Q1.

(a) **Prove that** $(n + 1)^3 = O(n^3)$,

Solution: Take C = 8, N = 1. Assuming that $n \geq N$,
$(n + 1)^3 \leq (2n)^3 \leq 8n^3$

(b) **Prove that**

$A_n = O(n)$, where

$$A_n = \begin{cases} 100, & \text{if } n < 10, \\ n + 1, & \text{otherwise.} \end{cases}$$

Solution: Take C = 2, N = 10. Assuming that $n \geq N$,
$An = n + 1 < n + n = 2n$.

Q2.

Prove that

(i) $n^2 + n + 1 = O(n^2)$,

(ii) $3 \cdot 2^n + 100 = O(2^n)$,

(iii) $e^n + e^{n+1} = O(e^n)$.

Solution.

(i) Take $C = 3$ and $N = 1$. We now claim that for all $n \geq N$,

$$n^2 + n + 1 \leq 3n^2.$$

Proof:

$$\begin{aligned} n^2 + n + 1 &\leq n^2 + n^2 + 1 && \text{Since } n \leq n^2 \text{ when } n \geq 1 \\ &\leq n^2 + n^2 + n^2 && \text{Since } 1 \leq n^2 \text{ when } n \geq 1 \\ &= 3n^2. \end{aligned}$$

(ii) Take $C = 103$ and $N = 1$. We now claim that for all $n \geq N$,

$$3 \cdot 2^n \leq 103 \cdot 2^n.$$

Proof:

$$\begin{aligned} 3 \cdot 2^n + 100 &\leq 3 \cdot 2^n + 100 \cdot 2^n && \text{Since } 2^n > 1 \text{ when } n \geq 1 \\ &= 103 \cdot 2^n \end{aligned}$$

(iii) Take $C = 4$ and $N = 1$. We now claim that for all $n \geq N$,

$$e^n + e^{n+1} \leq 4 \cdot e^n.$$

Proof:

$$\begin{aligned} e^n + e^{n+1} &= (e+1) \cdot e^n \\ &\leq 4 \cdot e^n && \text{Since } e + 1 \leq 4 \end{aligned}$$

The value of e is equal to approximately 2.71828.

Q3. I have 7 pairs of socks in my drawer, one of each color of the rainbow. How many socks do I have to draw out in order to guarantee that I have grabbed at least one pair? What if there are likewise colored pairs of gloves in there and I cannot tell the difference between gloves and socks and I want a matching set?

Answer:

Solution: After grabbing 7 socks, worst case scenario, I have grabbed a sock of each color. Thus, after grabbing one more sock, it has to match up with one of the previous socks so after grabbing 8 socks I am guaranteed to have a pair. For the second part, after grabbing 21 objects, it is possible that I have grabbed 3 items for each color and hence have gotten no sets yet. But the 22nd thing I grab must complete one of these 7 sets so after 22 items, I am guaranteed to have a matching set.

Q4. Three people are running for election. There are 202 people who vote. What is the minimum number of votes needed for someone to win the election?

Solution:
By pigeonhole, there exists a person who has gotten at least d202/3e = 68 votes. So, someone could win with a 67 − 67 − 68 split.

Q5. There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, what is the minimum number of different rooms that will be needed?

Solution:
There exists a time period that will have at least [677 / 38] = 18 classes during it. So 18 different rooms will be needed.

Q6. How many ways are there for 8 men and 5 women to stand in a line so that no two women stand next to each other?

Solution:
Once you place the men, there are 9 spots for the women. We can choose one for each woman to stand in and since the order matters, the final number is $P(8, 8) \cdot P(9, 5)$.

Q7. How many ways are there to choose a delegation out of 10 males and 10 females if the delegation is made up of 2 males and 3 females?

Answer.
$C(10, 2) \times C(10, 3)$

Q8. At a consultant mixer with 42 people, everyone shakes everyone else's hand exactly once. How many handshakes occur?

Solution: There exists one handshake between any two people, so one for each pair. There are $C(42, 2)$ different ways to choose a pair.

Q9. A committee of 5 people is to be chosen from 6 men and 4 women. In how many ways can this be done
(i) if there must be 3 men and 2 women on the committee
(ii) if it must consist of a majority of women?

Answer:
i) 6C3*4C2 =20*6= 120

ii)

Majority of women on a 5 person committee means 3+ women:

3 women, 2 men: $\binom{4}{3}\binom{6}{2} = 4 * 15 = 60$

4 women, 1 man: $\binom{4}{4}\binom{6}{1} = 1 * 6 = 6$

5 women, 0 men: $\binom{4}{5}\binom{2}{0} = 0 * 1 = 0$

60+6+0=66.

Q10. Race cars for a particular race are numbered sequentially from 12 to 115. What is the probability that a car selected at random will have a tens digit of 1?

Answer:

There are 104 integers from 12 to 115 inclusive. There are 8 integers from 12 to 19 and 6 integers from 110 to 115 for a total of 14 integers with a tens digit of 1. The probability of selecting a car with a tens digit of 1 is

14/104 = 7/52

Q11. Three friends play marbles each week. When they combine their marbles, they have 100 in total. 45 of the marbles are new and the rest are old. 30 are red, 20 are green, 25 are yellow, and the rest are white. What is the probability that a randomly chosen marble is new OR yellow?

Answer:

Prob(new OR yellow) = P(new) + P(yellow) - P(new AND yellow)

Prob(new) = $\frac{45}{100}$

Prob(yellow) = $\frac{25}{100}$

Prob(new AND yellow) = $\frac{45}{100} \times \frac{25}{100}$

so P(new OR yellow) = $\frac{45}{100} + \frac{25}{100} - \frac{45}{100} \times \frac{25}{100}$

$= \frac{70}{100} - \frac{9}{80}$

$= \frac{7}{10} - \frac{9}{80}$

$= \frac{56}{80} - \frac{9}{80} = \frac{47}{80}$