Q1:

Prove that the difference between 24^{12} and 15^{10} is a multiple of 3

Solution:

Step 1: Write the expression to find the difference

$$24^{12} - 15^{10}$$

Step 2: Take a out a factor of 3 from each term.

$$(24 \times 24^{11}) - (15 \times 15^{9})$$
$$(3 \times 8 \times 24^{11}) - (3 \times 5 \times 15^{9})$$

Step 3: Factorise by 3

$$(3 \times 8 \times 24^{11}) - (3 \times 5 \times 15^{9}) = 3([8 \times 24^{11}] - [5 \times 15^{9}])$$

We know that $3 \times (\text{any integer})$ is a multiple of 3 therefore,

$$3([8 \times 24^{11}] - [5 \times 15^{9}]) \text{ is a multiple of } 3$$

Q2:

Prove that $(n+2)^2 - (n-2)^2$ is divisible by 8 for any positive whole number n.

Step 1: Expand and simplify the expression

$$(n+2)^2 - (n-2)^2$$
$$(n^2 + 4n + 4) - (n^2 - 4n + 4)$$

We can see that the n^2 terms will cancel, as will the 4s, so all we're left with is

$$(n^2 + 4n + 4) - (n^2 - 4n + 4) = 4n - (-4n) = 8n$$

So, the whole expression simplifies to

$$8n$$

Step 2: Factorise the expression

$$8n = 8(n)$$

Now, if n is a whole number, then $8(n)$ must be divisible by 8. Thus, we have completed the proof.

Q3:

Prove that $(3n+1)^2+(n-1)^2$ is always even for any positive whole number n.

To answer this question, we will need to expand and simplify the expression given to us, so we can hopefully write it in a way that shows it is clearly divisible by 2 (since that's the definition of even). So, expanding the first bracket, we get

$$(3n+1)^2 = 9n^2+3n+3n+1 = 9n^2+6n+1.$$

Then, expanding the second bracket, we get

$$(n-1)^2 = n^2-n-n+1 = n^2-2n+1.$$

Adding the expansions together, we get

$$(9n^2+6n+1)+(n^2-2n+1) = 10n^2+4n+2$$

Is this an even number? Well, if we take a factor of 2 out of the expression:

$$2(5n^2+2n+1),$$

we see that since $5n^2+2n+1$ is a whole number because n is a whole number, the expression in question is equal to $2\times(\text{some whole number})$ and so must be even. Thus, we have completed the proof.

Q4: Prove for every positive integer n that

$$\sum_{j=n}^{2n-1}\left(2j+1\right) = 3n^2$$

Proof: by induction

BASIS $(n=1):\ \sum_{j=1}^{2\cdot 1-1}\left(2j+1\right)=(2\cdot 1+1)=3$

IND HYP: Assume that $\sum_{j=n}^{2n-1}\left(2j+1\right)=3n^2$

IND STEP: Then $\sum_{j=n+1}^{2n+1}\left(2j+1\right)=$

$$\sum_{j=n}^{2n-1}\left(2j+1\right)-[2n+1]+[2\cdot 2n+1]+[2(2n+1)+1]$$

$=3n^2-[2n+1]+[2\cdot 2n+1]+[2(2n+1)+1]$ by ind hyp

$=3n^2+6n+3$ by algebra

$=3(n+1)^2$ by more algebra

Q5:

Use mathematical induction to prove that for any integer n > 1, $\sum_{i=1}^{n}\frac{1}{i^2}<2-\frac{1}{n}$.

Solution:

Basis step: for n = 2: $\frac{1}{1^2}+\frac{1}{2^2}=1.25<2-\frac{1}{2}=1.5$

Inductive step: Assume it holds for any n, show it holds for n+1.

$$\sum_{i=1}^{n}\frac{1}{i^2}<2-\frac{1}{n};$$

$$\begin{aligned}\sum_{i=1}^{n+1}\frac{1}{i^2}&<2-\frac{1}{n}+\frac{1}{(n+1)^2}\\
&=2-\frac{(n+1)^2-n}{n(n+1)^2}=2-\frac{n^2+n+1}{n(n+1)^2}=2-\frac{n^2+n}{n(n+1)^2}-\frac{1}{n(n+1)^2}\\
&=2-\frac{n(n+1)}{n(n+1)^2}-\frac{1}{n(n+1)^2}=2-\frac{1}{n+1}-\frac{1}{n(n+1)^2}\\
&<2-\frac{1}{n+1}\end{aligned}$$

Q6: Prove that for every nonnegative integer n,

$$1^2+3^2+5^2+\cdots+(2n+1)^2=\frac{(n+1)(2n+1)(2n+3)}{3};$$

Solution:

Proof of the first formula by induction:

Basis: $n = 0$. The given formula turns into the correct equality $1^2 = \frac{1 \cdot 1 \cdot 3}{3}$. *Induction step:* Assuming that the given formula is true for n, we can prove that

$$1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 + (2n+3)^2 = \frac{(n+2)(2n+3)(2n+5)}{3}$$

as follows:

$$\begin{aligned}
1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 + (2n+3)^2 &= \frac{(n+1)(2n+1)(2n+3)}{3} + (2n+3)^2 \\
&= (2n+3)\left(\frac{(n+1)(2n+1)}{3} + (2n+3)\right) \\
&= (2n+3)\frac{2n^2+3n+1+6n+9}{3} \\
&= (2n+3)\frac{2n^2+9n+10}{3} \\
&= (2n+3)\frac{(n+2)(2n+5)}{3} \\
&= \frac{(n+2)(2n+3)(2n+5)}{3}.
\end{aligned}$$

Q7::

(a) Find a formula for

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)}$$

Answer: $\frac{n}{n+1}$.

½ , ½ + ⅙ = 4/6 =⅔, ⅔ + 1/12 = 9/12 = ¾ ……

(b) Prove the formula you conjectured in part (a).

Solution: we will prove the formula

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$$

by induction. *Basis:* When $n=0$, the formula turns into $0=\frac{0}{1}$. *Induction step:* assuming that the given formula holds for n, we can prove that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}+\frac{1}{(n+1)\cdot(n+2)}=\frac{n+1}{n+2}$$

as follows:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}+\frac{1}{(n+1)\cdot(n+2)}=\frac{n}{n+1}+\frac{1}{(n+1)\cdot(n+2)}$$

$$=\frac{n\cdot(n+2)+1}{(n+1)\cdot(n+2)}=\frac{n^2+2n+1}{(n+1)\cdot(n+2)}=\frac{(n+1)^2}{(n+1)\cdot(n+2)}=\frac{n+1}{n+2}.$$