$$f(x,y) = 2x^3 + 3xy^2$$

$$\boxed{F_{xy} = \frac{\partial^2 f}{\partial y \partial x}}$$

$$\frac{\partial f}{\partial x} = 6x^2 + 3y^2$$

$$\frac{\partial^2 f}{\partial x^2} = 12x$$

$$\frac{\partial^3 f}{\partial x^3} = 12$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial y} = 6xy$$

$$6xy \frac{\partial}{\partial x} = 6y$$

$$\frac{\partial f}{\partial x} = 6xy$$

$$\frac{\partial^2 f}{\partial x^2} = 6x$$

$$\frac{\partial^3 f}{\partial y^3} = 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \frac{\partial f}{\partial x}$$

$$\frac{\partial f}{\partial x} = 6x^2 + 3y^2$$

$$6x^2 + 3y^2 \frac{\partial}{\partial y} = 6y$$

F_{xy} and F_{yx} don't need to be equal. If there are partial derivatives for every xy then $F_{xy} = F_{yx}$

Laplace

$$U = (x^2 + y^2 + z^2)^{-\frac{1}{2}} \Rightarrow \frac{d^2 U}{dx^2} + \frac{d^2 U}{dy^2} + \frac{d^2 U}{dz^2} = 0$$

$$\frac{dU}{dx} = 2x \cdot \frac{-1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}} = -x(x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\frac{d^2 U}{dx^2} = -(x^2 + y^2 + z^2)^{-\frac{3}{2}} - x \cdot 2x(-\frac{3}{2})(x^2 + y^2 + z^2)^{-\frac{5}{2}}$$

$$\frac{dU}{dy} = 2y \cdot \frac{-1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}} = -y(x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\frac{d^2 U}{dy^2} = -(x^2 + y^2 + z^2)^{-\frac{3}{2}} - y \cdot 2y(-\frac{3}{2})(x^2 + y^2 + z^2)^{-\frac{5}{2}}$$

$$\frac{dU}{dz} = 2z \cdot \frac{-1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}} = -z \cdot (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\frac{d^2 U}{dz^2} = -(x^2 + y^2 + z^2)^{-\frac{3}{2}} - z \cdot 2z \cdot (-\frac{3}{2})(x^2 + y^2 + z^2)^{-\frac{5}{2}}$$

$$z = x^2 \arctan\left(\frac{y}{x}\right) \Rightarrow \frac{d^2 z}{dx\,dy}\bigg|_{(1,1)} = ? \qquad \frac{d}{dx}\left(\frac{dz}{dy}\right) = z_{yx}$$

$$\frac{dz}{dy} = x^2 \left(\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\right) = \frac{x \cdot x^2}{x^2 + y^2} = \frac{x^3}{x^2 + y^2}$$

$$\frac{d}{dx}\left(\frac{x^3}{x^2 + y^2}\right) = \frac{3x^2(x^2 + y^2) - 2x^4}{(x^2 + y^2)^2} = \frac{x^4 + 3x^2 y^2}{(x^2 + y^2)^2}$$

$$\frac{d^2 z}{dx\,dy}\bigg|_{(1,1)} = \frac{1+3}{(1+1)^2} = 1$$

Let's assign Δx as changes for x and Δy for y

$f(x,y)$

$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$

$\Delta x \to 0 \qquad \Delta y \to 0$

$\Delta x \cong dx \qquad \Delta y \cong dy$

$\Delta z = \frac{df}{dx} \cdot \Delta x + \frac{df}{dy} \cdot \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

where ε_1 is partial derivative for Δx and ε_2 for Δy

$\Delta x \to 0 \qquad \Delta y \to 0$

$\Delta z = \frac{df}{dx} dx + \frac{df}{dy} dy + \varepsilon_1 dx + \varepsilon_2 dy$

Important Part = Differentiation of f

$$z = f(x,y) = x^2 y - 3y \implies \Delta z = ? \quad dz = ?$$

$$\Delta x = -0,01 \qquad \Delta y = 0,02$$

$$\Delta z = \left[(x+\Delta x)^2 (y+\Delta y) - 3(y+\Delta y) \right] - (x^2 y - 3y)$$

$$\left[(x^2 + 2x\Delta x + (\Delta x)^2)(y+\Delta y) - 3(y+\Delta y) \right] - y(x^2 - 3)$$

$$y + \Delta y \left(x^2 + 2x\Delta x + (\Delta x)^2 - 3 \right)$$

$$yx^2 + y2x\Delta x + y(\Delta x)^2 - 3y + \Delta y x^2 + \Delta y 2x\Delta x + \Delta y (\Delta x)^2 - 3\Delta y - yx^2 + 3y$$

$$\underbrace{2xy\Delta x + x^2\Delta y - 3\Delta y}_{\text{important}} + \underbrace{2x\Delta x\Delta y + (\Delta x^2)y + \Delta x^2\Delta y}_{\text{ignored}}$$

$$\Delta z = 2xy\Delta x + (x^2 - 3)\Delta y$$

$$dz = 2xy\,dx + x^2 - 3\,dy \quad \cong$$

As we seen here when Δx and Δy
go to 0 then $dz \cong \Delta z$

$u = x^2 e^{y/x} \Rightarrow du = ?$

$du = \frac{du}{dx} dx + \frac{du}{dy} dy \cdots\cdots$

$dx = \left(2xe^{y/x} + \frac{y}{x^2} e^{y/x} x^2\right) dx$

$dy = (x e^{y/x}) dy$

$du = e^{y/x}(2x+y) dx + e^{y/x} dy$

Chain Rule

$z = f(x,y) \qquad x = g(r,s) \qquad y = h(r,s)$

$z \begin{cases} x \begin{cases} r \\ s \end{cases} \\ y \begin{cases} r \\ s \end{cases} \end{cases}$
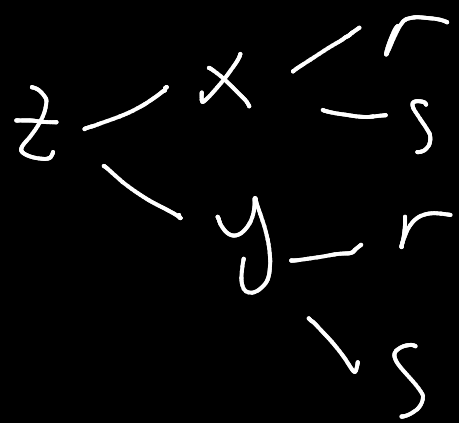
$\frac{df}{dr} = \frac{df}{dx} \cdot \frac{dx}{dr} + \frac{df}{dy} \cdot \frac{dy}{dr}$

$\frac{df}{ds} = \frac{df}{dx} \cdot \frac{dx}{ds} + \frac{df}{dy} \cdot \frac{dy}{ds}$

$$z = e^x \sin y \quad , \quad x = \ln t \quad , \quad y = t^2 \qquad \frac{dz}{dt} = ?$$

$$\frac{dz}{dx} = e^x \sin y$$

$$\frac{dz}{dy} = e^x \cos y$$

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{1}{t}$$

$$\frac{dy}{dt} = 2t$$

$$e^x \sin y \cdot \frac{1}{t} + e^x \cos y \cdot 2t$$

$$e^x \left(\frac{\sin y}{t} + \cos y \, 2t \right)$$

$$e^{\ln t} \left(\frac{\sin(t^2)}{t} + \cos(t^2) 2t \right)$$

$$\sin(t^2) + 2t^2 \cos(t^2)$$

$$T = x^3 - xy + y^3 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

$$\frac{dT}{dr} = ?$$

$$\frac{dT}{d\theta} = ?$$
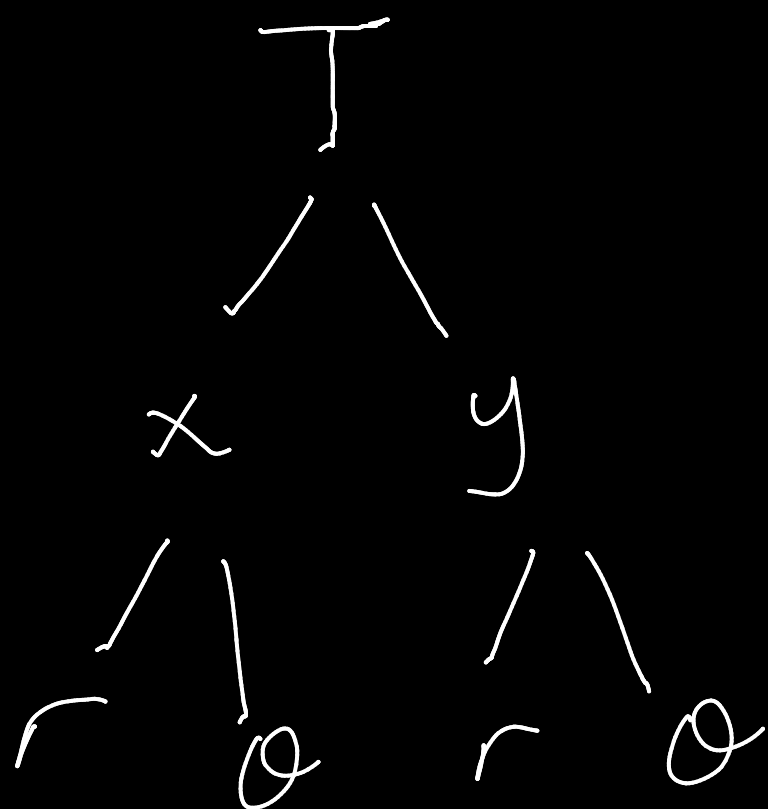
$$\frac{dT}{dr} = \frac{dT}{dx} \cdot \frac{dx}{dr} + \frac{dT}{dy} \cdot \frac{dy}{dr}$$

$$\frac{dT}{d\theta} = \frac{dT}{dx} \cdot \frac{dx}{d\theta} + \frac{dT}{dy} \cdot \frac{dy}{d\theta}$$

```
          T
         / \
        x   y
       /|   /\
      r θ  r  θ
```

$$\frac{dT}{dr} = (3x^2 - y)\cos\theta + (3y^2 - x)\sin\theta$$

$$\frac{dT}{d\theta} = (3x^2 - y) \cdot -r\sin\theta + (3y^2 - x) r\cos\theta$$

$$U = z \sin\left(\frac{y}{x}\right), \quad x = 3r^2 + s, \quad y = 4r - 2s^3, \quad z = 2r^2 - 3s^2$$

$$\frac{dU}{ds} = ? \qquad \frac{dU}{dr} = ?$$

$$\frac{dU}{ds} = \frac{dU}{dx} \cdot \frac{dx}{ds} + \frac{dU}{dy} \cdot \frac{dy}{ds} + \frac{dU}{dz} \cdot \frac{dz}{ds}$$

$$\frac{dU}{dr} = \frac{dU}{dx} \cdot \frac{dx}{dr} + \frac{dU}{dy} \cdot \frac{dy}{dr} + \frac{dU}{dz} \cdot \frac{dz}{dr}$$

$z = f(v)$ $v = x^2 y$ Prove that $x \dfrac{\partial z}{\partial x} = 2y \dfrac{\partial z}{\partial y}$

$$z$$
$$|$$
$$v$$
$$x \diagup \quad \diagdown y$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial y}{\partial y} = \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial v} = \frac{1}{2xy} \frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial v} = \frac{1}{x^2} \frac{\partial z}{\partial y}$$

$$\left. \right]$$

$$\frac{1}{2xy} \frac{\partial z}{\partial x} = \frac{1}{x^2} \frac{\partial z}{\partial y}$$

$$x \frac{\partial z}{\partial x} = 2y \frac{\partial z}{\partial y}$$

$$x^5 + y = t \qquad v = x^3 y$$

$$x^2 + y^3 = t^2 \qquad \frac{dv}{dt} = ?$$

Tree diagram:
$$v \to x \to t, \quad y \to t$$
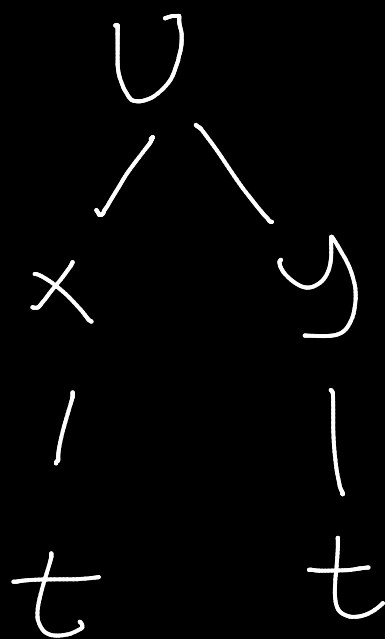
$$\frac{dv}{dt} = \underbrace{\frac{dv}{dx}}_{3x^2y} \cdot \frac{dx}{dt} + \underbrace{\frac{dv}{dy}}_{x^3} \cdot \frac{dy}{dt}$$

$$\frac{d}{dt}\left(x^5 + y = t\right) = 5x^4\frac{dx}{dt} + \frac{dy}{dt} = 1$$

$$\frac{d}{dt}\left(x^2 + y^3 = t^2\right) = 2x\frac{dx}{dt} + 3y^2\frac{dy}{dt} = 2t$$

$$\begin{bmatrix} 5x^4 & 1 \\ 2x & 3y^2 \end{bmatrix}\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix}\begin{bmatrix} 1 \\ 2t \end{bmatrix}$$

$$\frac{dx}{dt} = \frac{\begin{bmatrix} 1 & 1 \\ 2t & 3y^2 \end{bmatrix}}{\begin{bmatrix} 5x^4 & 1 \\ 2x & 3y^2 \end{bmatrix}} = \frac{3y^2 - 2t}{15x^4y^2 - 2x}$$

$$\frac{dy}{dt} = \frac{\begin{vmatrix} 5x^4 & 1 \\ 2x & 2t \end{vmatrix}}{\begin{vmatrix} 5x^4 & 1 \\ 2x & 3y^2 \end{vmatrix}} = \frac{10x^4t - 2x}{15x^4y^2 - 2x}$$

$$\frac{dv}{dt} = 3x^2y\left(\frac{3y^2 - 2t}{15x^4y^2 - 2x}\right) + x^3\left(\frac{10x^4t - 2x}{15x^4y^2 - 2x}\right)$$

Jacobian

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

$$F(u,v,w) = 0 \quad G(u,v,w) = 0 \quad H(u,v,w) = 0$$

$$\frac{\partial(F,G,H)}{\partial(u,v,w)} = \begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix}$$

$$F(x,y,u,v) = 0 \quad G(x,y,u,v) = 0$$

$$\frac{\partial u}{\partial x} = - \frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} \qquad \frac{\partial v}{\partial x} = - \frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$\frac{\partial u}{\partial y} = - \frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} \qquad \frac{\partial v}{\partial y} = - \frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$u^2 - v = 3x + y \qquad \frac{du}{dx}, \frac{du}{dy}, \frac{dv}{dx}, \frac{dv}{dy} = ?$$

$$v - 2u^2 = x - 2y$$

$$f(v, u, x, y) = 0 \Rightarrow u^2 - v - 3x - y = 0$$

$$g(v, u, x, y) = 0 \Rightarrow v - 2u^2 - x + 2y = 0$$

$$\frac{du}{dx} = -\frac{\begin{vmatrix} F_x & F_u \\ g_x & g_u \end{vmatrix}}{\begin{vmatrix} F_v & F_u \\ g_v & g_u \end{vmatrix}} = -\frac{\begin{vmatrix} -3 & -1 \\ -1 & -4u \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4 \end{vmatrix}} = -\frac{12u - 1}{-8u + 1}$$

$$\frac{du}{dy} = -\frac{\begin{vmatrix} F_y & F_u \\ g_y & g_u \end{vmatrix}}{\begin{vmatrix} F_v & F_u \\ g_v & g_u \end{vmatrix}} = -\frac{\begin{vmatrix} -1 & -1 \\ 2 & -4u \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4 \end{vmatrix}} = -\frac{4u + 2}{-8u + 1}$$

$$\frac{dv}{dx} = -\frac{\begin{vmatrix} F_v & F_x \\ g_v & g_x \end{vmatrix}}{\begin{vmatrix} F_v & F_u \\ g_v & g_u \end{vmatrix}} = -\frac{\begin{vmatrix} 2u & -3 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4 \end{vmatrix}} = -\frac{-2u + 3}{-8u + 1}$$

$$\frac{dv}{dy} = -\frac{\begin{vmatrix} F_v & F_y \\ g_v & g_y \end{vmatrix}}{\begin{vmatrix} F_v & F_u \\ g_v & g_u \end{vmatrix}} = -\frac{\begin{vmatrix} 2u & -1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4 \end{vmatrix}} = -\frac{4u + 1}{-8u + 1}$$

$$z^3 - xz - y = 0 \qquad \frac{d^2z}{dxdy} = ? \qquad$$

It is closed-form
but we have 2 independent but
1 dependent variable that's why
we can't use jacobian

$$\frac{dz}{dxdy} = \frac{d}{dx}\left(\frac{dz}{dy}\right)$$

$$\frac{dz}{dy} = 3z^2 \frac{dz}{dy} - x\frac{dz}{dy} - 1 = 0 \qquad \frac{dz}{dy} = \frac{1}{3z^2 - x}$$

We still
don't know

$$\frac{d}{dx}\left(\frac{1}{3z^2-x}\right) = \frac{d(3z^2-x)^{-1}}{dx} = -(3z^2-x)^{-2} \cdot \left(6z\frac{dz}{dx} - 1\right) = \frac{6z\frac{dz}{dx} - 1}{-(3z^2-x)^2}$$

$$\frac{dz}{dx} = 3z^2\frac{dz}{dx} - z - x\frac{dz}{dx} = 0 = \frac{dz}{dx} = \frac{z}{(3z^2-x)}$$

$$\frac{6z^3}{3z^2-x} - 1 \over -(3z^2-x)^2$$

$u = \sqrt{xy}$

is there a connection between them? If it's then how?

$v = e^{-xy} + xy$

$$\begin{matrix} u \\ \diagup \ \diagdown \\ x \quad y \end{matrix} \qquad \begin{matrix} v \\ \diagup \ \diagdown \\ x \quad y \end{matrix}$$
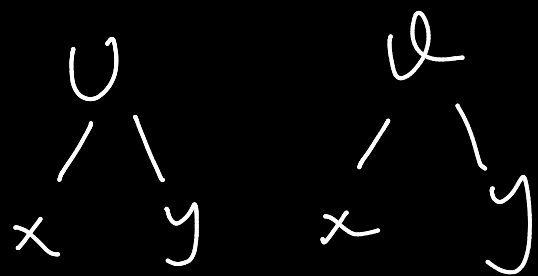
$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \dfrac{y}{2\sqrt{xy}} & \dfrac{x}{2\sqrt{xy}} \\ -y e^{-xy} + y & -x e^{-xy} + x \end{vmatrix} = 0$$

so there is a functional connection between u and v

$u^2 = xy$

$v = e^{-xy} + xy \Rightarrow v - e^{-u^2} + u^2 = 0$

$$x = r\cos\theta$$

$$0 = r\cos\theta$$

$$\theta = 90 \quad yada \quad r = 0$$

$$0 = r\sin\theta$$

$$\theta = 0 \quad yada \quad r = 0$$

Directional Derivative

$$\frac{dF}{dS} = \underset{\sim}{\nabla}F \cdot T$$

$F = x^2 y z^3$

$v = 0$

find directional derivative
through x, y, z

$x = e^{-v}$
$y = 2\sin v + 1$
$z = v - \cos v$

$\underset{\sim}{\nabla}F = 2xyz^3 \underset{\sim}{i} + x^2 z^3 \underset{\sim}{j} + x^2 y 3z^2 \underset{\sim}{k}$

$v = 0$
$\overline{x = 1}$
$y = 1$
$z = -1$

$\underset{\sim}{\nabla}F\Big|_{v=0} = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$

$T = \frac{T_0}{|T_0|} \qquad T_0 = \frac{dr}{dv}$

$r = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$r = e^{-v}\underset{\sim}{i} + (2\sin v + 1)\underset{\sim}{j} + (v - \cos v)\underset{\sim}{k}$

$\frac{dr}{dv} = -e^{-v}\underset{\sim}{i} + 2\cos v \underset{\sim}{j} + (1 + \sin v)\underset{\sim}{k}$

$\left(-2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right) \cdot \left(\frac{-i + 2\underset{\sim}{j} + k}{\sqrt{6}}\right)$

$T_0\Big|_{v=0} = -\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$

$= (2 - 2 + 3)\frac{1}{\sqrt{6}} = \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2}$

$T = \frac{-\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}}{\sqrt{6}}$

$F = 2x^3 y - 3 y^2 z$, $P(1,2,-1)$ $Q(3,-1,5)$

Find directional derivative of F for P through Q direction.

$\nabla F \cdot T$

$\nabla F = 6x^2 y \, \underset{\sim}{i} + (2x^3 - 6yz) \underset{\sim}{j} - 3y^2 \underset{\sim}{k}$

$\nabla F \big|_P = 12 \underset{\sim}{i} + 14 \underset{\sim}{j} - 12 \underset{\sim}{k}$

$T_0 = r_q - r_p = (3\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k}) - (\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}) = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}$

$T = \dfrac{T_0}{|T_0|} = \dfrac{2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}}{\sqrt{4+9+36}} = \dfrac{2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}}{7}$

$\nabla F \cdot T = (12 \underset{\sim}{i} + 14 \underset{\sim}{j} - 12 \underset{\sim}{k}) \cdot \left(\dfrac{2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}}{7} \right)$

$= \dfrac{24 - 42 - 72}{7} = \dfrac{-90}{7}$

what is the max value of directional derivative? $\left| \nabla F \big|_p \right| = \sqrt{12^2 + 14^2 + (-12)^2}$

Double Integral

$y = 2x^2$ $x = 1$ and $0x$ $f(x,y) = x^2 y - x + y$
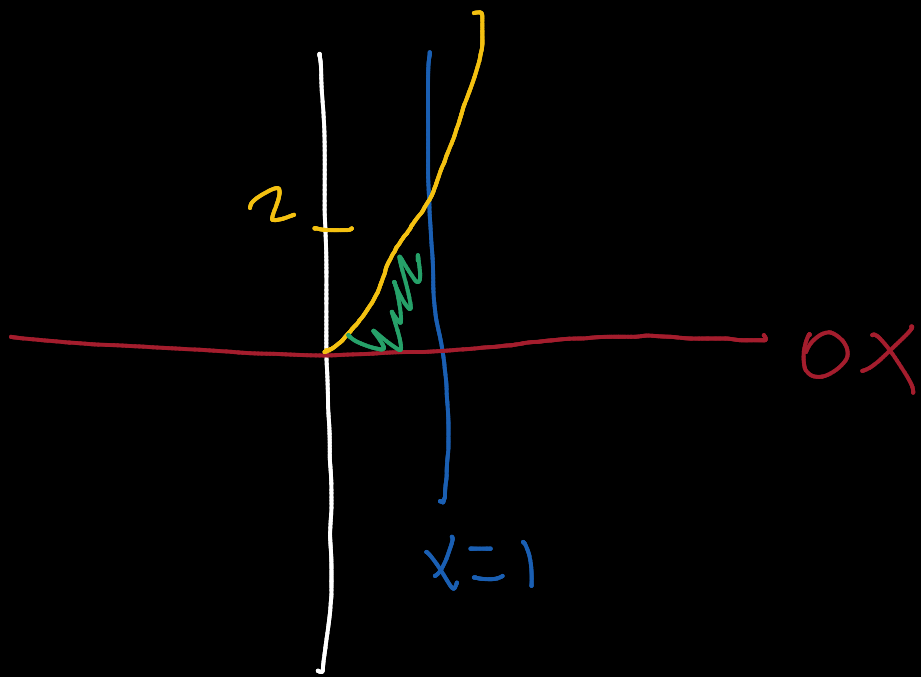
$$\int_0^1 \int_0^{2x^2} x^2 y - x + y \, dy \, dx$$

$$\int_0^1 \frac{x^2 y^2}{2} - xy + \frac{y^2}{2} \Big|_0^{2x^2}$$

$$\frac{x^2 (2x^2)^2}{2} - x(2x^2) + \frac{(2x^2)^2}{2}$$

$$\int_0^1 2x^6 - 2x^3 + 2x^4 \, dx$$

$$2\left(\frac{x^7}{7} - \frac{x^4}{4} + \frac{x^5}{5} \right) \Big|_0^1$$

$$2\left(\frac{1}{7} - \frac{1}{4} + \frac{1}{5} \right) = \frac{20 - 35 + 28}{70} = \frac{13}{70}$$

$$y \le 2x \quad, \quad x^2 - y \le 0 \qquad\qquad f(x,y) = x^3 + y$$

$y = 2x$
but shrinking
side

$x^2 = y$
but growing
side

$y = 2x$

$2x - x^2 = 0$

$x = 0$

$x = 2$
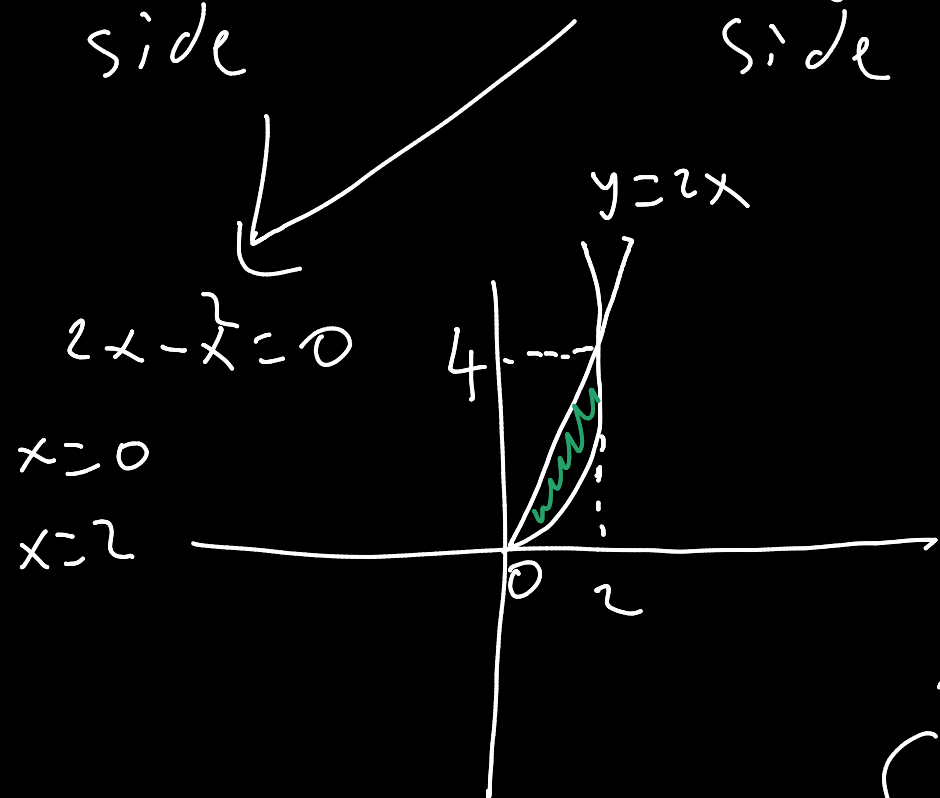
$$\int_0^2 \int_{x^2}^{2x} x^3 + y \; dy \, dx$$

$$\int_0^2 x^3 y + \frac{y^2}{2} \bigg|_{x^2}^{2x} \; dx$$

$$\int_0^2 \left(2x^4 + 2x^2\right) - \left(x^5 + \frac{x^4}{2}\right) dx$$

Area of the defined
place

$$\int_0^2 \int_{x^2}^{2x} dy \, dx$$

$$\frac{2x^5}{5} + \frac{2x^3}{3} - \frac{x^6}{6} - \frac{x^5}{10} \bigg|_0^2$$

$$\int_0^2 \left(2x - x^2\right) dx$$

$$\frac{64}{5} + \frac{16}{3} - \frac{64}{6} - \frac{32}{10}$$

$$x^2 - \frac{x^3}{3} \bigg|_0^2$$

$$= \frac{384 + 160 - 320 - 96}{30} = \frac{128}{30}$$

$$= 4 - \frac{8}{3}$$

$$= \frac{64}{15}$$

$$= \frac{4}{3} \; br$$

$y = 2\sqrt{x}$
$y = \sqrt{x}$
$y = x$

Area of the this

conjunction

$y = x$
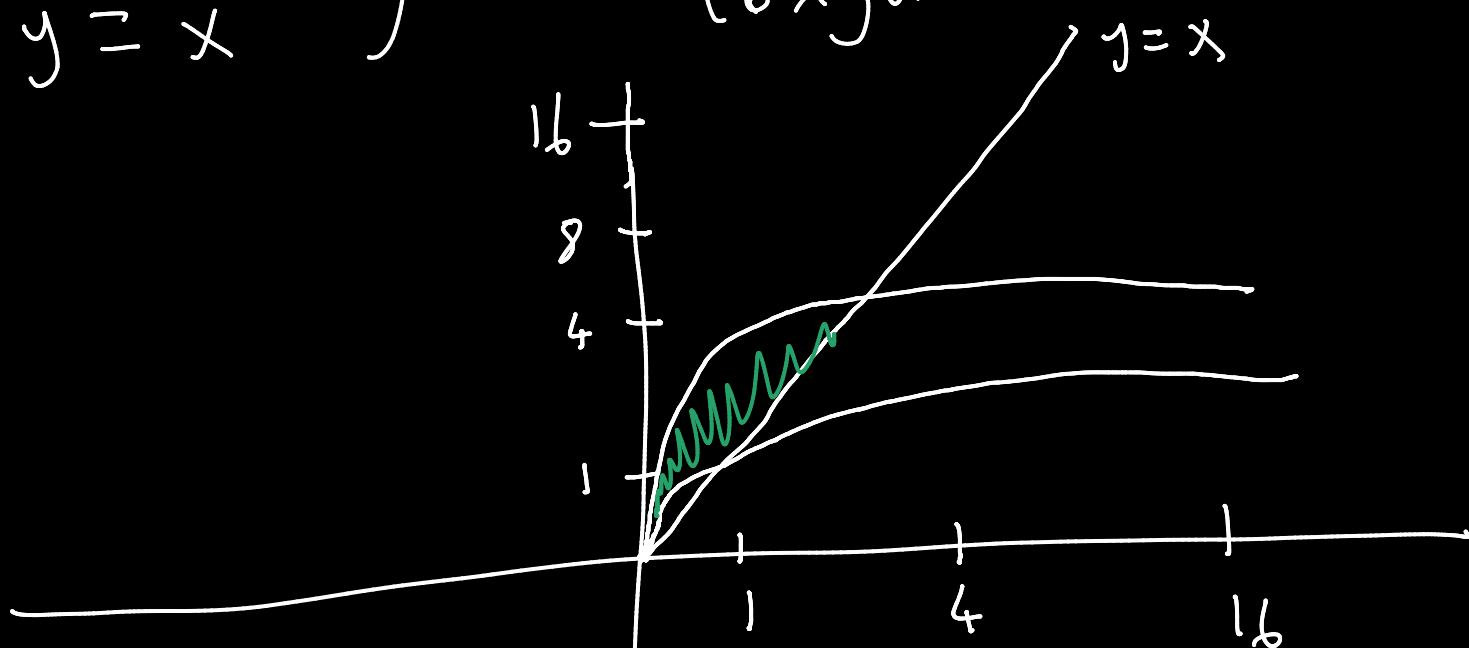
$$\int_{0}^{1}\int_{\sqrt{x}}^{2\sqrt{x}} dy\,dx + \int_{1}^{4}\int_{x}^{2\sqrt{x}} dy\,dx$$

$$\int_{0}^{1}\left(2\sqrt{x} - \sqrt{x}\right)dx + \int_{1}^{4}\left(2\sqrt{x} - x\right)dx$$

$$\frac{2x^{\frac{3}{2}}}{3}\Big|_{0}^{1} + \frac{4x^{\frac{3}{2}}}{3} - \frac{x^2}{2}\Big|_{1}^{4}$$

$$= \frac{2}{3} + \left(\frac{32}{3} - 8\right) - \left(\frac{4}{3} - \frac{1}{2}\right)$$

$$= \frac{2}{3} + \frac{8}{3} - \frac{5}{6} = \frac{4}{6} + \frac{16}{6} - \frac{5}{6} = \frac{15}{6} = \frac{5}{2}$$

$$x^2 + y^2 \leq 4x \qquad \text{area of this?}$$

$$x^2 + y^2 - 4x = 0$$

$$x^2 + y^2 - 4x + 4 - 4 = 0$$

$$x^2 - 4x + 4 + y^2 = 4$$

$$(x-2)^2 + y^2 = 4$$

$$\left((x-a)^2 + (y-b)^2 = r^2 \right)$$

$$y = \sqrt{4x - 4}$$

$$y = \sqrt{4 - (x-2)^2}$$

we choose
this for easier integs

$$2 \leftarrow x - 2 = v \qquad dx = dv$$

$$\frac{A}{2} = \int_0^4 \int_0^{\sqrt{4 - (x-2)^2}} dy \, dx = \int_{-2}^{2} \sqrt{4 - v^2} \, dv = \int_{-2}^{2} \sqrt{2^2 - v^2} \, dv$$

$$\sqrt{a^2 - x^2} \Rightarrow x = a\sin t$$
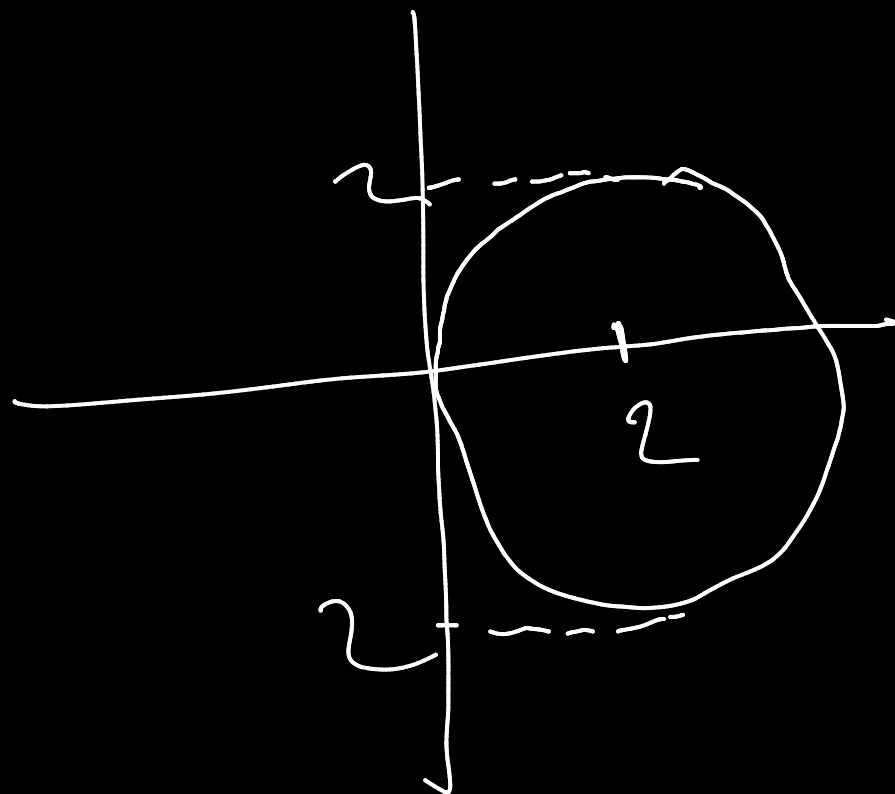
$$v = 2\sin t$$

$$dv = 2\cos t \, dt$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4 - (2\sin t)^2} \; 2\cos t \, dt$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4(1 - \sin^2 t)} \; 2\cos t \, dt$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \, dt = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + \cos 2t \, dt$$

$$2\left(t + \frac{\sin 2t}{2} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) \; 2\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 2\pi = \frac{A}{2}$$

$$A = 4\pi$$

$y^2 = 4 - x$

$y^2 = 4 - 4x$

Area ?
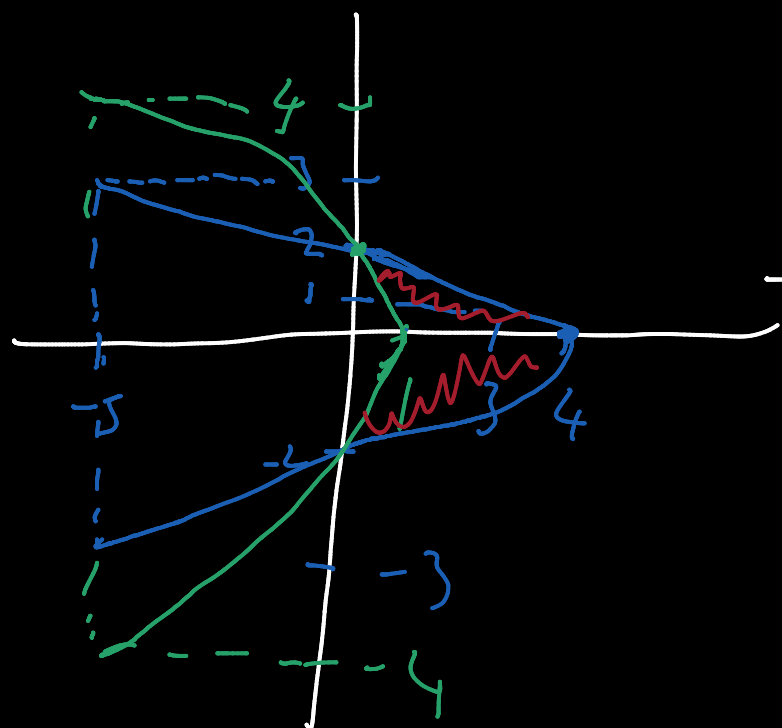
$x = \dfrac{4-y^2}{4}$ $x = 4-y^2$

$$\int_{-2}^{2} \int_{\frac{4-y^2}{4}}^{4-y^2} dx\, dy$$

$$= \int_{-2}^{2} 4 - y^2 - \left(\frac{4-y^2}{4}\right) dy$$

$$= \frac{1}{4}\int_{-2}^{2} 16 - 4y^2 - 4 + y^2 \; dy$$

$$= \frac{1}{4}\int_{-2}^{2} 12 - 3y^2 \, dy$$

$$= \frac{1}{4}\left(12y - y^3\right)\Big|_{-2}^{2}$$

$$= \frac{1}{4}\left(24 - 8\right) - \left(-24 + 8\right)$$

$$= 8 \quad \text{Area}$$

$x^2 + y^2 = a^2$

$x^2 + z^2 = a^2$

Volume?,

$z = \sqrt{a^2 - x^2}$

$y = \sqrt{a^2 - x^2}$

$x^2 + z^2 = a$

$x^2 + y^2 = a$
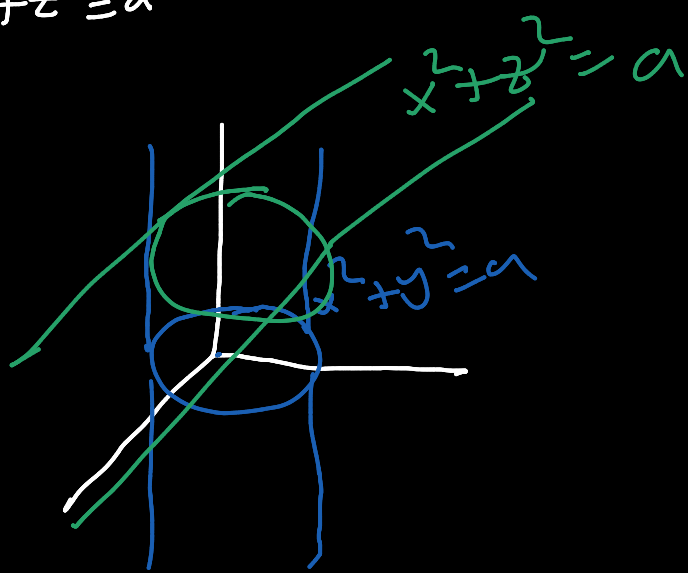
$$V = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2} \; dy \; dx$$

$$\int_{-a}^{a} \sqrt{a^2-x^2} \; y \Bigg|_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} = 2 \int_{-a}^{a} a^2 - x^2 \; dx$$

$$= 2\left(\left(a^2 x - \frac{x^3}{3}\right)\Bigg|_{-a}^{a}\right) = 2\left(\left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)\right)$$

$$= 2\left(2a^3 - 2\frac{a^3}{2}\right) = 4a^3 - \frac{4a^3}{3} = \frac{8a^3}{3}$$

$$z=0 \qquad x^2+y^2 \le 4x \qquad z+x=8 \qquad \text{Volume?}$$

$$y^2 = 4x - x^2$$

$$y = \pm\sqrt{4x-x^2}$$

$$\int_0^4 \int_{-\sqrt{4-(x-2)^2}}^{\sqrt{4-(x-2)^2}} (8-x)\,dy\,dx$$

$(x-2)^2 + y^2 = 4$

$y = \sqrt{4-(x-2)^2}$
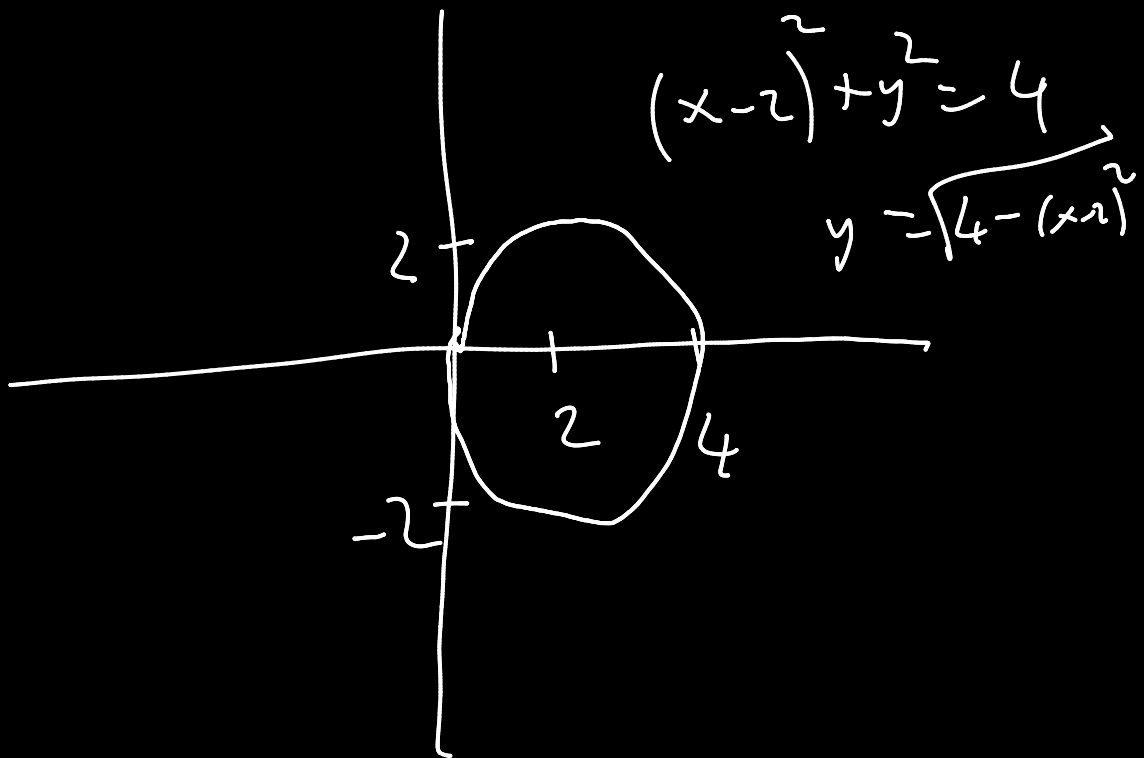
$$\int_0^4 (8y - xy) \Big|_{-\sqrt{4-(x-2)^2}}^{\sqrt{4-(x-2)^2}} dx = \int_0^4 \left(8\sqrt{4-(x-2)^2} - x\sqrt{4-(x-2)} - \left(-8\sqrt{4-(x-2)^2} + x\sqrt{4-(x-2)}\right)\right) dx$$

$$= \int_0^4 \left(16\sqrt{4-(x-2)^2} - 2x\sqrt{4-(x-2)^2}\right) = 2\int_0^4 (8-x)\left(\sqrt{4-(x-2)^2}\right) dx$$

$\sqrt{a^2-x^2}$

$x = a\sin t$

$dx = a\cos t\, dt$

$$16\int_0^4 \sqrt{4-(x-2)^2}\,dx \quad - \quad 2\int_0^4 x\sqrt{4-(x-2)^2}\,dx$$

$t \quad x-2 = 2\sin t$

$\begin{array}{c} x-2=v \\ dx=dv \end{array}$

$$16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4(1-\sin^2 t)}\,2\cos t\,dt$$

$$-2\int_{-2}^{2} (v+2)\sqrt{2^2-v^2}\,dv$$

$$64\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\,dt$$

$$-2\int_{-2}^{2} v\sqrt{4-v^2}\,dv - 4\int_{-2}^{2}\sqrt{4-v^2}\,dv$$

$v = 2\sin m$

$dv = 2\cos m\,dm$

$$-4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{4(1-\sin^2 m)}\,dm$$

$$64\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\,dt$$

$4-v^2 = \rho$

$-2v\,dv = d\rho$

$$\int_0^0 \rho^{\frac{1}{2}}d\rho = \frac{2\rho^{\frac{3}{2}}}{3} = \frac{2(4-v^2)^{\frac{3}{2}}}{3}\Big|_{v=-2}^{v=2} = 0$$

$$= -16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 m\,dm = -8\left(m+\cos 2m\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) = -8\pi$$

$$32\left(t + \frac{\sin 2t}{2}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)$$

$$= 32\left[\left(\frac{\pi}{2}+0\right) - \left(-\frac{\pi}{2}+0\right)\right]$$

$$= 32\pi \underline{\hspace{8cm}} 24\pi$$

Transformation with Double Integrals

$x^2 - y^2 = 1$

$x^2 - y^2 = 9$

$xy = 2$

$xy = 4$

$\iint (x^2 + y^2)\, dx\, dy = ?$

$x^2 - y^2 = v$

$v = 1$

$v = 9$

$2xy = w$

$w = 4$

$w = 8$
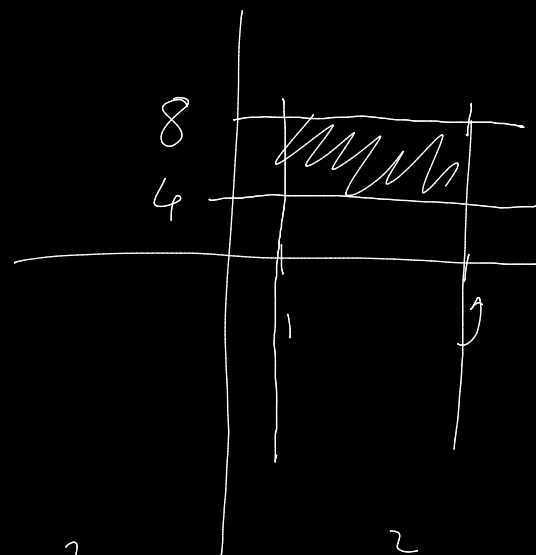
$v^2 + w^2 = \left(x^2\right)^2 - 2x^2 y^2 + \left(y^2\right)^2 + 4x^2 y^2$

$= \left(x^2 + y^2\right)^2$

$x^2 + y^2 = \sqrt{v^2 + w^2}$

$J = \dfrac{\partial(v, w)}{\partial(x, y)} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4x^2 + 4y^2$

$J = \dfrac{1}{4x^2 + 4y^2} = \dfrac{1}{4\sqrt{v^2 + w^2}}$

$\displaystyle \int_1^9 \int_4^8 \sqrt{v^2 + w^2} \, \frac{1}{4\sqrt{v^2 + w^2}} \, dw\, dv = \frac{1}{4} \int_1^9 4 \, dv = 8$

$x^2 + y^2 = 4$

$x^2 + y^2 = 9$

$x = r \cos \theta \qquad x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2$

$y = r \sin \theta$

$\iint \sqrt{x^2 + y^2} \, dx \, dy = ?$

$r = \sqrt{x^2 + y^2}$

$r = 2 \qquad \theta = 0$

$r = 3 \qquad \theta = 2\pi$

$J = \dfrac{d(x,y)}{d(r,\theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \left(\cos^2 \theta + \sin^2 \theta \right) = r$

$$\int_0^{2\pi} \int_2^3 r \cdot r \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{r^3}{3} \Big|_2^3 \, d\theta = \int_0^{2\pi} \left(9 - \frac{8}{3} \right) d\theta = \int_0^{2\pi} \frac{19}{3} \, d\theta$$

$$= \frac{38\pi}{3}$$

$x^2 + y^2 = 1 \qquad r = 1 \qquad \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \quad \frac{x}{y} = \tan\theta \quad J = \frac{d(x,y)}{d(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

$x^2 + y^2 = 16 \qquad r = 4$
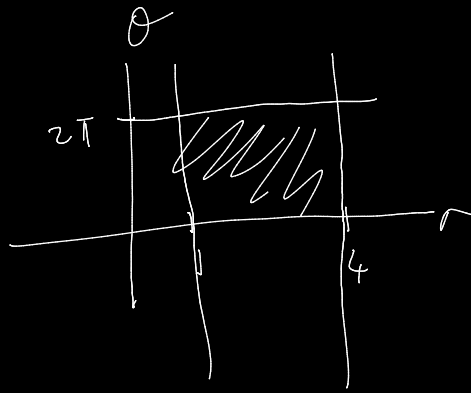
$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$\displaystyle\iint (x^2+y^2)^{\frac{1}{4}} \, dx \, dy \, ?$

$(x^2 + y^2)^{\frac{1}{4}} = \sqrt{r}$

$$\int_0^{2\pi} \int_1^4 \sqrt{r} \, r \, dr \, d\theta = \int_0^{2\pi} \int_1^4 r^{\frac{3}{2}} \, dr \, d\theta$$

$$\int_0^{2\pi} \frac{2r^{\frac{5}{2}}}{5} \Big|_1^4 \, d\theta = \frac{2}{5} \int_0^{2\pi} 31 \, d\theta$$

$$= \frac{124\pi}{5}$$

$$y = x^2$$
$$4y = x^2$$
$$2x = y^2$$
$$3x = y^2$$

$$\iint\limits_{B} dx\,dy = ?$$

$$\frac{y^2}{x} = v \qquad \frac{x^2}{y} = w$$

$$v = 2 \qquad w = 1$$
$$v = 3 \qquad w = 4$$
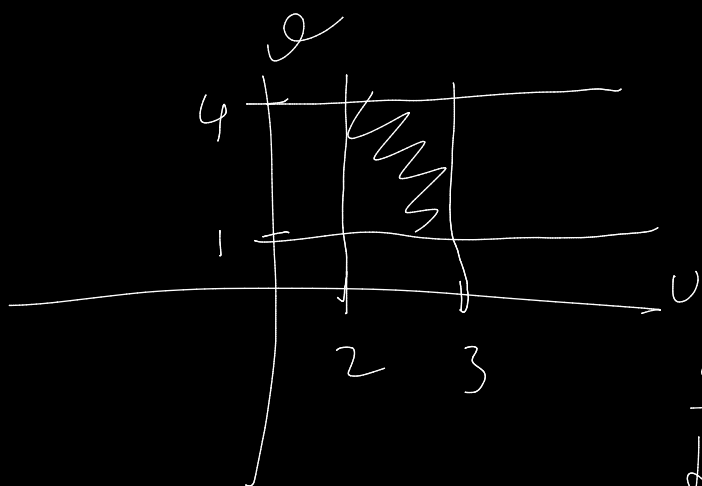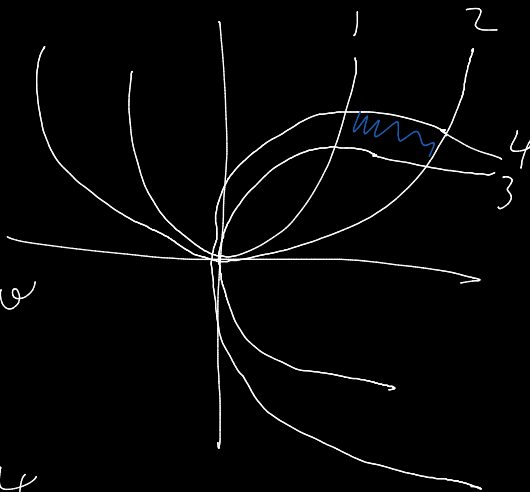
$$\frac{d(x,y)}{d(v,w)} \cdot \frac{d(v,w)}{d(x,y)} = 1$$

$$\frac{d(v,w)}{d(x,y)} = \begin{vmatrix} -y^2 x^{-2} & 2y\,x^{-1} \\ 2x\,y^{-1} & -x^2 y^{-2} \end{vmatrix} =$$

$$= \left(x^2 \cdot \frac{1}{x^2} \cdot y^2 \cdot \frac{1}{y^2} \right) - \left(4y \cdot \frac{1}{y} \cdot x \cdot \frac{1}{x} \right)$$

$$= 1 - 4 = 3$$

$$|J| = \frac{1}{3}$$

$$\int\limits_{2}^{3} \int\limits_{1}^{4} \frac{1}{3}\, dw\,dv$$

$$= \int\limits_{2}^{3} \left. \frac{w}{3} \right|_{1}^{4} dv$$

$$= \int\limits_{2}^{3} 1 \, dv$$

$$= 1$$

$y = 2x - 2$

$y = 2x - 4$

$y = -x + 2$

$y = -x + 4$

$u = x + y$

$w = 2x - y$

$$\iint\limits_{B} \underbrace{(2x-y)^2}_{w^2}\, dx\, dy\, ?$$

$u = 2 \qquad w = 2$

$v = 4 \qquad w = 4$

$$\frac{\partial(x,y)}{\partial(v,w)} \cdot \frac{\partial(v,w)}{\partial(x,y)} = 1$$

$$\frac{\partial(v,w)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ w_x & w_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3$$

$$\int_{2}^{4}\int_{2}^{4} \frac{w^2}{3}\, dw\, dv = \frac{1}{3}\int_{2}^{4} \frac{w^3}{3}\Big|_{2}^{4}\, dv = \frac{1}{9}\int_{2}^{4}(64 - 8)\, dv$$

$$\frac{1}{9}\left(56v\Big|_{2}^{4}\right) = \frac{1}{9}\left(224 - 112\right) = \frac{112}{9}$$

$$\iint\limits_{B} dx\, dy = \iint\limits_{B'} |J|\, dv\, dw$$

Mess Center of the Mess

Moment of Inertia

$M = \iint \delta(x,y)\, dx\, dy$

$\bar{x} = \dfrac{1}{M} \iint \delta(x,y)\, x\, dx\, dy$

$\bar{y} = \dfrac{1}{M} \iint \delta(x,y)\, y\, dx\, dy$

$I = \iint \delta(x,y)\, l^2\, dx\, dy$

Density is changing correspondes to square of distance between a corner of Mass ?

Center of the Mess ?

Moment of inertia correspondes to $x=4$ line ?

$\boxed{\text{Square}}$

$3\,cm$

$M = \int_0^3 \int_0^3 k\, x^2+y^2\, dy\, dx = k\iint_0^3 x^2+y^2\, dy\, dx = k\int_0^3 \left. x^2 y + \dfrac{y^3}{3}\right|_0^3 dx$

$= k\int_0^3 3x^2 + 9\, dx = 3k\left(\dfrac{x^3}{3} + 3x \left. \right)\right|_0^3 = 54k$

$\bar{x} = \dfrac{1}{54k} \int_0^3\int_0^3 k(x^2+y^2)\, x\, dx\, dy = \dfrac{1}{54}\int_0^3 \left. \dfrac{x^4}{4} + \dfrac{y^2 x^2}{2}\right|_0^3 dy = \dfrac{1}{54}\int_0^3 \dfrac{81}{4} + \dfrac{9y^2}{2}\, dy = \dfrac{1}{54}\left(\dfrac{81\cdot 3}{4} + \dfrac{9\cdot 9}{2}\right) = \dfrac{3^5+2\cdot3^4}{4\cdot27\cdot2} = \dfrac{3^4(5)}{8\cdot3} = \dfrac{15}{8}$

$\bar{y} = \dfrac{1}{54k}\int_0^3\int_0^3 k(x^2+y^2)\, y\, dx\, dy = \dfrac{1}{54}\iint_0^3 (x^2 y + y^3)\, dx\, dy = \dfrac{1}{54}\int_0^3 \left. \dfrac{x^3 y}{3} + xy^3\right|_0^3 dy = \dfrac{1}{54}\int_0^3\left(\dfrac{27y}{3} + 3y^3\right)dy = \dfrac{1}{18}\int_0^3 (y+3y^3)dy = \dfrac{1}{18}\left(\dfrac{3y^2}{2} + \dfrac{y^4}{4}\left.\right|_0^3\right) = \dfrac{1}{18}\left(\dfrac{27}{2} + \dfrac{81}{4}\right) = \dfrac{2\cdot3^3 + 3^4}{3^2\cdot2\cdot4} = \dfrac{3^3(5)}{3^2\cdot4} = \dfrac{15}{8}$

$I = \int_0^3\int_0^3 k(x^2+y^2)(4-x)^2\, dy\, dx = k\iint_0^3 (x^2+y^2)(16-8x+x^2)\, dy\, dx = k\iint_0^3\left(16x^2 + 16y^2 - 8x^3 - 8xy^2 + x^4 + x^2 y^2\right)dy\, dx =$

$16x^2 y + \dfrac{16y^3}{3}$
$-8x^3 y - \dfrac{8xy^3}{3}$
$x^4 y + \dfrac{x^2 y^3}{3}$

$48k\int_0^3 (x^2+3)\, dx = 48k\left(\dfrac{x^3}{3} + 3x\left.\right|_0^3\right) = 48k\cdot18$

$-24k\int_0^3 (x^3+3x)\, dx = -24k\left(\dfrac{x^4}{4} + \dfrac{3x^2}{2}\left.\right|_0^3\right) = -6k\cdot(3^4 + 2\cdot3^3)$

$3k\int_0^3 (x^4+3x^2)\, dx = 3k\left(\dfrac{x^5}{5} + x^3\left.\right|_0^3\right) = \dfrac{3k}{5}(3^5 + 5\cdot3^3)$

$\dfrac{1404k}{5}$

$y^2 = 4x+4$

$y^2 = 4-2x$ homogen plate.

center of the mass

moment of inertia

based on:

$4x+4 = 4-2x$

$6x=0$ $y^2=4$ $y=2$

$x = \dfrac{y^2-4}{4}$ $x = \dfrac{4-y^2}{2}$ $y=-2$

x axis)

$y=-3$ line

$\bar{x} = \dfrac{1}{m}\int x\cdot\delta(x,y)\,dxdy$

$\bar{y} = \dfrac{1}{m}\int y\cdot\delta(x,y)\,dxdy$

$\bar{x} = \dfrac{1}{8k}\int\limits_{-2}^{2}\int\limits_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}} x\cdot k\cdot dxdy$

$= \dfrac{1}{16}\int\limits_{-2}^{2}\left[\left(\dfrac{4-y^2}{2}\right)^2 - \left(\dfrac{y^2-4}{4}\right)^2\right]dy$

$= \dfrac{1}{16}\int\limits_{-2}^{2}\left[\dfrac{16-8y^2+y^4}{4} - \dfrac{y^4-8y^2+16}{16}\right]dy$

$= \dfrac{1}{16\cdot16}\int\limits_{-2}^{2}(64-32y^2+4y^4-y^4+8y^2-16)\,dy$

$= \dfrac{1}{16\cdot16}\int\limits_{-2}^{2}(48-24y^2+3y^4)\,dy$

$= \dfrac{1}{16\cdot16}\left(48y-8y^3+3\dfrac{y^5}{5}\right)\Big|_{-2}^{2}$

$= \dfrac{1}{16\cdot16}\left(96-64+\dfrac{3\cdot32}{5} - \left(-96+64-\dfrac{3\cdot32}{5}\right)\right) = \dfrac{1}{16\cdot16}\left(64+2\left(\dfrac{3\cdot32}{5}\right)\right) = \dfrac{1}{16\cdot16}\,64\left(\dfrac{4\cdot2}{5}\right)$

$\bar{x} = \dfrac{2}{5}$

$A = \int\limits_{-2}^{2}\int\limits_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}} dxdy = \int\limits_{-2}^{2}\dfrac{8-2y^2-y^2+4}{4}\,dy$

$= \dfrac{3}{4}\int\limits_{-2}^{2} 4-y^2\,dy = \dfrac{3}{4}\left(4y-\dfrac{y^3}{3}\right)\Big|_{-2}^{2}$

$= \dfrac{3}{4}\left(8-\dfrac{8}{3}+8-\dfrac{8}{3}\right) = \left(\dfrac{48-16}{3}\right)\cdot\dfrac{3}{4}$

$A = \dfrac{32\cdot3}{3\cdot4} = 8$ $m=8k$

$\bar{y} = \dfrac{1}{8k}\int\limits_{-2}^{2}\int\limits_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}} y\cdot k\cdot dxdy = \dfrac{1}{8}\int\limits_{-2}^{2} y\left(\dfrac{4-y^2}{2}-\dfrac{y^2-4}{4}\right)dy$

$= \dfrac{1}{8\cdot4}\int\limits_{-2}^{2}(8-2y^2-y^2+4)y\,dy = \dfrac{3}{8\cdot4}\int\limits_{-2}^{2}4y-y^3\,dy = \dfrac{3}{8\cdot4}\left(2y^2-\dfrac{y^4}{4}\Big|_{-2}^{2}\right)$

$= \dfrac{3}{8\cdot4}\left[(8-4)-(8-4)\right] = 0 = \bar{y}$

$I = \iint\delta(x,y)\,\ell^2\,dxdy$

$\int\limits_{-2}^{2}\int\limits_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}} k\cdot y^2\,dxdy = k\int\limits_{-2}^{2} y^2\left(\dfrac{4-y^2}{2}-\dfrac{y^2-4}{4}\right)dy$

$= \dfrac{k}{4}\int\limits_{-2}^{2}(8-2y^2-y^2+4)y^2\,dy = \dfrac{3k}{4}\int\limits_{-2}^{2} 4y^2-y^4\,dy = \dfrac{3k}{4}\left(\dfrac{4}{3}y^3-\dfrac{y^5}{5}\Big|_{-2}^{2}\right)$

$= \dfrac{3k}{4}\left[\left(\dfrac{32}{3}-\dfrac{32}{5}\right)-\left(-\dfrac{32}{3}+\dfrac{32}{5}\right)\right] = \dfrac{3k}{4}\left(\dfrac{64}{3}-\dfrac{64}{5}\right)$

$= \dfrac{3k}{4\cdot15}\left(4\left(\dfrac{16}{3}-\dfrac{16}{5}\right)\right) = \dfrac{k}{5}\cdot\left(16\cdot5-16\cdot3\right) = \dfrac{k}{5}\left(16(5-3)\right)$

$= \dfrac{32k}{5}\cdot4$

$\int\limits_{-2}^{2}\int\limits_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}} k(y+3)^2\,dxdy = k\int\limits_{-2}^{2}\int\limits_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}} y^2+6y+9\,dxdy = k\int\limits_{-2}^{2}\left(y^2+6y+9\right)x\Big|_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}}dy$

$k\int\limits_{-2}^{2}(y^2+6y+9)\left(\dfrac{4-y^2}{2}-\dfrac{y^2-4}{4}\right)dy = \dfrac{3k}{4}\int\limits_{-2}^{2}(y^2+6y+9)(4-y^2)\,dy$

$= \dfrac{3k}{4}\int\limits_{-2}^{2}(4y^2-y^4+24y-6y^3+36-9y^2)\,dy = \dfrac{3k}{4}\int\limits_{-2}^{2}(-y^4-6y^3-5y^2+24y+36)\,dy$

$= \dfrac{3k}{4}\left(-\dfrac{y^5}{5}-\dfrac{3y^4}{2}-\dfrac{5y^3}{3}+12y^2+36y\Big|_{-2}^{2}\right) = \dfrac{3k}{4}\left(\left(-\dfrac{32}{5}-24-\dfrac{40}{3}+48+72\right)-\left(\dfrac{32}{5}-24+\dfrac{40}{3}+48-72\right)\right)$

$= \dfrac{3k}{4}\left(-\dfrac{64}{5}+0-\dfrac{80}{3}+0+144\right)$

$= k\left(\dfrac{3\cdot-4\cdot4}{4\cdot5}-\dfrac{3\cdot4\cdot20}{4\cdot3}+\dfrac{3\cdot36\cdot4}{4}\right) = k\left(\dfrac{-48}{5}-20+108\right) = \left(\dfrac{-48+88}{5}\right)k$

$= \dfrac{(-48+440)k}{5} = \dfrac{392k}{5}$