Lines

Slope-Intercept Form Linear Equations

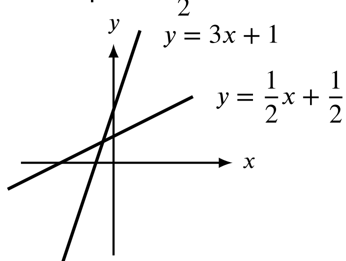
- Slope-Intercept Form Linear Equation:

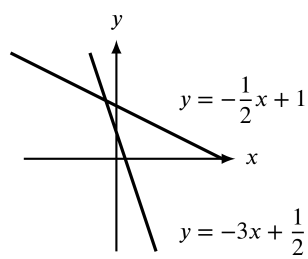
$$y = mx + b$$

- Total Amount $=$ (Amount per Thing $\times$ Number of Things) $+$ Starting Amount
  Example: Jabrill has \$5 (starting amount) and earns \$4 per shirt sold $\Rightarrow \quad j = 4s + 5$
- The four components of a linear equation in the form $y = mx + b$ are the y-value, the x-value, the coefficient m (slope), and the constant b (y-intercept). We can solve for any one of these when we know the other three.
- Slope Formula:

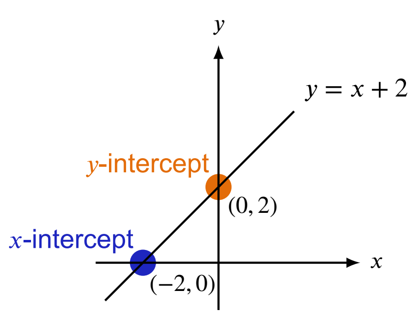
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

- A positive slope means that the y-value increases as the x-value increases. Steeper lines have higher slopes because the y-value goes up more as the x-value increases. A line with a slope of 3 is steeper than a line with a slope of $\frac{1}{2}$.
  A negative slope means that the y-value decreases as the x-value increases. Steeper lines have lower (more negative) slopes because the y-value decreases more as the x-value increases. A line with a slope of -3 is steeper than a line with a slope of $-\frac{1}{2}$.





- Horizontal lines have a slope of zero and their equations are of the form $y = c$, where c is a constant (the y-value is the same for all points on the line).
  Vertical lines have an undefined slope and their equations are of the form $x = c$, where c is constant (the x-value is the same for all points on the line).
- The y-intercept is the starting amount or the value of y when the x-value is 0.
  The x-intercept is the value of x when the y-value is equal to 0.



- You can find the equation of any line as long as you know any two points on the line or know the slope and any one point. If you are given the slope and y-intercept, then you already have everything you need to make the equation.

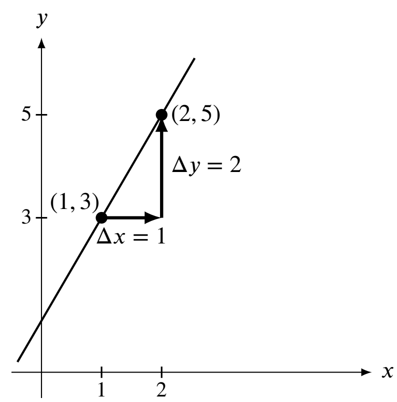
  Step 1: Use the Slope Formula to find the slope from two points: $(1, 3)$ and $(2, 5)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{2-1} = \frac{2}{1} = 2$$

  Step 2: Plug the slope and a point into the Slope-Intercept Form to find the y-intercept: $(1, 3)$ and $m = 2$

$$y = mx + b \quad \Rightarrow \quad 3 = 2(1) + b \quad \Rightarrow \quad 3 = 2 + b \quad \Rightarrow \quad 1 = b \quad \Rightarrow \quad \boxed{y = 2x + 1}$$

- Using up-and-over arrows with graphical representations to find Δy and Δx, or counting differences between coordinate values between rows of tables, is often less error prone than substituting the coordinates of points into the long form of the Slope Formula.



$\Delta x = 1$

x	y
1	3
2	5
3	7
4	9

$\Delta y = 2$

- When asked to interpret the meaning of a coefficient or constant in a Slope-Intercept Form linear equation, first determine which one of the four components of a linear equation it is (the y-value, the slope, the x-value, or the y-intercept).

  The y-intercept is the "starting amount" and is measured in the same units as the y-value.

  The slope is the rate of change of the y-value as the x-value changes and is measured in the units of the y-value divided by the units of the x-value (look for keywords like "per" which indicate division).

Standard Form Linear Equations

- Standard Form Linear Equation:

$$Ax + By = C$$

- Convert Standard Form to Slope-Intercept Form:

$$y = \frac{-A}{B}x + \frac{C}{B}$$

- Slope of a Line in Standard Form:

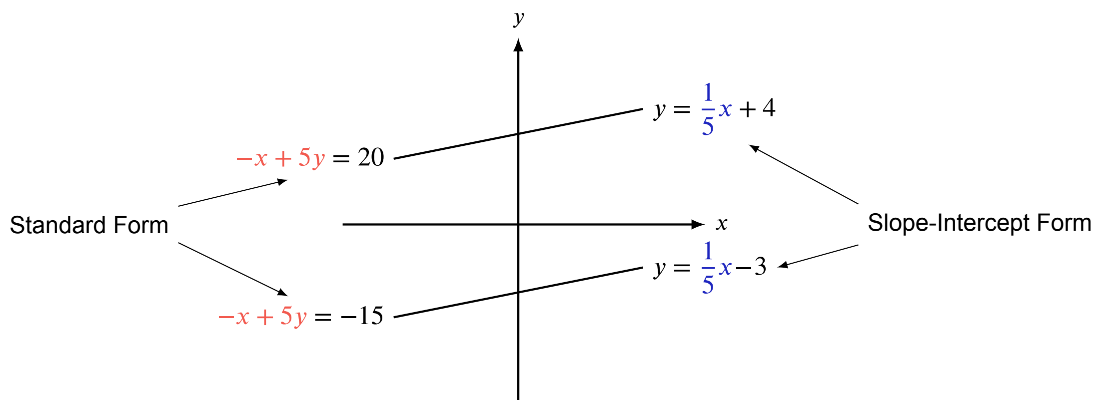
$$\frac{-A}{B}$$

- y-intercept of a Line in Standard Form:

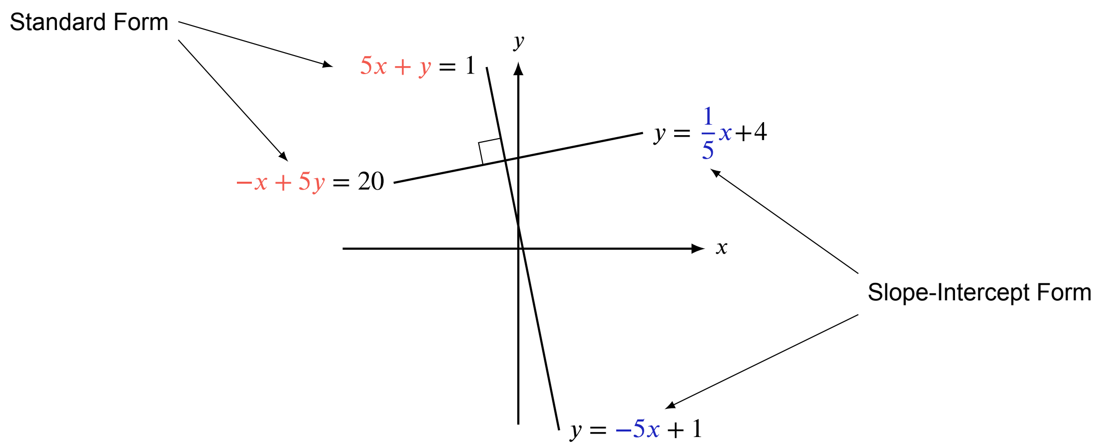
$$\frac{C}{B}$$

Parallel and Perpendicular Lines

- Parallel lines are the same distance apart everywhere and thus they never intersect. Parallel Lines have the same slope but different y-intercepts.
- Parallel lines in Standard Form will have the same x- and y-coefficients (or the ratio of x- and y-coefficients will be the same).



- Perpendicular lines cross at a right angle. The slope of each line in a pair of perpendicular lines is the **negative reciprocal** of the slope of the other line.
- Perpendicular lines in Standard Form will have the x- and y-coefficients swapped, and one of the coefficients will be multiplied by -1 (or the ratio of the x- and y-coefficients will be flipped and multiplied by -1).



Systems of Linear Equations

Substitution

- Solutions to systems of linear equations can be thought of as points where the graphs of the lines intersect. At these intersection points, both lines have the same x- and y- values.

- Set expressions for y equal to each other to find the solution point when both equations are already in Slope-Intercept Form. Use substitution when the equations are in different forms, particularly when one variable is already solved for in terms of the other.

$$\left.\begin{aligned} y &= 2x+1 \\ y &= x+4 \end{aligned}\right\} \Rightarrow \quad 2x+1=-x+4 \quad \Rightarrow \quad 3x=3 \quad \Rightarrow \quad x=1$$

$$\left.\begin{aligned} x+4y &= 20 \\ x &= 4y \end{aligned}\right\} \Rightarrow \quad (4y)+4y=56 \quad \Rightarrow \quad 8y=56 \quad \Rightarrow \quad y=7$$

Elimination

- Use elimination to solve most systems of linear equations, particularly when both equations are in Standard Form. Multiply one or both equations by numbers that will cause the coefficients of the variable you want to eliminate to cancel out when the equations are combined through addition or subtraction.

Solve for f:

$$\left.\begin{aligned} f+s &= 30 \\ 4f+6s &= 140 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} -6(f+s) &= -6(30) \\ 4f+6s &= 140 \end{aligned}\right\} \Rightarrow \begin{array}{rr} & -6f-6s=-180 \\ + & 4f+6s=140 \\ \hline & -2f \quad\; =-40 \end{array} \Rightarrow \quad f=20$$

Solve for x:

$$\left.\begin{aligned} -5x+2y &= 10 \\ 4x+5y &= 25 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} -5(-5x+2y) &= -5(10) \\ 2(4x+5y) &= 2(25) \end{aligned}\right\} \Rightarrow \begin{array}{rr} & 25x-10y=-50 \\ + & 8x+10y=50 \\ \hline & 33x \quad\; =0 \end{array} \Rightarrow \quad x=0$$

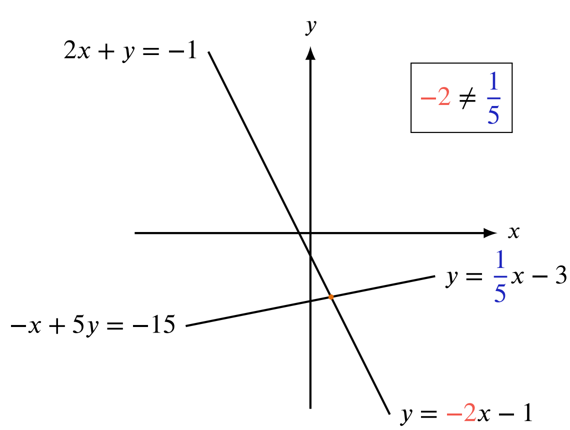
- Use combination without eliminating either variable if possible when the problem asks for an expression involving terms with both variables.

Find $x+y$:

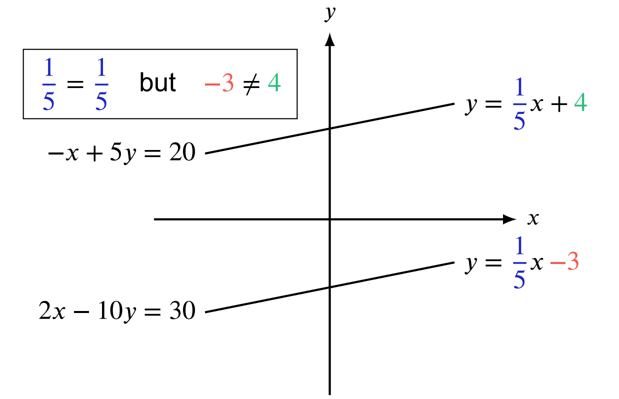
$$\begin{array}{rr} & 2x+3y=100 \\ + & 4x+3y=380 \\ \hline & 6x+6y=480 \end{array} \Rightarrow \quad x+y=80$$

Number of Solutions to Systems of Linear Equations

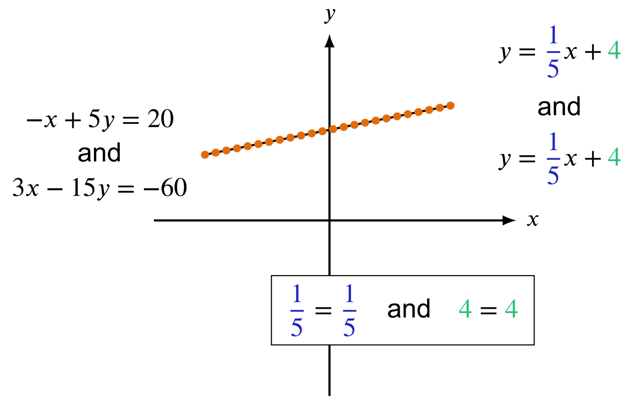
- A system of linear equations has **one solution when the two lines are *not* parallel (have different slopes) and thus cross at one point.**



- A system of linear equations has **no solutions when the lines are parallel** (have the same slopes and different y-intercepts), because the lines do not intersect.



- A system of linear equations has **infinite solutions when the two lines are exactly the same**. The **slopes and y-intercepts must be the same**.



- When asked to find unknown coefficient values in systems of linear equations in Standard Form that have no solutions or infinitely many solutions, use proportions of the ratios of the coefficients and constants to solve for unknown values.

No Solutions: $\begin{matrix} 12x - 4y = 2 \\ ax + 2y = 7 \end{matrix} \Rightarrow \frac{a}{12} = \frac{2}{-4} \Rightarrow a = 12\left(\frac{-1}{2}\right) \Rightarrow a = -6$

∞ Solutions: $\begin{matrix} 4x + 6y = 12 \\ -2x - 3y = b \end{matrix} \Rightarrow \frac{b}{12} = \frac{-3}{6} \Rightarrow b = 12\left(\frac{-1}{2}\right) \Rightarrow b = -6$

Alternatively, multiply one or both equations by scale factors so that any known coefficients or constants in the same position can be matched between the two equations.

No Solutions: $\begin{matrix} 8x - 2y = 1 \\ ax + 3y = 3 \end{matrix} \Rightarrow \begin{matrix} -3(8x - 2y) = -3(1) \\ 2(ax + 3y) = 2(3) \end{matrix} \Rightarrow \begin{matrix} -24x + 6y = -3 \\ 2ax + 6y = 6 \end{matrix} \Rightarrow \begin{matrix} 2a = -24 \\ a = -12 \end{matrix}$

∞ Solutions: $\begin{matrix} 3x - 9y = 12 \\ ax + by = 4 \end{matrix} \Rightarrow \begin{matrix} 3x - 9y = 12 \\ 3(ax + by) = 3(4) \end{matrix} \Rightarrow \begin{matrix} 3x - 9y = 12 \\ 3ax + 3by = 12 \end{matrix} \Rightarrow \begin{matrix} 3a = 3 \\ a = 1 \end{matrix}$ and $\begin{matrix} 3b = -9 \\ b = -3 \end{matrix}$

Linear Inequalities & Absolute Value

Linear Inequalities

- The pointy or small end of the inequality sign is the lesser side of the inequality; the open or big end of the inequality sign is the greater side of the inequality.

Name	Symbol	Usage
Less Than	$<$	The value on the left is less than the value on the right. The inequality $x < 5$ means that x can be any value from $-\infty$ up to, but **not** including, 5.
Less Than Or Equal To	$\leq$	The value on the left is less than or equal to (no more than) the value on the right. For example, $x \leq 5$, means that x can be any value from $-\infty$ up to 5, including 5 itself.
Greater Than	$>$	The value on the left is greater than the value on the right . The inequality $x > 5$ means that x can be any value greater than 5 up to ∞.
Greater Than Or Equal To	$\geq$	The value on the left is greater than or equal to (no less than) the value on the right. For example, $x \geq 5$ means that x can be any value from 5 (including 5 itself) up to ∞.
Absolute Value	$\|a\|$	Produces a non-negative value indicating how far a number is from 0. ($\|-5\| = 5$)

- Solve inequalities like you solve equations, but remember to flip the sign if you multiply or divide by a negative number.

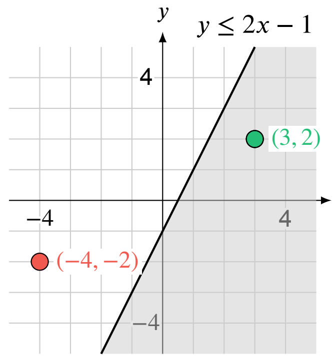
$$-8x + 3 \leq 59 \quad \Rightarrow \quad -8x \leq 56 \quad \Rightarrow \quad x \geq -7$$

- In real world problems, correct answers often have to be integers, particularly if the problem involves items that cannot be divided into fractional amounts (books, shirts, etc.).
- Compound inequalities can be solved in one step as long as you are careful about flipping both inequality signs if you multiply or divide by negative numbers. Just make sure to apply the same operation to all three parts of the compound inequality.

$$2 < -3x + 2 < 8 \quad \Rightarrow \quad 0 < -3x < 6 \quad \Rightarrow \quad \frac{0}{-3} > x > \frac{6}{-3} \quad \Rightarrow \quad 0 > x > -2$$

- When graphing linear inequalities in the xy-plane, use a solid line if there is a $\leq$ or $\geq$ sign in the inequality (because points on the line **are** solutions to the inequality), and use a dashed line if there is a $<$ or $>$ sign in the inequality (because points on the line **are NOT** solutions to the inequality). Shade **above** the line to show the solution region when the y-variable is **greater than** the linear expression. Shade **below** the line to show the solution region when the y-variable is **less than** the linear expression.

  Plug in the x- and y-coordinates of a point to the inequality to see if a point falls in the shaded solution region or is on a solid boundary line.

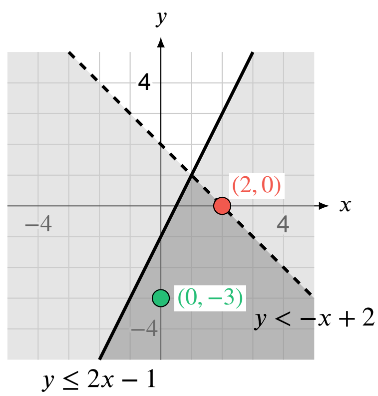


$y \leq 2x - 1$

Test $(3, 2)$		Test $(-4, -2)$
	$2 \leq 2(3) - 1$		$-2 \leq 2(-4) - 1$
	$2 \leq 6 - 1$		$-2 \leq -8 - 1$
✓	$2 < 5$	✗	$-2 \nleq -9$

Systems of Linear Inequalities

- Solutions to a system of linear inequalities must satisfy all the inequalities in the system. If the inequalities are graphed on the same xy-plane, the solutions are located within the overlapping shaded regions and on any solid lines bounding those regions.



$y \leq 2x - 1$
$y < -x + 2$

Test $(0, -3)$

$-3 \leq 2(0) - 1$
$-3 < -(0) + 2$
$\Downarrow$
✓ $\mathbf{-3 \leq -1}$
✓ $\mathbf{-3 < 2}$

Test $(2, 0)$

$0 \leq 2(2) - 1$
$0 < -(2) + 2$
$\Downarrow$
$0 \leq 4 - 1$
$0 < -2 + 2$
$\Downarrow$
✓ $\mathbf{0 \leq 3}$
✗ $\mathbf{0 \nless 0}$

Absolute Value

- You can solve absolute value equations by making two equations: one where the expression in the absolute value bars is set equal to the original value on the other side of the equation, and one where the expression is set equal to the negative of that value. For example, $|x| = 5$ yields both $x = 5$ and $x = -5$.

  Note that while the expression *within* the absolute value bars can have any value, positive or negative, **the absolute value of the expression can never be negative, by definition**. For example, while $|x| = 5$ yields two possible solutions ($x = 5$ and $x = -5$), the equation $|x| = -5$ yields **zero solutions** because the absolute value of an expression can never be negative.

- $|x + a| = b \quad \Rightarrow \quad \begin{cases} x + a = b \\ x + a = -b \end{cases}$

- $|x + a| > b \quad \Rightarrow \quad \begin{cases} x + a > b \\ x + a < -b \end{cases}$

- $|x + a| < b \quad \Rightarrow \quad \begin{cases} x + a < b \\ x + a > -b \end{cases} \quad \Rightarrow \quad -b < x + a < b$

Exponents

Exponent Rules

- Exponential terms consist of a base a and exponent b and are in the form a^b.
- Exponents tell you how many factors of the base should be multiplied by each other. For example, $2^5 = 2 \times 2 \times 2 \times 2 \times 2$ (there are 5 factors of 2 multiplied by each other).
- $a^b a^c = a^{b+c}$
- $\left(a^b\right)^c = a^{bc}$
- $\dfrac{a^b}{a^c} = a^{b-c}$
- $a^{-b} = \dfrac{1}{a^b}$ and $\dfrac{1}{a^{-b}} = a^b$
- $(ab)^c = a^c b^c$ BUT WATCH OUT FOR THIS MISTAKE: $(a+b)^c \neq a^c + b^c$
- $a^{\frac{b}{c}} = \sqrt[c]{a^b} = \left(\sqrt[c]{a}\right)^b$ Note that the rule will hold as long as the base, a, is non-negative. Rewrite radical expressions as terms with fractional exponents.

Methods of Solving Exponent Equations

- You can solve exponential equations by raising both sides to a power such that the exponents of the variable term will multiply to the correct power (the power you are looking for) on one side of the equation. For example if you are given the equation $x^{\frac{1}{5}} = 2$, you can solve for x^1 by raising both sides of the equation to the fifth power:

$$\left(x^{\frac{1}{5}}\right)^5 = 2^5 \quad \Rightarrow \quad x = 32$$

- You can write relatively large bases in terms of smaller bases in order to rewrite exponential terms with fractional exponents (without using a calculator) in order to match answer choices.

  For example, if we wanted to simplify the term $9^{\frac{3}{4}}$, we could rewrite the 9 as 3^2, allowing us to break the exponent up and rewrite $9^{\frac{3}{4}}$ as a radical expression with a base of 3.

$$9^{\frac{3}{4}} = \left(3^2\right)^{\frac{3}{4}} = 3^{2\left(\frac{3}{4}\right)} = 3^{\frac{3}{2}} = 3^{\left(1+\frac{1}{2}\right)} = 3 \cdot 3^{\frac{1}{2}} = 3\sqrt{3}$$

  It is also advantageous to write terms in another base whenever you have two different base numbers that share a common factor.

  For example, if we are given the equation $27^x \cdot 3^{4y} = 3^5$ and asked to solve for $3x + 4y$, we can combine the terms in the expression on the left side if we rewrite 27 as a base of 3 raised to the third power, allowing us to apply exponent rules to combine the terms.

$$27^x \cdot 3^{4y} = 3^5 \quad \Rightarrow \quad \left(3^3\right)^x \cdot 3^{4y} = 3^5 \quad \Rightarrow \quad 3^{3x} \cdot 3^{4y} = 3^5 \quad \Rightarrow \quad 3^{3x+4y} = 3^5$$

  Changing the base was useful in this example because it allowed us to write both sides of the equation as 3 raised to a power, which allows us to very easily see that $3x + 4y$ must be equal to 5 (since 3 to one power can't be equal to 3 to a different power).

Quadratics & Other Polynomials

Standard Form Polynomials

- The Standard Form for a univariate (single variable) polynomial puts the terms in order from highest to lowest power, so the Standard Form of a quadratic (second-degree polynomial) equation is

$$y = ax^2 + bx + c$$

where a, b, and c are constants.

- When adding or subtracting expressions, group and combine like terms of the same variable (same base and exponent).

$$(3x + 1 - 2y^2) - (2x + 4 - 5y^2)$$
$$3x - 2x + 1 - 4 - 2y^2 + 5y^2$$
$$x - 3 + 3y^2$$

- When multiplying expressions, distribute all terms in the first expression to all terms in the second expression.

$$(2x + 3)(x + 2) = 2x(x + 2) + 3(x + 2)$$
$$(2x + 3)(x + 2) = 2x(x) + 2x(2) + 3(x) + 3(2)$$
$$(2x + 3)(x + 2) = 2x^2 + 4x + 3x + 6$$
$$(2x + 3)(x + 2) = 2x^2 + 7x + 6$$

Solving Polynomial Equations

- If the two sides of an equation are polynomials written in Standard Form, and you are told that the equation is true for all values of the variable (that is, the polynomials are **equivalent**), you can match the coefficients of corresponding terms to find the values of any unknown coefficients.

$$-2x(5x + 1) + 6(5x + 1) = ax^2 + bx + c$$
$$-10x^2 - 2x + 30x + 6 = ax^2 + bx + c$$
$$-10x^2 + 28x + 6 = ax^2 + bx + c \Rightarrow$$

$a = -10$, $b = 28$, and $c = 6$

Factoring

- When factoring a quadratic expression in Standard Form where the coefficient of the x^2 term is 1, $x^2 + bx + c = 0$, we need to find two numbers, p and q, such that $p + q = b$ (the sum of the numbers is b) and $pq = c$ (the product of the numbers is c). The solutions to the equation are $-p$ and $-q$, and we can rewrite the expression in Factored Form. One of the factors is $(x + p)$ and the other factor is $(x + q)$.

$$(x + p)(x + q) = 0$$

For example, to find the factors of the quadratic expression in the equation $y = x^2 + 3x + 2$, we need to find two numbers that add to 3 and multiply to 2. The numbers 1 and 2 add to 3 (we know that $1 + 2 = 3$) and multiply to 2 (we know that $1(2) = 2$), so the factors are $(x + 1)$ and $(x + 2)$. The solutions are thus -1 and -2.

$$y = x^2 + 3x + 2$$
$$y = (x + 1)(x + 2)$$

- The sum of the solutions of a quadratic equation in Standard Form, $ax^2 + bx + c = 0$, is equal to

$$\frac{-b}{a}$$

- The product of the solutions of a quadratic equation in Standard Form, $ax^2 + bx + c = 0$, is equal to

$$\frac{c}{a}$$

Other Methods of Finding the Roots of Quadratics

- When the coefficient of the x^2 term is NOT 1 (when $a \neq 1$), first try dividing all of the terms by a to see if the resulting expression is easily factorable.

$$3x^2 - 12x - 15 = 0$$

$$\frac{3x^2 - 12x - 15}{3} = \frac{0}{3}$$

$$x^2 - 4x - 5 = 0$$

$$(x-5)(x+1) = 0$$

$$x = 5\,,\, x = -1$$

- Sometimes, when the coefficient of the x^2 term is NOT 1 (when $a \neq 1$), you can use the Product Sum (ac) Method, which is a generalized form of factoring.
When factoring a quadratic expression in Standard Form $ax^2 + bx + c = 0$ using the Product Sum (ac) Method, we need to find two numbers, r and s, such that $r + s = b$ (the sum of the numbers is b) and $rs = ac$ (the product of the numbers is ac). The solutions to the equation are $\frac{-r}{a}$ and $\frac{-s}{a}$, and we can rewrite the expression in Factored Form. One of the factors is $\left(x + \frac{r}{a}\right)$ and the other factor is $\left(x + \frac{s}{a}\right)$.

$$a\left(x + \frac{r}{a}\right)\left(x + \frac{s}{a}\right) = 0$$

$$4x^2 - x - 3 = 0$$

Product: $ac = 4(-3) = -12$
Sum: $b = -1$
The numbers r and s whose sum is -1 and whose product is -12 are 3 and -4, so $r = 3$ and $s = -4$.
The zeros are $x = \frac{-r}{a} = \frac{-3}{4}$ and $x = \frac{-s}{a} = \frac{-(-4)}{4} = 1$.
The factors are $\left(x + \frac{3}{4}\right)$ and $(x-1)$.

- If you cannot find numbers that satisfy the conditions for factoring, you can always use Completing the Square:

1. Divide both sides of the equation by the a coefficient to eliminate the coefficient of the x^2 term.
2. Move the constant to other side to isolate the x terms.
3. Replace the left side of the equation with $\left(x + \frac{b}{2}\right)^2$ (where b is the coefficient of the x term) and balance the extra constant on the other side by adding $\left(\frac{b}{2}\right)^2$ to the right side of the equation, pre-calculating $\frac{b}{2}$ so you don't have to do that twice.
4. Take the square root of both sides, being careful to include both positive and negative square roots.
5. Move the constant to right side of the equation to isolate and solve for x.
6. Enumerate the solutions.

- The Quadratic Formula can be used to find the solutions of quadratic equations in Standard Form, $0 = ax^2 + bx + c$.

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Completing the Square

$$\begin{aligned} x^2 - 6x + 4 &= 0 \\ x^2 - 6x &= -4 \\ (x-3)^2 &= -4 + 3^2 \\ (x-3)^2 &= -4 + 9 \\ (x-3)^2 &= 5 \\ x - 3 &= \pm\sqrt{5} \\ x &= 3 \pm \sqrt{5} \end{aligned}$$

Quadratic Formula

$$x^2 - 6x + 4 = 0$$

$$x = \frac{-(-6)}{2(1)} \pm \frac{\sqrt{(-6)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{6}{2} \pm \frac{\sqrt{36 - 16}}{2}$$

$$x = 3 \pm \frac{\sqrt{20}}{2}$$

$$\vdots$$

$$x = 3 \pm \sqrt{5}$$

- If you are given a system of equations consisting of one linear equation and one quadratic equation, or less commonly, two quadratic equations, you can use substitution to collapse the system of two equations into one quadratic equation which you can then solve.

$$\begin{aligned} y &= x^2 \\ y &= 8x + 20 \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} 8x + 20 &= x^2 \\ 0 &= x^2 - 8x - 20 \\ 0 &= (x-10)(x+2) \end{aligned}$$

- When an expression appears multiple times in a quadratic equation, you can probably shortcut the problem by factoring with respect to that expression rather than a single variable.

$$\begin{aligned} (x-4)^2 + 2(x-4) - 15 &= 0 \\ [(x-4) - 3][(x-4) + 5] &= 0 \\ (x-7)(x+1) &= 0 \\ x = 7, \quad x &= -1 \end{aligned}$$

Factoring Perfect Squares / Difference of Squares

- When a binomial expression is squared, the result takes one of the following forms:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

- When a binomial expression $(a + b)$ is multiplied by its conjugate, $(a - b)$, the result is equal to the **difference of squares** $a^2 - b^2$.

$$(a+b)(a-b) = a^2 - b^2$$

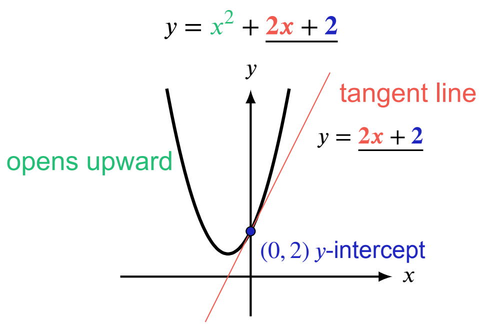
Graphs of Quadratics / Forms of Quadratic Equations

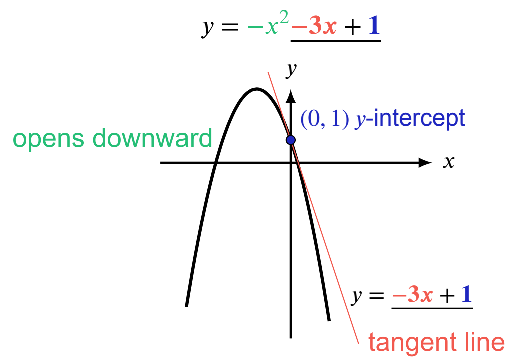
Standard Form

- For quadratics in Standard Form $y = ax^2 + bx + c$, the value of a (the x^2 coefficient) dictates the direction and elongation of the parabola. Positive values cause the graph to open upwards; negative values cause the graph to open downwards.

  The value of b (the x coefficient) is the slope of the tangent line through the y-intercept. The sign of the slope of this tangent line, in conjunction with the sign of the a coefficient, will indicate on which side of the y-axis the vertex lies.

  The value of c (the constant term) is the y-intercept.

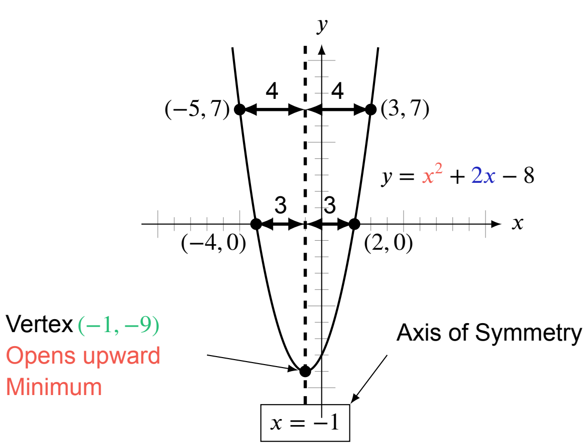




- Parabolas representing quadratic functions of the form $y = f(x)$ are symmetric around a vertical line called the axis of symmetry, which intersects the parabola at one point called the vertex. If the parabola opens upwards (a is positive), the vertex is the lowest point (it has the minimum y-value of any point); if it opens downwards (a is negative), the vertex is the highest point (maximum y-value).

  For a quadratic in Standard Form $y = ax^2 + bx + c$, the x-value of the vertex (and the equation of the axis of symmetry) is

$$x_v = \frac{-b}{2a}$$



$$x_v = \frac{-b}{2a} = \frac{-2}{2(1)} = \frac{-2}{2} = -1$$

Plug the x-value of the vertex into the Standard Form equation in order to easily find the y-value of the vertex. This also allows you to construct the Vertex Form of the parabola from the Standard Form.

$$y_v = (x_v)^2 + 2x_v - 8 = (-1)^2 + 2(-1) - 8 = 1 - 2 - 8 = -9$$

The vertex is $(-1, -9)$ and the value of the leading coefficient is $a = 1$, so the Vertex Form of the equation is $y = (x + 1)^2 - 9$.

- To rapidly convert a quadratic equation with an (implied) a-coefficient of 1 from Standard Form to Vertex Form, thus finding the vertex, use the Completing the Square procedure by replacing $x^2 + bx$ with $\left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$.

Complete the Square: $y = x^2 - 6x - 16$ $\Rightarrow$ $b = -6$, so $\frac{b}{2} = \frac{-6}{2} = -3$ $\Rightarrow$

$$y = \mathbf{x^2 - 6x} - 16$$
$$y = (\mathbf{x}-3)^2 - (-3)^2 - 16$$
$$y = (\mathbf{x - 3})^2 - \mathbf{9} - 16$$
$$y = (x-3)^2 - 25$$

Vertex $(3, -25)$

If there is a non-1 a-coefficient, replace $ax^2 + bx$ with $a\left(x+\frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2$,

Complete the Square: $y = -2x^2 + 4x - 8$ $\Rightarrow$ $a = -2$ and $b = 4$, so $\frac{b}{2a} = \frac{4}{2(-2)} = -1$ $\Rightarrow$

$$y = \mathbf{-2x^2 + 4x} - 8$$
$$y = -2(\mathbf{x}-1)^2 - [-2(-1)^2] - 8$$
$$y = \mathbf{-2(x - 1)^2 + 2} - 8$$
$$y = -2(x-1)^2 - 6$$

Vertex $(1, -6)$

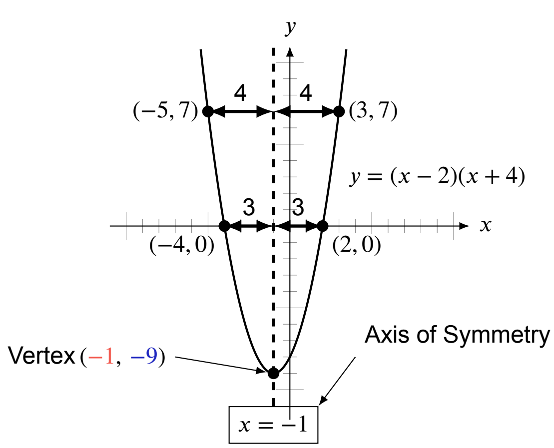
Factored Form

- The Factored Form of a quadratic is

$$y = a(x+p)(x+q)$$

where a, p, and q are constants.

- For quadratics in Factored Form $y = a(x+p)(x+q)$, $-p$ and $-q$ are roots. If $(x - z)$ is a factor of a quadratic, then z is a root of the function and vice versa.

- 

The x-value of the vertex, x_v, is exactly halfway between the two root values and can be found by averaging the two roots (add the roots and divide by 2).

$$x_v = \frac{\text{Sum of roots}}{2} = \frac{-4+2}{2} = \frac{-2}{2} = -1$$

Plug the x-value of the vertex, x_v, into the Factored Form in order to easily find the y-value of the vertex. This allows you to construct the Vertex Form of the parabola from the Factored Form.

$$y_v = (x_v - 2)(x_v + 4) = (-1-2)(-1+4) = (-3)(3) = -9$$

The vertex is $(-1, -9)$ and the value of the leading coefficient is $a = 1$, so the Vertex Form of the equation is $y = (x+1)^2 - 9$.

- To convert from Factored Form to Standard Form, expand the terms and recombine them in order of decreasing degree.

- The graphs of polynomials will "bounce" off of the x-axis when the exponent of a factor is even; they will go through the x-axis if the exponent of a factor is odd.

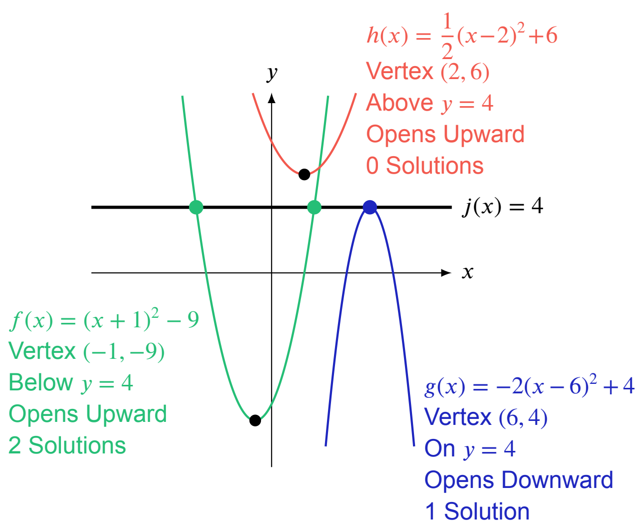
  Even-degree polynomials open upwards when the leading coefficient is positive (downwards when negative). Odd-degree polynomials' y-values go from $-\infty$ to ∞ from left to right when the leading coefficient is positive (vice versa when negative).

Vertex Form

- The Vertex Form of a quadratic is

$$y = a(x-h)^2 + k$$

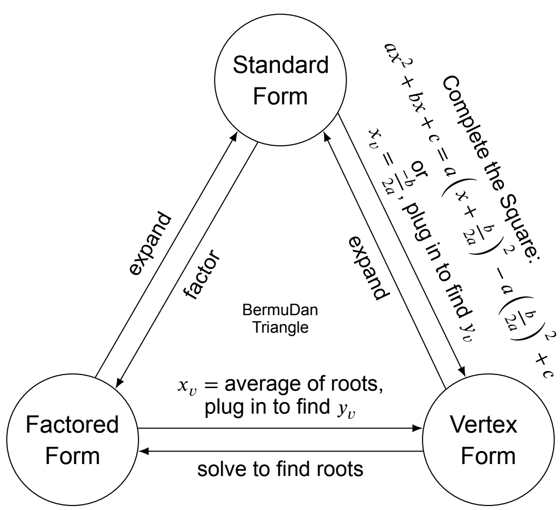
where a, h, and k are constants. The vertex of the graph is (h, k) and the value of a dictates the direction and elongation of the parabola.
The range of a function is the set of all y-values that can be produced by the function. For parabolas that open upwards (the a coefficient is positive), the range is all y-values greater than or equal to the y-value of the vertex. For parabolas that open downwards (the a coefficient is negative), the range is all y-values less than or equal to the y-value of the vertex.



- To convert from Vertex Form to Standard Form, expand the terms and recombine them in order of decreasing degree.

Features of the Forms of Quadratic Equations

- The Standard Form of a quadratic shows its y-intercept as a constant.
- The Factored Form of a quadratic shows its roots as constants.
- The Vertex Form of a quadratic shows its maximum or minimum value as a constant and shows the coordinates of its vertex as a pair of constants.



Number and Type of Zeros of Quadratics

- For quadratics in Standard Form, $y = ax^2 + bx + c$, where a, b, and c are constants, the discriminant

$$b^2 - 4ac$$

  indicates the number of real zeros.

  - When the discriminant is positive, there are 2 real zeros or roots.
  - When the discriminant is equal to 0, there is one real zero or root.
  - When the discriminant is negative, there are no real zeros or roots, but there are 2 complex zeros or roots, and they are the complex conjugates of one another.

- If a Standard Form quadratic is factorable (or is written in Factored Form to begin with), then it has either one (if the quadratic is a perfect square expression) or two real zeros or roots. There is no need to use the discriminant to check for the number of zeros if you can factor the quadratic.
- If a Vertex Form quadratic is given (or you can use Completing the Square to rewrite a Standard Form quadratic in Vertex Form), you can simply visualize or sketch the parabola to determine the number of intersections with the x-axis (these are x-intercepts, which will represent the zeros or roots) because you will know the position of the vertex and whether the parabola opens up or down. Once again, you can forego the use of the discriminant.

Radical & Rational Expressions

Radicals

- When a given equation has an expression in a radical sign, move everything to the other side of the equation before squaring both sides of the equation to eliminate the radical.

$$\sqrt{2x-2}+3=x-2 \quad\Rightarrow\quad \sqrt{2x-2}=x-5 \quad\Rightarrow\quad \left(\sqrt{2x-2}\right)^2=(x-5)^2 \Rightarrow \quad 2x-2=x^2-10x+25$$

- When squaring both sides of an equation, extraneous solutions are often created. Remember that even though both $x=2$ and $x=-2$ are valid solutions to the equation $x^2=4$, when an equation is given to you with an existing square root, the **value of that square root, by definition, is non-negative** (this is also true for all other even roots).
- For questions that ask you to pick the correct solution set for an equation containing a square root, you will need to double check any solutions you find algebraically by plugging those solutions back into the equation, so it may be more efficient to skip the algebra and instead plug in the answer choices.

  Which of the values in the set $\{0, 3, 9\}$ are solutions to the equation $\sqrt{2x-2}+3=x-2$?

Check solution $x=0$:

$$\sqrt{2(0)-2}+3=(0)-2$$
$$\sqrt{-2}+3=-2$$
$$✗ \quad \sqrt{-2}\neq -5$$

Check solution $x=3$:

$$\sqrt{2(3)-2}+3=(3)-2$$
$$\sqrt{6-2}+3=1$$
$$\sqrt{4}+3=1$$
$$✗ \quad 2+3\neq 1$$

Check solution $x=9$:

$$\sqrt{2x-2}+3=x-2$$
$$\sqrt{2(9)-2}+3=(9)-2$$
$$\sqrt{18-2}+3=7$$
$$\sqrt{16}+3=7$$
$$✓ \quad 4+3=7$$

Solving algebraically produces two possible solutions, but they need to be tested anyway, so we should go straight to testing values.

$$\sqrt{2x-2}+3=x-2$$
$$\sqrt{2x-2}=x-5$$
$$\left(\sqrt{2x-2}\right)^2=(x-5)^2$$
$$2x-2=x^2-10x+25$$
$$x^2-12x+27=0$$
$$(x-3)(x-9)=0$$
$$x=3,\ x=9$$

Rational Expressions & Remainder Theorem

- Rational expressions consist of one polynomial being divided by another. For some questions, you may need to combine terms through addition or subtraction. To do so, you may need to write terms so that they have common denominators just as you would with any fractions.

$$2+\frac{3}{x+5} \quad\Rightarrow\quad 2\left(\frac{x+5}{x+5}\right)+\frac{3}{x+5} \quad\Rightarrow\quad \frac{2x+10}{x+5}+\frac{3}{x+5} \quad\Rightarrow\quad \frac{2x+13}{x+5}$$

- The value of a rational function is undefined when the denominator is equal to 0. Use factoring if needed to determine what values of x will cause the denominator to be 0.

$$f(x)=\frac{x-1}{x^2+5x+6} \quad\Rightarrow\quad f(x)=\frac{x-1}{(x+2)(x+3)} \quad\Rightarrow$$ Undefined when $x=-2$ and $x=-3$

- Polynomial Long Division is very rarely necessary (there are always multiple routes through a problem), but you should have a handle on the process just in case.

$$
\begin{array}{r|rrr}
 & x & +5 & \\
\hline
x+3 & x^2 & +8x & +15 \\
 & -(x^2 & +3x) & \Downarrow \\
\hline
 & & 5x & +15 \\
 & & -(5x & +15) \\
\hline
 & & & 0
\end{array}
\qquad
\begin{array}{r|rrr}
 & & x & -4 \\
\hline
2x+1 & 2x^2 & -7x & -4 \\
 & -(2x^2 & +x) & \Downarrow \\
\hline
 & & -8x & -4 \\
 & & -(-8x & -4) \\
\hline
 & & & 0
\end{array}
\qquad
\begin{array}{r|rrrr}
 & 3x^2 & +2x & +9 & \\
\hline
3x-2 & 9x^3 & +0x^2 & +23x & -18 \\
 & -(9x^3 & -6x^2) & \Downarrow & \Downarrow \\
\hline
 & & 6x^2 & +23x & \Downarrow \\
 & & -(6x^2 & -4x) & \Downarrow \\
\hline
 & & & 27x & -18 \\
 & & & -(27x & -18) \\
\hline
 & & & & 0
\end{array}
$$

- The Polynomial Remainder Theorem states that when a polynomial $f(x)$ is divided by a binomial $x-r$, the remainder of the division, R, is equal to $f(r)$. Therefore, when $f(r)=0$, there is no remainder, and thus $f(x)$ is divisible by $x-r$.

  For example, if we wanted to check if $f(x)$ is divisible by $x-4$, we could plug 4 into $f(x)$. If $f(4)=0$, then $f(x)$ is divisible by $x-4$.

Imaginary & Complex Numbers

Imaginary and Complex Numbers

- There is an imaginary number i, and $i = \sqrt{-1}$. It follows that $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$; as with all bases, $i^0 = 1$. For powers higher than 4, the pattern repeats for every set of 4. For example, the next four powers of i are as follows:

$$i^5 = i$$
$$i^6 = -1$$
$$i^7 = -i$$
$$i^8 = 1$$

- The complex conjugate of a complex number $a + bi$ is $a - bi$. Multiplying a complex number by its conjugate will eliminate the imaginary part of the complex number, leaving a real number.
- If there is a complex number in the denominator of a fraction, multiply the numerator and denominator by the complex conjugate of the denominator to produce a real number in the denominator.

Ratios, Probability, and Proportions

Ratios and Probability

- A fraction $\frac{a}{b}$ is a form of ratio that can also be expressed with the notation $a : b$, which can be read as “a to b” or “a in b.” Sometimes ratios will be expressed as decimal numbers; for example, $\frac{1}{2}$ is equivalent to the decimal 0.5.
- Ratio notation can be used to show the relative values of any number of terms. For example, the side lengths of a triangle might be in a ratio of $3 : 4 : 5$.
- Given a ratio of $a : b$, you may need to make fractions for use in solving problems.

  Depending on the problem, it may be beneficial to use a **part-to-part** fraction of $\frac{a}{b}$ or $\frac{b}{a}$, which helps when comparing the amount of one part to the amount of another part. Other times, you might need to use a **part-to-whole** fraction like $\frac{a}{a+b}$ or $\frac{b}{a+b}$, where the denominator is the sum of the parts (the whole), which helps when comparing the amount of one part to the total, or whole, amount.

  For example, a snack mixture contains 2 parts of peanuts and 5 parts of pretzels, so there are a total of 7 parts in the mixture.

  Part-to-Part: the amount of peanuts is $\frac{2}{5}$ the amount of pretzels (the ratio of peanuts to pretzels is $\frac{2}{5}$)

  Part-to-Whole: the amount of peanuts is $\frac{2}{7}$ of the entire mixture (peanuts make up two-sevenths of the mixture)
- Reducing ratios works exactly like reducing fractions. Just as the fraction $\frac{4}{10}$ reduces to $\frac{2}{5}$, so too does the ratio $4 : 10$ reduce to $2 : 5$.

  The process works the same for ratios with more than two terms. For example, a ratio of $6 : 8 : 10$ can be reduced by dividing all of the terms by 2, resulting in a reduced ratio of $3 : 4 : 5$.
- $$\text{Probability} = \frac{\text{Desired Outcomes}}{\text{Possible Outcomes}}$$
- Choose values from tables very carefully, making sure to restrict yourself appropriately based on the selection criteria.

  Start by determining the **reference** total number of events or entities (people, things, etc.), which will act as the **denominator**. **This reference number is often found in an “if” statement in the question prompt**. For example, if a question contains the phrase, “If a plumber from California is chosen at random...” then the denominator in the fraction is the number of plumbers from California.

  The numerator must be smaller than the denominator because we are always looking for a subset of the reference group that was used for the denominator. Completing the example phrase we just started: “If a plumber from California is chosen at random, what is the probability that he will have more than 4 years of experience?” For the numerator, draw only from the pool of people who fit into the reference category—they must be plumbers from California. Then, we need just those plumbers from California *who have more than 4 years of experience*.

Proportions

- Proportions are formed by setting two ratios equal to each other. Proportions are easy ways to solve for values that have constant rates of change (instead of using linear equations).

  For example, if a table manufacturer makes tables of varying sizes but the length to width ratio of their tables is always 5 to 2 ($5:2$), then you can write the following proportion:

$$\frac{\text{length}}{\text{width}} = \frac{5}{2}$$

  We recommend writing proportions with the unknown in the numerator of the ratio on the left side. For example, the same table manufacturer wants to make a table with a length of 8 feet that conforms to the standard 2 to 5 width to length ratio. We can solve for the width of that table quickly using the following proportion.

$$\frac{\text{width}}{\text{length}} = \frac{2}{5} \quad \Rightarrow \quad \frac{w}{8\text{ ft}} = \frac{2}{5} \quad \Rightarrow \quad w = \frac{2(8\text{ ft})}{5} \quad \Rightarrow \quad w = \frac{16}{5}$$

Relative Change in Non-Linear Relationships

- Sometimes relationship problems involve squared or cubed relationships rather than linear relationships. For example, the area of circle is given by the formula $A = \pi r^2$, where A is the area and r is the radius. Doubling the radius does not double the area. Due to the squared relationship, doubling the radius actually quadruples the area of the circle.

$$A_{\text{original}} = \pi r^2 \quad \Rightarrow \quad A_{\text{new}} = \pi(2r)^2 = 4\pi r^2 \quad \Rightarrow \quad A_{\text{new}} = 4A_{\text{original}}$$

Percentages

Percentages

- In any situation involving percentages, there are three fundamental components:
  1. The reference, base, total, starting, or original amount of something
  2. The relative portion or share of the reference amount, which will be given or requested as a percentage
  3. The actual value of the relative portion or share, which will be in the same units as the reference value
- You can write basic percentage equations in either of these forms:

$$\frac{\text{portion of the total}}{\text{total}} = \text{relative amount} \quad \text{or} \quad \text{portion of total} = (\text{relative amount})(\text{total})$$

- You can always orient yourself in percentage problems by putting all the provided information into the form, "A is $P\%$ of B," which translates to the second equation form above written as

$$A = pB$$

  where the lowercase p is the decimal representation of P.
- Convert from decimal to percentage by multiplying the decimal by 100; this is easily accomplished by moving the decimal point two places to the right, adding zeros as needed.

  Convert from percentage to decimal by dividing the percentage by 100, easily done by moving the decimal point two places to the left, adding zeros as needed.

$$32\% \text{ is } 0.32 \text{ and } 1.24\% \text{ is } 0.0124$$

Percent Increase / Decrease

- When a value is **decreased** by a percentage, you need to multiply the value by the complementary decimal value of the percentage, which is **1 MINUS the decrease in decimal form**, to find the remaining amount. For example, if a problem tells you that something **loses, decreases by, declines by, is reduced by, is discounted by, or shrinks by 20%, you will need to multiply the number by 0.8, not 0.2, because the new value is $100\% - 20\% = 80\%$ of the original value**.
- When a value is **increased** by a percentage, you need to multiply the value by the decimal value of the percentage, which is **1 PLUS the increase in decimal form**, to get the total amount after the increase. For example, if a problem tells you that something **grows or increases by 20%, you will need to multiply the number by 1.2 because the new value is $100\% + 20\% = 120\%$ of the original value**.
- Percent More $\neq$ Percent Less
  It is important to note that 150 is 50% *more* than 100 because $1.5(100) = 150$, but 100 is obviously NOT 50% *less* than 150 because 50% (or half) of 150 is 75.

  Write your equations based on what the question tells you to write. Look for the words "is" or "equals" to tell you on which side of the equation to write certain terms. If x *is* 80% of y, write the equation $x = 0.8y$.

  It is INCORRECT to infer that if x is 20% less than y, then y is 20% more than x. Given the statement "x is 80% of y," you should NOT write $y = 1.2x$, but rather $x = 0.8y$, which means $y = \frac{x}{0.8}$ or $y = 1.25x$.

- You can calculate percent increases and decreases using the same equation $A = pB$, where A is the new amount and B is the original amount (read as "the new amount, A, is P percent of the old amount, B," where P is the percentage equivalent of the decimal p). The decimal p that is solved for is relative to 1 (indicating what percent A is of B).

  If $p = 1.2$, there was a 20% growth from B to A (that is, A is 120% of B).

  If $p = 0.85$, there was a 15% decrease from B to A (that is, A is only 85% of B, so B decreased by 15%).

- Percent Change Formula:

$$\text{Percent Change} = \frac{\text{New Value} - \text{Original Value}}{\text{Original Value}}$$

  The decimal is the percent change from the original value, and the sign (positive or negative) tells you whether the change is an increase or decrease.

- If a value changes by a certain percentage and then changes again by a certain percentage, you need to *multiply* the original value by the percentages, NOT add the percentages and then apply the result to the original value, because the reference value changes during the sequence.

  For example, if an item cost $100 initially, but the cost is reduced by 10% (which changes the price to $90), and then you use a 10% off coupon, you are taking 10% off of the new listed price of $90, not the original price. Therefore, the price you pay is 90% of 90% of $100 (you do NOT pay 80% of the original price):

$$\text{price} = 0.9(0.9)(100) = .9(90) = 81$$

Mixtures & Concentrations

- The percentage equations in solution and mixture problems are just linear equations with decimals.

  The amount of a substance in a solution/mixture is equal to the percentage of the solution that is made up of that substance times the total amount of the solution/mixture. For example, if there are A kilograms of a saline solution (a solution of salt in water) that is 10% salt by mass, then $0.1A$ is the amount of salt in that solution.

  When mixing solutions to form a new solution, add the amounts of a substance in the ingredients (the percentage of the solution that is made up of that substance times the amount of the solution) for the two solutions and set them equal to the amount of that substance in the resulting mixture (which, again, is the percentage of the solution that is made up of that substance times the total amount of the solution). For example, if we mix A kilograms of a 10% saline solution and B kilograms of a 20% saline solution to form a new 15% saline solution, we can write the following equation

$$0.1A + 0.2B = 0.15(A + B)$$

  because the amount of salt in the first solution is 10% of the total amount of that solution (which is A kilograms), the amount of salt in the second solution is 20% of the total amount of that solution (which is B kilograms), and the amount of salt in the new solution is 15% of the total amount of that solution (which is $A + B$ kilograms).

Exponential Relationships

Exponential Equations

- Exponential change is characterized by an initial value that is repeatedly multiplied or divided by the same amount.
- Most exponential equations you will have to write or interpret will deal with percent increase (growth) or decrease (decay) and will be of the form

$$y = a \cdot b^x$$

  where a is the initial value, b is the rate of change (the amount that the value is multiplied by over each interval), and x is the number of intervals (usually a time interval).
- In exponential equations where the initial equation's exponent is written in terms of one unit of measurement, but you are supplied with the period or interval variable or value in different units, you should use a proportion showing the relative values of the units to determine the value of the exponent when expressed in the original units.

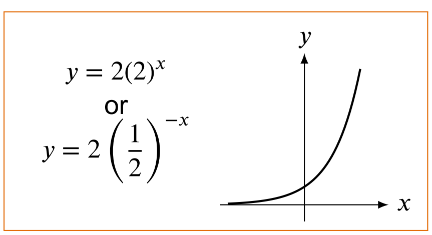
  For example, there are four quarters in one year, so we can use the proportion $\frac{q}{t} = \frac{4}{1}$, where q is the number of quarters and t is the number of years, to convert measurements in one unit to the other.

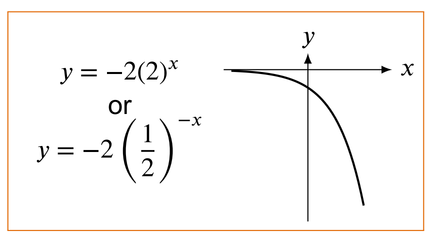
Graphs of Exponential Equations

- It is useful to know the general shapes of exponential graphs, where one side of the graph is almost horizontal and the other side is almost vertical. Plugging in test values and graphing a few points should give you a reasonable understanding of the graph for any particular exponential equation.

  In general for equations of the form $y = ab^{cx}$:

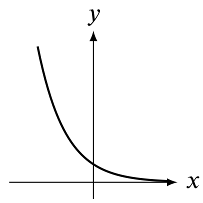
  - Positive values of a indicate that the graph has a positive y-intercept and will be entirely above the x-axis.
  - Negative values of a indicate that the graph has a negative y-intercept and will be entirely below the x-axis.
  - When $b \geq 1$ and c is positive AND when $0 < b < 1$ and c is negative, the graph levels off to the left side and goes to infinity (or negative infinity when a is negative) on the right side (the y-value changes slowly, then changes rapidly).



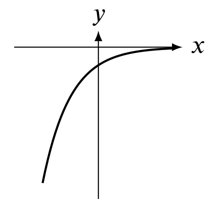


- When $b \geq 1$ and c is negative AND when $0 < b < 1$ and c is positive, the graph levels off to the right side and goes to infinity (or negative infinity when a is negative) on the left side (the y-value changes rapidly, then changes slowly).

$y = 2(2)^{-x}$
or
$y = 2\left(\frac{1}{2}\right)^x$



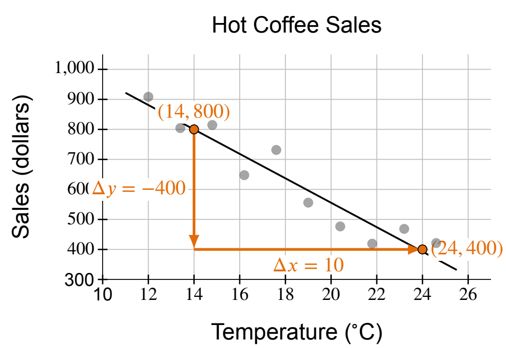
$y = -2(2)^{-x}$
or
$y = -2\left(\frac{1}{2}\right)^x$



Scatterplots and Line Graphs

Scatter Plots

- Find the equation of a line of best fit for a scatterplot in the same ways that you did for any regular line. If no line of best fit is drawn, try your best to draw one in that runs roughly through the center of the cluster of points.

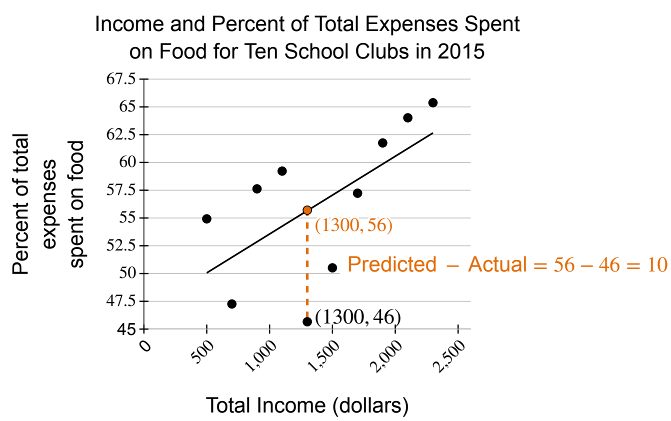


$$m \approx \frac{\Delta y}{\Delta x} = \frac{-400}{10} = -40$$

⇓ Substitute slope and point (14, 800) to solve for b :

$$y = mx + b$$
$$800 = -40(14) + b$$
$$800 - 560 = b$$
$$1{,}360 = b$$
$$\Downarrow$$
$$y = -40x + 1{,}360$$

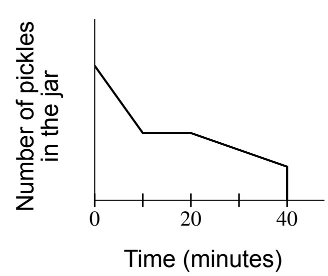
- Find the difference between actual and predicted values by seeing how far above or below the line of best fit a data point is.



- Do not make overly definite statements about what the slopes and intercepts of scatterplots tell us. They are merely models and sometimes only fit very roughly over particular intervals.

Reading Line Graphs

- Most line graph questions are simply related to the slope of the line on certain intervals. Steep lines mean something changed rapidly during an interval. Horizontal lines means something stayed the same for an interval. Vertical lines mean something changed instantaneously.
For example, Hannah eats pickles while she studies. She eats half of the pickles during the first 10 minutes of studying. After eating half of the pickles, she stops eating for the next 10 minutes. Then she gradually eats the pickles until she purposely spills all of the remaining pickles.



Functions

Evaluating Functions ("Plugging In")

- When plugging into a function $f(x)$, replace each instance of x with the value or expression that is being plugged into the function. Replace each x with a set of blank parentheses as an intermediate step if you have trouble plugging into the function.

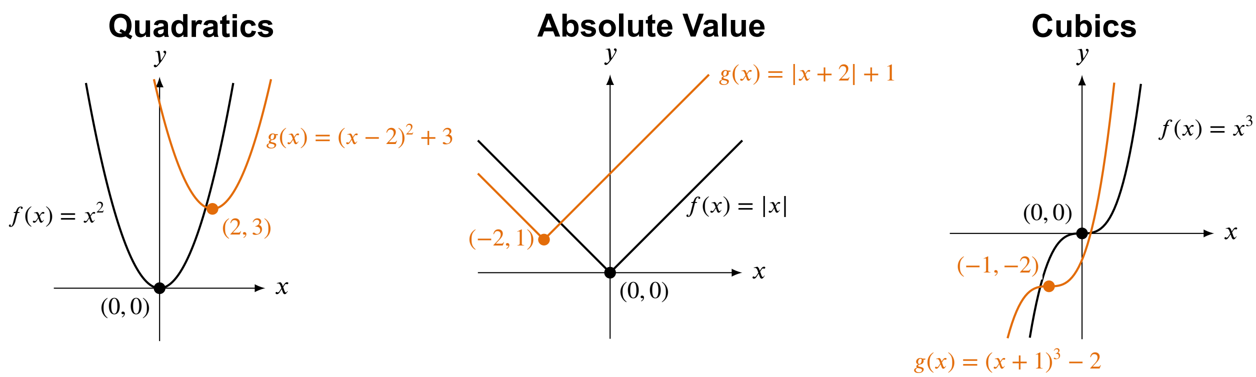
$$f(x) = x^2 - 2x + 1 \quad \Rightarrow \quad f(\quad) = (\quad)^2 - 2(\quad) + 1 \quad \Rightarrow \quad \begin{array}{l} f(-3) = (-3)^2 - 2(-3) + 1 \\ f(x+2) = (x+2)^2 - 2(x+2) + 1 \end{array}$$

- When working with composite functions, plug the input value into the inside function first and then plug the resulting value into the outside function.

$$\begin{array}{l} g(x) = 4x + 2 \\ h(x) = 3 - g(x) \end{array} \quad \Rightarrow \quad g(0) = 4(0) + 2 = 2 \quad \Rightarrow \quad h(0) = 3 - g(0) = 3 - 2 = 1$$

Function Shifting and Unfamiliar Graphs

- The graph of a function $g(x) = f(x - h) + k$ is shifted h units horizontally and k units vertically from the graph of $f(x)$.



- If you are given an unfamiliar graph type and are asked to find the corresponding function, plug identifiable points, such as the y-intercept or x-intercept, into each of the answer choices, eliminating any that don't fit all of the tested points.

Statistics

Data Sets: Mean, Median, Mode, and Range

- When trying to find the key properties of a data set, you may find it useful to rewrite the data set in order from least to greatest to help prevent mistakes.
- The **mean** is the average of the values in a data set. The mean is found by dividing the sum of the values in the data set by the size of the data set.

$$\text{Mean} = \frac{\text{Sum of Data Set}}{\text{Size of Data Set}}$$

- The **median** is located in the middle of the data set.

  For an odd-sized data set, the position of the median is found by adding 1 to the size of the data set and then dividing by 2. For example, the median of a data set with 49 values is found in position 25 (the 25th value in the data set) because $\frac{49+1}{2} = \frac{50}{2} = 25$.

  For an even-sized data set, the median is found by averaging the two middle values. The positions of these two values are found by dividing the size of the data set by 2; the result gives the position of the first middle value, and the other middle value is found at the next position in the data set. For example, the median of a data set with 100 values is found by averaging the values of the 50th and 51st values in the data set because $\frac{100}{2} = 50$.
- The **mode** of a data set is the value that appears most frequently in a data set.
- The **range** of a data set is the difference between the largest and smallest value in a data set. Subtract the smallest value from the largest value to find the range.

Process	**Set A:** $\{4, 2, 8, 10, 1, 9, 8\}$	**Set B:** $\{2, 0, 0, 4, 5, 5\}$
Increasing Order	$\{1, 2, 4, 8, 8, 9, 10\}$	$\{0, 0, 2, 4, 5, 5\}$
Mean	$\text{Mean}_A = \frac{1+2+4+2(8)+9+10}{7} = \frac{42}{7} = 6$	$\text{Mean}_B = \frac{2(0)+2+4+2(5)}{6} = \frac{16}{6} = \frac{8}{3}$
Median	$A = \{1, 2, 4, 8, 8, 9, 10\}$	$B = \{0, 0, 2, 4, 5, 5\}$ $\text{Median}_B = \frac{2+4}{2} = \frac{6}{2} = 3$
Mode	$\text{Mode}_A = 8$ This value appears twice, and no other value is repeated	No value appears more often than any other: there is **no mode**.
Range	$\text{Range}_A = 10 - 1 = 9$	$\text{Range}_B = 5 - 0 = 5$

Frequency Tables, Histograms, and Standard Deviation

- Frequency tables and histograms are used to show how many times particular values occur in a data set. From these representations, we are able to calculate all of the key properties of data sets. To find the mean, we need to find the sum of the data, which can be determined by multiplying each value in the data set by its frequency (or the height of its bar) and summing the results, then dividing by the number of samples. The median is found by counting the number of elements until you reach the position of the median (in the middle of the set).

Ages of **200** People Enrolled in a Hot Yoga Studio

Age	Frequency	
18	34	People 1–34
19	21	People 35–55
23	37	People 56–92
25	38	People 93–130
30	46	
45	24	

$$\text{Mean Age} = \frac{34(18) + 21(19) + 37(23) + 38(25) + 46(30) + 24(45)}{200}$$

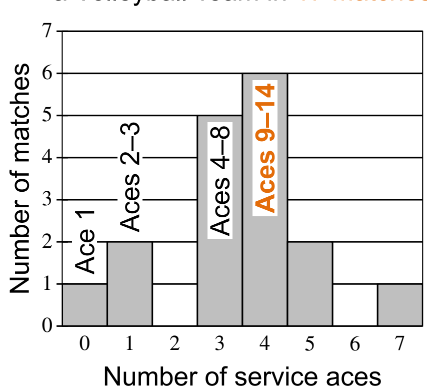
$$\text{Mean Age} = \frac{5{,}272}{200}$$

$$\text{Mean Age} = 26.36$$

There are 200 values in the set, so the median is the average of the 100th and 101st values in the set, both of which are 25 years old.

$$\text{Median Age} = 25$$

Number of Service Aces by a Volleyball Team in 17 Matches



$$\text{Mean Aces} = \frac{1(0) + 2(1) + 5(3) + 6(4) + 2(5) + 1(7)}{17}$$

$$\text{Mean Aces} = \frac{58}{17}$$

$$\text{Mean Aces} \approx 3.41$$

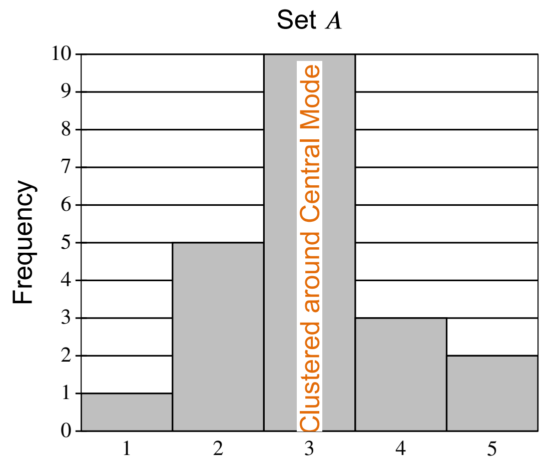
There are 17 values in the set, so the median is the 9th value in the set, which is 4 aces.

$$\text{Median Aces} = 4$$

- **Standard deviation** is a measure of how closely clustered the values in a data set are (how close to the mean of the data most of the values are). Tightly clustered data sets will have a lower standard deviation than will data sets that are more spread out.

Set A	
Value	Frequency
1	1
2	5
3	**10**
4	3
5	2

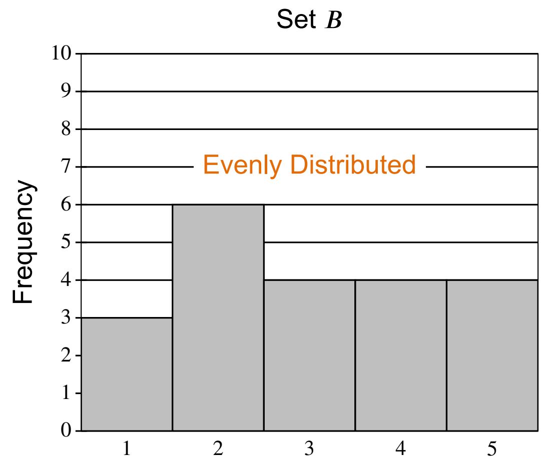
Clustered around Central Mode



Lower Standard Deviation

Set B	
Value	Frequency
1	3
2	6
3	4
4	4
5	4

Evenly Distributed

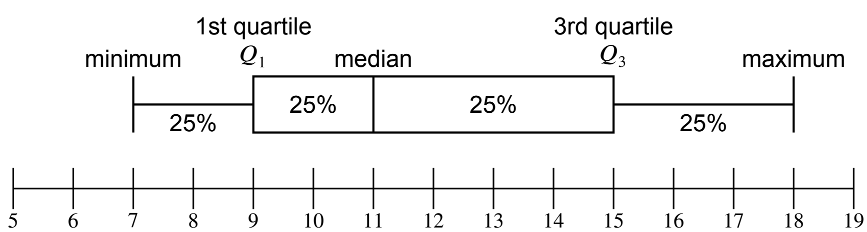


Higher Standard Deviation

Box Plots

- A box plot is formed by drawing short vertical lines for the median and quartiles Q_1 and Q_3 and then connecting the tops and bottoms of those lines to form a box. The outermost vertical lines mark the maximum and minimum values in the set. About 25% of the data set will be found between each vertical line on the box plot (each section contains roughly the same amount of data).

  For the box plot in the figure below, the median is 11; the first quartile is 9, and the third quartile is 15; the minimum is 7, and the maximum is 18.



Survey Design / Interpreting Results

- In order to draw valid conclusions from surveys, the samples must be sufficiently large (so far there has never been a sample that was too small on any released tests because, regardless of the population size, even seemingly small sample sizes are fairly accurate), and **most importantly, truly random**. If the subjects in a survey share some trait (other than just being members of the larger population), then the survey's results are restricted to just the population that shares that trait and cannot be applied to the larger population to which those subjects belong.
- The **margin of error** accounts for the potential difference between the true value for an entire population and the value found based on a survey sample. The true value for the whole population is most likely (but not definitely) found inside the margin of error around the value found for the survey. A larger sample size leads to a smaller margin of error.

  An ecologist selected a random sample of 50 beavers from a river and found that the mean weight of the beavers in the sample was 42 pounds (lbs), with an associated margin of error of 3.1 lbs. Therefore, any weight between $42 - 3.1 = 38.9$ lbs and $42 + 3.1 = 45.1$ lbs is a plausible value for the mean weight of all the beavers in the river.
- Do not make overly definitive claims based on a survey. The results only apply to the population that shares the traits common to those sampled. A cause-and-effect relationship is highly unlikely unless you are specifically told that all other variables have been controlled for. You are a lot safer merely making a claim just that there is a correlation between two things (rather than a cause-and-effect relationship).

Unit Conversions

Simple Conversions with Proportions

- When converting from one unit to another unit based on a direct conversion rate, you can simply use a proportion to solve the problem.

  EXAMPLE:
  Convert 3 inches to centimeters (1 inch $= 2.54$ cm):

$$\frac{\ell_{\text{cm}}}{\ell_{\text{in}}} = \frac{2.54\,\text{cm}}{1\,\text{in}} \quad \Rightarrow \quad \frac{\ell_{\text{cm}}}{3\,\text{in}} = \frac{2.54\,\text{cm}}{1\,\text{in}} \quad \Rightarrow \quad \ell_{\text{cm}} = \left(\frac{2.54\,\text{cm}}{1\,\cancel{\text{in}}}\right)(3\,\cancel{\text{in}}) \quad \Rightarrow \quad \ell_{\text{cm}} = 7.62\,\text{cm}$$

Factor-Label Method

- Use the Factor-Label method when a direct conversion from one unit to another is not available. String together unit conversion factors in order to cancel one unit at a time until the desired units are achieved.

  EXAMPLE:
  Convert 76 inches to meters (1 inch $= 2.54$ cm and 1 m $= 100$ cm):

$$76\,\cancel{\text{in}}\left(\frac{2.54\,\cancel{\text{cm}}}{1\,\cancel{\text{in}}}\right)\left(\frac{1\,\text{m}}{100\,\cancel{\text{cm}}}\right) = 1.93\,\text{m}$$

- Identify a **starting point** for word problems with unit conversions by picking out information that tells us about the **end goals**. In order to determine the steps needed after the starting point is identified, try to cancel or convert unwanted units, one-by-one, based on additional information in the problem.

The Distance-Speed-Time Equation

- The distance-speed-time equation is

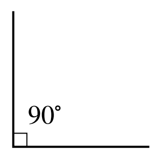
$$d = st$$

  where d is distance, s is speed, and t is time.

Angles, Triangles, and Trigonometry

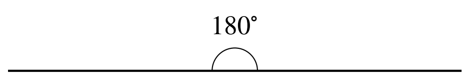
Angles

- Angles are formed by the intersection of two lines; the point at which the two lines meet is called the **vertex** (plural, **vertices**).
- Angles are most commonly measured in **degrees**, which is denoted by the degree symbol (°).
- A 90°angle is a **right angle** (the lines forming the angle are perpendicular). Right angles are often marked with a small square instead of an arc and angle measurement.



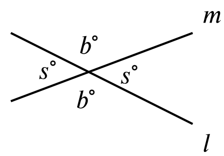
Any angles that together form a right angle sum to 90°. When two angles form a 90° angle they are called **complementary angles**.

- A 180° angle results in a straight line.



Any angles that together form a straight line sum to 180°. When two angles form a 180° angle they are called **supplementary angles**.

- Angles that measure more than 0° and less than 90° are called **acute angles**. Angles that measure more than 90° and less than 180° are called **obtuse angles**. Angles greater than 180° are called **reflex** angles.
- Whenever two lines intersect, four angles are formed, and the angles opposite each other, called **Vertical Angles**, are equal.



As previously explained, angles that form a line sum to $180°$. Therefore, $s + b = 180$.

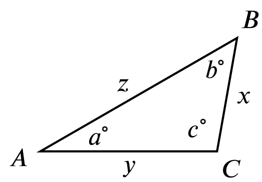
- To indicate congruence (equality of measure) for sides and angles, those features can be marked with dashes or arcs, called hatch (or hash or tick) marks. Sides marked with the same number of hatch marks are congruent, and angles marked with the same number of arcs are congruent.

  Parallel lines can be marked by arrow heads. Lines marked with the same number of arrow heads are parallel.
- When a diagonal (non-perpendicular) line runs across two parallel lines, eight angles are formed. The four larger (obtuse) angles are all equal to each other. Similarly, the four smaller (acute) angles are also equal to each other.

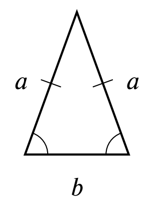
Triangles and Other Polygons

- **The interior angles of any triangle sum to 180°**. The length of each side is positively correlated with the size of the angle opposite that side, so the **largest side of a triangle is always across from the largest angle of the triangle, and the smallest side is across from the smallest angle**.

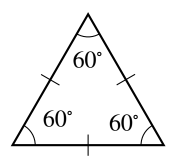
  In the triangle below, $a + b + c = 180$ because the sum of any triangle's interior angles is 180°. Since $\angle C$ is the largest angle, z is the longest side. Similarly, since $\angle A$ is the smallest angle, x is the shortest side.



- **In isosceles triangles, two of the angles are equal to each other, and the sides across from those angles are therefore also equal to each other.**



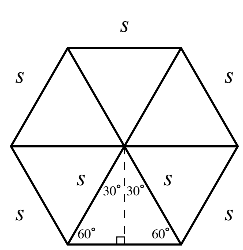
- **In equilateral triangles, all of the sides are equal to each other, and therefore all of the angles are also equal**. Because all three of the angles are equal and they must sum to 180°, each angle is $\frac{180^\circ}{3} = 60^\circ$.



- For a polygon with n sides and angles, the sum of the interior angles can be found using the following formula, which is very rarely needed:

$$\text{Sum of Interior Angles} = 180(n - 2)$$

- In a **regular polygon**, all sides are equal, and all angles are equal. The measure of any angle is equal to the sum of the interior angles divided by the number of angles.
- **Regular hexagons** (six-sided polygons) can be split into 6 equilateral triangles if we draw a line from the center of the figure to each of the six vertices.



Similar Triangles

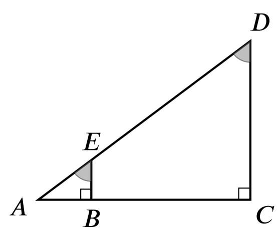
- Two triangles are called **Similar Triangles** when they have the same interior angles. Put another way, each angle in one triangle has a corresponding match in the other triangle.

In the figure above, $\triangle ABC$ is similar to $\triangle DEF$ because all of the angles in $\triangle ABC$ have a matching angle in $\triangle DEF$. Notice that the triangles are different sizes, however. **In similar triangles, all of the angles have a match, but the sides are not *necessarily* equal. However, sides across from matching angles are always in the same proportion to each other.**

In the figure shown above, $\overline{AB}$ corresponds to $\overline{DE}$; $\overline{BC}$ corresponds to $\overline{EF}$; and $\overline{AC}$ corresponds to $\overline{DF}$. In these particular triangles, the sides in $\triangle ABC$ are all twice the length of the corresponding sides in $\triangle DEF$.

- **If we know (or can determine) that there are two angles that are the same in two triangles, the third angle must also be the same** because the three angles in each triangle must sum to the same value (180°). This, in turn, means that **the two triangles are similar triangles**.
- Similar triangles are commonly depicted as one triangle inside of another, where you can show that the triangles must be similar because all the angles in both triangles will have a match.

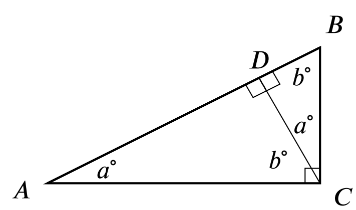
In the figure below, $\triangle ACD$ is similar to $\triangle ABE$ because they both contain $\angle A$ and a right angle, which means the third angles in the two triangles must also be equal.



- You can use part-to-whole or part-to-part ratios to solve for lengths in divided triangles. When a **triangle is divided by a line parallel to one side, the other two sides are divided proportionally**; this is the **Triangle Proportionality Theorem**.

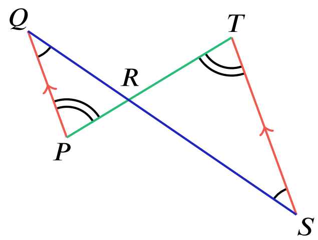
In the triangle above, the Triangle Proportionality Theorem tells us that $\frac{AE}{ED} = \frac{AB}{BC}$.

- Right triangles can be divided with a single line that creates three similar triangles. In the figure below, a line is drawn from the right angle vertex C that intersects the hypotenuse at a right angle at point D. The larger right triangle is split into two smaller right triangles, ΔADC and ΔCDB, both of which are similar to the original triangle ΔACB and thus to each other as well.



- Similar triangles arise in arrangements such as that shown in the figure below, with the two similar triangles touching at one vertex (where there is a vertical angle) and each having a side that is parallel to a side in the other triangle.

  In the figure below, the angles meeting at point R are vertical angles (and thus are equal to each other), and side $\overline{PQ}$ is parallel to side $\overline{ST}$.



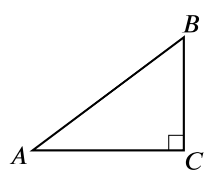
$$\Delta PQR \sim \Delta TSR$$
and
$$PQ : QR : RP = TS : SR : RT$$

Trigonometry

- In a right triangle, the side that is across from the right angle is called the **hypotenuse** ($\overline{AB}$ in the triangle below).

  The side across from an acute angle is called the **opposite** side. In the triangle below, side $\overline{BC}$ is the opposite side for $\angle A$, and side $\overline{AC}$ is the opposite side for $\angle B$.

  The side that forms an angle with the hypotenuse is called the **adjacent** side of that angle. In the triangle below, side $\overline{AC}$ is the adjacent side for $\angle A$, and side $\overline{BC}$ is the adjacent side for $\angle B$.



$$\sin A = \frac{BC}{AB} \quad \sin B = \frac{AC}{AB}$$
$$\cos A = \frac{AC}{AB} \quad \cos B = \frac{BC}{AB}$$
$$\tan A = \frac{BC}{AC} \quad \tan B = \frac{AC}{BC}$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

- The trig ratios can be remembered using the mnemonic **SOH-CAH-TOA**, which distills the information that **S**ine is **O**pposite over **H**ypotneuse, **C**osine is **A**djacent over **H**ypotenuse, and **T**angent is **O**pposite over **A**djacent.
- Trigonometric functions' values are derived from the side ratios **in a right triangle**, and their **values for a given angle don't change** if the angle appears in a figure that is not a right triangle; the **sine of an angle is the sine of that angle regardless of what type of figure** includes that angle. However, you can **only calculate these functions from a figure when the angle is part of a right triangle**.
- The sine and cosine of complementary angles have the following relationship:

$$\sin(x^\circ) = \cos(90^\circ - x^\circ) \quad \text{and} \quad \cos(x^\circ) = \sin(90^\circ - x^\circ)$$

  This is useful because the acute angles in a right triangle sum to 90°.

Perimeter & Area

- The **perimeter** of a polygon is the sum of the lengths of the outside edges.
- **Area** is a measure of the space that a shape covers on a plane. Area is measured in units of length squared.
- The area of a rectangle of length ℓ and width w is given by

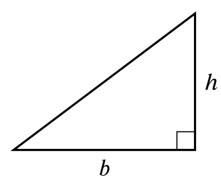
$$A = \ell w$$

The area of a square of side length s is given by

$$A = s^2$$

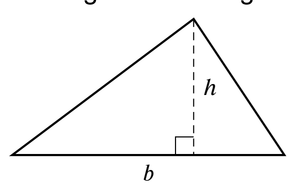
- For a triangle with a base length b and height h, the area A is found with the following formula:

$$A = \frac{1}{2}bh$$



- The height of a triangle is always perpendicular to the base.

To show the height for acute triangles (all angles smaller than 90°), you usually have to draw in a line that is perpendicular to the base and that divides the triangle into two right triangles.



For obtuse triangles (one angle is larger than 90°), depending on the orientation, you may need to draw the perpendicular line for the height outside of the triangle as in the first picture below. Alternatively, you can always rotate the triangle (as in the second picture below) so that the perpendicular height line goes into the largest angle.

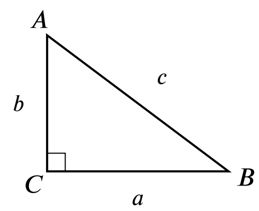
- If you need to find the area of other polygons, divide them into triangles and rectangles so that you can sum the areas of the individual sections.

Pythagorean Theorem & Special Right Triangles

- If you know two of the side lengths of a right triangle, you can use the **Pythagorean Theorem** to find the length of the third side.

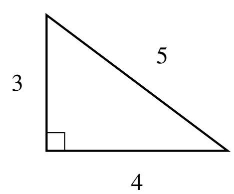
  The Pythagorean Theorem states the following for a right triangle with legs measuring a units and b units and hypotenuse measuring c units:

$$a^2 + b^2 = c^2$$



- There are some special right triangles in which all three sides of the triangle are whole numbers (or are in whole number ratios with each other). These are called **Pythagorean Triples**.
- The most common Pythagorean Triple is the $\mathbf{3:4:5}$ **triangle**.

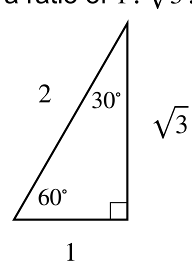
  In a right triangle with legs of length 3 and 4, the hypotenuse has length 5.



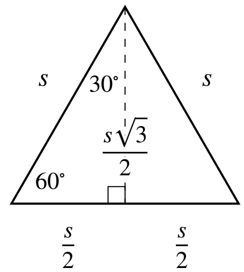
$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= 5^2 \\ 9 + 16 &= 25 \\ 25 &= 25 \end{aligned}$$

  The actual side lengths can be any set of values that conform to the $3:4:5$ ratio.

- Other Pythagorean Triples include $\mathbf{5:12:13}$ triangles (occasionally seen), and the far rarer $\mathbf{7:24:25}$ triangles, $\mathbf{8:15:17}$ triangles, and $\mathbf{20:21:29}$ triangles.
- The test makes extensive use of **angle-based special right triangles** because their sides are in easy-to-write-and-remember ratios to each other.
- The first important angle-based special right triangle is a **30-60-90 right triangle**, which has a 30°, 60°, and 90° angle. Its sides, from shortest to longest, are in a ratio of $1:\sqrt{3}:2$.

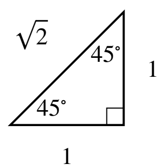


- **One of the most common ways that 30-60-90 triangles are hidden is in equilateral triangles**. If you divide an equilateral triangle with side lengths s in half by drawing a line from any vertex to its opposite side and intersecting that opposite side at a right angle, the triangle is divided into two 30-60-90 triangles where the short legs of these triangles have length $\frac{s}{2}$ and the height of the triangles is $\frac{s\sqrt{3}}{2}$.

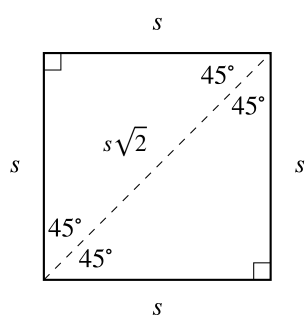


Note that this can be useful when dealing with regular hexagons, which can be divided into 6 equilateral triangles, which can then be divided into 30-60-90 triangles.

- The second important angle-based special right triangle is a **45-45-90 right triangle**. Since **two angles are the same, these triangles are isosceles, meaning that both legs are the same length**. The sides are in a ratio of $1:1:\sqrt{2}$.



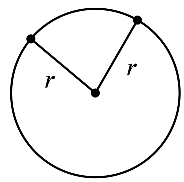
- **One of the most common ways that 45-45-90 triangles are hidden is in squares**. If you divide a square with side lengths s in half diagonally from one vertex to its opposite, the square is divided into two 45-45-90 triangles where the legs have length s and the hypotenuse has length $s\sqrt{2}$.



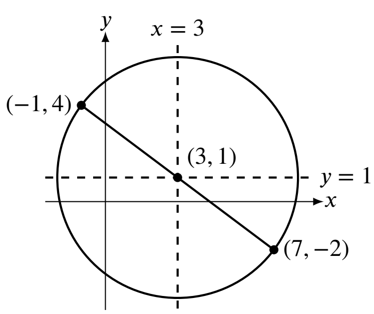
Circles & Volume

Circles & Angles

- A circle is a shape formed by **all of the points that are the same distance away from one central point** (the center of the circle). The **radius**, marked r in the figure below, is the distance from any point on the circle to the center point.



- A **diameter** is a line from one point on the arc to its opposite point on the arc that goes directly through the center of the circle and whose length, d, is equal to twice the radius: $d = 2r$.
- The center point of the circle is the midpoint between the endpoints of any diameter. This center point's x- and y-coordinates can be found by averaging the x- and y-coordinates, respectively, of the endpoints of any diameter.



$$x_{center} = \frac{(-1)+7}{2} \qquad y_{center} = \frac{4+(-2)}{2}$$

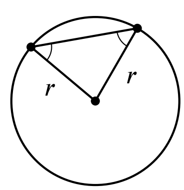
$$x_{center} = \frac{6}{2} \qquad y_{center} = \frac{2}{2}$$

$$x_{center} = 3 \qquad y_{center} = 1$$

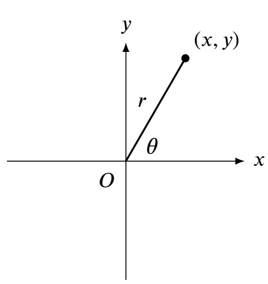
Center $= (x_{center}, y_{center})$

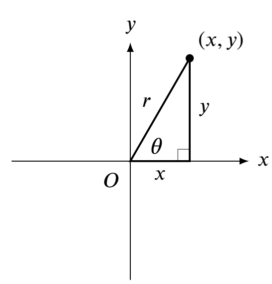
Center $= (3, 1)$

- If a triangle is formed with one vertex at the center of the circle and the other two vertices at points on the circle, then two sides of the triangle are equal to the radius, and thus the triangle is isosceles.

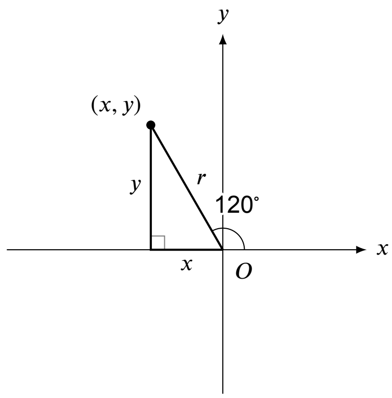


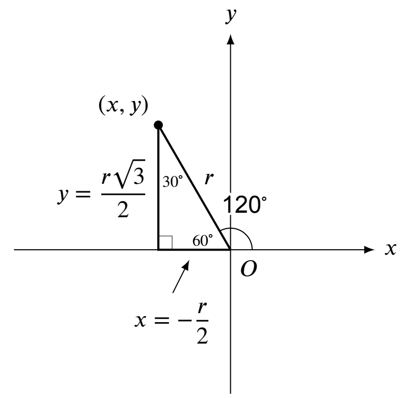
- When we graph a line segment of length r having one end at the origin, extending to a point (x, y), and lying at an angle θ measured relative to the positive x-axis, we can form a triangle as shown below. The horizontal leg has length x, and the vertical leg has length y. The values of x and y can be positive or negative, while r is always positive.





When determining absolute or relative side lengths for such a triangle, be sure to assign the correct sign. In the example below, the short leg lies along the negative x-axis, so its length will be negative. It is essential to have the proper signs for the side lengths when constructing trigonometric ratios for angles represented on the coordinate plane.

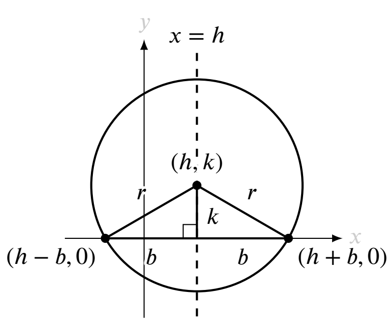


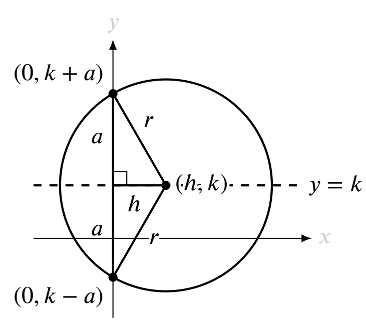


The angle ratios of a right triangle drawn on the coordinate plane are still calculated the same way (remember SOH-CAH-TOA).

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$

Pairs of points on the circle that have the same y-coordinate will be equidistant from the vertical axis of symmetry, and the distance between such points and the axis of symmetry is simply the difference in the points' x-coordinates and the axis of symmetry's (and thus the center point's) x-coordinate, as seen below, left. With respect to the horizontal axis of symmetry, the same principle applies for pairs of points that have the same x-coordinate, as seen on the right below.





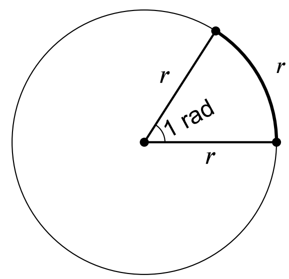
If we're not given the coordinates of the center point, but we have the coordinates of two points on the circle with either the same x- or y-value, **average the x- or y-coordinates of the two points on the circle to find the corresponding coordinate of the center point**.

Circumference, Arc Length, and Radians

- The circumference C of a circle with a radius r and diameter $d = 2r$ is found with the following equation:

$$C = 2\pi r \quad \text{or} \quad C = \pi d$$

- For an angle measuring 1 **radian**, the length of the intercepted arc is equal to the **radius**, by definition.



The number of radians of angle measure is defined as the number of radiuses in intercepted arc length, so if we know how many radians a central angle measures, we know how many radiuses the intercepted arc measures, and if we know how many radiuses long an arc measures, we know the measure in radians of the central angle that intercepts that arc.

- There are 360° in a circle, and there are 2π radians in a circle, so we can use the following proportion when we want to convert between the degrees and radians:

$$\frac{\text{degrees}}{\text{radians}} = \frac{360^\circ}{2\pi\text{ rad}} \qquad \text{or, equivalently} \qquad \frac{\text{degrees}}{\text{radians}} = \frac{180^\circ}{\pi\text{ rad}}$$

This gives:

$$\text{degrees} = \text{radians}\left(\frac{180}{\pi}\right) \qquad \text{and} \qquad \text{radians} = \text{degrees}\left(\frac{\pi}{180}\right)$$

- ***Radians* of angle produce *radiuses* of arc length**. Therefore, the formula for the length L of an arc of a circle with radius r intercepted by a central angle θ (measured in radians) is given by

$$L = r\theta$$

If you are given a central angle measure in radians, **do not convert this angle to degrees to calculate the intercepted arc length.**

- If you are given the measurement of the central angle in degrees, you can calculate the intercepted arc length by converting the angle from degrees to radians or you can find the arc's fraction of the whole circumference by using a proportion.

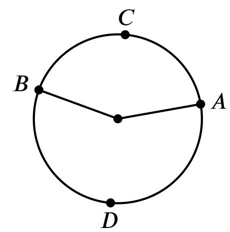
Since there are 360° in a circle, the fraction of the circle that is marked by a central angle measuring x° is $\frac{x}{360}$. The ratio of the arc length L to the whole circumference C is equal to the ratio of the central angle to 360°.

$$\frac{L}{C} = \frac{x}{360} \qquad \text{solving for } L \qquad L = \frac{x}{360}C$$

If you are not given the circumference, but you do know the radius, you can use the circumference formula and substitute $2\pi r$ for C to end up with the formula

$$L = \frac{x}{360}(2\pi r)$$

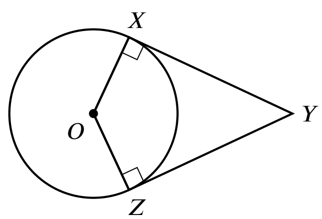
- The measure of an arc between points A and B on a circle can be denoted as $\overset{\frown}{AB}$. If there is a third point C between the points A and B, the arc may be represented as $\overset{\frown}{ACB}$. In the figure below, you may choose to refer to **minor** (smaller) **arc** $\overset{\frown}{ACB}$ in order to distinguish it from **major** (larger) **arc** $\overset{\frown}{ADB}$, which is the **complementary** arc.



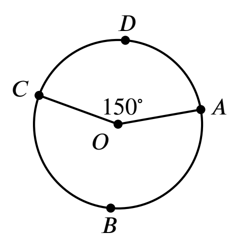
If there are only two points, and neither major nor minor is specified, assume the reference is to the minor arc.

- If you are told that lines are **tangent** to a circle, then you should note immediately that those lines **intersect the circle at exactly one point**, and a **radius drawn from the center of the circle to the point of tangency will intersect the tangent line at a right angle**.

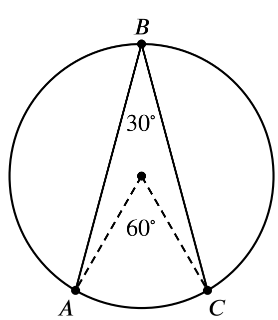
For example, in the figure below, lines $\overline{XY}$ and $\overline{ZY}$ are tangent to the circle.



- An arc can be specified by the corresponding central angle (in degrees or radians). For example, in the figure below, we can say that the arc $\overset{\frown}{ADC}$ measures 150° because the central angle forming that intercepted arc measures 150°.



- An **inscribed angle** is an angle with its vertex on the circumference of the circle. **The measure of an inscribed angle is half of the measure of the central angle of its intercepted arc**.



Area of a Circle

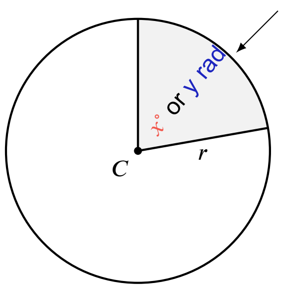
- The formula for the area A of a circle of radius r is the following:

$$A = \pi r^2$$

- The area of a sector (pie-slice-shaped portion) of a circle marked by a central angle θ is proportional to the ratio of the central angle to the angle measure of a full circle (360° or 2π radians).

$$\frac{A_{\text{sector}}}{A_{\text{circle}}} = \frac{\theta_{\text{degrees}}}{360^\circ} \quad \text{or} \quad \frac{A_{\text{sector}}}{A_{\text{circle}}} = \frac{\theta_{\text{radians}}}{2\pi}$$

Therefore, the area of a sector marked by a central angle measuring x° or y radians is found with the following equations:



$$\frac{A_{\text{sector}}}{A_{\text{circle}}} = \frac{x^\circ}{360^\circ}$$

or

$$\frac{A_{\text{sector}}}{A_{\text{circle}}} = \frac{y}{2\pi}$$

$$A_{sector} = \frac{x^\circ}{360}(\pi r^2) \quad \text{or} \quad A_{sector} = \frac{y \text{ rad}}{2\pi}(\pi r^2)$$

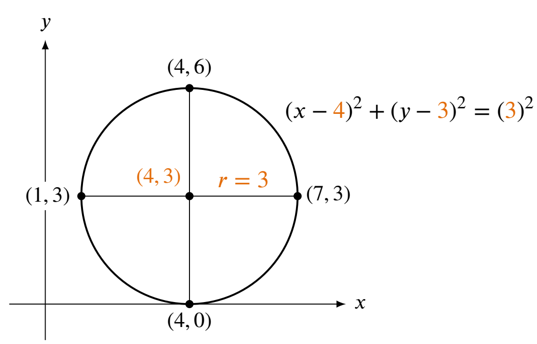
The Circle Equation

- The equation for all points (x, y) on a circle **centered at the origin** with radius r is derived from the Pythagorean Theorem:

$$x^2 + y^2 = r^2$$

For any circle with radius r, **centered at a point** $(\boldsymbol{h}, \boldsymbol{k})$, the equation for all points (x, y) on the circle is slightly modified to reflect the shifted center point:

$$(x - h)^2 + (y - k)^2 = r^2$$



All points strictly inside of a circle conform to the inequality $(x - h)^2 + (y - k)^2 < r^2$. All points strictly outside of a circle conform to the inequality $(x - h)^2 + (y - k)^2 > r^2$.

In order to write a circle equation in the form $ax^2 + bx + ay^2 + cy + d = e$ in Standard Form, $(x-h)^2 + (y-k)^2 = r^2$, follow these steps:

1. Isolate all constant terms on the opposite side of the equation from the x- and y-terms.
2. Divide both sides of the equation by a, the coefficient of the x^2 and y^2 terms, making sure to distribute the division across all terms. The x^2 and y^2 coefficients will always match for circle equations, and most often on the test will be 1, so this step will often be unnecessary.
3. Complete the square for both x- and y-variable polynomial expressions.

   If the x-coefficient is p and the y-coefficient is q, you should rewrite the x terms as $\left(x + \frac{p}{2}\right)^2$, rewrite the y terms as $\left(y + \frac{q}{2}\right)^2$, and add $\left(\frac{p}{2}\right)^2$ and $\left(\frac{q}{2}\right)^2$ to the other side of the equation.
4. The equation is now in Standard Form for circles, and you can therefore identify the center point and radius.

$$2x^2 + 12x + 2y^2 + 16y - 98 = 52$$

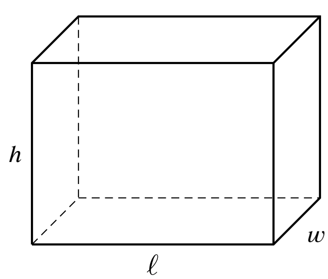
$2x^2 + 12x + 2y^2 + 16y = 150$	Isolate constants on the right side of the equation
$x^2 + 6x + y^2 + 8y = 75$	Divide by the common x^2 and y^2 coefficient
$(x+3)^2 + y^2 + 8y = 75 + 9$	Complete the square for the x-terms
$(x+3)^2 + (y+4)^2 = 75 + 9 + 16$	Complete the square for the y-terms
$(x+3)^2 + (y+4)^2 = 100$	
$(x+3)^2 + (y+4)^2 = 10^2$	

Volume

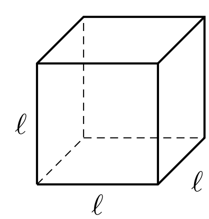
The **volume of a solid or three-dimensional figure is the amount of space that the object occupies** and is usually measured in units^3 (though the volume of fluids like water and oil might be measured in units like ounces or liters).

A **right rectangular prism** is a box (all faces meet at right angles) with 6 rectangular faces and dimensions of length ℓ, width w, and height h.

A **cube** is a special case of rectangular prism where all three dimensions are equal (there is one common side length ℓ) and therefore, every face is a square.



Rectangular Prism



Cube

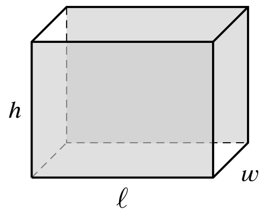
For right rectangular prisms, the volume formula is given to you in the info box at the beginning of test sections.

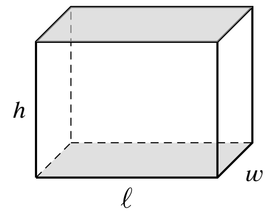
$$V = \ell w h$$

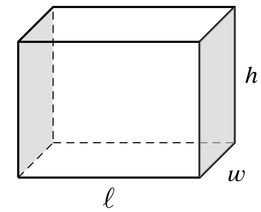
The volume of a cube with side length s is simplified because all three dimensions are equal:

$$V = s \cdot s \cdot s \qquad \text{or} \qquad V = s^3$$

➔ To find the surface area of a right rectangular prism, add up the areas of all 6 faces of the object. As shown in the figures below, there are two faces that are rectangles measuring ℓ by h, two faces that are rectangles measuring ℓ by w, and two faces measuring h by w.





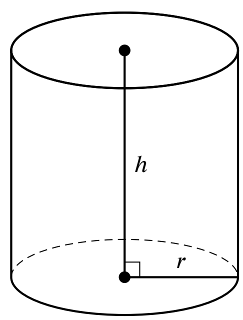


$$S = 2(\ell h + \ell w + wh)$$

For cubes with sides of length ℓ:

$$S = 6\ell^2$$

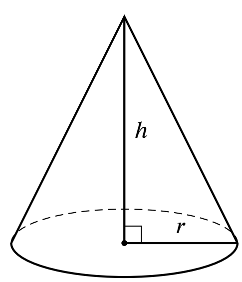
➔ A **right circular cylinder** is a three-dimensional figure with a circular base of radius r and height of h.



The volume of a cylinder is given to you in the info box at the beginning of math sections on the test, and it's the area of the circular base times the height of the object.

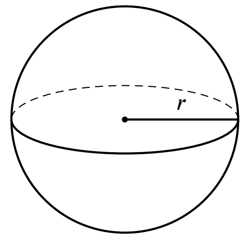
$$V_{cyl} = \pi r^2 h$$

➔ Right circular cones have a circular base with radius r, but unlike cylinders, cones taper to a point as they reach their height h.



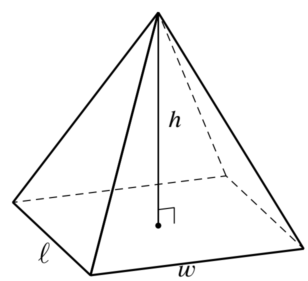
$$V = \frac{1}{3}\pi r^2 h$$

- A **sphere** is the shape formed by all points in three dimensions that are the same distance, r (a radius), away from the center point.



$$V = \frac{4}{3}\pi r^3$$

- A **right rectangular pyramid** has a rectangular base of dimensions ℓ by w and four triangular faces that meet at a point directly above the center of the base at a height of h.



$$V = \frac{1}{3}\ell w h$$

- The **density D of an object is equal to the mass of the object m divided by its volume V:**

$$D = \frac{m}{V}$$