Problem 1 (CO5): Nonregular Language (10 points)

Use the pumping lemma to **demonstrate** that L_1 and L_2 are not regular.

(a) $L_1 = \{w \in \{0,1\}^* : w = 0^{n!} \text{ where } n \geq 0\}$ (5 points)

(b) $L_2 = \{w \in \{0,1\}^* : w = 0^a 1^b 1^c 0^d \text{ where } a + b = c + d \text{ and } a, b, c, d \geq 0\}$ (5 points)

(a) Assume for the sake of contradiction that L_1 is regular. Then let p be the pumping length for L_1. Now we take the string

$$w = 0^{p!} \in L_1.$$

Then the length of w is $|w| = p! \geq p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq p$, and $xy^iz \in L_1$ for each $i \geq 0$. y consists of only 0s, so

$$xy^iz = 0^{p!+(i-1)|y|}$$

Then, for $i = 2$, xy^2z will be

$$xy^2z = xyyz = 0^{p!+|y|}.$$

Now, $|y| \leq p < p \cdot p!$, hence,

$$p! < p! + |y| < p! + p \cdot p! = p!(1+p) = (p+1)!$$

So $p! < p! + |y| < (p+1)!$, and the length of xy^2z is strictly between two consecutive factorials. Hence, it cannot be a factorial. Thus we get a contradiction! Hence, L_1 is not a regular language.

(a) **(Alternate Solution)** Assume for the sake of contradiction that L_1 is regular. Then let p be the pumping length for L_1. Now we take the string

$$w = 0^{p!} \in L_1.$$

Then the length of w is $|w| = p! \geq p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq p$, and $xy^iz \in L_1$ for each $i \geq 0$. y consists of only 0s, so

$$xy^iz = 0^{p!+(i-1)|y|}$$

Then, for $i = 2$, xy^2z will be

$$xy^2z = xyyz = 0^{p!+|y|}.$$

Since this string is in L_1, the length has to be a factorial. But the length is strictly greater than $p!$, since $|y| > 0$. Hence, the length is at least $(p+1)!$.

$$p! + |y| \geq (p+1)! \implies |y| \geq (p+1)! - p! = p \cdot p!$$

But we know $|xy| \leq p$, so that $|y| \leq p$. This is a contradiction, since $p < p \cdot p!$. Hence, L_1 is not a regular language.

(b) Assume for the sake of contradiction that L_2 is regular. Then let p be the pumping length for L_2. Now we take the string

$$w = 0^p 1 1 0^p \in L_2.$$

Then the length of w is $|w| = 2p + 2 \geq p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq p$, and $xy^i z \in L_2$ for each $i \geq 0$. $|xy| \leq p$, so y consists of only 0s, so

$$xy^i z = 0^{p+(i-1)|y|} 1 1 0^p.$$

We choose $i = 4$, so that

$$xy^4 z = 0^{p+3|y|} 1 1 0^p.$$

Since this is in L_2, one can write it as $0^a 1^b 1^c 0^d$ for $a + b = c + d$. By equating

$$0^{p+3|y|} 1 1 0^p = 0^a 1^b 1^c 0^d,$$

we get $a = p + 3|y|$, $d = p$ and $b + c = 2$. So $c - b \leq 2$. Furthermore,

$$c - b = a - d = 3|y| \geq 3,$$

as $|y| \geq 1$. So we get $c - b \leq 2$ and $c - b \geq 3$, which is a contradiction! Therefore, L_2 is not regular.

(b) **(Alternate Solutiion)** Assume for the sake of contradiction that L_2 is regular. Then let p be the pumping length for L_2. Now we take the string

$$w = 0^p 1^p 1^p 0^p \in L_2.$$

Then the length of w is $|w| = 4p \geq p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq p$, and $xy^i z \in L_2$ for each $i \geq 0$. $|xy| \leq p$, so y consists of only 0s, so

$$xy^i z = 0^{p+(i-1)|y|} 1^p 1^p 0^p.$$

We choose $i = 2p + 2$, so that

$$xy^{2p+2} z = 0^{p+(2p+1)|y|} 1^p 1^p 0^p.$$

Since this is in L_2, one can write it as $0^a 1^b 1^c 0^d$ for $a + b = c + d$. By equating

$$0^{p+(2p+1)|y|} 1^p 1^p 0^p = 0^a 1^b 1^c 0^d,$$

we get $a = p + (2p + 1)|y|$, $d = p$ and $b + c = 2p$. So $c - b \leq 2p$. Furthermore,

$$c - b = a - d = (2p + 1)|y| \geq 2p + 1,$$

as $|y| \geq 1$. So we get $c - b \leq 2p$ and $c - b \geq 2p + 1$, which is a contradiction! Therefore, L_2 is not regular.