DONE

Mathematical Induction

□ Discuss about mathematical induction ?

⇒ **Mathematical induction:** Mathematical induction is a technique for proving theorems such as $1+2+3+\dots+n = \frac{n(n+1)}{2}$. Mathematical theorem is used extensively to prove results about a large variety of discreate object.

For-example it is used to prove results about the complexity of algolithms, the correctness of certain types of computer program. theorem about graph and tree.

A proof by mathematical induction that P(n) is true for every positive integers n consists of two steps:

1. Basis step : The proposition P(1) is shown to be true.

2. Inductive step: The implication $P(n) \rightarrow P(n+1)$ is shown to true for every positive integer n.

Example 1: Use mathematical Induction that the sum of the first n odd integers is n^2.

⇒ Solution: Let $P(n)$ denote the proposition that the sum of first n odd positive integer is n^2 i.e

$P(n) = 1+3+5+ \cdots + (2n-1) = n^2$ ---- ①

Basis step:

Let $P(1)$ will be true for this statement ①

$L.H.S = 2n-1$ $\qquad$ $R.H.S = n^2$

$= 2.1 - 1$ $\qquad$ $= 1^2$

$= 1$ $\qquad$ $= 1$

Since $L.H.S = R.H.S$ so $P(1)$ is true

Induction step:

Assume that $P(n)$ is true that is assume

$1+3+5+ \cdots + (2n-1) = n^2$ ------ ②

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Under this assumption, it must be shown that $p(n+1)$ is true, namely that

$$1+3+5+ \cdots + (2n+1) = (n+1)^2 \quad \text{--- (3)}$$

Adding (2) No equation by $(2n+1)$ then we get

$$1+3+5+ \cdots + (2n-1)+(2n+1) = n^2+2n+1$$
$$= (n+1)^2 \quad \text{--- (4)}$$

Now rewriting the L.H.S of the statement (4)

$$1+3+5+ \cdots + (2n-1)+(2n+1) \cancel{= (n+1)^2}$$
$$= [1+3+5+ \cdots +(2n-1)] + (2n+1)$$
$$= n^2+2n+1 \quad [\text{From } (1)]$$
$$= (n+1)^2$$

So the equation (4) show that $p(n+1)$ is true.

Since $P(1)$ is true and the implication $p(n) \longrightarrow p(n+1)$ is true for all positive integers n, $P(n)$ is true for all positive integers n.

Example 4: Use mathematical induction to show that, $1+2+2^2+\cdots+2^n = 2^{n+1}-1$ for all non-negative integer.

⇒ Solution: Let $p(n)$ denote the proposition that $1+2+2^2+\cdots+2^n = 2^{n+1}-1$ for all non-negative integer n.

Basis step: Let $P(0)$ will be true for this statement.

$$L.H.S = 2^n \qquad\qquad R.H.S = 2^{n+1}-1$$
$$= 2^0 \qquad\qquad = 2^1 - 1$$
$$= 1 \qquad\qquad = 2-1 = 1$$

Since $L.H.S = R.H.S$ so $P(0)$ is true.

Inductive step: Assume that $p(n)$ is true that is assume,

$$1+2+2^2+\cdots+2^n = 2^{n+1}-1 \quad \text{---} (1)$$

Under this assumption it must be shown that $p(n+1)$ is true, namely that first

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$1+2+2^2+ - - - - - + 2^{n+1} = 2^{n+1+1} - 1$

$= 2^{n+2} - 1$ ---- ②

Adding equation ① by 2^{n+1} then we get

$1+2+2^2+ - - - - + 2^n + 2^{n+1} = 2^{n+1} + 2^{n+1} - 1$

$= 2^n . 2^1 + 2^n . 2^1 - 1$

$= 2^n (2+2) - 1$

$= 2^n . 4 - 1$

$= 2^n . 2^2 - 1$

$= 2^{n+2} - 1$ ---- ③

Now rewriting the L.H.S statement in ③

$1+2+2^2+ - - - + 2^n + 2^{n+1}$

$= [1+2+2^2+ - - - + 2^n] + 2^{n+1}$

$= 2^{n+1} - 1 + 2^{n+1}$ [From ①]

$= 2 . 2^{n+1} - 1$

$= 2^{n+2} - 1$

so the equation ③ show that $p(n+1)$ is true

The statement is proved for all non-negative integer n

Example-5: (Sum of Geometric Progressions)

Problem Use mathematical Induction to prove this formula for the sum of a finite number of terms of a geometric progression;

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r-1}$$

where $r \neq 1$

where n is a non-negative integer.

Solution:

Let $P(n)$ be a proposition that the sum of the first $(n+1)$ terms of a geometric progression in this formula is correct.

Basis step:

Let $P(0)$ will be true for this statement

$$L.H.S = ar^n$$
$$= ar^0$$
$$= a \cdot 1$$
$$= a$$

$$R.H.S = \frac{ar^{n+1} - a}{r-1}$$
$$= \frac{ar^{0+1} - a}{r-1}$$
$$= \frac{a(r-1)}{r-1} = a$$

Since L.H.S = R.H.S so $p(0)$ is true

Induvtive step:

Assume that $p(n)$ is true

that is assume

$$a+ar+ar^2+\cdots+ar^n = \frac{ar^{n+1}-a}{r-1} \quad \cdots (1)$$

Under this assumption it must be shown that $p(n+1)$ is true namely that

$$a+ar+ar^2+\cdots+ar^{n+1} = \frac{ar^{n+1+1}-a}{r-1} \quad \cdots (2)$$

$$= \frac{ar^{n+2}-a}{r-1}$$

Adding equation (1) by ar^{n+1} then we get

$$a+ar+ar^2+\cdots+ar^n+ar^{n+1} = \frac{ar^{n+1}-a}{r-1}+ar^{n+1}$$

$$= \frac{ar^{n+1}+(r-1)\,ar^{n+1}-a}{r-1}$$

$$= \frac{ar^{n+1}+ar^{n+2}-ar^{n+1}-a}{r-1}$$

$$= \frac{ar^{n+2} - a}{r-1}$$

$$= \frac{a(r^{n+2} - 1)}{r-1} \quad \text{------} \quad (3)$$

Now re-writing the L.H.S of the statement (3) then we get

$$a + ar + ar^2 + \cdots + ar^n + ar^{n+1}$$

$$= (a + ar + ar^2 + \cdots + ar^n) + ar^{n+1}$$

$$= \frac{ar^{n+1} - a}{r-1} + ar^{n+1} \quad [\text{From } (1)]$$

$$= \frac{ar^{n+1} - a + (r-1)(ar^{n+1})}{r-1}$$

$$= \frac{ar^{n+1} - a + ar^{n+2} - ar^{n+1}}{r-1}$$

$$= \frac{ar^{n+2} - a}{r-1}$$

$$= \frac{a(r^{n+2} - 1)}{r-1}$$

9 so the equation ③ show that $P(n+1)$ is true

The statement is proved for the sum of a finite number of term of geometric progression.

Example-8: show that if n is a positive integer then $1+2+\cdots+n = \frac{n(n+1)}{2}$

Solution: Let $P(n)$ be the proposition that the sum of the first n positive integer is $\frac{n(n+1)}{2}$

Basis step: Let $P(1)$ will be true for this statement

$L.H.S = n$ $\quad$ $R.H.S = \frac{1(1+1)}{2}$

$= 1$ $\quad$ $= \frac{2}{2} = 1$

Since $L.H.S = R.H.S$ so $P(1)$ is true

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Inductive step: Assume that $P(n)$ is true

that is assume

$$1+2+3+\cdots+n = \frac{n(n+1)}{2} \quad \cdots (1)$$

Under this assumption it must be shown that $P(n+1)$ is true, namely that

$$1+2+3+\cdots+(n+1) = \frac{(n+1)(n+2)}{2}$$

$$\cdots (2)$$

Adding equation (1) by $(n+1)$ then we get.

$$1+2+3+\cdots+n+(n+1) = \frac{n(n+1)}{2}+(n+1)$$

$$= \frac{n^2+n+2n+2}{2}$$

$$= \frac{n^2+3n+2}{2} \quad \cdots (3)$$

Now re-writing the L.H.S of the statement (2)

$$1+2+3+\cdots+n+(n+1)$$

$$= [1+2+3+\cdots+n]+n+1$$

$$= \frac{n(n+1)}{2}+n+1$$

$$= \frac{n^2+n+2n+2}{\cancel{n+}\,2}$$

$$= \frac{n^2+3n+2}{2}$$

So the equation ③ shown that $P(n+1)$ is true. So the statement is proved for all integer.

Example: Use mathematical induction to prove that $2^n < n!$ for every positive integer n with $n \geq 4$

Solution: Let $P(n)$ be the proposition that $2^n < n!$

Basis step: Let $P(4)$ will be true for this statement

$$L.H.S = 2^n = 2^4 = 16 \qquad R.H.S = n! = 4! = 24$$

Since $16 < 24$. So $P(4)$ is true

Inductive step: Assume that $p(n)$ is true

that is assume that $2^n < n!$

under this assumption it must be shown

that $p(n+1)$ is true namely

$$2^{n+1} < (n+1)! \quad - - - (1)$$

Multiplying both sides of the inequality

$2^n < n!$ by 2

$$\therefore 2 \cdot 2^n < 2n!$$

$$\Rightarrow 2^{n+1} < (n+1) \cdot n!$$

$$2^{n+1} < (n+1)!$$

this shows that $p(n+1)$ is true when

$p(n)$ is true. Hence it follows that

$2^n < n!$ is true for all integers

n with $n \geq 4$.

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Example: Use mathematical induction to prove the inequality $n < 2^n$ for all positive integer n

⇒ Solution: Let $p(n)$ be the proposition $n < 2^n$

Basis step: $p(1)$ is true, since $1 < 2^1 = 2$

Inductive step: Let $p(n)$ is true for the positive integer n. That is $n < 2^n$. We need to show that $p(n+1)$ is true.

i.e we need to show that $n+1 < 2^{n+1}$

Adding 1 to both sides of $n < 2^n$ we have,

$$n+1 < 2^n + 1$$
$$\leq 2^n + 2^n \quad \because 1 \leq 2^n$$
$$< 2 \cdot 2^n$$
$$\therefore n+1 < 2^{n+1}$$

$p(n+1)$ is true based on the assumption that $p(n)$ is true. By the principle of mathematical induction, it has been

~~shown that $n < 2^n$ is true for all positive integer.~~

Example 8: Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

$\Rightarrow$ solution: Let $p(n)$ denote the proposition $n^3 - n$ is divisible by 3.

Basis step: $p(1)$ is true, since $1^3 - 1 = 0$ is divisible by 3

Inductive step:

Assume that $p(n)$ is true that is assume $n^3 - n$ is divisible by 3. Under this assumption. It must be shown that $p(n+1)$ is true. namely that $(n+1)^3 - (n+1)$ is divisible by 3

Now, we have

$$(n+1)^3-(n+1) = (n+1)\{(n+1)^2-1\}$$
$$= (n+1)(n^2+2n)$$
$$= n^3+2n^2+n^2+2n$$
$$= n^3+3n^2+2n$$
$$= n^3+3n^2+3n+1-n-1$$
$$= n^3-n+3n^2+3n$$
$$= (n^3-n)+3(n^2+n)$$

Problem 2 to n^3:

Since, both terms in this sum are divisible by 3, it follows that $(n+1)^3-(n+1)$ is also divisible by 3.

By principle of mathematical induction n^3-n is divisible by 3 whenever n is a positive integer.