$$f(x,y) = 2x^3 + 3xy^2$$

$$F_{xy} = \frac{d^2 f}{dy\,dx}$$

$$\frac{df}{dx} = 6x^2 + 3y^2$$

$$\frac{d^2 f}{dx^2} = 12x$$

$$\frac{d^3 f}{dx^3} = 12$$

$$\frac{d^2 f}{dx\,dy} = \frac{d}{dx}\frac{df}{dy}$$

$$\frac{df}{dy} = 6xy$$

$$6xy\frac{d}{dx} = 6y$$

$$\frac{df}{dx} = 6xy$$

$$\frac{d^2 f}{dx^2} = 6y$$

$$\frac{d^3 f}{dy^3} = 0$$

$$\frac{d^2 f}{dy\,dx} = \frac{d}{dy}\frac{df}{dx}$$

$$\frac{df}{dx} = 6x^2 + 3y^2$$

$$6x^2 + 3y^2\frac{d}{dy} = 6y$$

F_{xy} and F_{yx} don't need to be equal. If there are partial derivatives for every xy then $F_{xy} = F_{yx}$

Laplace

$$v=(x^2+y^2+z^2)^{-\frac{1}{2}} \Rightarrow \frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}=0$$

$$\frac{dv}{dx}=2x\cdot\frac{-1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}}=-x(x^2+y^2+z^2)^{\frac{-3}{2}}$$

$$\frac{d^2v}{dx^2}=-(x^2+y^2+z^2)^{\frac{-3}{2}}-x\cdot 2x\left(\frac{-3}{2}\right)(x^2+y^2+z^2)^{\frac{-5}{2}}$$

$$\frac{dv}{dy}=2y\cdot\frac{-1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}}=-y(x^2+y^2+z^2)^{\frac{-3}{2}}$$

$$\frac{d^2v}{dy^2}=-(x^2+y^2+z^2)^{\frac{-3}{2}}-y\cdot 2y\left(\frac{-3}{2}\right)(x^2+y^2+z^2)^{\frac{-5}{2}}$$

$$\frac{dv}{dz}=2z\cdot\frac{-1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}}=-z\cdot(x^2+y^2+z^2)^{\frac{-3}{2}}$$

$$\frac{d^2v}{dz^2}=-(x^2+y^2+z^2)^{\frac{-3}{2}}-z\cdot 2z\cdot\left(\frac{-3}{2}\right)(x^2+y^2+z^2)^{\frac{-5}{2}}$$

$$z = x^2 \arctan\left(\frac{y}{x}\right) \Rightarrow \left.\frac{\partial^2 z}{\partial x \partial y}\right|_{(1,1)} = ?$$

$$\frac{d}{dx}\left(\frac{dz}{dy}\right) = z_{yx}$$

$$\frac{dz}{dy} = x^2 \left(\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\right) = \frac{x \cdot x^2}{x^2+y^2} = \frac{x^3}{x^2+y^2}$$

$$\frac{d}{dx}\left(\frac{x^3}{x^2+y^2}\right) = \frac{3x^2(x^2+y^2) - 2x^4}{(x^2+y^2)^2} = \frac{x^4 + 3x^2y^2}{(x^2+y^2)^2}$$

$$\left.\frac{\partial^2 z}{\partial x \partial y}\right|_{(1,1)} = \frac{1+3}{(1+1)^2} = 1$$

Let's assign Δx as changes for x and Δy for y

$f(x,y)$

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$\Delta x \to 0 \quad \Delta y \to 0$

$\Delta x \cong dx \quad \Delta y \cong dy$

$$\Delta z = \frac{df}{dx} \cdot \Delta x + \frac{df}{dy} \cdot \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 is partial derivative for Δx and ε_2 for Δy

$\Delta x \to 0 \quad \Delta y \to 0$

$$\Delta z = \frac{df}{dx} dx + \frac{df}{dy} dy + \varepsilon_1 dx + \varepsilon_2 dy$$

Important Part = Differentiation of f

$z = f(x,y) = x^2 y - 3y \Rightarrow \Delta z = ? \quad dz = ?$

$\Delta x = -0,01 \qquad \Delta y = 0,02$

$$\Delta z = \left[(x+\Delta x)^2 (y+\Delta y) - 3(y+\Delta y)\right] - (x^2 y - 3y)$$

$$\left[(x^2 + 2x\Delta x + (\Delta x)^2)(y+\Delta y) - 3(y+\Delta y)\right] - y(x^2 - 3)$$

$$y+\Delta y\left(x^2 + 2x\Delta x + (\Delta x)^2 - 3\right)$$

$$\cancel{yx^2} + y2x\Delta x + y(\Delta x)^2 \cancel{-3y} + \Delta y x^2 + \Delta y 2x\Delta x + \Delta y(\Delta x)^2 - 3\Delta y \cancel{- yx^2} \cancel{+3y}$$

$$\underbrace{2xy\Delta x + x^2 \Delta y - 3\Delta y}_{\text{important}} + \underbrace{2x\Delta x\Delta y + (\Delta x^2)y + \Delta x^2 \Delta y}_{\text{ignored}}$$

$$\Delta z = 2xy\,\Delta x + (x^2 - 3)\Delta y \cong$$

$$dz = 2xy\,dx + x^2 - 3\,dy$$

As we seen here when Δx and Δy go to 0 then $dz \cong \Delta z$

$v = x^2 e^{y/x} \Rightarrow dv = ?$ $\quad dv = \frac{dv}{dx}dx + \frac{dv}{dy}dy$ .....

$dx = (2xe^{y/x} + \frac{y}{x^2}e^{y/x}x^2)\,dx$

$dy = (xe^{y/x})\,dy$

$dv = e^{y/x}(2x+y)\,dx + e^{y/x}\,dy$

Chain Rule

$z = f(x,y) \quad x = g(r,s) \quad y = h(r,s)$

$z \begin{cases} x \begin{cases} r \\ s \end{cases} \\ y \begin{cases} r \\ s \end{cases} \end{cases}$

$$\frac{df}{dr} = \frac{df}{dx}\cdot\frac{dx}{dr} + \frac{df}{dy}\cdot\frac{dy}{dr}$$

$$\frac{df}{ds} = \frac{df}{dx}\cdot\frac{dx}{ds} + \frac{df}{dy}\cdot\frac{dy}{ds}$$

$z = e^x \sin y \quad , x = \ln t \quad , y = t^2 \qquad \frac{dz}{dt} = ?$

$\frac{dz}{dx} = e^x \sin y$

$\frac{dz}{dy} = e^x \cos y$

$\frac{dx}{dt} = \frac{1}{t}$

$\frac{dy}{dt} = 2t$

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$

$$e^x \sin y \cdot \frac{1}{t} + e^x \cos y \cdot 2t$$

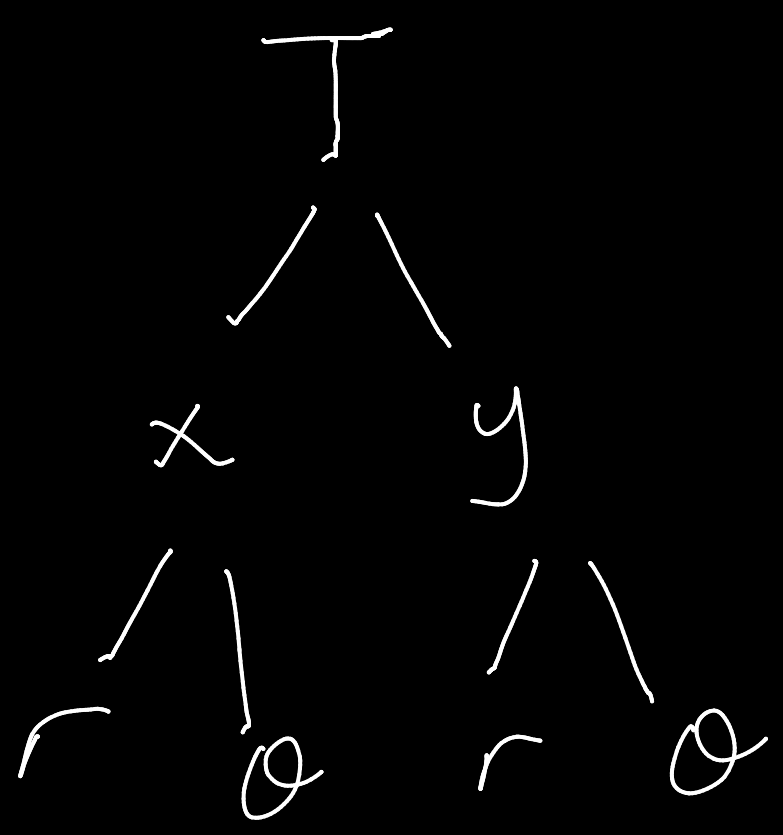
$$e^x \left(\frac{\sin y}{t} + \cos y \, 2t \right)$$

$$e^{\ln t} \left(\frac{\sin(t^2)}{t} + \cos(t^2) 2t \right)$$

$$\sin(t^2) + 2t^2 \cos(t^2)$$

$T = x^3 - xy + y^3$ $\quad x = r\cos\theta \quad y = r\sin\theta$

$\frac{dT}{dr} = ?$ $\quad \frac{dT}{d\theta} = ?$



$$\frac{dT}{dr} = \frac{dT}{dx}\cdot\frac{dx}{dr} + \frac{dT}{dy}\cdot\frac{dy}{dr}$$

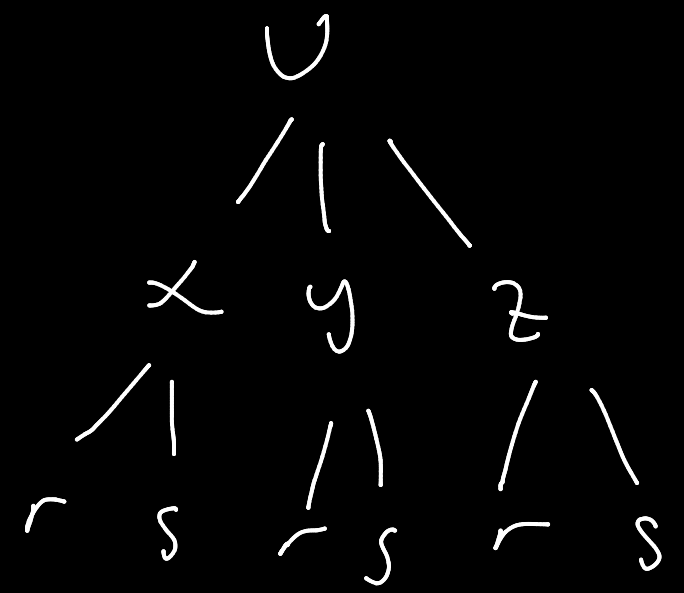
$$\frac{dT}{d\theta} = \frac{dT}{dx}\cdot\frac{dx}{d\theta} + \frac{dT}{dy}\cdot\frac{dy}{d\theta}$$

$$\frac{dT}{dr} = (3x^2 - y)\cos\theta + (3y^2 - x)\sin\theta$$

$$\frac{dT}{d\theta} = (3x^2 - y)\cdot -r\sin\theta + (3y^2 - x)\, r\cos\theta$$

$u = \sin^{-1}\left(\frac{y}{x}\right)$, $x = 3r^2 + s$, $y = 4r - 2s^3$, $z = 2r^2 - 3s^2$

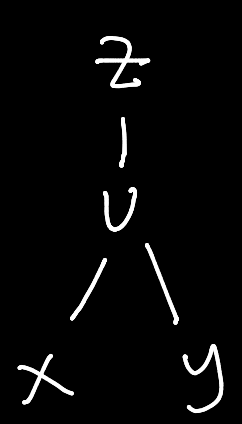
$\frac{du}{ds} = ?$ $\frac{du}{dr} = ?$



$$\frac{du}{ds} = \frac{du}{dx}\cdot\frac{dx}{ds} + \frac{du}{dy}\cdot\frac{dy}{ds} + \frac{du}{dz}\cdot\frac{dz}{ds}$$

$$\frac{du}{dr} = \frac{du}{dx}\cdot\frac{dx}{dr} + \frac{du}{dy}\cdot\frac{dy}{dr} + \frac{du}{dz}\cdot\frac{dz}{dr}$$

$z = f(v)$ $\quad v = x^2 y$ $\quad$ prove that $\quad x\frac{\partial z}{\partial x} = 2y\frac{\partial z}{\partial y}$



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial y}{\partial y} = \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial v} = \frac{1}{2xy}\frac{\partial z}{\partial x}$$

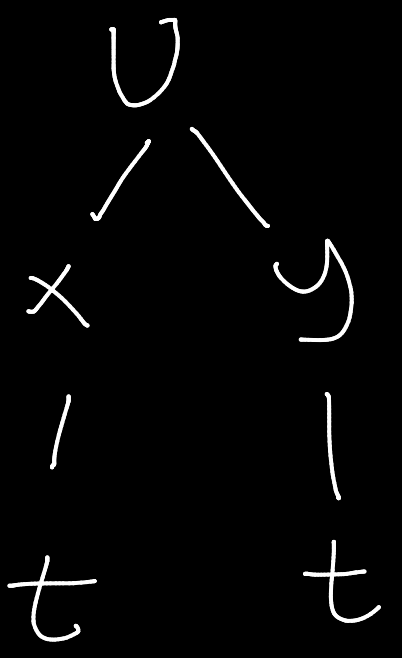
$$\frac{\partial z}{\partial v} = \frac{1}{x^2}\frac{\partial z}{\partial y}$$

$$\frac{1}{2xy}\frac{\partial z}{\partial x} = \frac{1}{x^2}\frac{\partial z}{\partial y}$$

$$x\frac{\partial z}{\partial x} = 2y\frac{\partial z}{\partial y}$$

$$x^5+y=t \qquad u=x^3y \qquad \frac{du}{dt}=?$$

$$x^2+y^3=t^2$$



$$\frac{du}{dt}=\frac{du}{\underbrace{dx}_{3x^2y}}\cdot\frac{dx}{dt}+\frac{du}{\underbrace{dy}_{x^3}}\cdot\frac{dy}{dt}$$

$$\frac{d}{dt}\left(x^5+y=t\right)=5x^4\frac{dx}{dt}+\frac{dy}{dt}=1$$

$$\frac{d}{dt}\left(x^2+y^3=t^2\right)=2x\frac{dx}{dt}+3y^2\frac{dy}{dt}=2t$$

$$\begin{bmatrix}5x^4 & 1\\ 2x & 3y^2\end{bmatrix}\begin{bmatrix}dx/dt\\ dy/dt\end{bmatrix}\begin{bmatrix}1\\ 2t\end{bmatrix}$$

$$\frac{dx}{dt}=\frac{\begin{bmatrix}1 & 1\\ 2t & 3y^2\end{bmatrix}}{\begin{bmatrix}5x^4 & 1\\ 2x & 3y^2\end{bmatrix}}=\frac{3y^2-2t}{15x^4y^2-2x}$$

$$\frac{dy}{dt}=\frac{\begin{vmatrix}5x^4 & 1\\ 2x & 2t\end{vmatrix}}{\begin{vmatrix}5x^4 & 1\\ 2x & 3y^2\end{vmatrix}}=\frac{10x^4t-2x}{15x^4y^2-2x}$$

$$\frac{du}{dt}=3x^2y\left(\frac{3y^2-2t}{15x^4y^2-2x}\right)+x^3\left(\frac{10x^4t-2x}{15x^4y^2-2x}\right)$$

Jacobian

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

$F(u,v,w)=0 \quad G(u,v,w)=0 \quad H(u,v,w)=0$

$$\frac{\partial(F,G,H)}{\partial(u,v,w)} = \begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix}$$

$F(x,y,u,v)=0 \quad G(x,y,u,v)=0$

$$\frac{du}{dx} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} \qquad \frac{dv}{dx} = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$\frac{du}{dy} = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} \qquad \frac{dv}{dy} = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$u^2 - v = 3x + y \qquad \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} = ?$$
$$v - 2u^2 = x - 2y$$

$$f(u, v, x, y) = 0 \Rightarrow u^2 - v - 3x - y = 0$$

$$g(u, v, x, y) = 0 \Rightarrow v - 2u^2 - x + 2y = 0$$

$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} -3 & -1 \\ -1 & -4v \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4 \end{vmatrix}} = -\frac{12u - 1}{-8u + 1}$$

$$\frac{\partial u}{\partial y} = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} -1 & -1 \\ 2 & -4v \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4 \end{vmatrix}} = -\frac{4u + 2}{-8u + 1}$$

$$\frac{\partial v}{\partial x} = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} 2u & -3 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4 \end{vmatrix}} = -\frac{-2u + 3}{-8u + 1}$$

$$\frac{\partial v}{\partial y} = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} 2u & -1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4 \end{vmatrix}} = -\frac{4u + 1}{-8u + 1}$$

$z^3 - xz - y = 0 \qquad \frac{d^2z}{dxdy} = ?$

It is closed-form but we have 2 independent but 1 dependent variable that's why we can't use Jacobian

$$\frac{dz}{dxdy} = \frac{d}{dx}\left(\frac{dz}{dy}\right)$$

$$\frac{dz}{dy} = 3z^2\frac{dz}{dy} - x\frac{dz}{dy} - 1 = 0 \qquad \frac{dz}{dy} = \frac{1}{3z^2 - x}$$

we still don't know

$$\frac{d}{dx}\left(\frac{1}{3z^2-x}\right) = \frac{d(3z^2-x)^{-1}}{dx} = -(3z^2-x)^{-2}\cdot\left(6z\frac{dz}{dx} - 1\right) = \frac{6z\frac{dz}{dx} - 1}{-(3z^2-x)^2}$$

$$\frac{dz}{dx} = 3z^2\frac{dz}{dx} - z - x\frac{dz}{dx} = 0 \Rightarrow \frac{dz}{dx} = \frac{z}{(3z^2-x)}$$

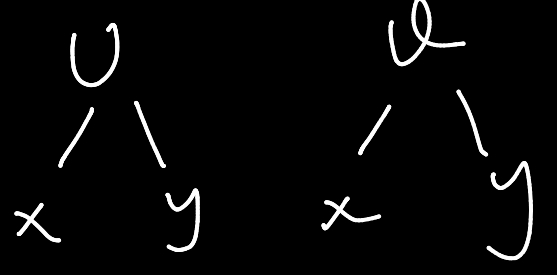
$$\frac{6z^2 - 1}{\frac{3z^2 - x}{-(3z^2-x)^2}}$$

$u = \sqrt{xy}$

$v = e^{-xy} + xy$

is there a connection between them?

If it's then how?



$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{y}{2\sqrt{xy}} & \frac{x}{2\sqrt{xy}} \\ -ye^{-xy}+y & -xe^{-xy}+x \end{vmatrix} = 0$$

so there is a functional connection between u and v

$u^2 = xy$

$v = e^{-xy} + xy \Rightarrow v - e^{-u^2} + u^2 = 0$

$x = r\cos\theta$

$\theta = r\cos\theta$

$\theta = \frac{\pi}{2}$ yada $r = 0$

$\theta = r\sin\theta$

$\theta = 0$ yada $r = 0$

Directional Derivative

$$\frac{dF}{ds} = \nabla F \cdot T$$

$F = x^2 y z^3$

$x = e^{-v}$

$y = 2\sin v + 1$

$z = v - \cos v$

$v = 0$

find directional derivative through x, y, z

$$\nabla F = 2xyz^3 \underset{\sim}{i} + x^2 z^3 \underset{\sim}{j} + x^2 y 3z^2 \underset{\sim}{k}$$

$$T = \frac{T_0}{|T_0|} \qquad T_0 = \frac{dr}{dv}$$

$v = 0$

$x = 1$

$y = 1$

$z = -1$

$$\nabla F\Big|_{v=0} = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$$

$$r = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$$

$$r = e^{-v}\underset{\sim}{i} + (2\sin v + 1)\underset{\sim}{j} + (v - \cos v)\underset{\sim}{k}$$

$$\frac{dr}{dv} = -e^{-v}\underset{\sim}{i} + 2\cos v \underset{\sim}{j} + (1 + \sin v)\underset{\sim}{k}$$

$$\left(-2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right) \cdot \left(\frac{-\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}}{\sqrt{6}}\right)$$

$$T_0\Big|_{v=0} = -\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$$

$$= (2 - 2 + 3)\frac{1}{\sqrt{6}} = \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2}$$

$$T = \frac{-\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}}{\sqrt{6}}$$

$F = 2x^3y - 3y^2z$ $\quad P(1,2,-1)$ $\quad Q(3,-1,5)$

Find directional derivative of F for P through Q direction.

$\nabla F \cdot T \qquad \nabla F = 6x^2y\,i + (2x^3 - 6yz)j - 3y^2k$

$$\nabla F\Big|_P = 12i + 14j - 12k$$

$$T_0 = r_q - r_p = (3i - j + 5k) - (i + 2j - k) = 2i - 3j + 6k$$

$$T = \frac{T_0}{|T_0|} = \frac{2i - 3j + 6k}{\sqrt{4+9+36}} = \frac{2i - 3j + 6k}{7}$$

$$\nabla F \cdot T = (12i + 14j - 12k) \cdot \left(\frac{2i - 3j + 6k}{7}\right)$$

$$= \frac{24 - 42 - 72}{7} = \frac{-90}{7}$$

what is the max value of directional derivative? $|\nabla F|_P| = \sqrt{12^2 + 14^2 + (-12)^2}$