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Mathmatical Induction

* Discuss about mathimatical induction.

Mathmatical Induction: Mathmatical induction is a technique for proving theorems. Mathmatical theorem is used extensively to prove results about a large variety of discreate object.

For example, it is used to prove results about the complexity of algorithms, the correctness of certain types of computer program theorem about graph and tree.

A proof by mathmatical induction that $p(n)$ is true for every positive integers n consists of two steps:

① Basis step: The proposition $p(1)$ is shown to be true.

② Inductive step: The implification $p(n) \to p(n+1)$ is shown to true for every positive integer n.

* Show that if n is a positive integer, then.

$$1+2+3+\cdots\cdots+n = \frac{n(n+1)}{2}$$

Let, $P(n) = 1+2+3+\cdots\cdots+n = \frac{n(n+1)}{2}$

for a positive integer.

Basis step:

$P(1)$ will be true

$$P(1) = \frac{1(1+1)}{2} = 1$$

The first element of series also 1.

So it is proved.

Inductive step:

We assume that $P(k)$ is true for an arbitrary positive integer k.

$$P(k) = 1+2+3+\cdots\cdots+k = \frac{k(k+1)}{2} \quad \text{—— (1)}$$

Under the assumption, it must be shown that $P(k+1)$ is true, namely that,

$$P(k+1) = 1+2+3+\cdots\cdots+k+(k+1) = \frac{(k+1)(k+1+1)}{2}$$

—— (2)

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from L.H.S of equation (2) :-

$1+2+3+\cdots\cdots+k+(k+1)$

$=(1+2+3+\cdots\cdots+k)+(k+1)$ [putting eqn 1]

$=\frac{k(k+1)}{2}+\frac{(k+1)}{}$

$=\frac{k(k+1)+2(k+1)}{2}$

$=\frac{(k+1)(k+2)}{2}$

$=$ R.H.S of eqn (2)

We have completed basis and inductive step, so by mathmatical induction we have proven that $1+2+3+\cdots+n=\frac{n(n+1)}{2}$ for all positive integer n.

Use mathmatical induction to show that

$1+2+2^2+\cdots\cdots 2^n = 2^{n+1}-1$ for all nonnegative integer n

Let $P(n)=1+2+2^2+\cdots+2^n=2^{n+1}-1$ for the integer n.

Basis step : $P(0)$ is true.

$P(0)=2^{0+1}-1=1$

The first element of series also 1.
so it is proved.

Inductive step: We assume that $p(k)$ is ~~true for all non negative integer k~~ an arbitrary non negative integer k.

$$p(k) = 1+2+2^2+\cdots+2^k = 2^{k+1}-1 \quad \text{——— (1)}$$

Under the assumption $p(k)$, it must be shown that $p(k+1)$ is also true,

$$p(k+1) = 1+2+2^2+\cdots+2^k+2^{k+1} = 2^{k+1+1}-1 \quad \text{——— (2)}$$

from equation (2) →

$$L.H.S = 1+2+2^2+\cdots+2^k+2^{k+1}$$

$$= [1+2+2^2+\cdots+2^k] + 2^{k+1}$$

$$= 2^{k+1}-1+2^{k+1} \quad \text{[putting the value of eqn (1)]}$$

$$= 2^{k+1}+2^{k+1}-1$$

$$= 2^{k+1}(1+1)-1$$

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$= 2^{k+1} \cdot 2^1 \cancel{+1} - 1$

$= 2^{k+1+1} - 1$

$= 2^{k+2} - 1$

$=$ R.H.S.

We have completed the basis step and inductive step, by mathmatical induction we have proven that $1+2+2^2+\cdots+2^n = 2^{n+1} - 1$

* Use mathmatical induction to prove the inequality

$$n < 2^n.$$

for all positive integer n.

Let $p(n) = n < 2^n$ for all positive integer n

Basis step: $p(n) = n < 2^n$

$p(1) = 1 < 2^1$

$= 1 < 2$

It is true.

<u>Inductive step</u> : We assume that

$p(k)$ is true

$$p(k) = k < 2^k \quad \text{———} \ 1$$

Under the assumption $p(k)$, it must be ~~true the~~ shown that $p(k+1)$ is true

$$p(k+1) = k+1 < 2^{k+1} \quad \text{———} \ 2$$

from equation 1 $\Rightarrow$

$$k < 2^k$$

$$\Rightarrow 2 \cdot k < 2^k \cdot 2$$

$$\Rightarrow \underline{k+k} < 2^{k+1}$$

$$\Rightarrow \underline{k+1} < 2^{k+1}$$

We completed the both basis step and inductive step by the principle of mathematical induction, we have shown that $n < 2^n$ is true for all positive integers n.

* Use mathmatical induction to prove that n^3-n is divisible by 3 whenever n is positive integer.

Let $p(n) = n^3 - \cancel{3}n$, where n is divisible by 3

<u>Basis step</u>: $p(1)$ is true.

$$p(n) = n^3 - n$$
$$p(1) = 1^3 - 1$$
$$= 0 \text{ (it's divisible by 3)}$$

So, its proved.

<u>Inductive step</u>: We assume that $p(k)$ is true.

$$p(k) = k^3 - k \quad \text{——} \ (1)$$

Under the assumption $p(k)$, it must be shown that $p(k+1)$ is true.

$$p(k+1) = (k+1)^3 - (k+1) \quad \text{——} \ (2)$$

from equation (2) $\Rightarrow$

$$(k+1)^3 - (k-1)$$

$(k+1)^3 - (k+1)$

$= (k^3+3k^2+3k+1) - (k+1)$

$= k^3+3k^2+3k-1-k+1$

$= (k^3-k) + 3(k^2+k)$

(k^3-k) is divisible by 3 (from eqn 1)
Second term $3(k^2+k)$ is clearly divisible by 3 because it is 3 times an integer.

We have completed the both basis and inductive step, by the principle of mathematical induction, we prove that n^3-n is divisible by 3 whenever n is a positive integer

* Use mathmatical induction to prove that $2^n < n!$ for every integer n with $n \geq 4$

Let $p(n) = 2^n < n!$

Basis step: $P(4)$ is to be true.

$p(4) = 2^4 < 4!$

$= 16 < 24$

It's true.

Inductive step: We assume that $p(k)$ is true for an arbitrary integer k with $k \geq 4$

$$p(k) = 2^k < k! \quad \text{——— (1)}$$

Under the assumption $p(k)$, we must shown that $p(k+1)$ is true also:

$$p(k+1) = 2^{k+1} < (k+1)! \quad \text{——— (2)}$$

from (1) $\Rightarrow$

$$2^k < k!$$

$$\Rightarrow 2 \cdot 2^k < 2 \cdot k!$$

$$\Rightarrow 2^{k+1} < (k+1)\, k!$$

$$\Rightarrow 2^{k+1} < (k+1)!$$

(proved)

We have completed the basis and inductive step, by using the principle of mathmatical induction we proved that $2^n < n!$ is true for all integer n with $n \geq 4$.

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* Use mathmatical induction to prove this formula for the sum of a finite number of terms of geometric progress with initial term a and common ratio r.

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \dots + ar^n = \frac{ar^{n+1} - a}{r-1}$$

where n is a non negative integer.

Let $p(n) = a + ar + ar^2 + \dots + ar^n = \frac{ar^{n+1} - a}{r-1}$

<u>Basis step</u> : p(0) is true.

$$p(0) = \frac{ar^{0+1} - a}{r-1}$$

$$= \frac{ar - a}{r-1}$$

$$= \frac{a(r-1)}{r-1}$$

$$= a$$

∴ The first element of series is also a so it is true. (proved)

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Inductive step: We assume ~~that~~ $p(k)$ is true for ~~all~~ an arbitrary non negative integer k.

$$p(k) = a + ar + ar^2 + \cdots + ar^k = \frac{ar^{k+1} - a}{r-1} \quad \text{——(1)}$$

Under then assumption $p(k)$, we must be show that $p(k+1)$ is true -

$$p(k+1) = a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+2} - a}{r-1} \quad \text{——(2)}$$

$$\frac{ar^{(k+1)} - a}{r-1} + ar^{(k+1)}$$

$$= \frac{ar^{(k+1)} - a + (r-1)\, ar^{k+1}}{(r-1)}$$

$$= \frac{ar^{k+1} + ar^{k+1}(r-1) - a}{(r-1)}$$

$$= \frac{ar^{k+1} + ar^{k+1+1} - ar^{k+1} - a}{(r-1)}$$

$$= \frac{ar^{k+1} + ar^{k+2} - ar^{k+1} - a}{(r-1)}$$

$$= \frac{a^{1+1-1} r^{k+1+k+2-k-1} - a}{r-1}$$

$$= \frac{ar^{k+2} - a}{r-1}$$

$\Rightarrow$

$\ldots ar^2 + \cdots + ar^k + ar^{k+1}$

$\ldots + ar^2 + \cdots + ar^k] + ar^{k+1}$

$\ldots + ar^{(k+1)}$

$\ldots + \dfrac{ar^{k+2} - ar^{k+1}}{(r-1)}$

$\ldots 2$

$\ldots - a \;\; = \text{R.H.S}$

$\ldots 1)$

(proved)

Inductive step: We assume that p(k) is true for ~~all~~ an arbitrary non negative integer k.

$$p(k) = a + ar + ar^2 + \cdots + ar^k = \frac{ar^{k+1} - a}{r-1} \quad \text{--- (1)}$$

Under the assumption p(k), we must be show that p(k+1) is true -

$$p(k+1) = a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+2} - a}{r-1} \quad \text{--- (2)}$$

from (2) ⇒

$$\text{L.H.S} = a + ar + ar^2 + \cdots + ar^k + ar^{k+1}$$

$$= [a + ar + ar^2 + \cdots + ar^k] + ar^{k+1}$$

$$= \frac{ar^{(k+1)} - a}{r-1} + ar^{(k+1)}$$

$$= \frac{ar^{(k+1)} - a}{(r-1)} + \frac{ar^{k+2} - ar^{k+1}}{(r-1)}$$

$$= \frac{ar^{k+2} - a}{(r-1)} = \text{R.H.S}$$

(proved)

We have completed the both basis and inductive step, so by mathmatical induction $p(n)$ is true for all nonnegative integers n.

* Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathmatical induction.

The sum of the first n positive odd integers for $n = 1, 3, 5, 7$ are,

$1 = 1 = 1^2$

$1+3 = 4 = 2^2$

$1+3+5 = 9 = 3^2$

$1+3+5+7 = 16 = 4^2$

From these values it is reasonable to conjecture that the sum of the first n positive odd integers is n^2, that is

$$1+3+5+\cdots+(2n-1) = n^2$$

Let $p(n) = 1+3+5+\cdots+(2n-1) = n^2$

Basis step: $p(1)$ is true.

$$p(n) = n^2$$
$$p(1) = 1^2$$

the first element of series also 1.

So, it is proved.

Inductive step: Assume that $p(k)$ is true for all positive odd integer.

$$p(k) = 1+3+5+\dots+(2k-1) = k^2 \quad \text{--- (1)}$$

Under the assumption, $p(k+1)$ will be true also,

$$p(k+1) = 1+2+3+\dots\dots+(2k-1)+(2k+1) = (k+1)^2 \quad \text{--- (2)}$$

from (2) $\Rightarrow$

$$LHS = 1+2+3+\dots\dots+(2k-1)+(2k+1)$$
$$= k^2+(2k+1)$$
$$= k^2+2k+1$$
$$= (k+1)^2$$
$$= R.H.S \text{ (proved)}$$

We have completed both basis and inductive

Step, by the principle of mathmatical induction we have proven the formula for the sum of first n positive odd integer and that is

$$1+3+5+\dots\dots+(2n-1)=n^2.$$

* Use mathmatical induction to prove that $7^{n+2}+8^{2n+1}$ is divisible by 57 for every non negative integer n.

Let $p(n) = 7^{n+2}+8^{2n+1}$ is divisible by 57

Basis step: $p(0)$ is true.

$$p(n) = 7^{n+2}+8^{2n+1}$$

$$p(0) = 7^{0+2}+8^{2\times0+1}$$

$$= 7^2+8$$

$$= 57 \text{ is divisible by } 57$$

inductive step: We assume that $p(k)$ is true for an arbitratry non negative integer k.

$$p(k) = 7^{k+2}+8^{2k+1} \quad \text{——— } ①$$

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Under the assumption $P(K)$, $P(K+1)$ is also true.

$$P(K+1) = 7^{K+1+2} + 8^{2(K+1)+1}$$

$$= 7^{K+3} + 8^{2K+1} \cdot 8^2$$

$$= 7 \cdot 7^{K+2} + 8^{2K+1} \cdot 64$$

$$\cancel{= 7(7^{K+2} + 8^{2K+1}) + 57 \cdot 8^{2K+1}}$$

$$= 7(7^{K+2}) + (57+7)\, 8^{2K+1}$$

$$= 7(7^{K+2} + 8^{2K+1}) + 57 \cdot 8^{2K+1}$$

Here, first term $7^{K+2} + 8^{2K+1}$ is divisible by 57 according to equation 1. Second term is also clearly divisible by 57 because it is 57 times an integer.

We, both have completed both the basis step and inductive step. by the principle of mathmatical induction we prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for all non negative integer n.