $V = IR$. Use this formula and your results from previous parts to determine the currents y_j at each of the edges.

$$I = \frac{V}{R} = \frac{9}{3} = 3\ A$$

The current at each node is equal to 3 A

(f) Use the framework developed above to determine the potential drops b_i and the currents y_j in all nodes and edges found in the circuit below:

We can consider the circuit to be a system of water canals, the source is a water pump, and the resistors are symbolized by pipes (the tightest the pipe, the higher the resistance for water to go through). The amount of water flowing around the system is the same, however, the intensity of the flow depends on who wide the pipes are (same with the current, the flow changes dependent on the resistant but the potential drop is the same regardless of the resistor capacity).

(* The incidence matrix of the circuit*)

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

Finding the potential drops b_i

$$A\ .\ x_i = b_i$$

In[23]:= $\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} . \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \\ x5 \\ x6 \end{pmatrix}$ // MatrixForm

Out[23]//MatrixForm=

$$\begin{pmatrix} -x1 + x2 \\ -x2 + x3 \\ -x3 + x4 \\ x1 - x4 \\ -x3 + x5 \\ -x5 + x6 \\ x4 - x6 \end{pmatrix} \qquad \begin{pmatrix} b1 \\ b2 \\ b3 \\ b4 \\ b5 \\ b6 \\ b7 \end{pmatrix}$$

We know (from the circuit) that the potential drops at the following points are:

The one containing a source

$b_2 = - x_2 + x_3 = 6$ V

$b_7 = x_4 - x_6 = 1$ V

The ones that have no source and no resistance

$b_4 = x_1 - x_4 = 0$ V

$b_5 = - x_3 + x_5 = 0$ V

The ones containing resistors (it depends on the current circulating in the edge)

$b_1 = - x_1 + x_2$

$b_3 = - x_3 + x_4$

$b_6 = - x_5 + x_6$

We can write the potentials x_i in terms of x1 and x2

x3 $\rightarrow$ 6 + x2

x4 $\rightarrow$ x1

x5 $\rightarrow$ 6 + x2

x6 $\rightarrow$ -1 + x1

b1 + b6 = -7

b1 + b3 = -6

Computing the current flow in the circuit:

In[1]:= $\texttt{Transpose}\left[\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}\right]$ // RowReduce // MatrixForm

Out[1]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The Matrix A^T has free variables at the non-pivot column 4, and 7. We can write the vector y_i as:

$$y = \begin{pmatrix} c1 \\ c1 \\ c1 - c2 \\ c1 \\ c2 \\ c2 \\ c2 \end{pmatrix} = c1 * \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c2 * \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

This suggests that there are three different currents flow in the circuit:

c1 , **c2** , and **c1-c2**

Let the upper loop be L1, lower loop L2, and L3 for the overall loop.
We know that the potential drops have to sum up to 0V in a closed loop, so we can write the system of equation to solve for the currents c1, c2

b1 = 3 c1

b3 = 2 (c1 - c2)

b6 = c2

b1 + b6 = -7

b1 + b3 = -6

c1 - c2 = c3

3 c2 + 5 (c3) = 6

4 (c2)+ 3 c3 = 7

In[24]:= $\text{Solve}\left[\begin{pmatrix} 5 & 3 \\ 3 & 4 \end{pmatrix}.\{cc3,\ cc2\} == \begin{pmatrix} 6 \\ 7 \end{pmatrix},\ \{cc3,\ cc2\}\right]$

Out[24]= $\left\{\left\{cc3 \to \frac{3}{11},\ cc2 \to \frac{17}{11}\right\}\right\}$

Furthermore, the current flowing in c1 would be the sum of c2 and c3

$$c_1 = \frac{20}{11}A \qquad c_2 = \frac{17}{11}A \qquad c_3 = \frac{3}{11}A$$

The potential drops are:

$$b_1 = \frac{20}{11} \cdot 3 = \frac{60}{11}V \qquad b_3 = \frac{17}{11} \cdot 2 = \frac{34}{11}V \qquad b_6 = \frac{3}{11} \cdot 1 = \frac{3}{11}V$$

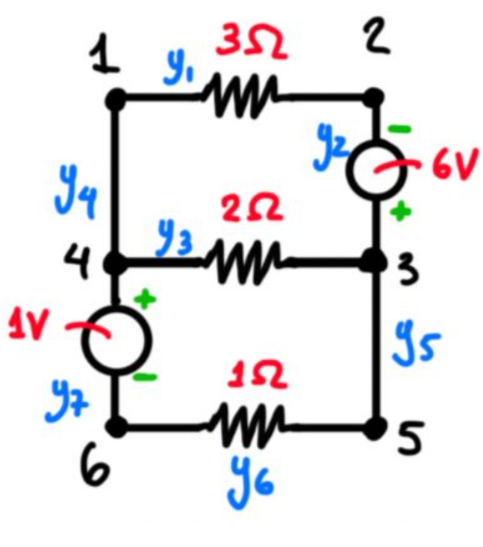


Figure 2: The circuit for Part (f).

Appendix:

#deduction: Solving the problem 2 requires deductive reasoning to disentangle the relationship between the form of the transformation matrix A and the regions that represent the unit square. I think I defined clearly the form of matrices that would produce a square, a segment, and an area of A using the givens provided.

#analogy: I used the analogy of the flow of water in a closed system to determine the validity of the circuit analysis in part (a) and (f). The similarity is that the current in a closed one loop is the same and it's proportional to the resistors that are in the loop (assuming that they're attached in series). The potential drops also sum up to 0.