* The limite is called double or Simultaneously limit

* $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = \lim\limits_{\substack{x\to x_0 \\ y\to y_0}} f(x,y)$

Limit in multivariable calculus are of two type:

① Simultaneous limit / Double limit

② Iterative limit.

① **Simultaneous limit:**

If we want to find simultaneous limit of $f(x,y)$ at (a,b). Then we write,

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

Date: also written,

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = L$$

* simultaneous limit এর ক্ষেত্রে (x,y) এর মান (a,b) কে একই সময় একই সাথে input করতে হয়।

Iterative limit:

If we want to find iterative limit of $f(x,y)$ at (a,b). then we write,

$$\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y) = L$$

$$= L$$

if L.H.S = R.H.S. then we say that iterative limit exists, Otherwise if L.H.S = R.H.S, then iterative limit does not exist.

Date: Day:

Ex-01:

Show that, $f(x,y) = \dfrac{xy}{x^2+y^2}$ is iterative at (0,0) but not simultaneous.

Sol:

For Simultaneous limit:

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

$$= \frac{0\times 0}{0+0}$$

$$= \frac{0}{0} \; ; \left\{\begin{array}{l}\text{where is in}\\ \text{indeterminate}\\ \text{form}\end{array}\right.$$

Hence, Simultaneous limit of $f(x,y)$ does not exist at (0,0).

Date: Day:

For iterative limit:

$$LHS = \lim_{x\to a} \lim_{y\to b} f(x,y)$$

$$= \lim_{x\to 0} \lim_{y\to 0} \frac{xy}{x^2+y^2}$$

$$= \lim_{x\to 0} \frac{x\times 0}{x^2+0^2}$$

$$= \lim_{x\to 0} \frac{0}{x^2} = \lim_{x\to 0} \times 0 = 0$$

Date: Day:

$$R.H.S = \lim_{y\to b} \lim_{x\to a} f(x,y)$$

$$= \lim_{y\to 0} \lim_{x\to 0} \frac{xy}{x^2+y^2}$$

$$= \lim_{y\to 0} \frac{0\times y}{0^2+y^2} = \lim_{y\to 0} \frac{0}{y^2}$$

$$= \lim_{y\to 0} 0 = 0$$

Since, L.H.S = R.H.S = 0

Hence, iterative limit of $f(x,y)$ exist at $(0,0)$.

Date: Day:

Home work:

Find iterative limit of the function:

(i) $f(x,y) = \dfrac{x^3+y^3}{x^3-y^3}$ at $(0,0)$

Sol: $L.H.S = \lim_{x\to 0} \lim_{y\to 0} \dfrac{x^3+y^3}{x^3-y^3}$

$$= \lim_{x\to 0} \frac{x^3+0^3}{x^3-0^3}$$

$$= \lim_{x\to 0} \frac{\cancel{x^3}}{\cancel{x^3}} = \lim_{x\to 0} (1)$$

Date: Day:

$$= 1$$

$$R.H.S = \lim_{y\to 0} \lim_{x\to 0} \frac{x^3+y^3}{x^3-y^3}$$

$$= \lim_{y\to 0} \frac{0^3+y^3}{0^3-y^3} = \lim_{y\to 0} \frac{\cancel{y^3}}{-\cancel{y^3}}$$

$$= \lim_{y\to 0} (-1)$$

$\therefore$ ~~L.S.~~ L.H.S $\neq$ R.H.S,

Hence, iterative limit of $f(x,y)$ dose not exist at $(0,0)$

Date: Day:

(ii) $f(x,y) = \dfrac{x^2+y^2}{(x+y)^2+2x^2y^2}$ at $(0,0)$

Sol:

$$L.H.S = \lim_{x\to 0} \lim_{y\to 0} \frac{x^2+y^2}{(x+y)^2+2x^2y^2}$$

$$= \lim_{x\to 0} \frac{x^2+0^2}{(x+0)^2+2x^2 \cdot 0^2}$$

$$= \lim_{x\to 0} \frac{x^2}{x^2+0} = \lim_{x\to 0} \frac{\cancel{x^2}}{\cancel{x^2}}$$

$$= \lim_{x\to 0} (1) = 1$$

$$R.H.S = \lim_{y\to 0} \lim_{x\to 0} \frac{x^2+y^2}{(x+y)^2+2x^2y^2}$$

$$= \lim_{y\to 0} \frac{0+y^2}{(0+y)^2+2\cdot 0 \cdot y^2}$$

$$= \lim_{y\to 0} \frac{y^2}{y^2+0} = \lim_{y\to 0} \frac{\cancel{y^2}}{\cancel{y^2}}$$

$$= \lim_{y\to 0} (1) = 1$$

Date: Day:

Since, $L.H.S = R.H.S = 1$

Hence, iterative limit of $f(x,y)$ exist at $(0,0)$.

(iii) $f(x,y) = \dfrac{x+y-1}{x+y+2}$ at $(0,1)$

Sol:

$$L.H.S = \lim_{x\to 0} \lim_{y\to 1} \frac{x+y-1}{x+y+2}$$

$$\cancel{= \lim_{x\to 0} \frac{1+y-1}{1+y+2}}$$

$$\cancel{= \lim_{x\to 0} \frac{y}{y+2}}$$

$$= \lim_{x\to 0} \frac{x+1-1}{x+1+2}$$

$$= \lim_{x\to 0} \frac{x}{x+3}$$

$$= \frac{0}{0+3} = 0$$

Date: Day:

$$R.H.S = \lim_{y\to 1} \lim_{x\to 0} \frac{x+y-1}{x+y+2}$$

$$= \lim_{y\to 1} \frac{0+y-1}{0+y+2}$$

$$= \lim_{y\to 1} \frac{y-1}{y+2}$$

$$= \frac{1-1}{1+2} = \frac{0}{3} = 0$$

Date: Day:

$L.H.S = R.H.S = 0$

Since, ~~L.H.S = R.H.S = 0~~

Hance, iterative limit of $f(x,y)$

dose exist at $(0,1)$.

(iv) $f(x,y) = \dfrac{3x+4y}{5x+2y}$ at $(0,0)$

Sol:

Date: Day:

$$L.H.S = \lim_{x\to 0} \lim_{y\to 0} \frac{3x+4y}{5x+2y}$$

$$= \lim_{x\to 0} \frac{3.x+4.0}{5x+2.0}$$

$$= \lim_{x\to 0} \frac{3x}{5x} = \lim_{x\to 0} \frac{3}{5}$$

$$= 3/5$$

$$R.H.S = \lim_{y\to 0} \lim_{x\to 0} \frac{3x+4y}{5x+2y}$$

$$= \lim_{y\to 0} \frac{3.0+4y}{5.0+2y}$$

Date: Day:

$$= \lim_{y\to 0} \frac{4y}{2y} = \lim_{y\to 0} (2)$$

$$= 2$$

Since, $L.H.S \neq R.H.S$

Hance, iterative limit of $f(x,y)$

dose not exist at $(0,0)$

(v) $f(x,y) = \frac{x^2 - y^2}{x^2y^2 - (x+y)^2}$ at $(0,0)$

$$L.H.S = \lim_{x\to 0} \lim_{y\to 0} \frac{x^2-y^2}{x^2y^2-(x+y)^2}$$

$$= \lim_{x\to 0} \frac{x^2\cdot 0 - 0^2}{x^2\cdot 0-(x+0)^2}$$

$$= \lim_{x\to 0} \frac{0\, x^2}{0 - x^2}$$

$$= \lim_{x\to 0} \frac{x^2}{-x^2} = \lim_{x\to 0}(-1)$$

$$= -1$$

$$R.H.S = \lim_{y\to 0} \lim_{x\to 0} \frac{x^2-y^2}{x^2y^2-(x+y)^2}$$

$$= \lim_{y\to 0} \frac{0-y^2}{0\cdot y^2-(0+y)^2}$$

$$= \lim_{y\to 0} \frac{-y^2}{-(y)^2} = \lim_{y\to 0}(1)$$

$$= 1$$

Since, L.H.S $\neq$ R.H.S

Hance, itenerative limit of $f(x,y)$ dose not exists at $(0,0)$.

Simultaneous limit:

Ex-01:

Suppose that, $f(x,y) = \frac{xy^2}{x^2+y^4}$

(i) Show that, $\lim_{(x,y)\to(0,0)} f(x,y)$ exists along $y=x$.

(ii) Show that, $\lim_{(x,y)\to(0,0)} f(x,y)$ exists along $y=\sqrt{x}$.

(iii) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist;

Sol:

(i) sol: Given, $f(x,y) = \frac{xy^2}{x^2+y^4}$

Now, along $y=x$,

$$f(x,y) = \frac{x\times x^2}{x^2+x^4}$$

$$= \frac{x^3}{x^2(1+x^2)} = \frac{x}{1+x^2}$$

www.uap.ac.bd

Date: Day:

NOW, $\lim_{(x,y)\to(0,0)} f(x,y)$ along $y=x$ is

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x}{1+x^2}$$

$$= \frac{0}{1+0}$$

$$= 0/1 = 0$$

Since, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$; which is

Date: Unige and finat. Day:

Hance, $\lim_{(x,y)\to(0,0)} f(x,y)$ exist along

$x=y$.

(ii) Sol: Given, $f(x,y) = \frac{xy^2}{x^2+y^4}$

Now, along $y=\sqrt{x}$,

$$f(x,y) = \frac{x(\sqrt{x})^2}{x^2+(\sqrt{x})^4} = \frac{x.x}{x^2+x^2}$$

$$= \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2}$$

Date: Day:

Now, $\lim_{(x,y)\to(0,0)} f(x,y)$ along $y=\sqrt{x}$

is $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{1}{2}$

$$= 1/2$$

Since, $\lim_{(x,y)\to(0,0)} f(x,y) = 1/2$; which is

Unique and finat.

Hance, $\lim_{(x,y)\to(0,0)} f(x,y)$ exist along

Date: Day:

$x=y$.

(iii) Sol:

Now, from (i) and (ii), we found that

$\lim_{(x,y)\to(0,0)} f(x,y) = 0$ along $y=x$ (line)

and $\lim_{(x,y)\to(0,0)} f(x,y) = 1/2$ along

$y=\sqrt{x}$ (curve).

Date: Day:

$\lim_{(x,y)\to(0,0)} f(x,y)$ is different

along $y=x$ and $y=\sqrt{x}$

(which are passing throw $(0;0)$);

Hence, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

H.W : 1 :

Date: Day:

if $f(x,y)=\frac{xy}{x^2+y^2}$, then show that,

(i) $\lim_{\substack{x\to0\\y\to0}} f(x,y)=0$ along $y=0$

(ii) $\lim_{\substack{x\to0\\y\to0}} f(x,y)=0$ along $x=0$

(iii) $\lim_{\substack{x\to0\\y\to0}} f(x,y)=0$ along $y=x^2$

Date: Day:

(iv) $\lim_{\substack{x\to0\\y\to0}} f(x,y)$ along $y=x$ limit exist

(v) $\lim_{\substack{x\to0\\y\to0}} f(x,y)$ dose not exist

(i) Sol:

Given,

$$f(x,y)=\frac{xy}{x^2+y^2}$$

Date: Day:

Now, along $y=0$,

$$f(x,y)=\frac{x\cdot0}{x^2+0^2}=\frac{0}{x^2}$$
$$=0$$

Now, $\lim_{\substack{x\to0\\y\to0}} f(x,y)=0$ along $y=0$,

then, L.H.S $=\lim_{\substack{x\to0\\y\to0}} f(x,y)=\lim_{\substack{x\to0\\y\to0}}(0)$

$=0=$ R.H.S

Date: Day:

Since, L.H.S = R.H.S = 0; which is Unique and limit.

Hance, $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does exist limit along $y=0$.

(ii) Sol:

Given, $f(x,y) = \dfrac{xy}{x^2+y^2}$

Date: Day:

Now, along $x=0$ $f(x,y)$ is,

$$f(x,y) = \frac{0\cdot y}{0^2+y^2} = \frac{0}{y^2} = 0$$

Now, $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ along $x=0$ is,

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{x\to 0\\ y\to 0}} (0) = 0$$

$\therefore$ L.H.S = R.H.S = 0; which is unique

Date: Day:

and limit.

Hance, $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does exist limit along $x=0$.

(iii) Given, $f(x,y) = \dfrac{xy}{x^2+y^2}$

Now, along ~~x=~~ $y=x^2$ is,

$$f(x,y) = \frac{x\cdot x^2}{x^2+x^4} = \frac{x^2 x}{x^2(1+x^2)}$$

Date: Day:

$$= \frac{x}{1+x^2}$$

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x}{1+x^2} = \frac{0}{1+0^2} = \frac{0}{1} = 0$$

Since, L.H.S = R.H.S = 0; which is unique and limit.

Hance, $\lim\limits_{\substack{x\to 0\\ x\to 0}} f(x,y)$ does exist limit along $y=x^2$.

Date: Day:

(iv) Given, $f(x,y) = \frac{xy}{x^2+y^2}$

now, along $y=x$ then $f(x,y)$ is

$$f(x,y) = \frac{x \cdot x}{x^2+x^2} = \frac{x^2}{2x^2} = \frac{1}{2}$$

$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ along $y=x$, then

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{x\to 0\\ y\to 0}} (1/2) = 1/2$$

Date: Day:

Since, $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 1/2$; which is

Unique and finat.

Hance, $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ " ~~limit~~ limit

doesexist along $y=x$. ~~(which are passing throw)~~

Date: Day:

(v) From (i), (ii), (iii) and (iv), we

found that

$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0$ along $x=0$ (line).

$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0$ along $y=0$ (line)

$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0$ along $y=x^2$ (~~line~~ curve)

Date: Day:

$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 1/2$ along $y=x$ (~~curve~~ line)

Here, $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ is different along

$x=0$, $y=0$, $y=x^2$ and $x=y$ (which

are passing throw)

Hance, $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ dose not exist.

Date: Day:

H-W : 2 :

if $f(x,y) = \frac{x^3+y^3}{x^3+y}$, then show that,

(i) $\lim_{\substack{x\to0\\y\to0}} f(x,y)$ exist along $y=mx$.

(ii) $\lim_{\substack{x\to0\\y\to0}} f(x,y)$ exist alone $y=-e^x x^2$

Date: Day:

(iii) $\lim_{\substack{x\to0\\y\to0}} f(x,y)$ does not exist

(i) Soln:

Given, $f(x,y) = \frac{x^3+y^3}{x^3+y}$

now, along $y=mx$ $f(x,y)$ is,

$$f(x,y) = \frac{x^3+(mx)^3}{x^3+mx} = \frac{x(x^2+m^3x^2)}{x(x^2+m)}$$

$$= \frac{x^2+m^3x^2}{x^2+m}$$

www.uap.ac.bd

Date: Day:

$\lim_{\substack{x\to0\\y\to0}} f(x,y)$ along $y=mx$ then

$$\lim_{\substack{x\to0\\y\to0}} f(x,y) = \lim_{\substack{x\to0\\y\to0}} \frac{x^2+m^3x^2}{x^2+m}$$

$$= \frac{0+m^3 \cdot 0}{0+m} = \frac{0}{m}$$

$$=0$$

since, $\lim_{\substack{x\to0\\y\to0}} f(x,y)=0$; which is uniqe

Date: Day:

and finat.

Hance, limit of $\lim_{\substack{x\to0\\y\to0}} f(x,y)$ does exist along $y=mx$.

www.uap.ac.bd

Date: Day:

(ii) Sol:

Given, $f(x,y) = \frac{x^3+y^3}{x^3+y}$

now, along $y = -e^x x^2$ is,

$$f(x,y) = \frac{x^3 + (-e^x x^2)^3}{x^3 + (-e^x x^2)}$$

$$= \frac{x^2(x - e^{3x}.x^4)}{x^2(x - e^x)}$$

Date: Day:

$$= \frac{x - e^{3x}.x^4}{x - e^x}$$

$\lim\limits_{\substack{x\to0\\y\to0}} f(x,y)$ along $y = -e^x x^2$ then

$$\lim_{\substack{x\to0\\y\to0}} f(x,y) = \lim_{\substack{x\to0\\y\to0}} \frac{x - e^{3x} x^9}{x - e^x}$$

$$= \frac{0 - e^{3.0}.0^9}{0 - e^0} = \frac{0 - 1.0}{-1}$$

$$= \frac{0}{-1} = 0$$

Date: Day:

Since,

$\lim\limits_{\substack{x\to0\\y\to0}} f(x,y) = 0$; which is uniqu

and finat.

Hance, limit of

$\lim\limits_{\substack{x\to0\\y\to0}} f(x,y)$ dose ~~not~~ exist along

$y = -e^x x^2$.

Date: Day:

(iii) Sol: $2x = 0$

now, $x^3 + y = 0$ Let, $y = mx^3$

$\Rightarrow -x^3 = y \Rightarrow y = mx^3$

Then, $\lim\limits_{\substack{x\to0\\y\to0}} \frac{x^3 + m^3 y^9}{x^3 + m\cancel{y}^3}$

$$= \lim_{\substack{x\to0\\y\to0}} \frac{x^3(1 + m^3 x^3)}{x^3(1+m)}$$

Date: Day:

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{1+m^3x^3}{1+m}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{1+m^3.0}{1+m}$$

$= \frac{1}{1+m}$; which is not unique.

Hence, $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3+y^3}{x^3+y}$ does not

Date: Day:

exist [showed]

~~H.W-(1)(v) $\frac{xy}{x^2+y^2}$~~

H.W-(1)(v) $\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2}$ limit

not exist.

Sol: now, $x^2+y^2=0 \Rightarrow x^2=-y^2$

$\Rightarrow x=my$

Date: Day:

Let, $x=my$

Then,

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{my.y}{m^2y^2+y^2} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{my^2}{y^2(1+m^2)}$$

$= \frac{m}{1+m^2}$; which is not

Date: Day:

unique. m gives different value along different curve, thus the limiting is not unique.

Hence, $\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{y^2+x^2}$ does not

exist.

Slide work:

Date: Day:

(1) Show,

$$\lim_{\substack{x\to0\\y\to0}} \frac{xy}{\sqrt{x^2+y^2}} \text{ exist}$$

Sol: now, $x^2+y^2=0 \Rightarrow x^2=-y^2$

$\Rightarrow x=my$

Let, $x=my$; m is orbitary number.

Then, $\cancel{\frac{xy}{\sqrt{x^2+}}}$ $\lim_{\substack{x\to0\\y\to0}} \frac{xy}{\sqrt{x^2+y^2}}$

Date: Day:

$$= \lim_{\substack{x\to0\\y\to0}} \frac{my\cdot y}{\sqrt{m^2y^2+y^2}} = \lim_{\substack{x\to0\\y\to0}} \frac{my^2}{y\sqrt{1+m^2}}$$

$$= \lim_{\substack{x\to0\\y\to0}} \frac{my}{\sqrt{1+m^2}} = \frac{m.0}{\sqrt{1+m^2}}$$

$$= \frac{0}{\sqrt{1+m^2}} = 0;$$ which is unique and finite.

Hence, $\lim_{\substack{x\to0\\y\to0}} \frac{xy}{\sqrt{x^2+y^2}}$ exist. $\boxed{\text{Showed}}$

Date: Day:

(2) Findout whether $\lim_{\substack{x\to0\\y\to0}} \frac{2xy^2}{x^2+y^4}$ exist or not.

Sol: now, $x^2+y^4=0 \Rightarrow x^2=-y^4$

$\Rightarrow x=-y^2$

$\Rightarrow x=my^2$

Let, $x=my^2$; m is orbitary number. Then,

$$\lim_{\substack{x\to0\\y\to0}} \frac{2xy^2}{x^2+y^4} = \lim_{\substack{x\to0\\y\to0}} \frac{2\cdot my^2\cdot y^2}{m^2y^4+y^4}$$

$$= \lim_{\substack{x\to0\\y\to0}} \frac{y^4(2m)}{y^4(1+m^2)} = \lim_{\substack{x\to0\\y\to0}} \frac{2m}{1+m^2}$$

$$= \frac{2m}{1+m^2};$$ which is not unique.

Since, m gives different value along different curve, thus the limiting is not unique. Hence, $\lim_{\substack{x\to0\\y\to0}} \frac{2xy^2}{x^2+y^4}$ does not exist.

H.W: (Class)

Date: Day:

(1) Show that,

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{2xy}{x^2+y^2} \text{ does not exist.}$$

Sol: now,

$$x^2+y^2=0 \Rightarrow y^2=-x^2$$

$$\Rightarrow y = mx$$

Let, $y=mx$; m is arbitrary number.

Date: Day:

$$\text{Then, } \lim_{\substack{x\to 0\\ y\to 0}} \frac{2xy}{x^2+y^2} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{2x.mx}{x^2+m^2x^2}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2(2m)}{x^2(1+m^2)} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{2m}{1+m^2}$$

$$= \frac{2m}{1+m^2}\text{; which is not unique.}$$

Since, m gives different value along different line ~~curve~~. thus the limiting is not unique.

Hance, $\lim \frac{2xy}{\ldots}$ does not exist.

Date: Day:

(2) Show that, $\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^2}{x^2+y^4}$ does not exist.

Sol: now,

$$x^2+y^4=0$$

$$\Rightarrow -y^4 = x^2$$

$$\Rightarrow my^2 = x$$

Let, $x= my^2$; m is arbitrary number.

Date: Day:

$$\text{Then, } \lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^2}{x^2+y^4}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{my^2.y^2}{m^2y^4+y^4} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{y^4 m}{y^4(1+m^2)}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{m}{1+m^2} = \frac{m}{1+m^2}\text{; which}$$

is not unique.

Since, m gives different value along different curve. thus the limiting is not unique.

Hance, $\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^2}{x^2+y^4}$ does not exist.

Date: Day:

③ find $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{xy(x^2-y^2)}{x^2+y^2}$ exist or not.

Sol: now, $x^2+y^2=0 \Rightarrow x^2=-y^2$
$\Rightarrow x = my$.

Let, $x=my$; m is arbitrary number.

Then, $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{xy(x^2-y^2)}{x^2+y^2}$

$= \lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{my.y(m^2y^2-y^2)}{m^2y^2+y^2}$

$= \lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{y^4\{m(m^2-1)\}}{y^2(m^2+1)}$

$= \lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{y^2 m(m^2-1)}{(m^2+1)}$

$= \dfrac{0.m(m^2-1)}{(m^2+1)} = \dfrac{0}{m^2+1}$

$=0$; which is unique and finite.

Hence, $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{xy(x^2-y^2)}{x^2+y^2}$ does exist.

Date: From Cont Slide → Simultaneous pro.   Day:

1. Evaluate : $\lim\limits_{(x,y)\to(1,0)} \dfrac{(x-1)^2 \ln x}{(x-1)^2+y^2}$

Sol:
now, $(x-1)^2+y^2=0$

$\Rightarrow (x-1)^2 = -y^2$

$\Rightarrow x = my+1$

Let, $x= my+1$; m is arbitrary number.

Then,

$\lim\limits_{(x,y)\to(1,0)} \dfrac{(x-1)^2 \ln x}{(x-1)^2+y^2}$

$= \lim\limits_{(x,y)\to(1,0)} \dfrac{(my+1-1)^2 \ln(my+1)}{(my+1-1)^2+y^2}$

$= \lim\limits_{(x,y)\to(1,0)} \dfrac{m^2y^2 \ln(my+1)}{m^2y^2+y^2}$

$= \lim\limits_{(x,y)\to(1,0)} \dfrac{y^2\ m^2 \ln(my+1)}{y^2(m^2+1)}$

$= \lim\limits_{(x,y)\to(1,0)} \dfrac{m^2 \ln(my+1)}{m^2+1}$

$= \dfrac{m^2 \ln(m.0+1)}{m^2+1}$

Date: Day:

$$= \frac{m^2 \ln(1)}{} = \frac{m^2 \ln 1}{m^2+1}$$

$$= \frac{m^2 \times 0}{m^2+1} = \frac{0}{m^2+1}$$

$= 0$; which is

Unique and finite

Hence, $\lim\limits_{(x,y)\to(0,1)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2}$ does exist.

2. Evaluate: $\lim\limits_{(x,y)\to(0,0)} \frac{x \sin y}{2x^2+1}$

Date: Day:

Sol: now, $2x^2+1=0 \Rightarrow 2x^2=-1$

$\Rightarrow x^2 = -1/2$

$\Rightarrow x = m/\sqrt{2}$

Let, $x = m/\sqrt{2}$; m is arbitrary number. Then,

$$\lim_{(x,y)\to(0,0)} \frac{x \sin y}{2x^2+1}$$

$$= \lim_{(x,y)\to(0,0)} \frac{m/\sqrt{2} \sin y}{\cancel{2}\frac{m^2}{\cancel{2}}+1}$$

www.uap.ac.bd

Date: Day:

$$\lim_{(x,y)\to(0,0)} \frac{m/\sqrt{2} \sin y}{m^2+1}$$

$$= \lim_{(x,y)\to(0,0)} \frac{m/\sqrt{2} \sin 0^\circ}{m^2+1}$$

$= \frac{m\sqrt{2} \cdot 0}{m^2+1} = \frac{0}{m^2+1} = 0$; which is unique and finite.

Hence, $\lim\limits_{(x,y)\to(0,0)} \frac{x \sin y}{2x^2+1}$ does exist

Date: Day:

3. Evaluate: $\lim\limits_{(x,y)\to(0,0)} \frac{2x^2}{x^2-y^2+x}$ exist or not.

Sol: now, $x^2-y^2+x=0$

$\Rightarrow x^2+x=y^2$

$\Rightarrow y = m\sqrt{x^2+x}$

Let, $y = m\sqrt{x^2+x}$ then,

$$\lim_{(x,y)\to(0,0)} \frac{2x^2}{x^2-y^2+x}$$

www.uap.ac.bd

Date: Day:

$$= \lim_{(x,y)\to(0,0)} \frac{2x^2}{x^2-\left(m\sqrt{x^2+x}\right)^2+x}$$

$$= \lim_{(x,y)\to(0,0)} \frac{2x^2}{x^2-(x^2-x)+x}$$

$$= \lim_{(x,y)\to(0,0)} \frac{2x^2}{0} = \frac{2\cdot 0}{0} = \frac{0}{0}$$

~~Here, $\frac{0}{0}$ is undefined.~~

~~Hence, $\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2-y^2+x}$ does not exist.~~

Date: Day:

$$= \lim_{(x,y)\to(0,0)} \frac{2x^2}{x^2-m^2x^2-m^2x+x}$$

$$= \lim_{(x,y)\to(0,0)} \frac{2x^2}{x(x-m^2x-m^2+1)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{2x}{x-m^2x-m^2+1}$$

$$= \frac{2\cdot 0}{0-m^2\cdot 0-m^2+1}$$

$$= \frac{0}{-m^2+1} = 0$$ which is finite and unique

Date: Day:

Hence, $\lim_{(x,y)\to(0,0)} \frac{2x^2}{x^2-y^2+x}$ does exist.

Continuity

Continuity of a function of two Variable:

A function is said to be continous at the point (a,b) if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that,

$$|f(x,y) - f(a,b)| < \varepsilon \text{ for } |f(x,y) - f(a,b)| < \delta.$$

Date: Day:

Example-01:

if $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} ; & (x,y) \neq 0 \\ 0 ; & (x,y) = 0 \end{cases}$

the prove that $f(x,y)$ is discontinous at $(0,0)$

Sol:

~~A function is~~ ,

(1) $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

now, $x^2+y^2=0 \Rightarrow x^2=-y^2$

$\Rightarrow x = my$; m is any arbitrary number,

Let, $x=my$ then,

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{m^2y^2-y^2}{m^2y^2+y^2}$$

Date: Day:

$$= \lim_{(x,y)\to(0,0)} \frac{y^2(m^2-1)}{y^2(m^2+1)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{m^2-1}{m^2+1} = \frac{m^2-1}{m^2+1}$$; which is not unique.

Since, m gives differont value along differont ~~value along limit~~ line. thus the limiting is not unique.

Hance, $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

Thought it does not exist, so $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ is dis continous at $(0,0)$. $\boxed{\text{Showed}}$

Example-2:

if $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & ; \text{when } (x,y) \neq (0,0) \\ 0 & ; \text{when } (x,y) = (0,0) \end{cases}$

Prove that $f(x,y)$ is continuous at $(0,0)$.

Sol:

① $\lim\limits_{(x,y)\to(a,b)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2}$

now, $x^2+y^2=0 \Rightarrow x^2=-y^2 \Rightarrow x=my$

Let, $x=my$; where m is any arbitrary number.

Then, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} \dfrac{m^2y^2y^2}{m^2y^2+y^2}$

$= \lim\limits_{(x,y)\to(0,0)} \dfrac{m^2y^2.y^2}{y^2(m^2+1)} = \lim\limits_{(x,y)\to(0,0)} \dfrac{m^2y^2}{m^2+1}$

$= \dfrac{m^2.0}{m^2+1} = \dfrac{0}{m^2+1} = 0$; which is finite and unique.

② $f(x,y) = (0,0) = 0$

$f(a,b) = f(0,0) = 0$

③ Here,

$\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2} = f(0,0) = 0$

Hence, $f(x,y)$ is continuous at $(0,0)$.

H.W on continuity of function of two variables.

① if $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & ; \text{if } (x,y) \neq (0,0) \\ 0 & ; \text{if } (x,y) = (0,0) \end{cases}$

Then prove that $f(x,y)$ is discontinuous at $(0,0)$.

Date: Day:

Sol:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$$

now, $x^2+y^2=0 \Rightarrow x=my$

Let, $x=my$; m is any arbirary number,

Then, $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$

$= \lim_{(x,y)\to(0,0)} \frac{2\cdot my\cdot y}{m^2y^2+y^2}$

Date: Day:

$= \lim_{(x,y)\to(0,0)} \frac{\cancel{y^2}\, 2m}{\cancel{y^2}(m^2+1)}$

$= \lim_{(x,y)\to(0,0)} \frac{2m}{m^2+1}$

$= \frac{2m}{m^2+1}$; which is not Unique.

Since, m gives different value along different line. thus the limiting is not unique.

www.uap.ac.bd

Date: Day:

Hence, $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ dose not exist. So, it is $f(x,y)$ is discontinuous

$\boxed{\text{Proved}}$

② Investigate the continuity of the function.

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0) \end{cases}$$

Date: ~~Then~~ at the point Day: $(x,y)=(0,0)$.

Sol: $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2}$

now, $x^2+y^2=0 \Rightarrow x=my$

Let, $x=my$; m is any arbirary number.

Then, $\lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{my\cdot y(m^2y^2-y^2)}{m^2y^2+y^2}$

$= \lim_{(x,y)\to(0,0)} \frac{\cancel{y^2}\cdot m(m^2y^2-y^2)}{\cancel{y^2}(m^2+1)}$

www.uap.ac.bd

Date: Day:

$$= \lim_{(x,y)\to(0,0)} \frac{x(x^2y^2-y^2)}{x^2+1}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x(x^2\cdot 0-0)}{x^2+1}$$

$$= \frac{x\cdot 0}{x^2+1} = 0$$; which is unique and finite.

Then

$\therefore f(a,b) = f(0,0) = 0$

Here, $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$

Date: Day:

So, we can say, ~~it is a~~ $f(x,y)$ is continous at $(0,0)$.

Date: Day:

Partial Derivatives:

$$f_x(x,y) = \frac{\delta f}{\delta x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \frac{\delta f}{\delta y} = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

$$f_{xy}(x,y) = \frac{\delta}{\delta y} f_x(x) = \lim_{k\to 0} \frac{f_x(x,y+k) - f_x(x,y)}{k}$$

$$f_{yx}(x,y) = \frac{\delta}{\delta x}(f_y) = \lim_{h\to 0} \frac{f_y(x+h,y) - f_y(x,y)}{h}$$

Date: Day:

$$f_{xx}(x,y) = \frac{\delta}{\delta x}(f_x) = \lim_{h\to 0} \frac{f_x(x+h,y) - f_x(x,y)}{h}$$

$$f_{yy}(x,y) = \frac{\delta}{\delta y}(f_y)$$

Example: 01:

If $f(x,y) = x^2 + xy + y^2$, then using analytical difinition find $f_x(-1,1)$, $f_y(2,5)$ and $f_{xy}(-1,1)$

Date: Day:

Sol^n

Given, $f(x,y) = x^2+xy+y^2$ ---- (1)

Then by analytical difination of partial derivation,

$$\frac{\delta}{\delta x} f(x,y) = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$\therefore f_x(x,y) = \cancel{\lim_{h\to 0} (x^2+xy\ (x+h)^2+(}$$

$$= \lim_{h\to 0} \frac{(x+h)^2+(x+h)y+y^2-(x^2+xy+y^2)}{h}$$

Date: Day:

$$= \lim_{h\to 0} \frac{\cancel{x^2}+h^2+2xh+\cancel{xy}+hy+\cancel{y^2}-\cancel{x^2}-\cancel{y^2}-\cancel{xy}}{h}$$

$$= \lim_{h\to 0} \frac{h^2+2xh+hy}{h} = \lim_{h\to 0} \frac{\cancel{h}(h+2x+y)}{\cancel{h}}$$

$$= \frac{0+2x+y}{0} = 2x+y$$

$\therefore f_x(x,y) = 2x+y$ so, $f_x(-1,1) = -2+1$
$= -1$

Date: Day:

$$\frac{\delta f}{\delta y} = f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}$$

$$\therefore f_y(x,y) = \lim_{k\to 0} \frac{x^2+x(y+k)+(y+k)^2-(x^2+xy+y^2)}{k}$$

$$= \lim_{k\to 0} \frac{\cancel{x^2}+\cancel{xy}+kx+\cancel{y^2}+k^2+2ky-\cancel{x^2}-\cancel{y^2}-\cancel{xy}}{k}$$

$$= \lim_{k\to 0} \frac{\cancel{k}(x+2y+k)}{\cancel{k}} = x+2y+0$$

$$= x+2y$$

$\therefore f_y'(x,y) = x+2y$

Date: Day:

$f_y(2,5) = 2+2\times 5 = 2+10 = 12$

Or,

$$f_y(2,5) = \lim_{k\to 0} \frac{\cancel{(2)^2}+\cancel{2\cdot 5}+k\cdot 2+\cancel{5^2}+k^2+10k-\cancel{2^2}-\cancel{5^2}}{k}$$

$$= \lim_{k\to 0} \frac{\cancel{k}(2+10+k)}{\cancel{k}} = 12+0 = 12$$

$\therefore f_y(2,5) = 12$

$2x+y$

Date: Day:

$f_{xy}(x,y) = \frac{\delta}{\delta y} f_x =$

$\lim_{k\to 0} \frac{f_x(x,y+k) - f_x(x,y)}{k}$

$= \lim_{k\to 0} \frac{2x+y+k-2x-y}{k}$

$= \lim_{k\to 0} \frac{k\,1}{k} = \lim_{k\to 0} (1)$

$= 1$

$\therefore f_{xy}(x,y) = 1$ so, $f_{xy}(-1,1) = 1$

Date: Day:

or, {either this or that,}

$f_{xy}(-1,1) = \lim_{k\to 0} \frac{-2+1+k-(-2+1)}{k}$

$= \lim_{k\to 0} \frac{\cancel{-2}+\cancel{1}+\cancel{k}+\cancel{2}-\cancel{1}}{\cancel{k}}$

$= \lim_{k\to 0} 1 = 1$ (h)

H.W-01:

If $f(x,y) = x^2 + 3xy + 2y^3$, that find $f_{xx}(1,2)$ and $f_{xy}(1,2)$.

$f_{xy}(1,2)$.

www.uap.ac.bd

Date: Day:

Sol:

Given that,

$f(x,y) = x^2 + 3xy + 2y^3$

Then by analytical difination of partial derivation,

$\frac{\delta f}{\delta x} = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$

$= \lim_{h\to 0} \frac{(x+h)^2 + 3(x+h)y + 2y^3 - (x^2+3xy+2y^3)}{h}$

Date: Day:

$= \lim_{h\to 0} \frac{x^2+h^2+2xh+3xy+3hy+2y^3-x^2-2y^3-3xy}{h}$

$= \lim_{h\to 0} \frac{h^2+2xh+3hy}{h} = \lim_{h\to 0} \frac{\cancel{h}(h+2x+3y)}{\cancel{h}}$

$= 0 + 2x+3y = 2x+3y$

$\therefore f_x(x,y) = 2x+3y$

$\frac{\delta}{\delta x} f_x = f_{xx}(x,y) = \lim_{h\to 0} \frac{f_x(x+h,y) - f_x(x,y)}{h}$

$\therefore f_{xx}(1,2) = \lim_{h\to 0} \frac{2(1+h)+3\cdot 2-(2\cdot 1+3\cdot 2)}{h}$

www.uap.ac.bd

Date: Day:

$= \lim\limits_{h\to 0} \frac{\cancel{2}+2h+\cancel{6} \cancel{-2-6}}{h}$

$= \lim\limits_{h\to 0} (2) = 2$

$\therefore f_{xx}(1,2) = 2$

$\frac{\delta}{\delta y} f_y = f(x,y) = \lim\limits_{k\to 0} \frac{f_x(x,y+k) - f_x(x,y)}{k}$

$\therefore f_{xy}(x,y) = \lim\limits_{k\to 0} \frac{x^2+3x(y+k)+2(y+k)^2-x^2-3xy-2y^2}{k}$

Date: Day:

$= \lim\limits_{k\to 0} \frac{3xk+4yk+2k^2}{k}$

$= 3x+4y+0 = 3x+4y$

$f_{x,y}(1,2) = 3+8 = 11$ (a)

$= 11$ (Ans)

$f_{xy}(x,y) = \lim\limits_{k\to 0} \frac{f_x(x,y+k) - f_x(x,y)}{k}$

$= \lim\limits_{k\to 0} \frac{\cancel{2x}+\cancel{3y}+3\cancel{k}-\cancel{2x}-\cancel{3y}}{\cancel{k}}$

$= 3$ $f_{xy}(1,2) = 3$.

Date: Day:

Differentiability of a function of two variables

Definition:

The function $f(x,y)$ is said to be differentiable at the point (a,b) if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k)-f(a,b)-hf_x(a,b)-kf_y(a,b)}{\sqrt{h^2+k^2}} = 0.$$

Date: Day:

Example-01:

Show that the function,

$$f(x,y) = \begin{cases} \frac{x^2y^5}{x^2+y^2}; & \text{when } (x,y)\neq(0,0) \\ 0; & \text{when } (x,y)=(0,0) \end{cases}$$

is ~~continuous~~, Possess partial derivatives at $(0,0)$ but not differentiable at $(0,0)$.

Date: Day:

Soln:

By analytical definition of partial derivatives,

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,k) - f(x,y)}{h}$$

$$\therefore f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h^3-0}{h^2-0} - \frac{0-0}{0-0}}{h}$$

$$= \lim_{h\to 0} \frac{h^3/h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}}$$

$$= \lim_{h\to 0} (1) = 1$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{1}{k}\left[\frac{-k^3}{k^2} - 0\right]$$

$$= \lim_{k\to 0} \frac{1}{\cancel{k}} \cdot -\cancel{k} = \lim_{k\to 0} (-1)$$

$$= -1$$

Date: Day:

$$\therefore L.H.S = \lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k) - f(a,b) - hf_x(a,b) - kf_y(a,b)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}}\left[f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)\right]$$

$$= \lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}}\left[\frac{h^3-k^3}{h^2+k^2} - h + k\right]$$

$$= \lim_{(h,k)\to(0,0)} \frac{1}{(h^2+k^2)^{3/2}}\left[h^3 - k^3 - h(h^2+k^2) + k(h^2+\ldots\right.$$

$$= \lim_{(h,k)\to(0,0)} \frac{1}{(h^2+k^2)^{3/2}}\left[\cancel{h^3} - \cancel{k^3} - \cancel{h^3} - hk^2 + kh^2 + \cancel{k^3}\right]$$

$$= \lim_{(h,k)\to(0,0)} \frac{hk(h-k)}{(h^2+k^2)^{3/2}} \quad \ldots\ldots (1)$$

now, let, $h = mk$ | now, $h^2+k^2=0$, $h = mk$

Then, $$\frac{hk(h-k)}{(h^2+k^2)^{3/2}} = \frac{mk^3(m-1)}{(m^2k^2+k^2)^{3/2}}$$

$$= \frac{mk^3(m-1)}{k^3(m^2+1)^{3/2}} = \frac{m(m-1)}{(m^2+1)^{3/2}}$$

$$\lim_{(h,k)\to(0,0)} \frac{m(m-1)}{(m^2+1)^{3/2}} = \frac{m(m-1)}{(m^2+1)^{3/2}}$$

Date: Day:

which is not ~~not~~ unique.

Hance, $\lim_{(h,k)\to(0,0)} \frac{hk(h-k)}{(h^2+k^2)^{3/2}}$ does not exist.

Hance, the function $f(x,y)$ is not differenciable at $(0,0)$.

H.W:

(1) if the function, $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{\sqrt{x^2+y^2}} ; (x,y) \neq (0,0) \\ 0 ; (x,y) = (0,0) \end{cases}$

Date: Day:

The examine the diffrenentiability of $f(x,y)$ at $(0,0)$

Sol: by analytical defenition of & partial derivatives,

$$f_x(x,y) = \lim_{h\to0} \frac{f(x+h,y) - f(x,y)}{h}$$

Date: Day:

$$\therefore f_x(0,0) = \lim_{h\to0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h\to0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to0} \frac{\frac{h.0(h^2-0^2)}{\sqrt{h^2+0}} - 0}{h}$$

$$= \lim_{h\to0} \frac{0}{\sqrt{h^2}} / h = \lim_{h\to0} 0/h$$

$$= \lim_{h\to0} 0 = 0$$

Date: Day:

$$f_y(x,y) = \lim_{k\to0} \frac{f(x, y+k) - f(x,y)}{k}$$

$$f_y(0,0) = \lim_{k\to0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to0} \frac{1}{k} \left[\frac{0.k(0-k^2)}{\sqrt{0+k^2}} - 0 \right]$$

$$= \lim_{k\to0} \frac{0}{k.k} = \lim_{k\to0} 0 = 0$$

Date: Day:

L.H.S =

$$\lim_{(h,k)\to(a,b)} \frac{f(a+h,b+k)-f(a,b)-hf_x(a,b)-kf_y(a,b)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}}\left[\frac{hk(h^2-k^2)}{\sqrt{h^2+k^2}} - h\cdot 0 - k\cdot 0\right]$$

$$= \lim_{(h,k)\to(0,0)} \frac{1}{(h^2+k^2)}\left[hk(h^2-k^2)\right]$$

Date: Day:

$$= \lim_{(h,k)\to(0,0)} \frac{hk(h^2-k^2)}{(h^2+k^2)} \quad \text{------ (1)}$$

Let, $h = mk$

now, $(h^2+k^2) = 0$

$\Rightarrow h = mk$

Then, $\frac{mk\cdot k(m^2k^2-k^2)}{(m^2k^2+k^2)}$

$$= \frac{mk^4(m^2-1)}{k^2(m^2+1)} = \frac{mk^2(m^2-1)}{(m^2+1)}$$

Date: Day:

$$\lim_{(h,k)\to(0,0)} \frac{mk(m^2-1)}{(m^2+1)}$$

$$= \frac{m\cdot 0(m^2-1)}{(m^2+1)} = \frac{0}{(m^2+1)} = 0;$$

which is unique and finate.

Hence, the $\lim_{(h,k)\to(0,0)} \frac{hk(h^2-k^2)}{(h^2+k^2)^{3/2}}$ dose exist.

Hence, the function $f(x,y)$ is differenciable at $(0,0)$

Date: Day:

(ii) Let, $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}; & (x,y) \neq (0,0) \\ 0; & (x,y) = (0,0) \end{cases}$

Then show that both $f_x(0,0)$ and $f_y(0,0)$ both exist but $f(x,y)$ is discontinuous at $(0,0)$.

[Continuous slide H.W.1]

Date: Day:

Sol:

By analytical definition of partial partial derivaties.

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_x(x,y) = \lim_{h\to 0} \frac{\frac{(x+h)y}{(x+h)^2+y^2} - \frac{xy}{x^2+y^2}}{h}$$

Date: Day:

$$= \lim_{h\to 0} \frac{\frac{(x^2+y^2)(x+h)y - xy\{(x+h)^2+y^2\}}{\{(x+h)^2+y^2\}(x^2+y^2)}}{h}$$

$$= \lim_{h\to 0} \left\{ \frac{(x^2+y^2)(xy+hy) - xy(x^2+2xy+y^2+y^2)}{(x^2+2xy+h^2+y^2)(x^2+y^2)} \right\}$$

$$\therefore xy^3$$

$$= \lim_{h\to 0} \frac{x^3y + x^2yh + xy^3 + hy^3 - x^3y - 2x^2y^2 - xy^3}{x^4 + 2x^3y + h^2x^2 + x^2y^2 + x^2y^2 + 2xy^3 + h^2y^2 + y^4}$$

$$= \lim_{h\to 0} \frac{x^2yh + hy^3 - 2x^2y^2 - xy^2}{x^4 + 2x^3y + h^2x^2 + 2x^2y^2 + 2xy^3 + h^2y^2 + y^4}$$

Date: Day:

$$= \frac{-2x^2y^2 - xy^2}{x^4+2x^3y+2x^2y^2+2xy^3+y^4}$$

$$\therefore f_x(x,y) = \frac{-2x^2y^2 - xy^2}{x^4+y^4+2x^3y+2xy^3+2x^2y^2}$$

$$f_x(0,0) = \frac{-0-0}{0+0+0+0+0}$$

$$= 0$$

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

Date: Day:

$$= \lim_{k\to 0} \frac{\frac{x(y+k)}{x^2+(y+k)^2} - \frac{xy}{x^2+y^2}}{k}$$

$$= \lim_{k\to 0} \left\{\frac{(xy+xk)(x^2+y^2) - xy(x^2+y^2+2ky+k^2)}{(x^2+y^2+k^2+2ky)(x^2+y^2)}\right\}/k$$

$$= \lim_{k\to 0} \frac{(x^3y + x^3k + xy^3 + xy^2k) - x^3y - xy^3 - 2kxy^2 - k^2xy}{x^4+x^2y^2+k^2x^2+2kx^2y+x^2y^2+y^4+k^2y^2+2ky^3}$$

$$\therefore f_y(0,0) = \frac{0}{0} = 0$$

www.uap.ac.bd

x^2+y^2

now,

$$x^2+y^2=0$$

$$\Rightarrow my = x$$

Then,

$$\frac{my \cdot y}{m^2y^2+y^2} = \frac{my^2}{y^2(m^2+1)}$$

$$= \frac{m}{(m^2+1)}$$

Date: Day:

$\lim\limits_{(h,k)\to(0,0)} \frac{m}{(m^2+1)} = \frac{m}{m^2+1}$; which is not unique.

So, it doesn't exist.

Hance, we can say $f(x,y)$ is discontinous at $(0,0)$

Date: Day:

(iii) Let, $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$

Show that, $f_y(x,0) = x$; $f_x(0,y) = -y$

and $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Sol:

$$f_y(x,0) = \lim_{k\to 0} \frac{f(x,0+k) - f(x,0)}{k}$$

Date: Day:

$$= \lim_{k\to 0} \frac{\frac{xk(x^2-k^2)}{x^2+k^2} - \frac{x.0(x^2-0)}{x^2+0}}{k/1}$$

$$= \lim_{k\to 0} \frac{xk(x^2-k^2)}{(x^2+k^2).k}$$

$$= \frac{x(x^2-0)}{x^2} = x$$

$$f_x(0,y) = \lim_{h\to 0} \frac{f(0+h,y) - f(0,y)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{hy(h^2-y^2)}{x^2+h^2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{y(h^2-y^2)}{x^2+h^2} = \frac{-y^3}{y^2} = -y$$

Date: Day:

$\cancel{f_{xy}(0,0)=}$

$\cancel{f_x(0,0) = \lim_{h\to 0} \frac{f(h+0,0)-f(0,0)}{h}}$

$= \lim_{h\to 0} \frac{\frac{h\cdot 0(h^2-0)}{h^2+0}}{h} = 0$

$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,0+k)-f_x(0,0)}{k}$

$= \lim_{k\to 0} \frac{-k-0}{k} = \lim_{k\to 0} \frac{-k}{k}$

$= \lim_{k\to 0} (-1) = -1$

Date: Day:

$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h}$

$= \lim_{h\to 0} \frac{h-0}{h} = \lim_{h\to 0} (1)$

$= 1$

$\therefore f_{xy}(0,0) \neq f_{yx}(0,0)$ [showed]

Chain rule

Date: Day:

$z = f(u(x,y), v(x,y))$

$\frac{\delta z}{\delta x} = \frac{\delta f}{\delta u} \times \frac{\delta u}{\delta x} + \frac{\delta f}{\delta v} \times \frac{\delta v}{\delta x}$

$\frac{\delta z}{\delta y} = \frac{\delta f}{\delta u} \times \frac{\delta u}{\delta y} + \frac{\delta f}{\delta v} \times \frac{\delta v}{\delta y}$

Ex: Suppose that, $z = f(u,v)$, where $u = e^x \cos y$, $v = e^x \sin y$. Show that,

(i) $\frac{\delta z}{\delta x} = u \frac{\delta f}{\delta u} + v \frac{\delta f}{\delta v}$

Date: Day:

(ii) $\frac{\delta z}{\delta y} = -v \frac{\delta f}{\delta u} + u \frac{\delta f}{\delta v}$

Sol: (1)

We know,

$\frac{\delta z}{\delta x} = \frac{\delta f}{\delta u} \times \frac{\delta u}{\delta x} + \frac{\delta f}{\delta v} \times \frac{\delta v}{\delta x}$

$= \frac{\delta f}{\delta u} \times \frac{\delta (e^x \cos y)}{\delta x} + \frac{\delta f}{\delta v} \times \frac{\delta}{\delta x}(e^x \sin y)$

$= \frac{\delta f}{\delta u} \cdot e^x \cos y + \frac{\delta f}{\delta x} e^x \sin y$

$= u \frac{\delta f}{\delta u} + v \frac{\delta f}{\delta v}$ [showed]

(ii) L.H.S =

$$\frac{\delta z}{\delta y} = \frac{\delta f}{\delta u}\times\frac{\delta u}{\delta y} + \frac{\delta f}{\delta v}\times\frac{\delta v}{\delta y}$$

$$= \frac{\delta f}{\delta u}\,-(e^{x}\sin y) + \frac{\delta f}{\delta v}(e^{x}\cos y)$$

$$= -v\frac{\delta f}{\delta u} + u\frac{\delta f}{\delta u}$$

$= R.H.S$ [showed]

Ex: Suppose That, $u = f(x-y,\ y-z,\ z-x)$

show that, $u_x + u_y + u_z = 0$

Sol: $p = x-y$, $q = y-z$, $R = z-x$

$$\therefore u = f(p(x,y),\ q(y,z),\ R(z,x))$$

$$u_x = \frac{\delta u}{\delta x} = \frac{\delta f}{\delta p}\cdot\frac{\delta p}{\delta x} + \frac{\delta f}{\delta R}\cdot\frac{\delta R}{\delta x}$$

$$= \frac{\delta f}{\delta p}\cdot 1 + \frac{\delta f}{\delta R}(-1)$$

$$= \frac{\delta f}{\delta p} - \frac{\delta f}{\delta R}$$

$$u_y = \frac{\delta u}{\delta y} = \frac{\delta f}{\delta p}\cdot\frac{\delta}{\delta y}(x-y) + \frac{\delta f}{\delta q}\frac{\delta}{\delta y}(y-z)$$

$$= \frac{\delta f}{\delta p}(-1) + \frac{\delta f}{\delta q}\cdot 1$$

$$= -\frac{\delta f}{\delta p} + \frac{\delta f}{\delta q}$$

$$u_z = \frac{\delta u}{\delta z} = \frac{\delta f}{\delta q}\cdot\frac{\delta}{\delta z}(y-z) + \frac{\delta f}{\delta R}\frac{\delta}{\delta z}(z-$$

$$= \frac{\delta f}{\delta q}(-1) + \frac{\delta f}{\delta R}$$

$$= -\frac{\delta f}{\delta q} + \frac{\delta f}{\delta R}$$

$$L.H.S = u_x + u_y + u_z$$

$$= \cancel{\frac{\delta f}{\delta p}} - \cancel{\frac{\delta f}{\delta R}} - \cancel{\frac{\delta f}{\delta p}} + \cancel{\frac{\delta f}{\delta q}} - \cancel{\frac{\delta f}{\delta q}} + \cancel{\frac{\delta f}{\delta R}}$$

$= 0 = R.H.S$

$\therefore L.H.S = R.H.S$ [Proved]

www.uap.ac.bd

H.W

Date: Day:

(1) if $\omega = f\left(\frac{y-x}{xy}, \frac{z-y}{yz}\right)$, then

Show that,

$$x^2 \frac{\delta\omega}{\delta x} + y^2\frac{\delta\omega}{\delta y} + z^2 \frac{\delta \omega}{\delta z} = 0$$

Sol: $p = \frac{y-x}{xy}$, $q = \frac{z-y}{yz}$

$$\frac{\delta w}{\delta x} = \frac{\delta f}{\delta p}\cdot\frac{\delta}{\delta x}\left(\frac{y-x}{xy}\right) \cancel{+ \frac{\delta f}{\delta q}\cdot\frac{\delta}{\delta x}(}$$

$$= \frac{\delta f}{\delta p}\cdot\frac{xy(-1)-(y-x)y}{x^2y^2}$$

Date: Day:

$$= \frac{\delta f}{\delta p}\, \frac{-xy - y^2 + xy}{x^2y^2}$$

$$= \frac{-1}{x^2}\,\frac{df}{\delta p} \Rightarrow x^2\frac{\delta\omega}{\delta x} = -\frac{\delta f}{\delta p}$$

$$\frac{\delta w}{\delta y} = \frac{\delta f}{\delta p}\,\frac{\delta}{\delta y}\left(\frac{y-x}{xy}\right) + \frac{\delta f}{\delta q}\,\frac{\delta}{\delta y}\left(\frac{z-y}{yz}\right)$$

$$= \frac{\delta f}{\delta p}\,\frac{xy - xy + x^2}{x^2y^2} + \frac{\delta f}{\delta q}\,\frac{-yz - z^2 + yz}{y^2z^2}$$

$$= \frac{\delta f}{\delta p}\,\frac{1}{y^2} - \frac{\delta f}{\delta q}\,\frac{1}{y^2}$$

$$y^2\frac{\delta w}{\delta y} = \frac{\delta f}{\delta p} - \frac{\delta f}{\delta q}$$

Date: Day:

$$\frac{\delta\omega}{\delta z} = \frac{\delta f}{\delta q}\,\frac{\delta}{\delta z}\left(\frac{z-y}{yz}\right)$$

$$= \frac{\delta f}{\delta q}\,\frac{yz - yz + y^2}{y^2z^2}$$

$$\Rightarrow z^2\frac{\delta w}{\delta z} = \frac{\delta f}{\delta q}$$

$$L.H.S = x^2\frac{\delta\omega}{\delta x} + y^2\frac{\delta\omega}{\delta y} + z^2\frac{\delta\omega}{\delta z}$$

$$= -\cancel{\frac{\delta f}{\delta p}} + \cancel{\frac{\delta f}{\delta p}} - \cancel{\frac{\delta f}{\delta q}} + \cancel{\frac{\delta f}{\delta q}}$$

Date: Day:

$$= 0 = R.H.S.$$

∴ L.H.S = R.H.S [Proved]

(2) if $u = f(\overset{p}{x^2+2yz}, \overset{q}{y^2+2zx})$ then

Show that, $(y^2-zx)\frac{\delta u}{\delta x} + (x^2-yz)\frac{\delta u}{\delta y} + (z^2-xy)\frac{\delta u}{\delta z} = 0$

Sol: ~~$\frac{\delta u}{\delta x} = \frac{\delta f}{\delta p}(-z) + \frac{\delta f}{\delta q}2x$~~

Date: Sol:

$$\cancel{\frac{\delta u}{\delta x} = \frac{\delta f}{\delta p} 2x^2 +}\ \delta$$

$$\frac{\delta u}{\delta x} = \frac{\delta f}{\delta p} \times 2x + \frac{\delta f}{\delta q} \cdot 2z$$

$$\frac{\delta u}{\delta y} = \frac{\delta f}{\delta p} 2z + \frac{\delta f}{\delta q} 2y$$

$$\frac{\delta u}{\delta z} = \frac{\delta f}{\delta p} 2y + \frac{\delta f}{\delta q} 2x$$

$$L.H.S = (y^2 - zx)\frac{\delta u}{\delta x} + (x^2 - yz)\frac{\delta u}{\delta y} + (z^2 - xy)\frac{\delta u}{\delta z}$$

Date: Day:

$$= (y^2 - xz)\left(2x\frac{\delta f}{\delta p} + 2z\frac{\delta f}{\delta q}\right)$$

$$+ (x^2 - yz)\left(2z\frac{\delta f}{\delta p} + 2y\frac{\delta f}{\delta q}\right)$$

$$+ (z^2 - xy)\left(2y\frac{\delta f}{\delta p} + 2x\frac{\delta f}{\delta q}\right)$$

$$= \cancel{2xy^2\frac{\delta f}{\delta p}} + \cancel{2y^2z\frac{\delta f}{\delta q}} - \cancel{2x^2z\frac{\delta f}{\delta p}}$$

$$- \cancel{2xz^2\frac{\delta f}{\delta q}} + \cancel{2x^2z\frac{\delta f}{\delta p}} + \cancel{2x^2y\frac{\delta f}{\delta q}}$$

$$- \cancel{2yz^2\frac{\delta f}{\delta q}} - \cancel{2yz^2\frac{\delta f}{\delta p}} + \cancel{2yz^2\frac{\delta f}{\delta p}}$$

$$+ \cancel{2z^2x\frac{\delta f}{\delta q}} - \cancel{2xy^2\frac{\delta f}{\delta p}} - \cancel{2x^2y\frac{\delta f}{\delta q}}$$

Date: Day:

$$= 0 = R.H.S$$

$\therefore L.H.S = R.H.S$ [Showed]

(8) if $z = z(x,y)$ and $x = e^u + e^{-v}$; $y = e^{-u} + e^v$, then show that,

$$\frac{\delta z}{\delta u} - \frac{\delta z}{\delta v} = x\frac{\delta z}{\delta x} - y\frac{\delta z}{\delta y}.$$

Sol: $$\frac{\delta z}{\delta u} = \frac{\delta z}{\delta x} \cdot e^u + \frac{\delta z}{\delta y}(-e^{-u})$$

$$= e^u\frac{\delta z}{\delta x} - e^{-u}\frac{\delta z}{\delta y}$$

Date: Day:

$$\frac{\delta z}{\delta v} = -e^{-v}\frac{\delta z}{\delta x} + e^v\frac{\delta z}{\delta y}$$

$$L.H.S = \cancel{(e^u + e^{-v})} e^u \frac{\delta z}{\delta x}$$

$$= e^u\frac{\delta z}{\delta x} - e^{-u}\frac{\delta z}{\delta y} + e^{-v}\frac{\delta z}{\delta x} - e^v\frac{\delta z}{\delta y}$$

$$= (e^u + e^{-v})\frac{\delta z}{\delta x} - (e^{-u} + e^v)\frac{\delta z}{\delta y}$$

$$= x\frac{\delta z}{\delta x} - y\frac{\delta z}{\delta y} = R.H.S$$

$\therefore L.H.S = R.H.S$ [Proved]

Homogenous function

Homogenous function এর ক্ষেত্রে

Degree = প্রতি পদের variable এর মোটের ঘাত। (Homogenous degree).

Ex: $f(x,y) = x^5 + 4x^3y^2 + 2x^4y + y^5$

[Homogenous degree = 5]

$f(x,y) = x^5 + 2x^2y^2 + 2x^3y^2 + 2$

[non-homogenous]

Ex-1:

Verify Euler's theorem for $u(x,y,z) = x^3+y^3+z^3$

Sol: To prove Euler's theorem, u suppose to be homogenous function, degree $= n$.

So, $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} + z\frac{\delta u}{\delta z} = nu$

$f(x,y,z) = x^3+y^3+z^3$

u is a homogenous function And degree is 3. So, $n = 3$

$L.H.S = x\frac{\delta}{\delta x}(x^3+y^3+z^3) + y\frac{\delta}{\delta y}(x^3+y^3+z^3) + z\frac{\delta}{\delta z}(x^3+y^3+z^3)$

$= 3x^3 + 3y^3 + 3z^3$

$= 3(x^3+y^3+3z^3)$

$= 3u(x,y,z)$

$= R.H.S$

Thus, $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} + z\frac{\delta u}{\delta z} = 3u$

So, its ~~~~ verify Euler's theorem.

Date: Day:

② Verify Euler's theorem for $u(x,y,z) = x^2+y^2+z^2$

Sol: u is a homogeneous function and $n=2$

Then, $x\frac{\delta}{\delta x}(x^2+y^2+z^2)+y\frac{\delta}{\delta y}(x^2+y^2+z^2)$
$+z\frac{\delta}{\delta z}(x^2+y^2+z^2)$

$= 2x^2+2y^2+2z^2 = 2(x^2+y^2+z^2)$
$=2u$

$\therefore$ L.H.S = R.H.S [Proved]

Date: Day:

④ $u(x,y,z) = \frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}$
$= x^2y^{-1}+y^2z^{-1}+z^2x^{-1}$

$n=1$

L.H.S $= x\frac{\delta}{\delta x}(x^2y^{-1}+y^2z^{-1}+z^2x^{-1})+y\frac{\delta}{\delta y}(x^2y^{-1}+y^2z^{-1}+z^2x^{-1})$
$+z\frac{\delta}{\delta z}(x^2y^{-1}+y^2z^{-1}+z^2x^{-1})$

$= x(2x-x^{-2})+y(-y^{-2}+2y)+$
$z(-z^{-2}+2z)$

$= 2x^2-x^{-1}-y^{-1}+2y^2-z^{-1}+2z^2$

$=2(x^2+y^2+z^2)-(\frac{1}{x}+\frac{1}{y}-\frac{1}{z})$

Date: Day:

$\neq$ RHS

[Not proved]

③ $u=(x,y,z) = \frac{xy^2}{z}+\frac{yz^2}{x}+\frac{zx^2}{y}$

$n=1+2-1=2$

~~$L.H.S = x(y^2z^{-1})+y(z^2x^{-1})+(x^2y^{-1})z$~~

~~$=\frac{xy^2}{z}+\frac{zy}{x}+\frac{x^2z}{y}+$~~

~~$=\frac{yz^2}{x}+$~~

Date: Day:

$L.H.S = x\left(\frac{y^2}{z}-\frac{yz^2}{x^2}+2\frac{xz}{y}\right)+$
$y\left(2\frac{xy}{z}+\frac{z^2}{x}-\frac{zx^2}{y^2}\right)+$
$z\left(-\frac{xy^2}{z^2}+\frac{2yz}{x}+\frac{x^2}{y}\right)$

$=2\left(\frac{xy^2}{z}+\frac{yz^2}{x}+\frac{zx^2}{y}\right)$

$=2u(x,y,z)$

$=$ R.H.S [Proved]

Date: Day:

(2) Verify Euler's theorem for $u(x,y,z) = x^2+y^2+z^2$

Sol: U is a homogeneous function and $n=2$

Then, $x\frac{\partial}{\partial x}(x^2+y^2+z^2) + y\frac{\partial}{\partial y}(x^2+y^2+z^2)$
$+ z\frac{\partial}{\partial z}(x^2+y^2+z^2)$

$= 2x^2+2y^2+2z^2 = 2(x^2+y^2+z^2) = 2u$

$\therefore$ L.H.S = R.H.S [Proved]

Date: Day:

(4) $u(x,y,z) = \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}$

$= x^2y^{-1} + y^2z^{-1} + z^2x^{-1}$

$n = 1$

L.H.S $= x\frac{\partial}{\partial x}(x^2y^{-1}+y^2z^{-1}+z^2x^{-1}) + y\frac{\partial}{\partial y}(x^2y^{-1}+y^2z^{-1}+z^2x^{-1})$
$+ z\frac{\partial}{\partial z}(x^2y^{-1}+y^2z^{-1}+z^2x^{-1})$

$= x(2x - x^{-2}) + y(-y^{-2}+2y) +$
$z(-z^{-2}+2z)$

$= 2x^2 - x^{-1} - y^{-1} + 2y^2 - z^{-1} + 2z^2$

$= 2(x^2+y^2+z^2) - \left(\frac{1}{x}+\frac{1}{y}-\frac{1}{z}\right)$

Date: Day:

$\neq$ RHS

[Not proved]

(3) $u=(x,y,z) = \frac{xy^2}{z} + \frac{yz^2}{x} + \frac{zx^2}{y}$

$n = 1+2-1 = 2$

~~L.HS $= x(y^2z^{-1}) + y(z^2x^{-1}) + (x^2y^{-1})z$~~

~~$= \frac{xy^2}{z} + \frac{xzy}{x} + \frac{x^2z}{y} +$~~

~~$= \frac{yz^2}{x} +$~~

Date: Day:

L.H.S $= x\left(\frac{y^2}{z} - \frac{yz^2}{x^2} + 2\frac{xz}{y}\right) +$
$y\left(\frac{2xy}{z} + \frac{z^2}{x} - \frac{zx^2}{y^2}\right) +$
$z\left(-\frac{xy^2}{z^2} + \frac{2yz}{x} + \frac{x^2}{y}\right)$

$= 2\left(\frac{xy^2}{z} + \frac{yz^2}{x} + \frac{zx^2}{y}\right)$

$= 2u(x,y,z)$

= R.H.S [Proved]

Maximum or Minimum of a function

Date: Day:

Suppose, that,

we have a function $f(x,y)$

Step-1: find, $\frac{\delta f}{\delta x}$, $\frac{\delta f}{\delta y}$

$r = \frac{\delta^2 f}{\delta x^2}$, $s = \frac{\delta^2 f}{\delta x \delta y}$, $t = \frac{\delta^2 f}{\delta y^2}$

step-2: for critical point,

$\frac{\delta f}{\delta x} = 0$ —— (1) and $\frac{\delta f}{\delta y} = 0$ —— (2)

Date: Day:

After solving (1) and (2) you will Obtain some point (x,y) ∴ Let, $(x,y) = (a,b)$

Step-03: now, Put $(x,y) = (a,b)$ at r,s,t

(i) if $r<0$, $t<0$ and $rt - s^2 > 0$ then $f(x,y)$ has maximum at maximum value $= f(a,b)$

Date: Day:

(ii) if $r>0$, $t>0$ and $rt - s^2 > 0$ then $f(x,y)$ has minimum value at $(x,y) = (a,b)$, minimum value is $= f(a,b)$

(iii) if neither of the condition at (i) and (ii) satisfied then $f(x,y)$ has neither minimum or maximum at (a,b). Here (a,b) is saddle point.

Date: Day:

Ex- find Maximum or minimum of the function.

$$f(x,y) = x^3 + y^3 + 3x - 12y + 20$$

Sol: $\frac{\delta f}{\delta x} = 3x^2 + 3$; $\frac{\delta f}{\delta y} = 3y^2 - 12$

$$s = \frac{\delta f}{\delta x \delta y} = \frac{\delta}{\delta x}(3y^2 - 12) = 0$$

$r = \frac{\delta^2 f}{\delta x^2} = \frac{\delta}{\delta x}(6x) = 6$; $t = \frac{\delta^2 f}{\delta y^2} = 6y$

Date: Day:

For critical point,

$\frac{\delta f}{\delta x} = 0$ $\frac{\delta f}{\delta y} = 0$

$\Rightarrow 3x^2 - 3 = 0$ $\Rightarrow 3y^2 - 12 = 0$

$\Rightarrow x = \pm 1$ $\Rightarrow y = \pm 2$

So, Critical Point, $(1,2), (1,-2), (-1,2)$ and $(-1,-2)$

At, $(x,y) = (1,2)$:

$r = 6 \cdot 1 = 6 > 0$ $t = \cancel{12} \cdot 2 = 12 > 0$

Date: Day:

$rt - s^2$

$= 6 \cdot 12 - 0 = 72 > 0$

$\therefore f(x,y)$ has minimum value is

$f(1,2) = 1 + 8 - 3 - 24 + 20 = 2$ minimum

At $(x,y) = (1,-2)$

$r = 6 > 0$ $t = -12 < 0$

~~$rt - s^2 = -72 < 0$~~ $s = 0$

Thus, $f(x,y)$ has no maximum on minimum value at $(1,-2)$.

$\therefore (1,-2)$ is a saddle point

Date: Day:

At $(x,y) = (-1,2)$:

$r = -6 < 0$ $t = 12 > 0$ $s = 0$

Thus, $f(x,y)$ has no maximum on minimum value at $(-1,2)$.

$\therefore (-1,2)$ is a saddle point

At $(x,y) = (-1,-2)$:

$r = -6 \cancel{>} 0$ $t = -12 < 0$ $rt - s^2 = 72 - 0 = 72 > 0$

Date: Day:

$\therefore f(x,y)$ has maximum value is

$f(-1,-2) = -1 - 8 - 3 + 24 + 20$

$= 32$.

$\therefore$ maximum value, $= 32$.

Lagrange Multipliers:

Date: Day:

① Use lagrange multipliers to find three real numbers whose sum is 12 and the sum of whose the sum of whose squars is minimum.

Sol:

Let, the three real numbers are x, y and z.

Date: Day:

Where,

$$x+y+z=12$$

$$\Rightarrow x+y+z-12=0 \quad \text{--- (i)}$$

So, $g(x,y,z)=x+y+z-12$ --- (ii)

We have to minimize,

$$f(x,y,z)=x^2+y^2+z^2 \quad \text{--- (iii)}$$

For extremum,

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow \frac{\delta f}{\delta x}\hat{i}+\frac{\delta f}{\delta y}\hat{j}+\frac{\delta f}{\delta z}\hat{k}=\lambda\left[\frac{\delta g}{\delta x}\hat{i}+\frac{\delta g}{\delta y}\hat{j}+\frac{\delta g}{\delta z}\hat{k}\right]$$

Date: Day:

$$\Rightarrow 2x\hat{i}+2y\hat{j}+2z\hat{k}=\lambda[\hat{i}+\hat{j}+\hat{k}]$$

$$\Rightarrow 2x=\lambda \;;\; 2y=\lambda \;;\; 2z=\lambda$$

[সহগ সমীকৃত করে]

$$\Rightarrow 2x=2y=2z$$

$$\Rightarrow x=y=z \quad \text{--- (iv)}$$

implise in equation (i) $\Rightarrow$

$$x+x+x-12=0$$

Date: Day:

$$\Rightarrow 3x=12$$

$$\Rightarrow x=4$$

$\therefore y=4$ and $z=4$

$\therefore$ Three real number is, 4, 4 and

4. (m)

Date: Day:

② For a rectangulen whose peramenter is 20 m. Use the lagrange multiplies method to find the dimension that will maximize area.

Sol:

Let,

the two dimension are x and y.

where,

$$2(x+y)=20$$

Date: Day:

$$\Rightarrow x+y=10 \Rightarrow x+y-10=0 \quad ----①$$

$$g(x,y)=x+y-10 \quad ----②$$

$$f(x,y)=xy \quad ----③$$

For extremum,

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow \hat{i}\frac{\delta f}{\delta x}+\hat{j}\frac{\delta f}{\delta y}+\hat{k}\frac{\delta f}{\delta z}=\lambda\left[\frac{\delta g}{\delta x}\hat{i}+\hat{j}\frac{\delta g}{\delta y}+\hat{k}\frac{\delta g}{\delta z}\right]$$

Date: Day:

$$\Rightarrow y\hat{i}+x\hat{j}=\lambda[\hat{i}+\hat{j}]$$

$$\Rightarrow y=\lambda\ ;\ x=\lambda$$

$$\Rightarrow y=x \quad ---④$$

implies in equation ① =>

$$x+x-10=0 \Rightarrow 2x-10=0$$

$$\Rightarrow x=5$$

$x=5$, ④ =>

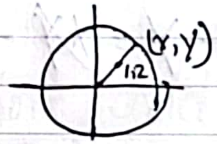
$$y=5$$

Date: So, Day:

The two dimension $x=5$ m and $y=5$ m.

③ Find the points on the circle $x^2+y^2=8$ which are ~~at~~ closest and furthest from the point $(1,2)$

Sol:



Supposed that,

The point is (x,y)

Date: Day:

Where,

$$x^2+y^2-80=0 \text{ --- } ①$$

$$g(x,y)= x^2+y^2-80 \text{ --- } ⑪$$

Now, the distance between the points (x,y) and $(1,2)$ is

$$d=\sqrt{(x-1)^2+(y-2)^2}$$

$$\Rightarrow d^2=(x-1)^2+(y-2)^2$$

$$\Rightarrow f(x,y)= (x-1)^2+(y-2)^2 \text{ --- } ③$$

Date: Day:

For extremum,

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow 2(x-1)\hat{i} + 2(y-2)\hat{j} = \lambda 2x\hat{i} + 2y\lambda\hat{j}$$

$$\Rightarrow (x-1)=x\lambda \quad ; \quad (y-2)= y\lambda$$

$$\Rightarrow \frac{x-1}{x}=\lambda \quad ; \quad \frac{y-2}{y}=\lambda$$

$$\Rightarrow \frac{x-1}{x}=\frac{y-2}{y} \Rightarrow \cancel{xy}-y=\cancel{yx}-2x$$

$$\Rightarrow \cancel{x=2y} \text{ ..... } ④$$

$$\Rightarrow y=2x \text{ --- } ④$$

Date: Day:

Implies ~~of~~ on ① $\Rightarrow$

$$x^2+4x^2=80$$

$$\Rightarrow x^2=16 \Rightarrow x=\pm 4$$

$\cancel{y=\pm 8}$ $x=4,\ y=8$

$$x=-4,\ y=-8$$

So, points are $(4,8)$ and $(-4,-8)$

At $(4,8)$,

Date: Day:

$$g(4,8)= (4-1)^2+(8-2)^2$$

$$= 9+36=45$$

At $(-4,-8)$,

$$g(-4,-8)=(-4-1)^2+(-8-2)^2$$

$$=25+100$$

$$=125$$

$\because 125 > 45$, So, $(4,8)$ is Closest point and $(-4,-8)$ is furthest point.

Date: Multiple Intneget Day:

Ex-12

$$\int_0^{\ln 2}\int_0^1 xye^{xy^2}\,dy\,dx = \frac{1}{2}[1-\ln 2]$$

Sol: L.H.S $= \displaystyle\int_0^{\ln 2}\int_0^1 xy\, e^{xy^2}\,dy\,dx$

$$= \int_0^{\ln 2} x\int_0^1 ye^{xy^2}\,dy\,.\,dx$$

Date: Day:

$[2y\,dy = dy^2]$

$$= \frac{1}{2}\int_0^{\ln 2} x\int_0^1 2y\,dy\, e^{xy^2}\,dx$$

$$= \frac{1}{2}\int_0^{\ln 2} x\int_0^1 e^{xy^2}\,dy^2\,dx$$

$$= \frac{1}{2}\int_0^{\ln 2} x\left[\frac{e^{xy^2}}{x}\right]_0^1 dx$$

Date: Day:

$$= \frac{1}{2}\int_0^{\ln 2} x\left[\frac{e^x}{x}-\frac{1}{x}\right]dx$$

$$= \frac{1}{2}\int_0^{\ln 2} (e^x-1)\,dx$$

$$= \frac{1}{2}\left[(e^{\ln 2}-e^0)-(\ln 2-0)\right]$$

$$= \frac{1}{2}\left[1-\ln 2\right] = R.H.S$$

Date: $\therefore$ L.H.S = R.H.S [showed] Day:

H.W:

① $\displaystyle\int_1^2\int_0^1\int_{-1}^1 (x^2+y^2+z^2)\,dx\,dy\,dz$

Sol:

$$= \int_1^2 dz\int_0^1 dy\int_{-1}^1 (x^2+y^2+z^2)\,dx$$

$$= \int_1^2 dz\int_0^1 dy\left[\frac{x^3}{3}+y^2x+z^2x\right]_{-1}^1$$

$$= \int_1^2 dz\int_0^1\left(\frac{1}{3}+y^2+z^2+\frac{1}{3}+y^2+z^2\right)dy$$

$$= \int_1^2 dz \int_0^1 \left(\frac{2}{3} + 2y^2 + 2z^2\right) dy$$

$$= 2\int_1^2 dz \int_0^1 \left(\frac{1}{3} + y^2 + z^2\right) dy$$

$$= 2\int_1^2 dz \left[\frac{y}{3} + \frac{y^3}{3} + z^2 y\right]_0^1$$

$$= 2\int_1^2 \left(\frac{1}{3} + \frac{1}{3} + z^2 - 0 - 0 - 0\right) dz$$

$$= 2\int_1^2 \left(\frac{2}{3} + z^2\right) dz$$

$$= 2\left[\frac{2}{3}z + \frac{z^3}{3}\right]_1^2$$

$$= 2\left[\frac{4}{3} + \frac{8}{3} - \frac{2}{3} - \frac{1}{3}\right]$$

$$= \frac{2}{\cancel{3}} \times \cancel{9}\,3 = 6$$ (ans)

(2) $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z}\, dx\, dy\, dz$

Sol^n:

$$= \int_0^1 dz \int_0^1 dy \int_0^1 (e^{x+y+z})\, dx$$

$$= \int_0^1 dz \int_0^1 dy \int_0^1 e^x . e^{y+z}\, dx$$

$$= \int_0^1 dz \int_0^1 dy . e^{y+z}[e-1]$$

$$= [e-1]\int_0^1 dz \int_0^1 e^y . e^z\, dy$$

$$= [e-1]\int_0^1 dz . e^z [e-1]$$

$$= [e-1]^2 \int_0^1 e^z\, dz = [e-1]^3$$

$$= e^3 - 3e^2 + 3e - 1$$ (ans)

(3) $\int_1^3 \int_{1/x}^{1} \int_0^{\sqrt{xy}} xyz \, dz \, dy \, dx$

Sol:

$= \int_1^3 dx \int_{1/x}^{1} dy \int_0^{\sqrt{xy}} xyz \, dz$

$= \int_1^3 dx \int_{1/x}^{1} dy \cdot xy \left[\frac{z^2}{2}\right]_0^{\sqrt{xy}}$

$= \frac{1}{2} \int_1^3 dx \int_{1/x}^{1} dy \cdot xy \cdot xy$

$= \frac{1}{2} \int_1^3 dx \int_{1/x}^{1} x^2 y^2 \, dy$

$= \frac{1}{2} \int_1^3 dx \cdot x^2 \left[\frac{y^3}{3}\right]_{1/x}^{1}$

$= \frac{1}{6} \int_1^3 dx \; x^2 \left(1 - \frac{1}{x^3}\right)$

$= \frac{1}{6} \int_1^3 \left(x^2 - \frac{1}{x}\right) dx$

$= \frac{1}{6} \left[\frac{x^3}{3} - \ln x\right]_1^3$

$= \frac{1}{6} \left[\frac{27}{3} - \ln 3 - \frac{1}{3} + 0\right]$

$= \frac{1}{\cancel{6}_3} \cdot \frac{\cancel{26}^{13}}{3} - \frac{1}{6} \ln 3$

$= \frac{13}{9} - \frac{1}{6} \ln 3$ (Ans)

H.W:

1. $\iint_R xy \, dy \, dx$, where R is the quadrant of the circle $x^2+y^2=a^2$ when, $x>=0$ and $y>=0$

Sol:

Since, R is the first quadrant of $x^2+y^2=a^2 \Rightarrow y=\sqrt{a^2-x^2}$

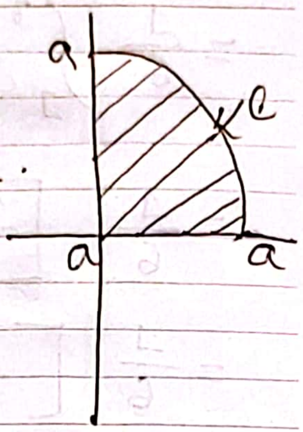
www.uap.ac.bd

Date: Day:

Hence,

x varise from 0 to a.

and y varise from 0 to $\sqrt{a^2-x^2}$



$$\iint_R xy\,dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

Date: Day:

$$= \int_0^a x\,dx \int_0^{\sqrt{a^2-x^2}} xy\,dy = \int_0^a x\,dx \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}}$$

$$= \int_0^a \frac{x}{2}\left[a^2-x^2-0\right]dx$$

$$= \frac{1}{2}\int_0^a (xa^2-x^3)\,dx = \frac{1}{2}\left\{a^2\left[\frac{x^2}{2}\right]_0^a - \left[\frac{x^4}{4}\right]_0^a\right\}$$

Date: Day:

$$= \frac{1}{2}\left\{a^2\left[\frac{a^2}{2}-0\right] - \left[\frac{a^4}{4}-0\right]\right\}$$

$$= \frac{1}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right] = \frac{1}{4}\left[a^4-\frac{a^4}{2}\right]$$

$$= \frac{a^4}{8} \text{ (Ans)}$$

Jacobian:

Let,

$u = u(x,y,z)$

Date: Day:

$v = v(x,y,z)$

$w = w(x,y,z)$

$$J = \frac{\delta(u,v,w)}{\delta(x,y,z)}$$

$$= \begin{vmatrix} \delta u/\delta x & \delta u/\delta y & \delta u/\delta z \\ \delta v/\delta x & \delta v/\delta y & \delta u/\delta z \\ \delta w/\delta x & \delta w/\delta y & \delta w/\delta y \end{vmatrix}$$

$\therefore du\,dv\,dw = |J|\,dx\,dy\,dz$

Date: Day:

Example-01:

By transforming to polar co-ordinates show that,

$$\int_0^\infty \int_0^\infty dx\,dy = \int_0^\infty \int_0^{\pi/2} r\,d\theta\,dr$$

Sol: We know that,

Date: Day:

in polar co-ordinate system,

$$x(r,\theta) = x = r\cos\theta$$

$$y(r,\theta) = y = r\sin\theta$$

we know that,

$$dx\,dy = |J|\,dr\,d\theta \quad \text{--- (i)}$$

$$J = \frac{\delta(x,y)}{\delta(r,\theta)} = \begin{vmatrix} \frac{\delta x}{\delta\theta} & \frac{\delta x}{\delta r} \\ \frac{\delta x}{\delta\theta} & \frac{\delta y}{\delta r} \end{vmatrix}$$

Date: Day:

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(1) = r$$

implisthat on (i) ⇒

$$dx\,dy = |J|\,dr\,d\theta$$

Date: Day:

$$\Rightarrow dx\,dy = r\,dr\,d\theta \quad \text{--- (ii)}$$

$$\Rightarrow \iint dx\,dy = \iint r\,dr\,d\theta \quad \text{--- (iii)}$$

When,

$x=0$; $r\cos\theta = 0 \Rightarrow r=0$ or $\cos\theta = 0$

$r=\theta$ or $\theta = 90^\circ$

$x=\infty$; $r\cos\theta = \infty \Rightarrow r=\infty$ $[\cos\theta \neq \infty]$

When,

$x=0$; $r\sin\theta = 0 \Rightarrow r=0$ or $\theta = 00^\circ$

$x=\infty$; $r\sin\theta = \infty \Rightarrow r=\infty$ $[\sin\theta \neq \infty]$

Date: Day:

implies in (iii) $\Rightarrow$

$$\int_0^\infty\int_0^\infty dx\,dy = \int_0^{\pi/2}\int_0^\infty r\,dr\,d\theta$$

$$\int_0^\infty\int_0^\infty dx\,dy = \int_0^\infty\int_0^{\pi/2} r\,d\theta\,dr$$

$\boxed{\text{Showed}}$

Date: Day:

Example-28

Transform $\iiint dx\,dy\,dz$ into spherical polar co-ordinate system.

Soln: we know that,

$$x = r\cos\phi\,\sin\theta = x(r,\theta,\phi)$$

$$y = r\sin\phi\,\sin\theta = y(r,\theta,\phi)$$

$$z = r\,\cancel{\sin}\,\cos\theta = z(r,\theta,\phi)$$

Date: Day:

$$J = \frac{\delta(x,y,z)}{\delta(r,\theta,\phi)}$$

$$= \begin{vmatrix} \delta x/\delta r & \delta x/\delta\theta & \delta x/\delta\phi \\ \delta y/\delta r & \delta y/\delta\theta & \delta y/\delta\theta \\ \delta z/\delta r & \delta z/\delta\theta & \delta z/\delta\phi \end{vmatrix}$$

$$= \begin{vmatrix} \cos\phi\,\sin\theta & r\cos\phi\,\cancel{\sin}\cos\theta & -r\sin\phi\,\sin\theta \\ \sin\phi\,\sin\theta & r\cos\phi\,\sin\phi & r\cos\phi\,\sin \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

Date: Day:

$= \cos\phi\,\sin\theta(0 + r^2\sin^2\theta\cos\phi) -$

$r\cos\phi\cos\theta(0 - r\cos^2\theta\,\overset{\cos\phi}{\cancel{\sin\theta}}) +$

$(-r\sin\phi\,\sin\theta)(-r\sin^2\theta\,\sin\phi - r\cos^2\theta\,\sin\phi)$

~~$= r^2[\sin^3\theta\,\cos^2\phi + \cos^3\theta\,\cos^2\phi\,\sin$~~

Date: Day:

$= r^2 \left[\sin^3\theta (\cos^2\phi + \sin^2\phi) + \sin\theta \cos^2\theta (\cos^2\phi + \sin^2\phi) \right]$

$= r^2 \left[\sin\theta (\sin^2\theta + \cos^2\theta) \times (1) \right]$

$= r^2 \sin\theta$

So, $dx\,dy\,dz = |J|\, dr\, d\theta\, d\phi$
$= r^2 \sin\theta\, dr\, d\theta\, d\phi$

Date: Day:

Hance,

$$\iiint dx\,dy\,dz = \iiint r^2 \sin\theta\, dr\, d\theta\, d\phi$$

(shw)

H.W. 3 by using the transformation $x+y=u$, $y=uv$ then show that,

$$\int_0^1 \int_0^{1-x} e^{\frac{y}{y+x}}\, dy\, dx = \frac{1}{2}(e-1)$$

Date: Day:

Sol:

Given,

$x+y=u$

$\Rightarrow x = u - y$

$\Rightarrow x(u,v) = u - uv = u(1-v)$

$y(u,v) = y = uv$

The jacobin will be,

$$\frac{\delta(x,y)}{\delta(u,v)} = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$= u - \cancel{uv} + \cancel{uv}$

$= u$

Date: Day:

So, $dx\,dy = |J|\,du\,dv$

$\Rightarrow dx\,dy = u\,du\,dv$

$\Rightarrow \int_0^1\int_0^{1-x} e^{y/y+x}\,dy\,dx =$

$$\iint_R u e^{\cancel{u}v/\cancel{u}}\,dv\,du$$

$$= \iint_R u e^{v}\,dv\,dy$$

Date: Day:

When,

$x = 0$ then, $0 = u(1-v)$ then $u = 0,\ v = 1$

~~$x = 1$ then, $1 = u(1-v)$ then~~

$y = 0$ then, $u = 0 \quad v = 0$

$y = 1 - x$ then, $u = 1,\ v = 1$

$x = 1$ then, $u = 1,\ v = 0$

Date: Day:

$$\therefore \int_0^1\int_0^{1-x} e^{y/y+x}\,dy\,dx = \int_0^1\int_0^1 u e^{v}\,dv\,du$$

$$\therefore \int_0^1\int_0^1 u e^{v}\,dv\,du$$

L.H.S $= \int_0^1 u\,[e^{v}]_0^1\,du$

Date: Day:

$$= \int_0^1 u\,[e^1 - e^0]\,du$$

$$= \int_0^1 (e^1 - 1)\,u\,du$$

$$= (e-1)\left[\frac{u^2}{2}\right]_0^1 = (e-1)\left[\frac{1}{2} - 0\right]$$

$$= \frac{e-1}{2} = \text{R.H.S [Showed]}$$

www.uap.ac.bd

Set 01 : (a) Iterative

(b) Continuity

Set 02 : (a) Partial derivatives

(b) Differentiability 10

Set 03 : (a) chain rule

(b) Homogeneous function (Euler's)

Set 04 : (a) Maximum and minimum

(b) Lagrange Multipliers.

or

Set 05 : (a) Multiple integral

(b) Transformations using Jacobian