CSE331: Automata and Computability
Quiz on: Context-Free Grammar

Construct CFG for the following grammars.

a) $L = \{w \in \{(,)\}^* : w \text{ is a valid parenthesis.}\}$

$S \rightarrow (S) \mid SS \mid \varepsilon$

d) $L = \{w \in \{a, b, 1\}^* : a^n 1^{2n+m+3} b^m, \text{ where } n, m \geq 0\}$

$a^n 1^{2n+m+3} b^m$

$\Rightarrow \underbrace{a^n 1^{2n}} \overbrace{1^m 1^3 b^m}$ or $\underbrace{a^n 1^{2n}} 1^3 \overbrace{1^m b^m}$

$S \rightarrow PQ$
$P \rightarrow aP11 \mid \varepsilon$
$Q \rightarrow 1Qb \mid 111$

$S \rightarrow P111Q$
$P \rightarrow aP11 \mid \varepsilon$
$Q \rightarrow 1Qb \mid \varepsilon$

b) Convert the following Regular Expressions into a CFG: $((aa+bc)^* a)^* + cb$

$((\overbrace{aa+bc}^{F})^* a)^* + \underbrace{cb}_{Y}$

X, P, M

$S \rightarrow X \mid Y$
$X \rightarrow PX \mid \varepsilon$
$P \rightarrow Ma$
$M \rightarrow FM \mid \varepsilon$
$F \rightarrow aa \mid bc$
$Y \rightarrow cb$

c) $L = \{w \in \{0, 1\}^* : \text{length of } w \text{ is odd and the mid is } 1.\}$

$S \rightarrow 0S0 \mid 0S1 \mid 1S0 \mid 1S1 \mid 1$

or, $S \rightarrow XSX \mid 1$
$X \rightarrow 0 \mid 1$

Wrong Solve:

~~$S \rightarrow A1A$~~
~~$A \rightarrow 0A \mid 1A \mid \varepsilon$~~

d) $L = \{w \in \{0,1\}^*: w$ is an odd-length palindrome.$\}$

$S \rightarrow 0S0 \mid 1S1 \mid 0 \mid 1$

d) $L = \{w \in \{0,1\}^*: w$ is an even-length palindrome.$\}$

$S \rightarrow 0S0 \mid 1S1 \mid \varepsilon$

e) $L = \{w \in \{0, 1\}^*:$ the length of w is three more than the multiple of four.$\}$

$A \rightarrow MMMMA \mid MMM$

$M \rightarrow 0 \mid 1$

f) $L = \{w \in \{0, 1\}^*:$ the length of w is two more than the multiple of three.$\}$

$A \rightarrow MMMA \mid MM$

$M \rightarrow 0 \mid 1$

g) $L1 = \{w \in \{0, 1\}^*$, length of w is odd.$\}$
$L2 = \{ w1\#w2$, where $w1, w2 \in L1$ and $|w1| = |w2|\}$.
Construct a CFG for L2.

$S \rightarrow XXSXX \mid X\#X$

$X \rightarrow 0 \mid 1$

h) $L1 = \{w \in \{0, 1\}^*:$ count of 0 is a multiple of three.$\}$
$L2 = \{ w1\#w2$, where $w1 \in \{0,1\}^*, w2 \in L1$ and $|w1| = |w2|\}$.
Construct a CFG for L2.

$A \rightarrow XA1 \mid XB0 \mid \#$

$B \rightarrow XB1 \mid XC0$

$C \rightarrow XC1 \mid XA0$

i) $L = \{w \in \{a, b, c, d\}^*: a^n c^m d^m b^{3n}$, where $n \geq 0$ and $m \geq 1 \}$

$a^n c^m d^m b^{3n}$

$S \rightarrow aSbbb \mid X$

$X \rightarrow cXd \mid cd$

$m \geq 1$

Alternate solution

$S \rightarrow aSbbb \mid cXd$

$X \rightarrow cXd \mid \varepsilon$

j) $L = \{w \in \{0, 1\}^*: 0^n 1^m$, where $m \geq 3n\}$
[Hint: First, try to solve for m=3n]

if $m = 3n$

$0^n 1^{3n}$

since $m \geq 3n$

$0^n \_ 1^{3n}$ can have more 1s

$m = 3n$

$S \rightarrow 0S111 \mid A$

$A \rightarrow A1 \mid \varepsilon$

k) L = {w ∈ {a, b, c, d}*: $a^n c^{m+3} d^m b^{3n}$, where $n \geq 0$ and $m \geq 1$ }

$$a^n c^{m+3} d^m b^{3n}$$

$$\Rightarrow a^n c^m c^3 d^m b^{3n}$$

$S \rightarrow aSbbb \mid cAd \rightarrow m \geq 1$

$A \rightarrow cAd \mid c$

Alternate solution:

$S \rightarrow aSbbb \mid A_{m \geq 1}$

$A \rightarrow cAd \mid c\underbrace{ccc}_{c^3}d$

l) L3 = {w ∈ {a, b, 1}*: $a^n 1^{k+3} b^m$, where $k=2n+m$ and n, m, $k \geq 0$}

$$a^n 1^{K+3} b^m$$

$$\rightarrow a^n 1^{2n+m+3} b^m$$

Previously solved,
Same as Qs:

m) L = {w ∈ {0, 1}*: length of w is a multiple of six.}

$S \rightarrow XXXXXXS \mid \varepsilon$

$X \rightarrow 0 \mid 1$

n) L = {w ∈ {0, 1}*: w1##w2, where w1,w2 ∈ {0, 1}* and |w1| = |w2|.}

$S \rightarrow XSX \mid \#\#$

$X \rightarrow 0 \mid 1$

o) L1 ={ w ∈ {0, 1}*: length of w is multiple of six}
L2 = {w ∈ {0, 1}*: w1##w2, where w1,w2 ∈ {0, 1}* and |w1| = |w2|.}
L = L1 ∩ L2, Construct a CFG for L.

$S \rightarrow XXXSXXX \mid XX\#\#XX$

$X \rightarrow 0 \mid 1$

multiple of Six

Since ## in fixed, need to adjust the terminate length

p) L = {w ∈ {0, 1}*: w contains at least three 1s.}

$$(0+1)^*1(0+1)^*1(0+1)^*1(0+1)^*$$

$S \rightarrow X1X1X1X$

$X \rightarrow 0X \mid 1X \mid \varepsilon$

q) L1 = {w ∈ {0, 1}*: w contains at least three 1s.}
L2 = {w ∈ {0, 1}*: w1#w2, where w1,w2 ∈ {0, 1}* and |w1| = |w2|.}
L = L1 ∩ L2, Construct a CFG for L.

found no 1

0 0 1 0 0 1 # 1 1 0 1 0 0

found one 1

S → 0S0 | 1A0 | 0A1 | 1B1

found two 1s

A → 0A0 | 1B0 | 0B1 | 1C1

found three or more 1s

B → 0B0 | 1C0 | 0C1 | 1C1

C → XCX | #

more than three 1s

X → 0|1

r) L1 = {w ∈ {0, 1}*: w contains exactly three 1s.}
L2 = {w ∈ {0, 1}*: w1#w2, where w1,w2 ∈ {0, 1}* and |w1| = |w2|.}
L = L1 ∩ L2, Construct a CFG for L.

found no 1

0 0 1 0 0 1 0 # 0 0 1 0 0 0 0

found one 1

S → 0S0 | 1A0 | 0A1 | 1B1

found two 1s

A → 0A0 | 1B0 | 0B1 | 1C1

found three 1s

B → 0B0 | 1C0 | 0C1

C → 0C0 | #

s) L1 = {w ∈ {0, 1}*: w contains exactly three 1s.}
L2 = {w ∈ {0, 1}*: w1#w2, where w1 ∈ {0, 1}*, w2 ∈ L1, and |w1| = |w2|.}
Construct a CFG for L2.

00011011#00101100

S → XS0 | XA1

A → XA0 | XB1

B → XB0 | XC1

C → XC0 | #

C → 0|1

Take a look at the following grammar and solve the following problems:

$$P \rightarrow 00P \mid 01P \mid 10P \mid 11P \mid X$$
$$X \rightarrow 00Y \mid 10Y$$
$$Y \rightarrow 0Y \mid 1Y \mid \varepsilon$$

a) Show three different parse trees for the string "10010010"

b) Show the leftmost derivation for the parse trees

a) First Parse Tree

10010010

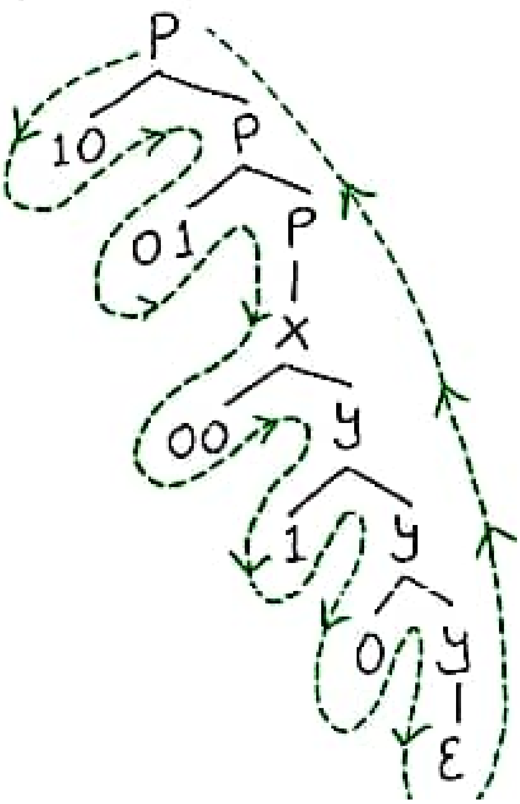
optimal checking if we have parsed the given string correctly

b) Leftmost Derivation for the parse tree shown at (a)

$P \rightarrow \underline{X}$
$\rightarrow 10\underline{Y}$
$\rightarrow 100\underline{Y}$
$\rightarrow 1001\underline{Y}$
$\rightarrow 10010\underline{Y}$
$\rightarrow 100100\underline{Y}$
$\rightarrow 1001001\underline{Y}$
$\rightarrow 10010010\underline{Y}$
$\rightarrow 10010010\varepsilon$

c) Second Parse Tree

10010010

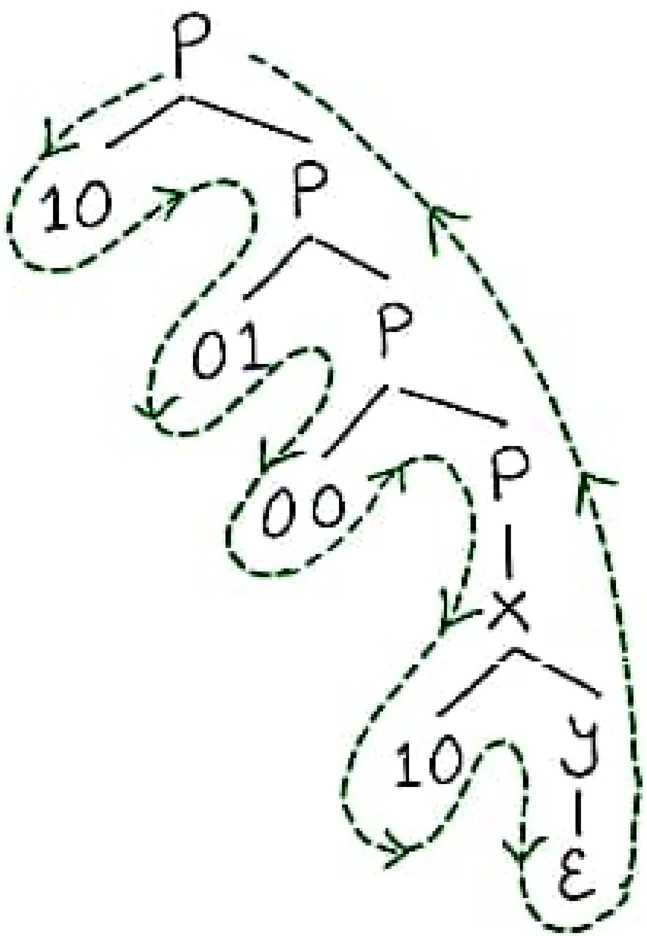


d) Leftmost Derivation for the parse tree shown at (c)

$P \rightarrow 10\underline{P}$
$\rightarrow 1001\underline{P}$
$\rightarrow 1001\underline{X}$
$\rightarrow 100100\underline{Y}$
$\rightarrow 1001001\underline{Y}$
$\rightarrow 10010010\underline{Y}$
$\rightarrow 10010010\varepsilon$

e) Third Parse Tree

10010010



f) Leftmost Derivation for the parse tree shown at (e)

$$
\begin{aligned}
P &\rightarrow 10\underline{P} \\
&\rightarrow 1001\underline{P} \\
&\rightarrow 100100\underline{P} \\
&\rightarrow 100100\underline{X} \\
&\rightarrow 10010010\underline{Y} \\
&\rightarrow 10010010\varepsilon
\end{aligned}
$$

Derivation, Parse tree, Ambiguity

Qa)

$S \longrightarrow ASB \mid SS \mid TSS \mid SAS \mid A$

$A \longrightarrow ASS \mid BS \mid B$

$B \longrightarrow 00 \mid 11 \mid 01 \mid 1$

a) left derivation of 00010111

$S \longrightarrow \underline{S}AS$
$\longrightarrow \underline{S}SAS$
$\longrightarrow \underline{A}SAS$
$\longrightarrow \underline{B}SAS$
$\longrightarrow 00\underline{S}AS$
$\longrightarrow 00\underline{A}AS$
$\longrightarrow 00\underline{B}AS$
$\longrightarrow 0001\underline{A}S$
$\longrightarrow 0001\underline{B}S$
$\longrightarrow 000101\underline{S}$
$\longrightarrow 000101\underline{A}$
$\longrightarrow 000101\underline{B}$
$\longrightarrow 00010111$

[underlined variables were derived in the next substitution]

↓

No need to show in the answer script. If you want you may.

[Multiple derivation is possible for this grammar and the string]

b) right derivation for 00010111

$$
\begin{aligned}
S &\longrightarrow SA\underline{S} \\
&\longrightarrow SAA \\
&\longrightarrow SA\underline{B} \\
&\longrightarrow S\underline{A}11 \\
&\longrightarrow S\underline{B}11 \\
&\longrightarrow \underline{S}0111 \\
&\longrightarrow S\underline{S}0111 \\
&\longrightarrow S\underline{A}0111 \\
&\longrightarrow SB0111 \\
&\longrightarrow \underline{S}010111 \\
&\longrightarrow \underline{A}010111 \\
&\longrightarrow \underline{B}010111 \\
&\longrightarrow 00010111
\end{aligned}
$$

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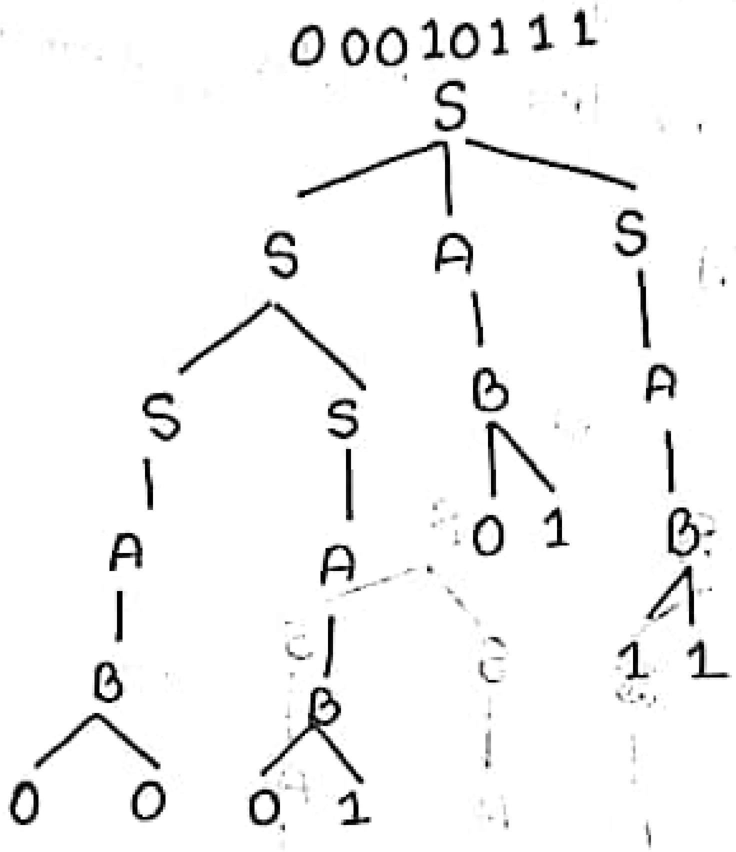
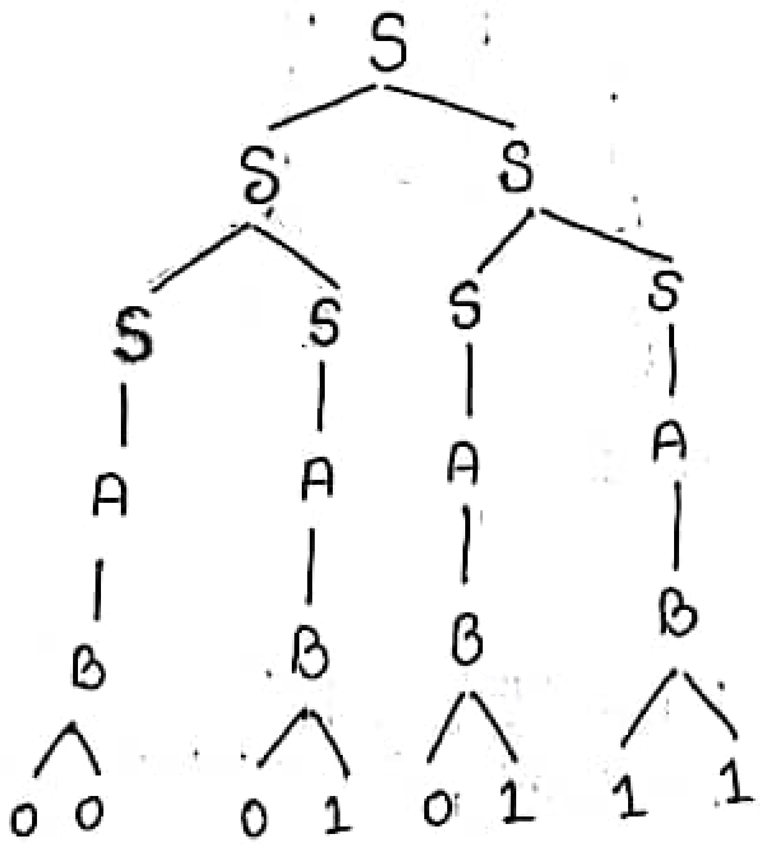
c) Parse Tree for the derivation given in (a)



d) Parse Tree for the derivation given in (b)

Same as 'C'

If you were asked to show that the Grammar is ambiguous. then you have to show another left Parse tree for this grammar on the given string.



[There are multiple possible parse tree apart from this.]

Qb) $A \rightarrow A1 \mid 0A1 \mid 01$

a) leftmost derivation for 001111

$A \rightarrow 0A1$

$\rightarrow 0A11$

$\rightarrow 0A111$

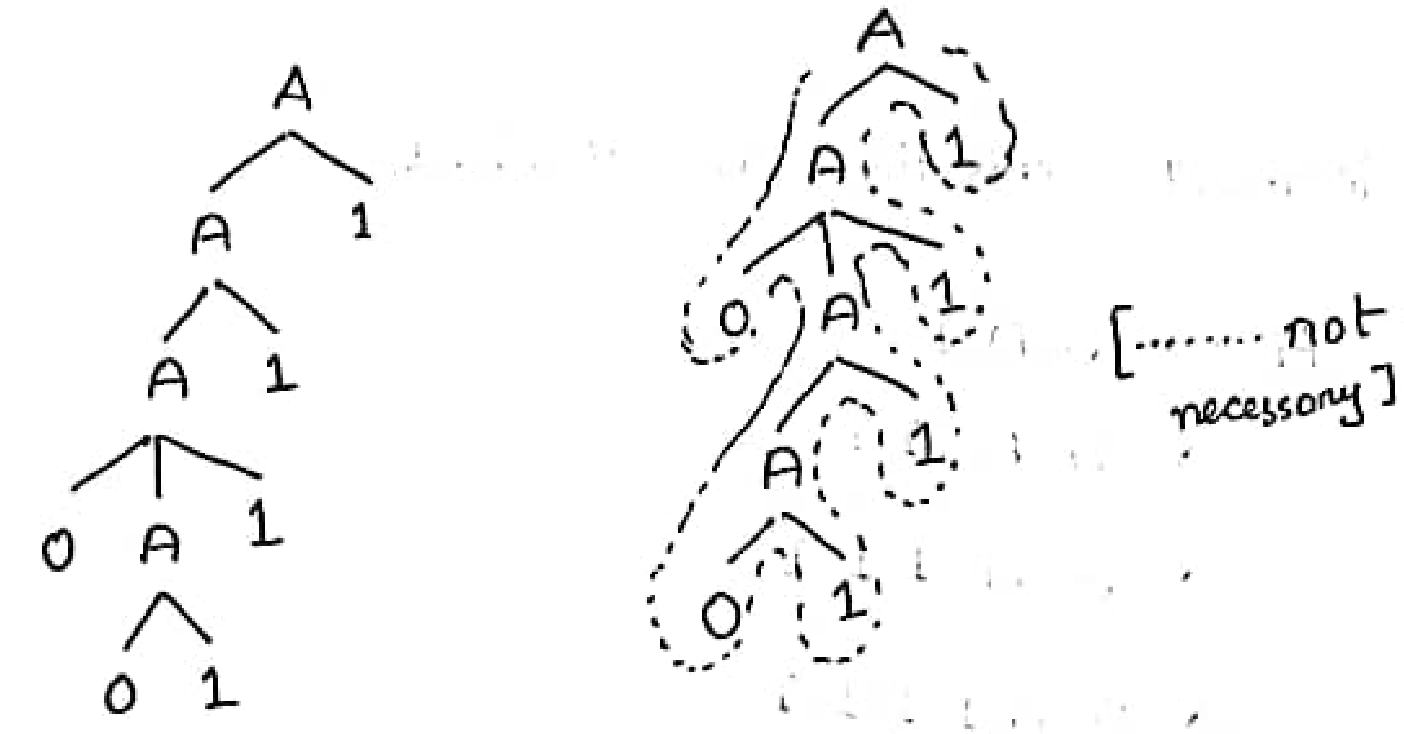
$\rightarrow 001111$

b) Parse tree for the derivation in (a):

[-------- not necessary]

[........ helps to check if we have ~~de~~ Parsed the given string]

c) Two more Parse trees

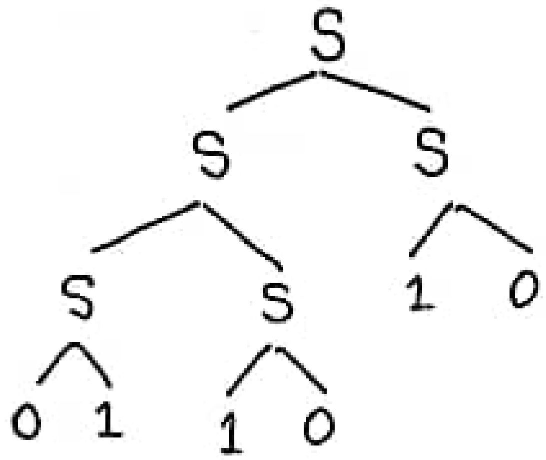


d) 000111 or 011111.

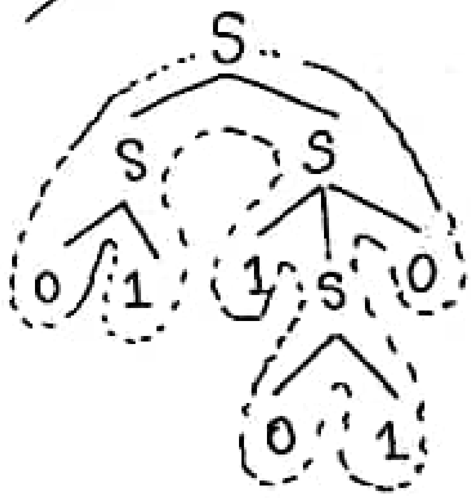
Qc) $S \rightarrow 0S1 \mid 1S0 \mid SS \mid 01 \mid 10$

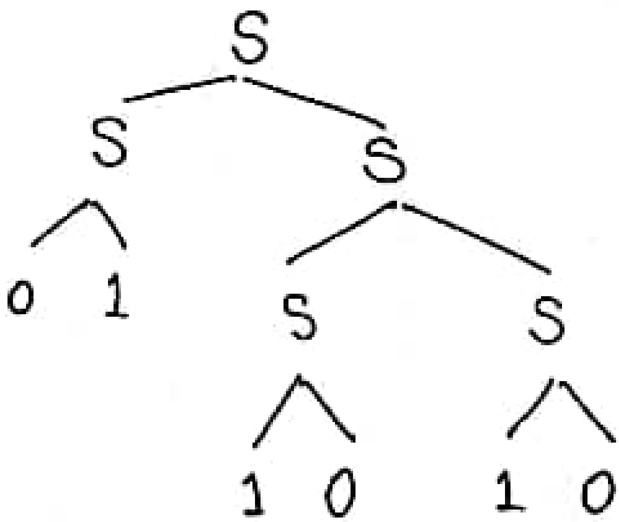
a) Two different parse tree for 011010.

1)



2)





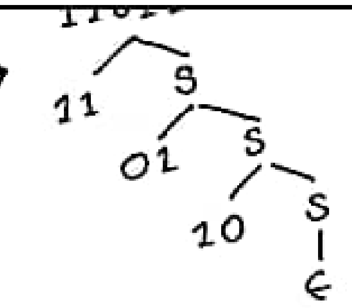
b) 000111 or 111000 or 001110 or 011100

or 110001 or 100011.

Consider the following two languages.

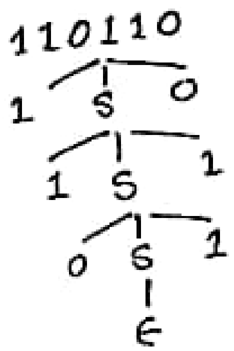
$L1 = \{w \in \{0,1\}^*$: The length of w is even.$\}$
$L2 = \{w \in \{0,1,\#\}^*: w = x\#1^n\#0^{2n+1}$ where $x \in L1$ and $n \geq 0\}$

a) Design a context-free grammar whose language is L1. [Points 6]

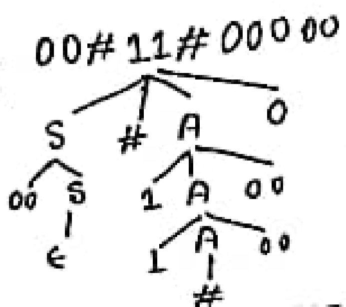


Solution 1: $S \rightarrow 00S \mid 01S \mid 10S \mid 11S \mid \epsilon$

Solution 2: ~~an~~, $S \rightarrow 0S0 \mid 0S1 \mid 1S0 \mid 1S1 \mid \epsilon$



b) Design a context-free grammar whose language is L2. [Points 4]



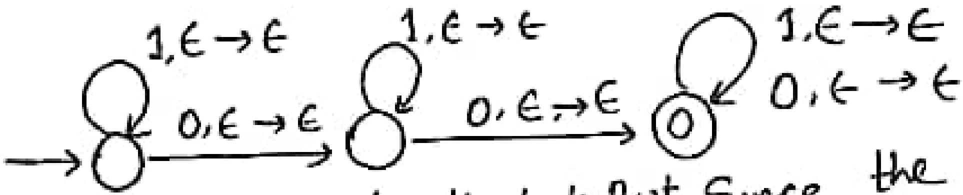
$W = X\#1^n\#0^{2n+1}$

$= X\#1^n\#0^{2n}0$

$S' \rightarrow S\#A0$
$S \rightarrow 00S \mid 01S \mid 10S \mid 11S \mid \epsilon$
$A \rightarrow 1A00 \mid \#$

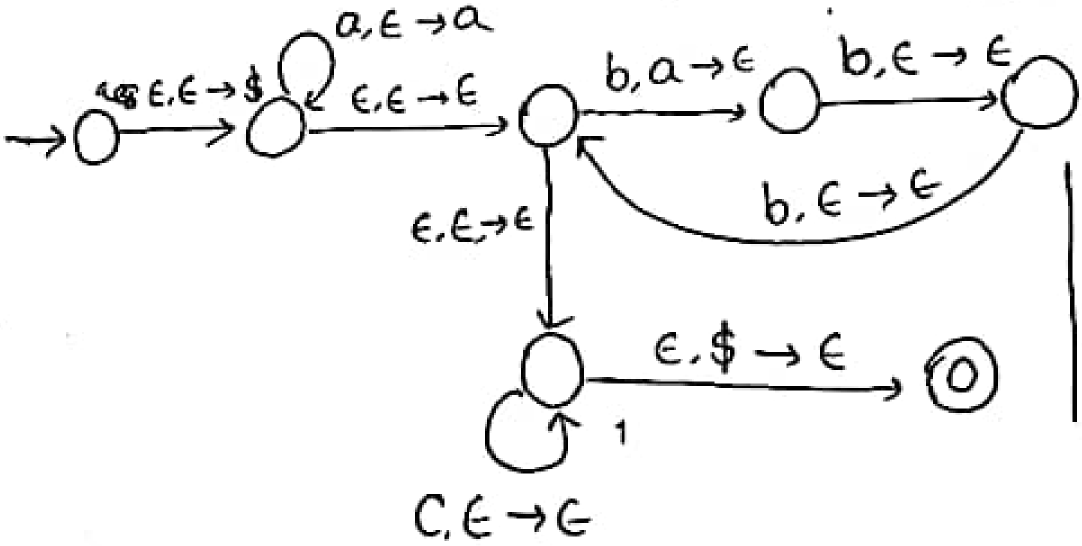
Problem 2: PDF

a) Give a PDA for the language $L = \{ w \in \{0,1\}^*$: w contains at least two 0s.$\}$ [4 Points]



you may push in the stack. But, since the language is regular, no need to use the stack.

b) Give a PDA for the language $L = \{ w \in \{a,b,c\}^* : w = a^nb^{3n}c^m$ where $n,m \geq 0 \}$ [6 Points]



Solution 2:
You may also push three 'a' by reading one 'a' in the stack.

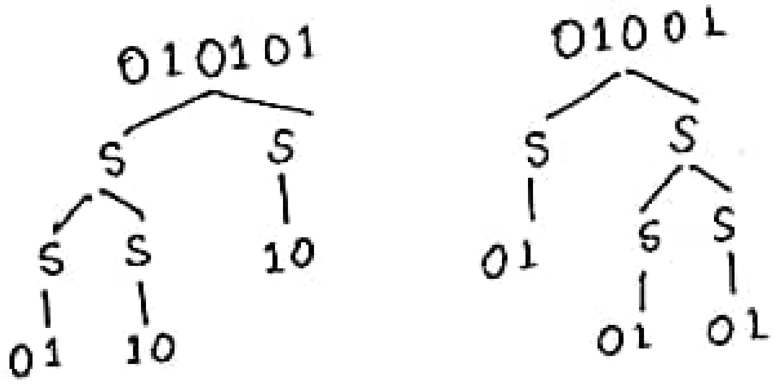
a) Consider the following context free grammar:

$$S \to 0S1 \mid 1S0 \mid SS \mid 01 \mid 10$$

i) Give a left derivation for the string 011010. [2 Points]

$S \longrightarrow SS$
$\longrightarrow 01S$
$\longrightarrow 01SS$
$\longrightarrow 0110S \longrightarrow 011010$

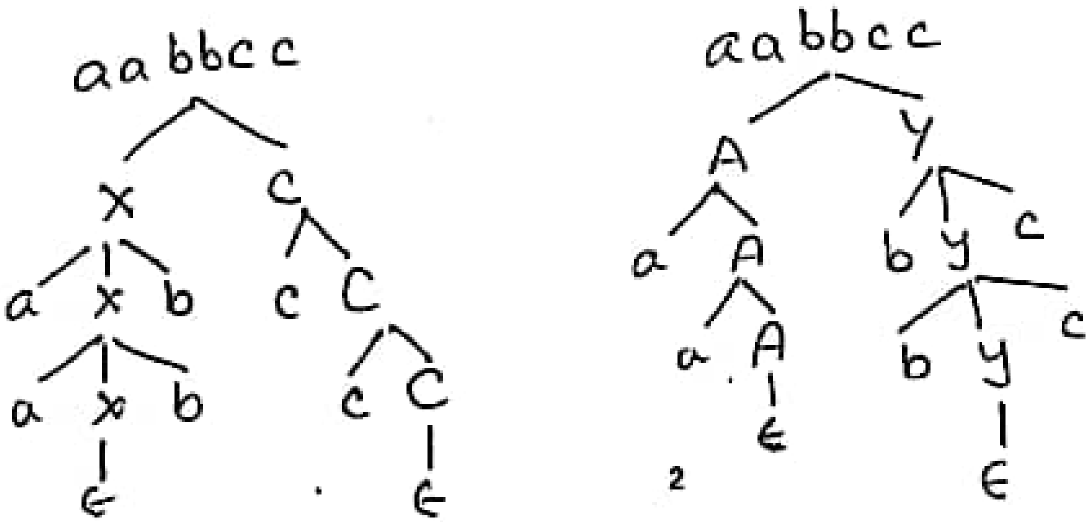
ii) Show that the grammar above is ambiguous by demonstrating two different parse trees for 010101. [3 Points]



b) Consider the following context free grammar:

$$S \to XC \mid AY$$
$$X \to aXb \mid \varepsilon$$
$$Y \to bYc \mid \varepsilon$$
$$A \to aA \mid \varepsilon$$
$$C \to cC \mid \varepsilon$$

Show that the grammar above is ambiguous by finding a length 6 string with two parse trees. [5 Points]



Question 03

a) w starts with '0' and the length of w is even.

$S \to 0AB$ $\qquad 0(\Sigma\Sigma)^*(0+1)$

$A \to 00A \mid 01A \mid 10A \mid 11A \mid \epsilon$

$B \to 0 \mid 1$

b) Every second letter in w is b

$((a+b)\,b)^* \cancel{(\epsilon \mid a \mid b)} (\epsilon + a + b)$

$S \to AB$

$A \to XA \mid \epsilon$

$X \to PQ$

$P \to a \mid b$

$Q \to b$

$B \to a \mid b \mid \epsilon$

Solution 2 :

$S \to abS \mid bbS \mid A$ $\qquad$ _b_b_b_

$A \to a \mid b \mid \epsilon$

c) The length of ω is divisible by ~~3~~ three.

$(\Sigma\Sigma\Sigma)^*$
$\rightarrow ((a+b)(a+b)(a+b))^*$ } No need to write ~~it~~ in the answer script (Optional)

$S \rightarrow AS \mid \epsilon$

$A \rightarrow BBB$

$B \rightarrow a \mid b$

d) ω starts and ends with different letters.

$a(a+b)^*b + b(a+b)^*a$

$S \rightarrow aAb \mid bAa$

$A \rightarrow aA \mid bA \mid \epsilon$

e) the number of a is at least the number of b in ω

$S \rightarrow \underbrace{aSbS \mid bSaS}_{\text{ensuring \#'b's equal to \#'a's, means the amount of b's won't exceed the amount of 'a's}} \mid \overset{\text{more 'a's}}{aS} \mid \epsilon$

Similar Qs:

ω contain more ~~b's~~ 'a's than ~~as~~ 'b's

$S \rightarrow XaX$

$X \rightarrow aXbX \mid bXaX \mid ~~\text{}~~ aX \mid \epsilon$

$L = \{ w \in \{a,b\}^* :$ the length of w is two more than multiple of six$\}$

$$S \longrightarrow XXXXXXS \mid XX$$

$$X \rightarrow a \mid b$$

d) $L = \{ \omega \in \{0,1\}^* : \omega = 0^n 1^{2n+3},\ n \geq 0 \}$

$$\omega = 0^n 1^{2n+3}$$

$$= 0^n 1^{2n} 1^3$$

$$S \longrightarrow AB$$

$$A \longrightarrow 0A11 \mid \epsilon$$

$$B \longrightarrow 111$$

f) $w = 0^n 1^n$, where n is odd.

wrong solution, since this doesn't ensure equal numbers of 0s & 1s. ←

$$\boxed{\begin{aligned} &S \to AB \quad \times \\ &A \to 0X \\ &X \to 00X \mid \epsilon \\ &B \to 1Y \\ &Y \to 11Y \mid \epsilon \end{aligned}}$$

$S \to 00S11 \mid A$

$A \to 01$

g) $w = 0^i 1^j 2^k$, where $j \geq 2i + 3k$

* first try to solve $j = 2i + 3k$

$0^i 1^j 2^k$

$\to 0^i 1^{2i+3k} 2^k$

$\to \underbrace{0^i 1^{2i}} \, \underbrace{1^{3k} 2^k}$

the additional 1s ($j>$) will occur in this place

$S \to ABC$

$A \to 0A11 \mid \epsilon$

$B \to 1B \mid \epsilon$

$C \to 111C2 \mid \epsilon$

h) $W = 0^i 1^j 2^k 3^m$, where $i = m$ and $j \geq 3k + 2$

$S \rightarrow 0S3 \mid X$

$X \rightarrow 111X2 \mid Y$

$Y \rightarrow 1Y \mid 11$

first try to solve $j = 3k+2$, $i = m$

$0^i 1^j 2^k 3^m$

$\rightarrow 0^i \, 1^{3k+2} \, 2^k \, 3^i$

$\rightarrow 0^i \, 1^{3k} \cdot \underline{1^2} \, 2^k \, 3^i$

the additional or more 1s ($j >$) will come from here.

i) $W = 0^i 1^j 2^k$, where $i > 2j + 3k$

$S \rightarrow 000S\cancel{k}2 \mid X$

$X \rightarrow 00X1 \mid Y$

$Y \rightarrow 0Y \mid 0$ Since [illegible] can't have equal we are not giving the ϵ transition

first try to solve $i = 2j + 3k$

$0^i 1^j 2^k$

$\rightarrow 0^{2j+3k} 1^j 2^k$

$\rightarrow 0^{3k} \, 0^{2j} \, 1^j \, 2^k$

We need $>$, here the # 0s ~~will be~~ one equal currently Hence, we have to increase the amount of 0s

J) A = ω contains at least two 0s

$L = \omega : \omega = 0^{3i} v\, 1^{2i}, \; v \in A, \; i \geq 0$

$S \rightarrow 000S11 \mid A$

$A \rightarrow X0X0X$

$X \rightarrow 0X \mid 1X \mid \epsilon$

$$\underbrace{\_\_\_}_{\text{any combinations of 0s and 1s}} \overbrace{0 \;\_\_\_\; 0}^{\text{atleast two 0s}} \;\_\_\_$$

K)

$L1 = \{ \omega \in \{0,1\}^* : \omega \text{ contains } 11 \}$

$L2 = \{ x \# y : x \in \{0,1\}^*, \ y \in L1, \ |x| = |y| \}$

$011 \ \boxed{10}0010 \# 0110 \ \boxed{11}010$

$S \rightarrow XSX \mid A$

continuing

keep the match until we are getting 11 in y.

$A \rightarrow 00B11 \mid 01B11 \mid 10B11 \mid 11B11$

we have found first 11 in y. Now to keep the length same we have to match 11 with all possible two length of 0,1 strings

$B \rightarrow XBX \mid \#$

$X \rightarrow 0 \mid 1$

Next, match the rest of the things.

L) $L = \{ \omega_1, \omega_2 \in \{0,1\}^* : \omega_1 \# \omega_2$ where number of 0s in ω_1 is equal to number of 1s in $\omega_2 \}$

What if at the same time $|\omega_1| = |\omega_2|$?

0 1 1 0 1 0 1 # 0 0 0 1 1 0 0 0 0 1 0

Note: here $1S \neq S1$

$$S \rightarrow 0S1 \mid 1S \mid S0 \mid \#$$

Since we are generating the 1s in ω_1.

Since we are only interested in the 0s in ω_1. And, we have to keep the count equal to 1s in ω_2. See, when we are having a 0 in ω_1, at the same time we are having a 1 in ω_2 → this is how we are equalizing the counts of 0 in ω_1 and 1s in ω_2.

See the example: Has to produce/generate the 1s in ω_1 and 0s in ω_2. Please note, the question doesn't ask for $|\omega_1| = |\omega_2|$

m) $L = \{w_1, w_2 \in \{0,1\}^* : w_1 \# w_2$ where length of w_2 is double of $w_1\}$.

$$S \longrightarrow ASAA \mid \#$$

$$A \longrightarrow 0 \mid 1$$

0 1 1 0 # 1 1 0 1 1 0 0 0

Regular Expression to CFG:

a) $\overbrace{(\underbrace{\underbrace{(ab)^*}_{B} + \underbrace{\underbrace{(a+b^3)}_{D}\underbrace{cb}_{E}}_{C}}_{A})^*}^{S}$

$S \to AS \mid \epsilon$

$A \to B \mid C$

$B \to abB \mid \epsilon$

$C \to DE$

$D \to a \mid bbb$

$E \to cb$

b) $\overbrace{(\overbrace{\underbrace{a^*b^*}_{P} + \underbrace{(ac+b^*c)b}_{Q}}^{R})^*}^{S}$

$S \to RS \mid \epsilon$
$R \to P \mid Q$
$P \to AB$

$A \to aA \mid \epsilon$

$B \to bB \mid \epsilon$

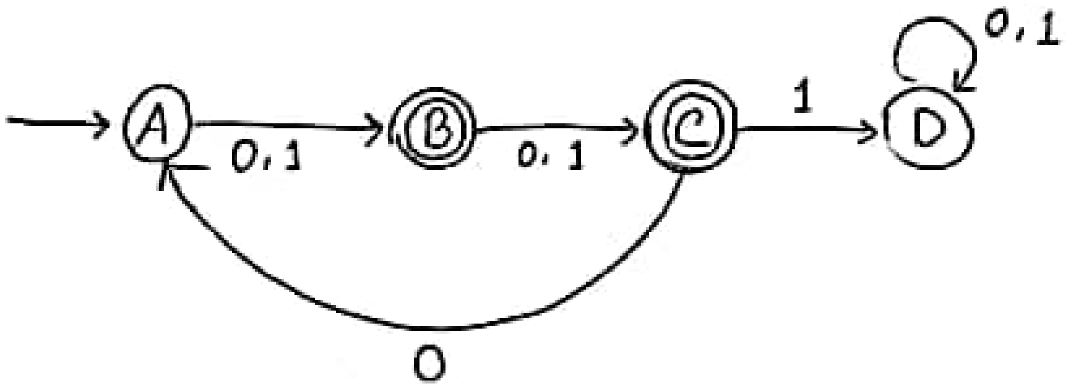
$Q \to Zb$

$Z \to X \mid Y$

$X \to ac$

$Y \to Bc$

c)



$A \rightarrow 0B \mid 1B$

$B \rightarrow 0C \mid 1C \mid \varepsilon$

$C \rightarrow 0A \mid 1D \mid \varepsilon$

$D \rightarrow 1D \mid 0D$

It's also okay if you do not include the production ~~to the~~ rules which are connected to D, since 'D' is a dead state.