Pumping Lemma:

1. L = {w ∈ {a,b}*: w is an even length palindrome}. Proof L is not a regular language.

Assume for the sake of contradiction that L is a regular language. Then let P be the pumping length for L. Now we take the string

$$w = a^p bba^p \in L$$

[Note, you may also choose $a^p b^p b^p a^p$ as w. However $a^p a^p$, $a^p ba^p b$, $a^p b^p$, they are not valid w]

Then the length of |w| = 2p+2 > p. So w can be split into xyz such that |y| > 0, |xy| ≤ P and $xy^i z \in L$ for each i ≥ 0.

Since |xy| ≤ P, and the first P characters of w are all a, we can conclude that y consists of only a.

Then, for i=2, xy^2z will be

$$xy^2z = xyyz = a^{p+|y|}bba^p \notin L$$

The string is not a Palindrome, since if we reverse the string, $a^p bba^{p+|y|}$ and $a^{p+|y|}bba^p$ is not the same. Thus ,we get a contradiction here. Hence, L is not a regular language.

2. L = {w ∈ {a,b}*: w is an odd length palindrome}. Proof L is not a regular language.

Assume for the sake of contradiction that L is a regular language. Then let P be the pumping length for L. Now we take the string

$$w = a^p ba^p \in L$$

Then the length of |w| = 2p+1 > p. So w can be split into xyz such that |y| > 0, |xy| ≤ P and $xy^i z \in L$ for each i ≥ 0.

Since |xy| ≤ P, and the first P characters of w are all a, we can conclude that y consists of only a.

Then, for i=2, xy^2z will be

$$xy^2z = xyyz = a^{p+|y|}ba^p \notin L$$

The string is not a Palindrome, since if we reverse the string, $a^p ba^{p+|y|}$ and $a^{p+|y|}ba^p$ is not the same. Thus ,we get a contradiction here. Hence, L is not a regular language.

3. L = {w ∈ { (,) }*: w is a valid parenthesis}. Proof L is a nonregular language.

Assume for the sake of contradiction that L is a regular language. Then let P be the pumping length for L. Now we take the string

$$w = (^p\,)^p \in L$$

Then the length of $|w| = 2p \geq p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq P$ and $xy^iz \in L$ for each $i \geq 0$.
Since $|xy| \leq P$, and the first P characters of w are all opening parentheses, we can conclude that y consists of only (.
Then, for i=2, xy^2z will be

$$xy^2z = xyyz = (^{p+|y|}\,)^p \notin L$$

The string is not a valid parenthesis sequence. Since $|y| > 0$, there are more opening parentheses than closing parentheses. Thus, we get a contradiction here. Hence, L is not a regular language.

4. L = {w ∈ { a, b, c }*: $w = a^i b^j c^k$, where i = j+k and i, j, k ≥ 0}. Proof L is a non-regular Language.

Assume for the sake of contradiction that L is a regular language. Then let P be the pumping length for L. Now we take the string

$$w = a^{2p}\, b^p\, c^p \in L$$

Then the length of $|w| = 4p \geq p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq P$ and $xy^iz \in L$ for each $i \geq 0$.

Since $|xy| \leq P$, and the first P characters of w are all a, we can conclude that y consists of only a.
Then, for i=2, xy^2z will be

$$xy^2z = xyyz = a^{2p+|y|}\, b^p\, c^p \notin L$$

Since $|y| > 0$, $2p+|y| > p+p$, means the amount of a in w is more than the amount of total b and c.Thus, we get a contradiction here. Hence, L is not a regular language.

5. $L = \{w \in \{0\}^* : w = 0^{n^3}$, where $n \geq 0\}$. Proof L is a non-regular Language.
[note: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$]

<u>Solution Idea 1:</u>

Assume for the sake of contradiction that L is a regular language. Then let P be the pumping length for L. Now we take the string

$$w = 1^{p^3} \in L$$

Then the length of $|w| = p^3 \geq p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq P$ and $xy^iz \in L$ for each $i \geq 0$.

Then, for i=2, xy^2z will be

$$xy^2z = xyyz = 1^{p^3+|y|}$$

Since the string is in L, $p^3 + |y|$ is a perfect cube number, which is, of course larger than p^3. The next perfect cube larger than p^3 is $(p + 1)^3$. So, we have

$$p^3 + |y| \geq (p + 1)^3$$
$$\Rightarrow p^3 + |y| \geq p^3 + 3p^2 + 3p + 1$$
$$\Rightarrow |y| \geq 3p^2 + 3p + 1$$

On the other hand, $|xy| \leq P$ gives us $|y| \leq P$. So,

$$p \geq |y| \geq 3p^2 + 3p + 1$$

This is clearly contradicts since $3p^2 + 3p + 1 > p$. Hence L is not a regular language.

<u>Solution Idea 2:</u>

Assume for the sake of contradiction that L is a regular language. Then let P be the pumping length for L. Now we take the string

$$w = 1^{p^3} \in L$$

Then the length of $|w| = p^2 \geq p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq P$ and $xy^iz \in L$ for each $i \geq 0$.

Then, for i=2, xy^2z will be

$$xy^2z = xyyz = 1^{p^3+|y|}$$

Now,

$$p^3 < p^3 + |y| \leq p^3 + p < p^3 + 3p^2 + 3p + 1 = (p + 1)^3$$

It is not possible to have a perfect cubic number between two consecutive perfect cubic numbers p^3 and $(p + 1)^3$. Thus, we have a contradiction. Hence, L is not a regular language.

6. L = {w ∈ {0}*: $w = 1^{n^2}$, where n ≥ 0}. Proof L is a non-regular Language.

Solution Idea 1:

Assume for the sake of contradiction that L is a regular language. Then let P be the pumping length for L. Now we take the string

$$w = 1^{p^2} \in L$$

Then the length of $|w| = p^2 \geq p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq P$ and $xy^iz \in L$ for each $i \geq 0$.

Then, for i=2, xy^2z will be

$$xy^2z = xyyz = 1^{p^2+|y|}$$

Since the string is in L, $p^2 + |y|$ is a perfect square number, which is, of course larger than p^2. The next perfect square larger than p^2 is $(p+1)^2$. So, we have

$$p^2 + |y| \geq (p+1)^2$$
$$\Rightarrow p^2 + |y| \geq p^2 + 2p + 1$$
$$\Rightarrow |y| \geq 2p + 1$$

On the other hand, $|xy| \leq P$ gives us $|y| \leq P$. So,

$$p \geq |y| \geq 2p + 1$$

This is clearly contradicts since $2p + 1 > p$. Hence L is not a regular language.

Solution Idea 2:

Assume for the sake of contradiction that L is a regular language. Then let P be the pumping length for L. Now we take the string

$$w = 1^{p^2} \in L$$

Then the length of $|w| = p^2 \geq p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq P$ and $xy^iz \in L$ for each $i \geq 0$.

Then, for i=2, xy^2z will be

$$xy^2z = xyyz = 1^{p^2+|y|}$$

Now,

$$p^2 < p^2 + |y| \leq p^2 + p < p^2 + 2p + 1 = (p+1)^2$$

It is not possible to have a perfect square number between two consecutive perfect square numbers p^2 and $(p+1)^2$. Thus, we have a contradiction. Hence, L is not a regular language.