Constrained Optimization

CS164 - Spring 2020

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Contents

1 Problem description

We are given a dataset of N points in R^2, namely $(x_1, y_1), (x_2, y_2), ..., (x_N, y_N)$. This assignment will look at finding lines of best fit for these points by solving various optimization problems. We seek to find a line of the form:

$$y = \theta_1 \times x + \theta_2$$

that best fits our given data. Defining a parameter vector:

$$\Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

and the data vectors by:

$$X = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_N & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_N \end{bmatrix}$$

the fit discrepancy between the line and the data can be expressed compactly by the vector:

$$Y - X\Theta$$

The classic method to fit this data is by using a least-squares estimator, where we compute the parameters to solve

$$\min \parallel Y - X\Theta \parallel_2$$

In this case, an analytical solution can be found, since the optimization problem is unconstrained:

$$\Theta^* = \left(X^T X \right)^{-1} X^T Y$$

However, the least-squares estimator can be influenced unduly by outlying datapoints. We can also consider two other measures of discrepancy that may provide a more robust fit. The L_1-norm cost:

$$\min \parallel Y - X\Theta \parallel_1$$

penalizes the total absolute value of the deviation of all datapoints from the line of best fit, whereas the L_∞-norm cost:

$$\min \parallel Y - X\Theta \parallel_\infty$$

penalizes the maximum deviation over all datapoints. Both of these problems can be expressed as linear programs. In this assignment, we will first investigate the nature of solutions to LPs in general, and then solve these two minimization problems for a synthetic dataset.

2 Solution

A linear program can be written in the form:

objective:

$$\min \left[c^T x \right]$$

subject to:

$$Ax \leq b \quad \rightarrow \quad Ax - b \leq 0$$

for matrices $A \in R^{m \times n}$, $b \in R^{m \times 1}$ and $c \in R^{n \times 1}$

Given a vector $\lambda \in R^{n \times 1}$, the Langragian of the LP is written as:

$$\mathcal{L}(x, \lambda) = c^T x + \lambda^T (Ax - b)$$

For x^* to be a local minimum there exists a unique λ^* s.t., the KKT conditions are:

- Stationarity: $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0 \;\rightarrow\; \nabla c^T x^* + (\lambda^*)^T \nabla (Ax^* - b) = 0$
- Complemantary slackness: $(\lambda^*)^T (Ax^* - b) = 0$
- Primal feasibility: $Ax^* - b \leq 0$
- Dual feasibility: $\lambda^* \geq 0$

Complimentary slackness condition presents two cases:

- $\lambda^* > 0 \rightarrow$ the optimum would lie on the border of the feasible set since the constraint is active

  $$Ax^* - b = 0 \;\rightarrow\; Ax^* = b$$

- $\lambda^* = 0 \rightarrow$ which implies that the condition is inactive at the optimum. In such a case, the stationarity condition would be written as:

  $$\nabla c^T x^* = 0 \;\rightarrow\; c^T = 0$$

  The problem is equivalent to an unconstrained optimization and the optimum lies interior of the feasible set. In general, constraints are supposed to push the optimum away from the optimal value of the unconstrained objective function. Therefore, we would expect $\lambda^* > 0$

- L_1 norm: corresponding to the Manhattan distance which translate the problem into minimizing the sum of the absolute values as follow:

$$\min \parallel Y - X\Theta \parallel_1 \rightarrow \min \sum_{i=1}^{N} \mid Y_i - X_i\Theta \mid$$

  Introducing a slack variable will serve as a boundary for the absolute value. The problem then can be rewritten as:

$$\forall s_i \geq 0 : \min \sum_{i=1}^{N} s_i$$

$$\text{s.t} \quad -s_i \leq \mid Y_i - X_i\Theta \mid \leq s_i$$

  The LP form would be:

  objective

$$\min \ 1^T s_i$$

  subject to

$$X_i\Theta - Y_i - s_i \leq 0$$

$$Y_i - X_i\Theta - s_i \leq 0$$

  The smaller the value of the slack variable s_i, the tightest the interval between $-s_i$ and s_i which would then minimize the value $\mid Y\_i - X\_i\Theta \mid$. The problem still an LP form but with two inequality constraints.

- L_∞ norm: corresponds to finding the maximum of the absolute values of $\mid Y - X\Theta \mid$ such as:

$$\min \parallel Y - X\Theta \parallel_\infty \iff \max \{\mid Y_1 - X_1\Theta \mid, \mid Y_2 - X_2\Theta \mid, .., \mid Y_N - X_N\Theta \mid\}$$

  Using a variable to bound the greatest absolute value would transform the problem to be written as:

$$\min \ s$$

$$\text{s.t} \ \ \forall i \quad -1^T s \leq \mid Y_i - X_i\Theta \mid \leq 1^T s$$

  The LP form would be:

  objective

$$\min \ s$$

  subject to

$$X_i\Theta - Y_i - 1^T s \leq 0$$

$$Y_i - X_i\Theta - 1^T s \leq 0$$

3 Regression L_1 and L_∞ using CVXPY

Some code is given below to generate a synthetic dataset. Using CVX, solve two linear programs for computing the regression line for l_1 and l_∞ regression. Plot the lines over the data to evaluate the fit.

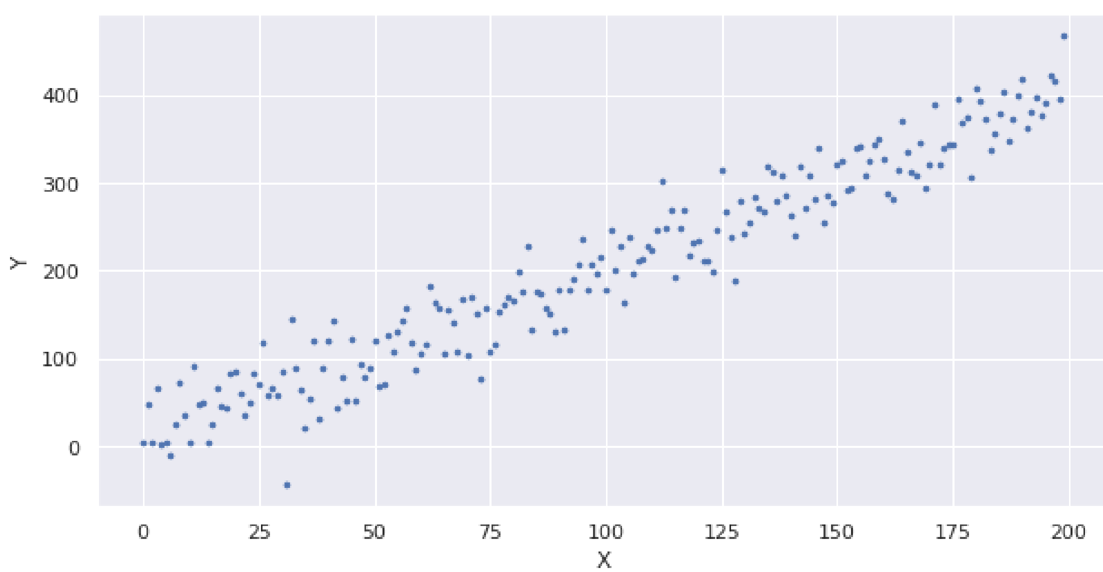
In [2]:
```
# L_1 and L_∞ regression using cvxpy
import numpy as np
import cvxpy as cvx
import seaborn as sns; sns.set()
import matplotlib.pyplot as plt

# Actual parameter values
theta1_act = 2
theta2_act = 5

# Number of points in dataset
N = 200
# Noise magnitude
mag = 30

# Datapoints
x = np.arange(0, N)
y = theta1_act * x + theta2_act *np.ones([1, N]) + np.random.normal(0, mag, N)

# Scatter plot of data
plt.figure(figsize=(10, 5))
plt.scatter(x, y, s=7); plt.xlabel('X'); plt.ylabel('Y')
plt.show()
```



```
In [0]: def plot_fit(x, y, theta, norm):
            plt.figure(figsize=(10, 5))
            slope, intercept = theta.value[0][0], theta.value[1][0]
            line = lambda k: k * slope + intercept
            plt.title('Least-error line with {}-norm'.format(norm), fontsize=20)
            plt.scatter(x, y, s=7); plt.xlabel('X'); plt.ylabel('Y')
            plt.plot(x, line(x), color='red', linewidth=.5)
```

3.1 L_1 norm

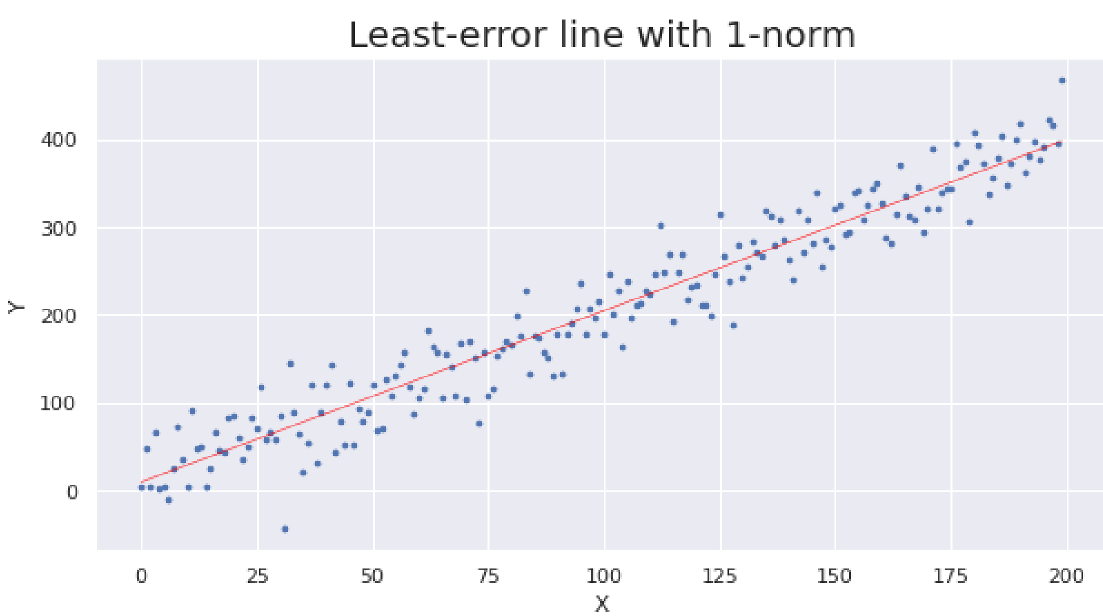
```
In [4]: theta_1 = cvx.Variable(shape=(2, 1))

        Y = y.T
        X = np.column_stack((x, np.ones(N)))
        S = cvx.Variable(shape=(N, 1))

        objective_1 = cvx.Minimize(cvx.atoms.norm1(S))
        constraints = [Y - X @ theta_1 >= -S, Y - X @ theta_1 <= S]
        problem_1 = cvx.Problem(objective_1, constraints)
        optimal_1 = problem_1.solve(verbose=False) # True
```

```
Optimal value:  4820.12001
Line slope: 1.95101
Intercept: 9.13685
```

```
In [5]: plot_fit(x, y, theta_1, 1)
```



3.2 L_∞ norm

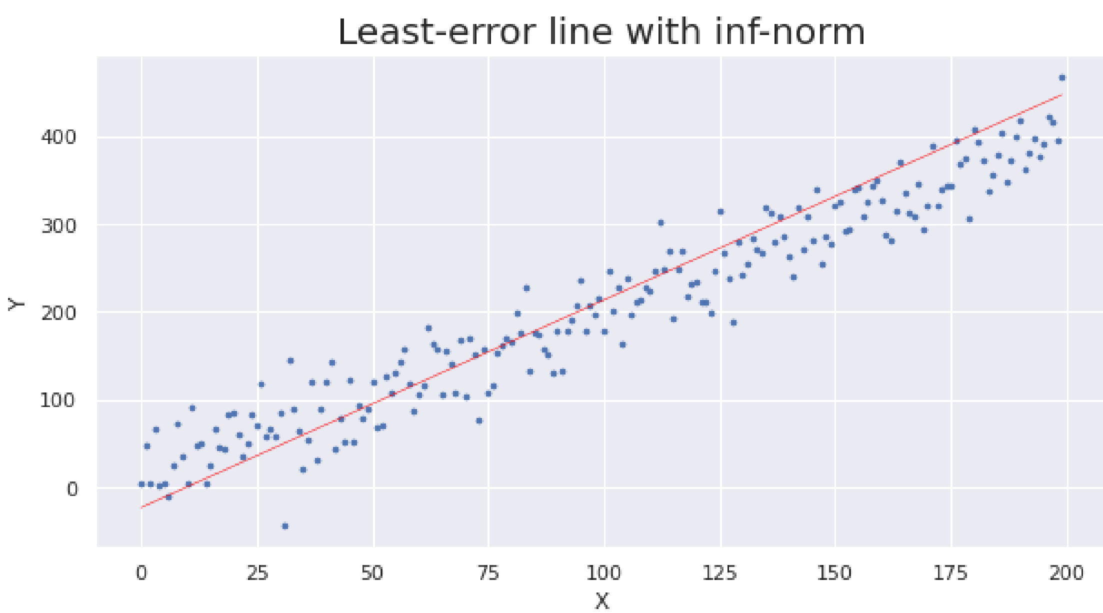
```
In [6]: theta_inf = cvx.Variable(shape=(2, 1))

        Y = y.T
        X = np.column_stack((x, np.ones(N)))

        objective_inf = cvx.Minimize(cvx.atoms.norm_inf(Y - X * theta_inf))
        problem_inf = cvx.Problem(objective_inf)
        optimal_inf = problem_inf.solve()
```

```
Optimal value:  93.28316
Line slope: 2.36313
Intercept: -23.34792
```

```
In [7]: plot_fit(x, y, theta_inf, np.inf)
```



3.3 P-norm

In [0]:
```
# what if we capture the optimal value as we change the norm [Minkowski distance]
def norm_regression(N, x_values, y_values, norm):
    theta = cvx.Variable(shape=(2, 1))
    Y = y_values.T
    X = np.column_stack((x_values, np.ones(N)))
    S = cvx.Variable(shape=(N, 1))

    if norm==1: objective = cvx.Minimize(cvx.atoms.norm1(S))
    else: objective = cvx.Minimize(cvx.atoms.Pnorm(S, p=norm))
    constraints = [Y - X @ theta >= -S, Y - X @ theta <= S]
    problem = cvx.Problem(objective, constraints)

    return problem.solve(), theta.value[0][0], theta.value[1][0]
```

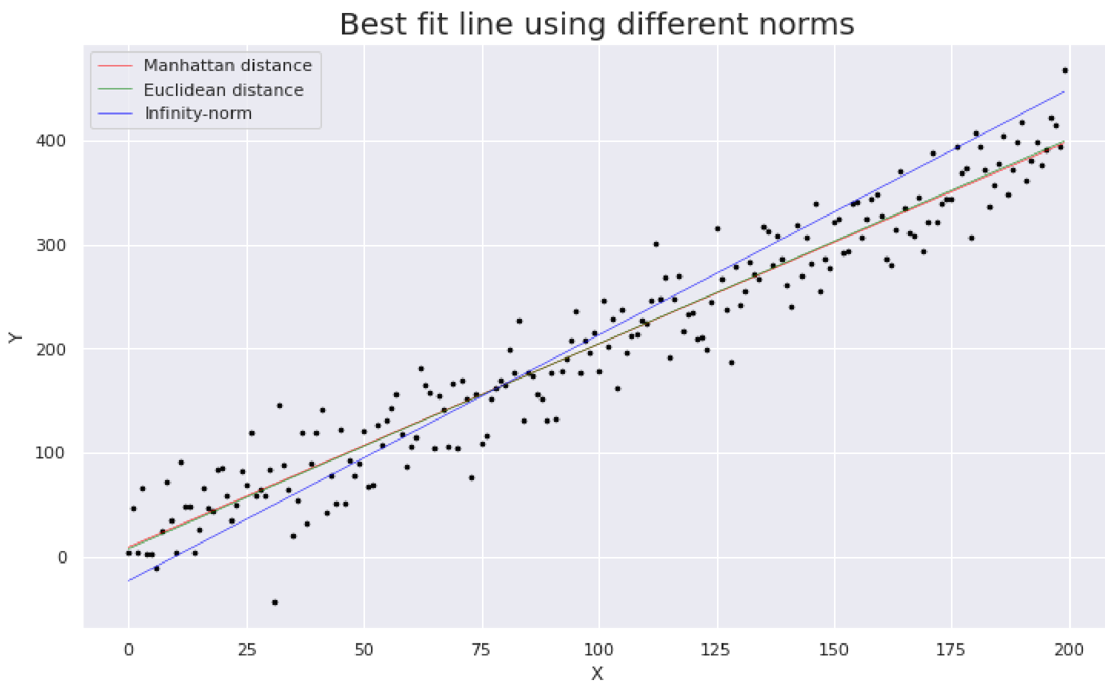
In [0]:
```
norm_optimality = []
for _ in range(1, 21):
    norm_optimality.append(norm_regression(200, x, y, _))
```

In [10]:
```
import pandas as pd
norm_optimality = pd.DataFrame(norm_optimality,
columns=['Optimal value', 'Slope', 'Intercept'])
norm_optimality.index += 1
norm_optimality.head(10)
```

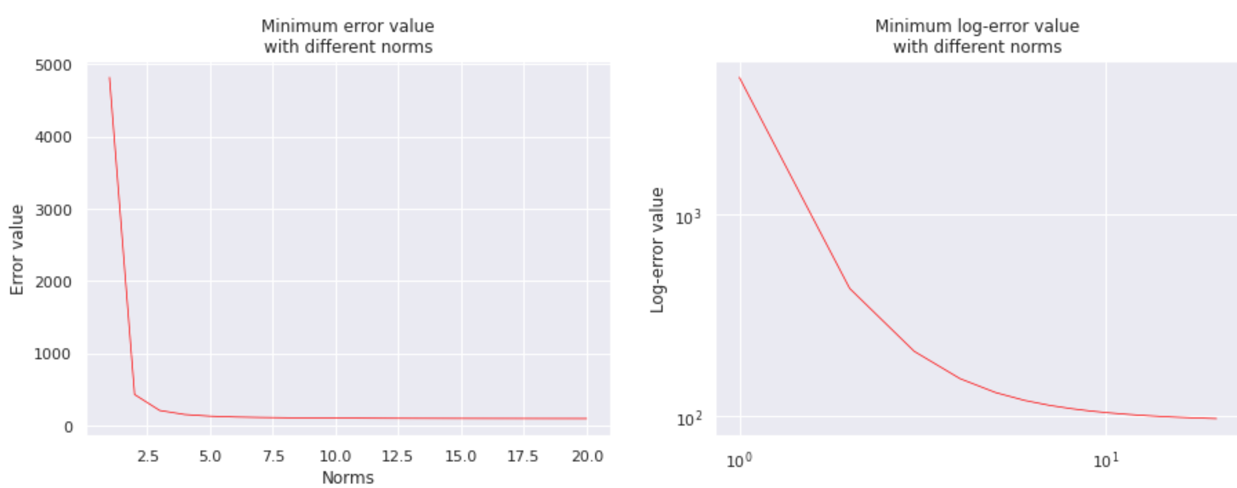
Out[10]:

	Optimal value	Slope	Intercept
1	4820.120007	1.951006	9.136853
2	430.561084	1.968524	7.527359
3	209.677544	1.980097	5.906714
4	153.390761	1.999338	3.172261
5	130.849820	2.024193	-0.241882
6	119.606653	2.049113	-3.609517
7	113.124618	2.070217	-6.442469
8	108.984011	2.086679	-8.648076
9	106.142374	2.099135	-10.321503
10	104.090373	2.108482	-11.588568

In [13]:
```
# Plotting the 1-norm, 2-norm, and infinity-norm
plt.figure(figsize=(12, 7))
plt.scatter(x, y, s=7, color='black')
line_1 = lambda k: k * norm_optimality['Slope'][1] + norm_optimality['Intercept'][1]
line_2 = lambda k: k * norm_optimality['Slope'][2] + norm_optimality['Intercept'][2]
line_inf = lambda k: k * theta_inf.value[0][0] + theta_inf.value[1][0]
plt.plot(x, line_1(x), color='red', linewidth=.5, label='Manhattan distance')
plt.plot(x, line_2(x), color='green', linewidth=.5, label='Euclidean distance')
plt.plot(x, line_inf(x), color='blue', linewidth=.5, label='Infinity-norm')
plt.legend(); plt.xlabel('X'); plt.ylabel('Y')
plt.title('Best fit line using different norms', fontsize=20)
plt.show()
```




```
In [12]: plt.figure(figsize=(15, 5))
         plt.subplot(1,2,1)
         plt.title('Minimum error value\nwith different norms')
         plt.xlabel('Norms'); plt.ylabel('Error value')
         plt.plot(list(range(1, 21)), norm_optimality['Optimal value'],
                  linewidth=.7, color='red')
         plt.subplot(1,2,2)
         plt.title('Minimum log-error value\nwith different norms')
         plt.ylabel('Log-error value')
         plt.loglog(list(range(1, 21)), norm_optimality['Optimal value'],
                    linewidth=.7, color='red')
         plt.show()
```



3.4 References

- Boyd, S., & Vandenberghe, L. (2015). Convex optimization. Cambridge: Cambridge University Press. Retrieved from: https://course-resources.minerva.kgi.edu/uploaded_files/mke/YDzxkr/boyd2004-cvx.pdf
- Calafiore, G., & Ghaoui, L. E. (2014).Optimzation models. Cambridge: Cambridge. Retrieved from: https://vel.life/Optimization.Models.pdf