Discrete Mathmatics

Logic and proofs

<u>Proposition</u>: A proposition is a declarative statement that is either true or false, but not both.

ex: a) Toronto is the capital of Canada.

b) $1+2=4$

example (a) and (b) are proposition.

<u>Propositional logic</u>: The area of logic that deals with proposition is called the propositional logic.

Converse statement are created by interchanging the roles of hypothesis and conclusion of the original conditional statement.

Let p and q are two pro-

he proposition $q \to p$ is called

of $p \to q$.

is raining, then the ground is wet

raining

ground is wet

converse: If the ground is wet, then it is raining.

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Proposition: A proposition is a declarative statement that is either true or false, but not both.

ex: a) Toronto is the capital of Canada.

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Propositional logic: The area of logic that deals with proposition is called the propositional logic.

Converse: Let p and q are two proposition. The proposition $q \rightarrow p$ is called the converse of $p \rightarrow q$.

ex: If it is raining, then the ground is wet

p — It is raining

q — The ground is wet

converse: If the ground is wet, then it is raining.

Inverse: Let p and q are two propositions. The proposition $\neg p \to \neg q$ is called the inverse of $p \to q$.

ex: If a number is positive, then it is greater than zero.

p — A number is positive
q — It is greater than zero.

se: If a number is not positive, then it is not greater than zero.

Contrapositive: Let p and q are two propositions. The proposition $\neg q \to \neg p$ is called the contrapositive of $p \to q$.

ex: If it is sunny, then Jony goes for a run.

p — It is sunny.
q — Jony goes for a run.

contrapositive: If Jony doesn't go for a run, then it is not sunny.

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Inverse statement is created by negating both the hypothesis and conclusion of the original ~~conclu-~~ ~~Conclusion~~ statement conditional

and q are two propositions $\neg p \to \neg q$ is called the inverse

ber is positive, then it is greater than zero.

p — A number is positive

q — It is greater than zero.

inverse: If a number is not positive, then it is not greater than zero

The contrapositive is created by inter-changing the negative hypothesis and negative conclusion of the original conditional statement.

ve: Let p and q are two

s. The proposition $\neg q \to \neg p$

ne ~~or~~ contrapositive of $p \to q$.

is sunny, then Jony goes

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q — Jony goes for a run.

contrapositive: If Jony doesn't go for a run, then it is not sunny.

How an English sentence be translated into logical expression:

Translating an english sentence into a logical expression involves creating a representation that captures the meaning of the sentence using propositional logic or predicate logic and ~~po~~propositional connectives.
Once we have translated sentences from English into logical expressions, we can analyze these logical expressions to determine their truth values, we can manipulate them, and we can use rules of intercference to ~~reason~~ reason about them.

ex: "You can't ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

Let, p — You can ride the roller coaster
q — You are under 4 feet tall
r — You are older than 16 years old.

Then the sentence can be translated into

$$(q \land \neg r) \longrightarrow \neg p$$

Define Tautology : A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a tautology.

ex: Let p be a proposition.

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

Here, $p \vee \neg p$ is a tautology because it is always true. No matter "p" is true or false, $p \vee \neg p$ is always true.

* Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Let p and q be proposition.

p	q	$(p \wedge q)$	$(p \vee q)$	$(p \wedge q) \rightarrow (p \vee q)$
F	F	F	F	T
F	T	F	T	T
T	F	F	T	T
T	T	T	T	T

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Let p and q be proposition.

[illegible]	[illegible]	$(p \wedge q)$	$(p \vee q)$	$(p \wedge q) \rightarrow (p \vee q)$
[illegible]	[illegible]	F	F	T
[illegible]	[illegible]	F	T	T
[illegible]	[illegible]	F	T	T
[illegible]	[illegible]	T	T	T

$(p \wedge q) \rightarrow (p \vee q)$

$\equiv \neg(p \wedge q) \vee (p \vee q)$

$\equiv \neg p \vee \neg q \vee p \vee q$

$= (\neg p \vee p) \vee (\neg q \vee q)$

$= T \vee T$

$= T$ (tautology)

Here, compound ~~pot~~ proposition $(p \wedge q) \rightarrow (p \vee q)$ is always true, no matter what the truth values of propositional variables. So, we can called it tautology.

(showed).

Universal quantifier: Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the domain of ~~discourse~~ discourse, often just reffered to as the domain. Such a statement is expressed using universal quantification.

The universal quantification of $p(x)$ is the statement

"$p(x)$ for all values of x in the domain".

$\forall x \; P(x)$

$\forall \rightarrow$ Universal quantifier.

The notation $\forall x \; P(x)$ denotes the universal quantification of $P(x)$.

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"p(x) for all values of x in the domain".

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Existential Quantifier: Many mathmatical statements assert that there is an element with a certain property. Such statements are expressed using existential quantification.

The existential quantification of $P(x)$ is the proposition

"There exist an element x in the domain such that $P(x)$".

$\exists_x P(x)$

$\exists x \longrightarrow$ Existential quantifier

The notation $\exists x P(x)$ denotes the existential quantification of $P(x)$.

Lemma: A lemma is a pre-theorem/result which is needed to prove a theorem.

Corollary: A corollary is a past theorem or result which follows ~~direct~~ directly from a theorem.

Conjecture: A conjecture is a statement that is being proposed to be a true statement. Many times it are to be false, So they are not theorems.

Lemma: A lemma is a proven or ~~auxilary~~ established auxilary theorem in mathmatics or formal logic.

Corollary: A corollary is a statement or theorem that can be directly derived from a previous proven theorem/result.

Conjecture: A conjecture is a statement that is proposed as a possible solution to a problem, but it has not been proven or established with certainly.

Negation: Let p be a proposition. $\neg p$ is called negation of p which simply state that:

"It is not the case that p"

~~If p is positive then $\neg p$ is negative~~.

If p is true then $\neg p$ is false

If p is false then $\neg p$ is true.

proposition: Rahim & karim lived togther.

Negation: It is not the case that Rahim & karim lived together.

(AND) Conjunction: Let p and q are two propositions. Conjunction of p and q is denoted by $p \wedge q$. When both p and q are true then only the compound proposition $p \wedge q$ is true, otherwise false.

ex: "12 is divisible by 3 and 3 is prime".

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

(OR) Disjunction: Let p and q are proposition. Disjunction of p and q is denoted by $p \vee q$. When p and q both are false then only the compound proposition $p \vee q$ is false, otherwise true.

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Contradiction: A statement that is always false is known as contradiction.

q	$\neg q$	$q \wedge \neg q$
F	T	F
T	F	F

Here, $q \wedge \neg q$ is contradiction because it is always false, no matter q is true or false, $q \wedge \neg q$ is always false.

(conditional statement) Implication: Let p and q be propositions. The conditional statement $p \to q$ if "p then q" is the proposition that is false when p true and q false, otherwise true.

p — Rita learn Math
q — Rita will find a good job.
$p \to q$: If Rita ~~fin~~ learn Math then she will get a good job.

p	q	$p \to q$
F	F	~~F~~ T
F	T	~~F~~ T
T	F	F
T	T	T

Define the rule of ~~inter~~ inference: We can always use a truth table to show that an argument form is valid. We do this by showing that whenever the premises are true, the ~~cu~~ conclusion must also be true.

For example, when an arguments form involve 10 different proposition variables, to use truth table to show this argument form is valid requires $2^{10} = 1024$ different rows. Fortunately we do not have to resort to truth tables. Instead we can first establish the validity of some relatively simple argument forms, called ~~full~~ rules of inference.

Universal Instantiation:

Universal instantiation is the rule of inference used to conclude that $P(c)$ is true, where c is a particular member of the domain given the premise $\forall x P(x)$

Universal instantiation is used when we conclude from the statement "All women are wise" that "Lisa is wise". Where Lisa is a member of the domain of all women.

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Universal Instantiation:-
This rule is used to conclude that $P(c)$ is true when $\forall x P(x)$ is true.

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Universal Generalization: Universal generalization is the rule of inference that states that $\forall x P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the domain. Universal generalization is used when we show that $\forall x P(x)$ is true by taking an arbitrary element c from the domain and showing that $P(c)$ is true. The element c that we select must be an arbitrary, and not a specific, element of the domain.

Universe of discourse: Many mathmatical statements assert that a property is true for all values of a variable in a particular domain called the domain of discourse or universe of discourse.

* Using truth table prove that $p \to q$ and $\neg p \vee q$ are logically equivalent.

p	q	$\neg p$	$p \to q$	$\neg p \vee q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

Universal Generalization:
This rule states that $\forall x\, P(x)$ is true, given the premise, $P(c)$ is true for an arbitrary c.

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x\, P(x)}$$

...eneralization: Universal generali- ... rule of inference that states ... is true, given the premise that ... for all elements c in the domain. ...eneralization is used when we ... $\forall x P(x)$ is true by taking an arbitrary ... from the domain and showing that P(c) is true. The element c that we select must be an arbitrary, and not a specific, element of the domain.

Universe of discourse: Many mathmatical statements assert that a property is true for all values of a variable in a particular domain called the domain of discourse or Universe of discourse.

* Using truth table prove that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

p	q	¬p	p→q	¬p∨q
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

The truth value of $p \to q$ and $\neg p \vee q$ are same. That's why $p \to q$ and $\neg p \vee q$ are logically equivalent.

(proved)

* Let $Q(a,b)$ denote the statement "$x=y-2$". Write the truth values of the propositions $Q(3,5)$ and ~~(Q)~~ $Q(2,5)$.

$$Q(a,b) \Rightarrow x = y-2$$
$$Q(3,5) \Rightarrow 3 = 5-2$$
$$\Rightarrow 3 = 3 \quad (T)$$

for $Q(3,5)$, $x=y-2$ is true, the truth value of $Q(x,y)$ is True.

$$Q(2,5) \Rightarrow x = y-2$$
$$\Rightarrow 2 = 5-2$$
$$\Rightarrow 2 = 3 \quad (F)$$

for $Q(2,5)$, $x=y-2$ is not true, the truth value of $Q(2,5)$ is false.

* $\exists x \forall y \forall z\ ((F(x,y)\ F(y,z)\ F(z,x)) \to \neg F(y,z))$

~~F(x,y) means into~~ translate this expression into English statement, where $F(a,b)$ means a and b are friends and the domain for x, y, z consist of all student in your varsity.

$F(x,y)$ means, x and y are friends
$F(z,x)$ means, x and z are friends.
$F(y,z)$ means, y and z are friends
$\neg F(y,z)$ means, y and z are not friends.

The statement $\exists x \forall y \forall z\ ((F(x,y)\ F(y,z)\ F(z,x) \to \neg F(y,z)))$

says that, there is a student x such as for all student y and for all students z if x and y are friends and x and z are friends then y and z are not friends.

* Let $Q(x)$ be the statement "$x<3$". What is the truth value of $\forall x Q(x)$, where the universe of discourse consist of all real numbers

$Q(x) \Rightarrow x<3$
$Q(1) \Rightarrow 1<3$ (T)
$Q(2) \Rightarrow 2<3$ (T)
$Q(3) \Rightarrow 3<3$ (F)

$Q(x)$ is not true for every real number x because, for instance $Q(3)$ is false.
So, the truth value is false.

* Show that $(\neg Q \wedge (P \to Q)) \to \neg P$ is tautology.

$(\neg Q \wedge (P \to Q)) \to \neg P$
$\equiv (\neg Q \wedge \neg P \vee Q) \to \neg P$
$\equiv \neg(\neg Q \wedge \neg P \vee Q) \vee \neg P$
$\equiv \neg(\neg Q) \vee \neg(\neg P) \wedge \neg Q \vee \neg P$
$\equiv Q \vee P \wedge \neg Q \vee \neg P$
$\equiv (Q \wedge \neg Q) \vee (P \vee \neg P)$
$\equiv F \vee T$
$\equiv T$

Here $(\neg Q \wedge (P \to Q) \to \neg P$ is always true. So it is tautology.

* Show that $(P \wedge Q) \rightarrow (P \vee Q)$ is a tautology

$$(P \wedge Q) \rightarrow (P \vee Q)$$

$$\equiv \neg(P \wedge Q) \vee (P \vee Q)$$

$$\equiv \neg P \vee \neg Q \vee P \vee Q$$

$$\equiv \neg P \vee P \vee \neg Q \vee Q$$

$$\equiv T \vee T$$

$$\equiv T$$

Here, $(P \wedge Q) \rightarrow (P \vee Q)$ is always true. So we call it a tautology.

* Let $Q(x, y)$ denote the statement "$x = y + 3$" What are the truth values of the proposition $Q(1,2)$ and $Q(3,0)$?

$$Q(x, y) \Rightarrow x = y + 3$$

$$Q(1,2) \Rightarrow 1 = 2 + 3$$

$$\Rightarrow 1 = 5 \text{ (F)}$$

For $Q(1,2)$, $x = y + 3$ is not true. So the truth value is false

$$Q(3,0) \Rightarrow 3 = 0 + 3$$

$$\Rightarrow 3 = 3 \text{ (T)}$$

for $Q(3,0)$, $x=y+3$ is true. So the truth value is true.

Free Variable: An occurance of a variable that is not bounded by a quantifier (either a universal quantifier or existential quantifier) is said to be free variables.

ex: $f(x) = 3x-1$

In this function x is free variable because you put in the place of x is any value without limitation.

Bound variables: An occurance of a variable that is bounded said to be bound variable.

example: $\sum_{x=1}^{4}(x+4)$

Here, x is bound variable, x goes for 1 to 4 in this summation.

Modus ponens: In propositional logic modus ponens is a deductive argument form and rule of inference.

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

* What are the negation of statements $\forall x\,(x^2 > x)$ and $\exists x\,(x^2 = 2)$?

$\forall x\,(x^2 > x)$

$= \neg \forall x\,(x^2 > x)$

$= \exists x\,\neg(x^2 > x)$

$= \exists x\,(x^2 \leq x)$

The negation of $\forall x\,(x^2 > x)$ is $\neg\forall x\,(x^2 > x)$, which is equivalent to $\exists x\,\neg(x^2 > x)$. This can be written as $\exists x\,(x^2 \leq x)$.

$\exists x\,(x^2 = 2)$

$= \neg\exists x\,(x^2 = 2)$

$= \forall x\,\neg(x^2 = 2)$

$= \forall x\,(x^2 \neq 2)$

The negation of $\exists x\,(x^2 = 2)$ is $\neg\exists x\,(x^2 = 2)$ which is equivalent to $\forall x\,\neg(x^2 = 2)$. This can be written as $\forall x\,(x^2 \neq 2)$.

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* Show that $\neg \forall x (P(x) \to Q(x))$ and $\exists x (P(x) \wedge \neg Q(x))$ are logically equivalent.

$$\neg \forall x \, (P(x) \to Q(x))$$
$$\equiv \exists x \neg (P(x) \to Q(x))$$
$$\equiv \exists x \neg (\neg P(x) \vee Q(x))$$
$$\equiv \exists x (\neg(\neg P(x)) \wedge \neg Q(x))$$
$$\equiv \exists x \, (P(x) \wedge \neg Q(x))$$

(Showed)

* Translate into English the statement

$$\forall x \forall y \, ((x>0) \wedge (y<0)) \to (xy<0))$$

This statement says that, for every real number of x and every real number of y, if $x>0$ and $y<0$ then $xy<0$.

This statement says that for all real numbers x and y if x is positive and y is negative than xy is negative.

This can be stated more succinctly as "~~P~~ The product of a positive real number and negative real number is always negative real number.

* Define predicate with example:

predicate : A predicate is an expression of one/more variables determined or some specific domain.

ex: consider $E(x,y)$ denote "$x=y$" consider $M(x,y)$ denote x is

Compound proposition: Compound proposition is a proposition formed by combining two or more simple proposition.

Connectives : The logical operator that are used to form compound proposition is called connectives.

ex: $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$

EXOR / XOR :

P	q	$P \oplus q$
F	F	F
F	T	T
T	F	T
T	T	F

odd numer of T for (T) otherwise F

Biconditional: Let p and q be proposition. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q". The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values and is false otherwise.

ex: $p \rightarrow$ You can take the flight
$q \rightarrow$ You buy a ticket.

You can take the flight if and only if you buy a ticket.

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Contingency: A compound proposition that is neither a tautology nor a contradiction is called a contingency.

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

$\Rightarrow$ contingency.

Logical Equivalent: The compound proposition p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology.

The notation $p \equiv q$ denotes that p and q are logically equivalent.

* What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integer not exceeding 4.

$$\forall x P(x) \Rightarrow x^2 < 10$$

$$P(1) \Rightarrow 1^2 < 10 \quad (T)$$

$$P(2) \Rightarrow 2^2 < 10 \quad (T)$$

$$P(3) \Rightarrow 3^2 < 10 \quad (T)$$

$$P(4) \Rightarrow 4^2 < 10 \quad (F)$$