CS164 9.1 PCW

Applications of the KKT conditions

Pre-Class Work

In this exercise, we will show how the KKT conditions can be used directly to find a globally optimal solutions to a simple optimization problem.
Consider a problem of the form:

$\max_{x,y} f(x, y) = \text{xy}$

subject to :

$x + y^2 \leqslant 2$

$x, y \geqslant 0$

(1) The cost function in this case is continuous and bounded over a compact region, so that we know that there exists a globally optimal solution. Even though the objective is not strictly concave, the KKT conditions can still be applied in this case, but are only necessary and not sufficient for a global optimum. Hence, there may be multiple points that satisfy all of the KKT conditions. Write out the KKT conditions for this function by defining an appropriate number of Lagrange multipliers.

In[3]:=
```
objective[x_, y_] := x * y
constraint1[x_, y_] := x + y^2 - 2
constraint2[x_, y_] := -x
constraint3[x_, y_] := -y
```

In[7]:=
```
Lagrange[x_, y_, λ1_, λ2_, λ3_] :=
 objective[x, y] + λ1 (constraint1[x, y]) + λ2 (constraint2[x, y]) + λ3 (constraint3[x, y])
D[Lagrange[x, y, λ1, λ2, λ3], x]
D[Lagrange[x, y, λ1, λ2, λ3], y]
D[Lagrange[x, y, λ1, λ2, λ3], λ1]
D[Lagrange[x, y, λ1, λ2, λ3], λ2]
D[Lagrange[x, y, λ1, λ2, λ3], λ3]
```

Out[8]= $y + \lambda 1 - \lambda 2$

Out[9]= $x + 2\, y\, \lambda 1 - \lambda 3$

Out[10]= $-2 + x + y^2$

Out[11]= $-x$

Out[12]= $-y$

We assume that the constraints are not active by setting $\lambda i = 0$

In[∘]:=
```
D[Lagrange[x, y, 0, 0, 0], x]
D[Lagrange[x, y, 0, 0, 0], y]
```

Out[∘]= y

Out[∘]= x

(2) Try various combinations of the multipliers being nonzero and solve for the corresponding x and y. HINT: all constraints cannot be active simultaneously.

We ignored the constraint that x and y have to be positive which then yielded two solutions for the system of equations

$\lambda 1 = 0 \mid \lambda 2 > 0 \mid \lambda 3 > 0$

In[∘]:=
```
Solve[D[Lagrange[x, y, 0, λ2, λ3]] == 0, {x, y, λ1, λ2, λ3}]
```

Out[∘]= $\left\{\left\{\lambda 3 \to \frac{x\,(y - \lambda 2)}{y}\right\}, \{x \to 0, y \to 0\}, \{y \to 0, \lambda 2 \to 0\}\right\}$

$\lambda 1 > 0 \mid \lambda 2 = 0 \mid \lambda 3 > 0$

In[∘]:=
```
Solve[D[Lagrange[x, y, λ1, 0, λ3]] == 0, {x, y, λ1, λ2, λ3}]
```

Out[∘]= $\left\{\left\{\lambda 3 \to \frac{x\,y - 2\,\lambda 1 + x\,\lambda 1 + y^2\,\lambda 1}{y}\right\}, \{x \to 2, y \to 0\}, \{y \to 0, \lambda 1 \to 0\}\right\}$

$\lambda 1 > 0 \mid \lambda 2 > 0 \mid \lambda 3 = 0$

In[∘]:=
```
Solve[D[Lagrange[x, y, λ1, λ2, 0]] == 0, {x, y, λ1, λ2, λ3}]
```

Out[∘]= $\left\{\left\{\lambda 2 \to \frac{x\,y - 2\,\lambda 1 + x\,\lambda 1 + y^2\,\lambda 1}{x}\right\}, \{x \to 0, \lambda 1 \to 0\}, \left\{x \to 0, y \to -\sqrt{2}\right\}, \left\{x \to 0, y \to \sqrt{2}\right\}\right\}$

$\lambda 1 = 0 \mid \lambda 2 = 0 \mid \lambda 3 > 0$

In[∘]:=
```
Solve[D[Lagrange[x, y, 0, 0, λ3]] == 0, {x, y, λ1, λ2, λ3}]
```

Out[∘]= $\{\{y \to 0\}, \{\lambda 3 \to x\}\}$

$\lambda 1 = 0 \mid \lambda 2 > 0 \mid \lambda 3 = 0$

In[∘]:=
```
Solve[D[Lagrange[x, y, 0, λ2, 0]] == 0, {x, y, λ1, λ2, λ3}]
```

Out[∘]= $\{\{x \to 0\}, \{\lambda 2 \to y\}\}$

$\lambda 1 > 0 \mid \lambda 2 = 0 \mid \lambda 3 > 0$

In[∘]:=
```
Solve[D[Lagrange[x, y, λ1, 0, λ3]] == 0, {x, y, λ1, λ2, λ3}]
```

Out[∘]= $\left\{\left\{\lambda 3 \to \frac{x\,y - 2\,\lambda 1 + x\,\lambda 1 + y^2\,\lambda 1}{y}\right\}, \{x \to 2, y \to 0\}, \{y \to 0, \lambda 1 \to 0\}\right\}$

$\lambda 1 > 0 \mid \lambda 2 = 0 \mid \lambda 3 = 0$

In[∘]:=
```
Solve[D[Lagrange[x, y, λ1, 0, 0]] == 0, {x, y, λ1, λ2, λ3}]
```

Out[∘]= $\left\{\left\{x \to \frac{2\,\lambda 1 - y^2\,\lambda 1}{y + \lambda 1}\right\}, \{y \to 0, \lambda 1 \to 0\}, \left\{y \to -\sqrt{2}, \lambda 1 \to \sqrt{2}\right\}, \left\{y \to \sqrt{2}, \lambda 1 \to -\sqrt{2}\right\}\right\}$

(3) State the globally optimal solution to the optimization problem by checking the objective function value at all of the points that satisfy the KKT conditions.

As mentioned in part two, there are only two cases where the solution is feasible. However, the solution with a negative value for y doesn't satisfy the constraint that it has to be positive. Therefore, the system has a single solution:

In[17]:=
```
NMaximize[{objective[x, y], constraint1[x, y] ≤ 0,
  constraint2[x, y] ≤ 0, constraint3[x, y] ≤ 0}, {x, y}]
```

Out[17]=
```
{1.08866, {x → 1.33333, y → 0.816497}}
```