Region of Convergence and Common Z-Transforms

April 18, 2020

Recall from Last Slide-set

Definition of a Z-transform

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

...where z, the independent variable, is a complex variable.

- The Z-transform is the discrete-time equivalent of the Laplace transform
- The Discrete-Time Fourier Transform (DTFT) of a signal x[n] is equal to its z-transform evaluated at each point on the unit circle of the z-plane described by the trajectory $z = e^{j\Omega}$.

Region of Convergence (ROC)

- The definition of the Z-Transform contains a power series. The negative exponent on z allows for the sum of the terms to converge

Condition for a Region of Convergence

$$\sum_{n=-\infty}^{\infty} \left| x[n]z^{-n} \right| < \infty$$

- X(z) cannot help us determine x(t) alone!

Definition of the Region of Convergence

The ROC is a collection of all points on the z-plane in which the sum converges.

Example: Simple Z-Transform of a Signal

$$x[n] = \{3.7, 1.3, -1.5, 3.4, 5.2\} \longrightarrow x[n=0] = 3.7$$

The Z-Transform is :

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\ &= x[0]z^{0} + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + x[4]z^{-4} \\ &= 3.7 + 1.3z^{-1} - 1.5z^{-2} + 3.4z^{-3} + 5.2z^{-4} \end{aligned}$$

... which yields a polynomial with a degree of -1.

Example: Simple Z-Transform of a Signal

As you may have noticed, the coefficients on each term contain the information of the time-domain samples.

Taking $z_1 = 1 + 2j$, the value of the transform is:

$$
\begin{aligned}
X(1+2j) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\
&= 3.7 + 1.3(1+2j)^{-1} - 1.5(1+2j)^{-2} + 3.4(1+2j)^{-3} \\
&+ 5.2(1+2j)^{-4} \\
&= 3.7 + \frac{1.3}{(1+2j)} - \frac{1.5}{(1+2j)^2} + \frac{3.4}{(1+2j)^3} + \frac{5.2}{(1+2j)^4} \\
&= 3.7826 - j0.0259
\end{aligned}
$$

Does this work for all values of z?

Unit Impulse

$$x[n] = \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Since our only non-zero sample is at n = 0, the Z-Transform becomes:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = x[0]z^{0} = 1$$

The impulse signal yields a constant value (no exponent). Because of this, we can say that the sum converges at every point on the z-plane. In other words, **the ROC is the entire z-plane**.

Shifted Unit Impulse

$$x[n-k] = \delta[n-k] = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases} \textit{where } k \neq 0, \in \mathbb{Z}$$

Using the same procedure as before, we get:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = x[k]z^{-k} = \boxed{z^{-k}}$$

Looking at the cases for k:

$$\begin{cases} k < 0: & \textit{transform does not converge at infinite radius} \\ k > 0: & \textit{transform does not converge at origin } (0+0j) \end{cases}$$

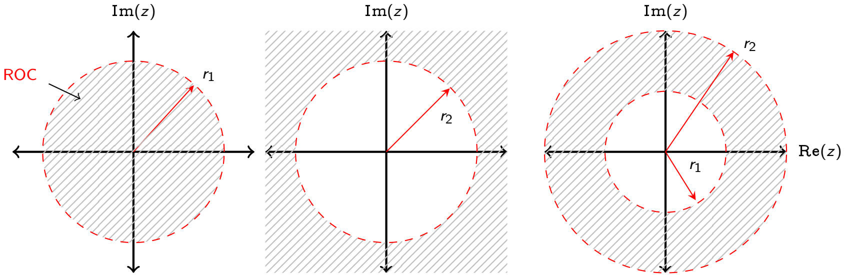
Otherwise, the transform converges at **all points on the z-plane.**

Here are the functions we discussed and their corresponding ROCs

Function	**Region of Convergence**
$\delta[n]$	entire z-plane
$\delta[n-k]$	entire z-plane except for $z = 0$ and $z = \infty$
x[n]	entire z-plane except for $z = 0$

Note: $x[n]$ is a signal with given values. (refer to the first example)
Next, we will look at the ROC for causal and non-causal signals.

Nature of ROC



Possible ROC Shape for a signal :

$$\begin{aligned} r &< r_1 &&: \text{ Inside a circle} \\ r &> r_2 &&: \text{ Outside a circle} \\ r_1 < r &< r_2 &&: \text{ Between 2 circles} \end{aligned}$$

Unit Step Function

$$x[n] = u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Again, we apply the z-transform to the function, but now the lower limit for the sum is 0 since $u[n] = 0$ for all $n < 0$:

$$X(z) = \sum_{n=0}^{\infty} u[n] z^{-n}$$

Since $u[n] = 1$ for all $n \geq 0$, we can remove the u[n] term from the summation.

$$X(z) = \sum_{n=0}^{\infty} z^{-n}$$

But what do we do from here?

Unit Step Function

The formula for an infinite geometric series is :

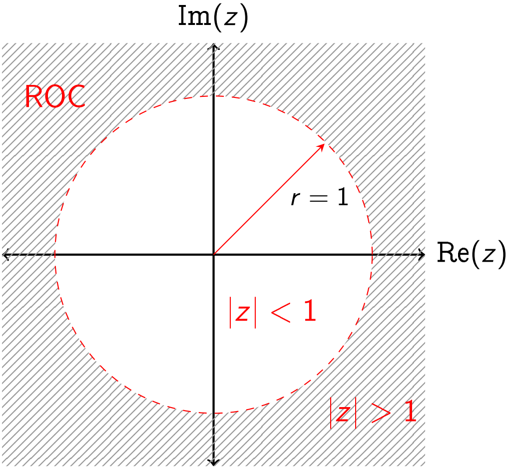
$$X(z) = \sum_{n=0}^{\infty} z^{-n}$$

$$= \frac{1}{1 - z^{-1}} = \boxed{\frac{z}{z - 1}}$$

...which converges for : $|z^{-1}| < 1$ and $|z| > 1$

The ROC of a Unit Step Function is the collection of points outside of a **circle of radius 1.**

Unit Step Function



Causal Signal

$$x[n] = a^n u[n] = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$ Where a is any real or complex value.

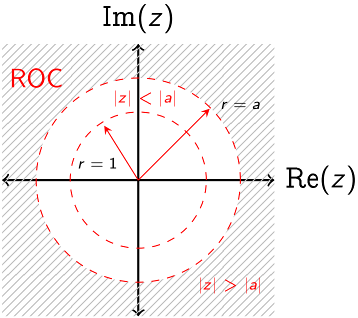
$$X(z) = \sum_{n=-\infty}^{\infty} a^n z^{-n} = \sum_{n=-\infty}^{\infty} (az^{-1})^n$$

And similarly, we can find that :

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} az^{-n} \\ &= \frac{1}{1 - az^{-1}} = \boxed{\frac{z}{z-a}} \end{aligned}$$

...which converges for : $|az^{-1}| < 1$, $\frac{|a|}{|z|} > 1$, $|z| > |a|$

Causal Signal



The left figure shows the ROC for a causal signal with $|a| > 1$, and the figure on the right shows the same for $|a| < 1$

Non Causal Signal

$$x[n] = -a^n u[-n-1] = \begin{cases} -a^n & n < 0 \\ 0 & n \geq 0 \end{cases}$$

Where a is any real or complex value.

Note that: $u[-n-1] = \begin{cases} 1 & n < 0 \\ 0 & n \geq 0 \end{cases}$

If we change the upper limit of the summation to $n = -1$, the $u[-n-1]$ term becomes obsolete.

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} \\ &= -\sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= -\sum_{n=-\infty}^{-1} (az^{-1})^n \end{aligned}$$

Non Causal Signal

If we let m = -n, our expression becomes :

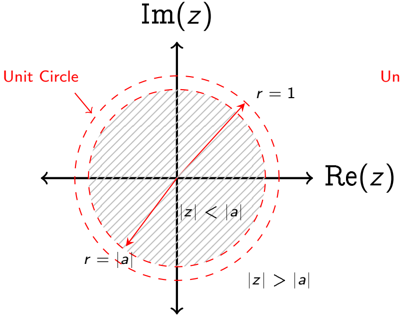
$$X(z) = -\sum_{m=-1}^{\infty} (a^{-1}z)^m$$

We will now apply another variable change of k = m - 1, so that we get k = 0 in our lower summation limit :

$$\begin{aligned} X(z) &= -\sum_{k=0}^{\infty} (a^{-1}z)^{k+1} \\ &= -a^{-1}z\sum_{k=0}^{\infty} (a^{-1}z)^k \\ &= -a^{-1}z\left(\frac{1}{1-a^{-1}z}\right) \end{aligned} \qquad \boxed{= \frac{z}{z-a}}$$

...which converges for : $|a^{-1}z| < 1$, $\frac{|z|}{|a|} < 1$, $|z| < |a|$

Non Causal Signal



The left figure shows the ROC for a causal signal with $|a| > 1$, and the figure on the right shows the same for $|a| < 1$

Zeros and Poles

Let's say we are given a Z-transform of :

$$X(z) = \frac{z}{z-a}$$

- We don't know if the original signal is causal or non-causal, since we are not given the conditions for z and a.

What we do know is that the transform is generally expressed in the form :

$$X(z) = K\frac{B(z)}{A(z)}$$

...where $B(z)$ and $A(z)$ are polynomials, K is the gain factor.

Zeros and Poles

Using the factored form for the polynomials, we get :

$$X(z) = K\frac{(z-z_1)(z-z_2)...,(z-z_M)}{(z-p_1)(z-p_2)...,(z-p_N)}$$

...where M and N are the orders of the polynomials on the numerator and denominator respectively.

- The roots of the polynomials in the numerator are **zeros**.
- The roots of the polynomials in the denominator are **poles**.

Why are they important?

Summary of Region of Convergence

- ROC is circular shaped: inside of a circle, outside of a circle or between two circles
- ROC cannot contain poles
- For a causal signal, the ROC is $|z| > r_1$
- For a non causal signal, the ROC is $|z| < r_2$
- For a signal that is neither causal or non causal, the ROC is $r_1 < |z| < r_2$

Properties of Z-Transforms

1. Linearity
2. Time Shifting
3. Time Reversal
4. Differentiation
5. Convolution

Linearity

For any two signals with their respective transforms :

$$x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) \quad x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z)$$

And two constants: α_1 , α_2 , we can find the Z-transform:

$$\begin{aligned}
\mathcal{Z}\{\alpha_1 x_1[n] + \alpha_2 x_2[n]\} &= \sum_{n=-\infty}^{\infty} (\alpha_1 x_1[n] + \alpha_2 x_2[n]) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} \alpha_1 x_1[n] z^{-n} + \sum_{n=-\infty}^{\infty} \alpha_2 x_2[n] z^{-n} \\
&= \alpha_1 \cdot \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} + \alpha_2 \cdot \sum_{n=-\infty}^{\infty} x_2[n] z^{-n} \\
&\boxed{= \alpha_1 \cdot \mathcal{Z}\{x_1[n]\} + \alpha_2 \cdot \mathcal{Z}\{x_2[n]\}}
\end{aligned}$$

Time Shifting

Given the transform pair :

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z)$$

For a time shifted signal x[n-k] :

$$\mathcal{Z}\{x[n-k]\} = \sum_{n=-\infty}^{\infty} x[n-k] \cdot z^{-n}$$

If we define $m = n - k$:

$$= \sum_{m=-\infty}^{\infty} x[m] \cdot z^{-(m+k)}$$

$$= z^{-k} \sum_{m=-\infty}^{\infty} x[m] \cdot z^{-m}$$

$$\boxed{= z^{-k} \cdot X(z)}$$

Time Reversal

For a reversed time signal x[-n] :

$$\mathcal{Z}\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n] \cdot z^{-n}$$

If we define $m = -n$:

$$\mathcal{Z}\{x[-n]\} = \sum_{m=+\infty}^{-\infty} x[m] \cdot z^{m}$$

We can switch the limits of the summation, and factor the 'm' out of the exponent to get :

$$\mathcal{Z}\{x[-n]\} = \sum_{m=-\infty}^{+\infty} x[m] \cdot (z^{-1})^{m} \qquad \boxed{= z^{-k} \cdot X(z^{-1})}$$

Differentiation

Starting with the definition :

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

We can differentiate both sides with respect to z :

$$\frac{d}{dz}[X(z)] = \frac{d}{dz}[\sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}]$$

Since summation and differentiation are both linear, the order can be reversed.

$$\frac{d}{dz}[X(z)] = \sum_{n=-\infty}^{\infty} \frac{d}{dz}[x[n] \cdot z^{-n}]$$

Differentiation

Now, by differentiating the term inside the sum :

$$= \sum_{n=-\infty}^{\infty} -nx[n]z^{-n-1}$$

$$= -z^{-1} \sum_{n=-\infty}^{\infty} -nx[n]z^{-n}$$

$$\frac{d}{dz}[X(z)] = \frac{1}{(-z)} \sum_{n=-\infty}^{\infty} -nx[n]z^{-n}$$

$$\boxed{(-z) \cdot \frac{d}{dz}[X(z)] = \mathcal{Z}\{nx[n]\}}$$

Convolution

The convolution of two discrete time signals is given by :

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k] \cdot x_2[n-k]$$

By applying the Z-transform :

$$\begin{aligned} \mathcal{Z}\{x_1[n] * x_2[n]\} &= \mathcal{Z}\{ \sum_{k=-\infty}^{\infty} x_1[k] \cdot x_2[n-k]\} \\ &= \sum_{n=-\infty}^{\infty} (\sum_{k=-\infty}^{\infty} x_1[k] \cdot x_2[n-k])z^{-n} \end{aligned}$$

Convolution

Changing the order of summation and multiplication gives us :

$$= \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k] z^{-n}$$

The inner summation term represents the Z-transform of $x_2[n-k]$, which means :

$$\mathcal{Z}\{x_1[n] * x_2[n]\} = \sum_{k=-\infty}^{\infty} x_1[k] z^{-k} \cdot X_2(z)$$

$$= X_2(z) \cdot \sum_{k=-\infty}^{\infty} x_1[k] z^{-k} \quad \boxed{= X_1(z) \cdot X_2(z)}$$