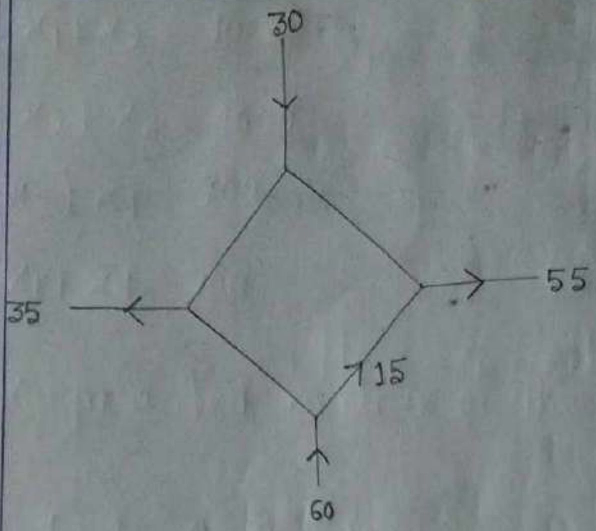
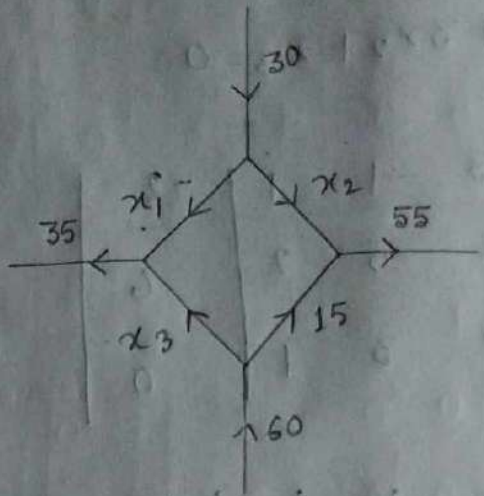
Application of Linear System

1. Network Analysis using Linear System





Node	Flow in	Flow Out
A	30	x_1+x_2
B	x_1+x_3	35
C	60	x_3+15
D	x_2+15	55

Equations are :

$x_1+x_2 = 30$

$x_1+x_3 = 35$

$x_3+15 = 60$

$x_2+15 = 55.$

Solve Equation :

$x_3 = 60-15 = 45$

$x_1+45 = 35$

$\Rightarrow x_1 = -10$

$x_2 = 55-15$

$= 40.$

Equations are :

$x_1+x_2 = 1000$

$x_2+x_3 = 1000$

$x_3+x_4 = 700$

$x_1+x_4 = 700$

Augmanted matrix of the system is :

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1000 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 1 & 0 & 0 & 1 & 700 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1000 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 0 & -1 & 0 & 1 & -300 \end{array}\right] R_4' = R_4 - R_1$$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1000 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 0 & 0 & 1 & 1 & 700 \end{array}\right] R_4' = R_4 + R_2$$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1000 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 7000 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_4' = R_4 - R_3$$

$x_1 + x_2 = 1000$

$x_2 + x_3 = 1000$

$x_3 + x_4 = 700$

Let, $x_4 = t$

$x_3 = 700 - t$

$x_2 = 1000 - (700 - t)$

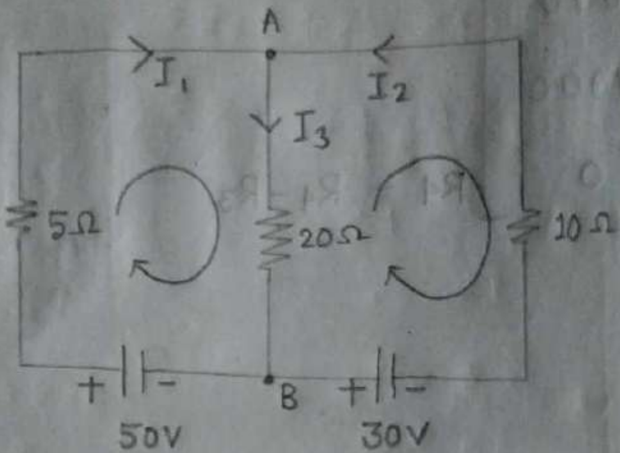
$\Rightarrow x_2 = 300 + t$

$x_1 = 1000 - 300 - t$

$\quad = 700 - t$

Where, $0 \leq t \leq 700$.

A circuit with three closed loops



Node	Flow in	Flow out
A	I_1+I_2	I_3
B	I_3	I_1+I_2

Equations are:

$5I_1+20I_3=-50$	$I_1+4I_3=10$
$10I_2+20I_3=-30$	$I_2+2I_3=-3$
$5I_1-10I_2+80=0$	$I_1-2I_2+16=0$

Balancing Chemical Equation:

$$CH_4 + O_2 \rightarrow CO_2 + H_2O$$

Balanced version of equation : $\underset{x_1}{CH_4} + \underset{x_2}{2O_2} \rightarrow \underset{x_3}{CO_2} + \underset{x_4}{2H_2O}$

	left		Right
Carbon =	x_1	=	x_3
Hydrogen =	$4x_1$	=	$2x_4$
Oxygen =	$2x_2$	=	$2x_3 + x_4$

$x_1 - x_3 = 0$

$2x_1 - x_4 = 0$

$2x_2 - 2x_3 - x_4 = 0$

Augmented Matrix of the system is :

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 2 & -1 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right] R_2' = R_2 - 2R_1$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & 2 & -1 & 0 \end{array}\right] R_3 \rightarrow R_2$$

Equations are : $x_1 - x_3 = 0$

$$2x_2 - 2x_3 - x_4 = 0$$

$$2x_3 - x_4 = 0$$

Let, $x_3 = t$ $\qquad$ $x_1 = t$

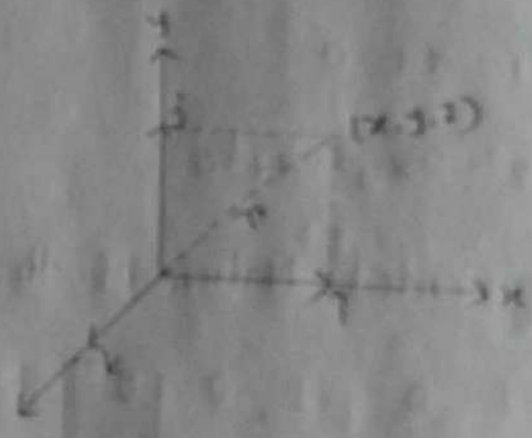
$x_4 = 2t$ $\qquad$ $x_2 = 2t$

$x_2 = \dfrac{x_4 + 2x_3}{2}$ $\qquad$ $x_3 = t$

$\qquad$ $x_4 = 2t$.

$= \dfrac{2t + 2t}{2} = 2t$

Vector Space



$r = x\hat{i} + y\hat{j} + z\hat{k} = (x, y, z)$

2 dimensional এ space হলো

n dimensional এ - n

কোনো ক্ষেত্রের Domain যদি Vector হয় সেটা হবে Vector space

Linear Combination: If $S = \{v_1, v_2, v_3, \dots v_n\}$ is a set of vectors in n space and v_1, v_2, v_n can be expressed by one vector $\vec{v}$ such that $\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_n\vec{v_n}$ where $c_1, c_2, \dots c_n$ are constants then v is called the linear Combination of $\vec{v_1}, \vec{v_2} \dots \vec{v_n}$.

Problem : 1 Consider $\vec{u} = (1,2,-1)$, $\vec{v} = (6,4,2)$ in R^3. Show that, $\vec{w} = (9,2,7)$ is a linear combination of $\vec{u}$ and $\vec{v}$. Also show that, $\vec{w} = (4,-1,8)$ is not linear combination of $\vec{u}$ and $\vec{v}$.

Solution: From the definition:

$$\vec{w} = c_1\vec{u_1} + c_2\vec{v_2}$$

$$\Rightarrow (9,2,7) = c_1(1,2,-1) + c_2(6,4,2)$$

$$\Rightarrow (9,2,7) = (c_1, 2c_1, -c_1) + (6c_2, 4c_2, 2c_2)$$

$$\Rightarrow (9,2,7) = c_1 + 6c_2, 2c_1 + 4c_2, -c_1 + 2c_2$$

Equations are,

$$c_1 + 6c_2 = 9$$
$$2c_1 + 4c_2 = 2$$
$$-c_1 + 2c_2 = 7$$

Augmented Matrix of this system:

$$\left[\begin{array}{cc|c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{array}\right] \begin{array}{l} \\ R_2' = R_2 - 2R_1 \\ R_3' = R_3 + R_1 \end{array}$$

$$= \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3' = R_3 + R_2 \end{array}$$

$$c_1 + 6c_2 = 9$$
$$-8c_2 = -16$$
$$c_2 = 2$$
$$c_1 + 6 \times 2 = 9$$
$$c_1 = -3$$

Part 2

$\vec{w}' = c_1\vec{u} + c_2\vec{v}$

$(4,-1,8) = c_1(1,2,-1) + c_2(6,4,2)$

$\Rightarrow (4,-1,8) = (c_1, 2c_1, -c_1) + (6c_2, 4c_2, 2c_2)$

$\Rightarrow (4,-1,8) = c_1 + 6c_2,\ 2c_1 + 4c_2,\ -c_1 + 2c_2$

Equations are:

$c_1 + 6c_2 = 4$

$2c_1 + 4c_2 = -1$

$-c_1 + 2c_2 = 8$

Augmented matrix of this system is:

$$\left[\begin{array}{cc|c} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 12 \end{array}\right] \begin{array}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_3 + R_1 \end{array}$$

$$= \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 0 & 3 \end{array}\right] R_3' = R_3 + R_2$$

$0 = 3$. Which is impossible. The system has no solution

So, $\vec{w}$ can't be expressed as a linear combination of $\vec{u}$

Dependancy and Independancy

If $c_1 = c_2 = \dots = c_n = 0$ then vectors are independent.

Otherwise it will be dependant.

Problem: 1 Determine whether $\vec{v}_1 = (1,2,2,-1)$, $\vec{v}_2 = (4,9,9,-4)$, $\vec{v}_3 = (5,8,9,-5)$ are linearly dependant or independant.

Solution: $c_1\vec{v}_1 + c_2\vec{v} + c_3\vec{v}_3 = 0$

$\Rightarrow c_1(1,2,2,-1) + c_2(\cancel{5,8}\ 4,9,9,-4) + c_3(5,8,9,-5) = 0$

$\Rightarrow (c_1, 2c_1, 2c_1, -c_1) + (4c_2, 9c_2, 9c_2, -4c_2) + (5c_3, 8c_3, 9c_3, -5c_3) = 0$

$\Rightarrow (c_1 + 4c_2 + 5c_3, 2c_1 + 9c_2 + 8c_3, 2c_1 + 9c_2 + 9c_3, -c_1 - 4c_2 - 5c_3) = 0$

Equations are:

$c_1 + 4c_2 + 5c_3 = 0$

$2c_1 + 9c_2 + 8c_3 = 0$

$2c_1 + 9c_2 + 9c_3 = 0$

$-c_1 - 4c_2 - 5c_3 = 0$

<u>Problem :2</u> Determine wheather $\vec{V_1} = (0,3,1,-1)$, $\vec{V_2} = (6,0,5,1)$. $\vec{V_3} = (4,-7,1,3)$ are linearly dependant and independant.

Solution : $c_1\vec{V_1} + c_2\vec{V_2} + c_3\vec{V_3} = 0$

$$\Rightarrow c_1(0,3,1,-1) + c_2(6,0,5,1) + c_3(4,-7,1,3) = 0$$

$$\Rightarrow (3c_1, c_1, -c_1) + (6c_2, 5c_2, c_2) + (4c_3, -7c_3, c_3, 3c_3) = 0$$

$$\Rightarrow (6c_2 + 4c_3, 3c_1 - 7c_3, c_1 + 5c_2 + c_3, -c_1 + c_2 + 3c_3) = 0$$

Equations are : $6c_2 + 4c_3 = 0$

$$3c_1 - 7c_3 = 0$$

$$c_1 + 5c_2 + c_3 = 0$$

$$-c_1 + c_2 + 3c_3 = 0$$

Augmented matrix of the system is :

$$\left[\begin{array}{ccc|c} 0 & 6 & 4 & 0 \\ 3 & 0 & -7 & 0 \\ 1 & 5 & 1 & 0 \\ -1 & 1 & 3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 3 & 0 & -7 & 0 \\ 0 & 6 & 4 & 0 \\ 1 & 5 & 1 & 0 \\ -1 & 1 & 3 & 0 \end{array}\right] R_1 \rightarrow R_2 \text{ (Swap)}$$

$$= \left[\begin{array}{ccc|c} 3 & 0 & -7 & 0 \\ 0 & 6 & 4 & 0 \\ 0 & 15 & 10 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right] \begin{array}{l} \\ \cancel{R_2' =} \\ R_3' = 3R_3 - R_1 \\ R_4' = 3R_4 + R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 3 & 0 & -7 & 0 \\ 0 & 6 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3' = 6R_3 - 15R_2 \\ R_4' = 3R_4 - R_2 \end{array}$$

$3c_1 - 7c_3 = 0$

$6c_2 + 4c_3 = 0$

Now, $c_2 = 2t$

$c_3 = -3t$

$3c_1 - 7c_3 = 0$

$3c_1 = 7c_3$

$3c_1 = -21t$

$c_1 = -21t,$

$c_2 = 2t$

$c_3 = 3t$

Since, $c_1 \neq c_2 \neq c_3$, so, the vectors are dependant.

Eigen Value and Eigen Vector

Eigen value and Eigen Vector করতে হবে অবশ্যই matrix থাকতে হবে যা square matrix হবে এবং ঐ square matrix এর একটি Identity matrix নিব।

A= square matrix

I= square matrix identity matrix.

λ = যে কোনো একটি constant.

λI = করলে একটি matrix পাবো

Characteristic equations:

$[\lambda I - A] = 0$

Problem 1 Find the value ~~of~~ and vectors of the matrix.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

For eigen value, λ

Characteristics equations $[\lambda I - A] = 0$

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \right| = 0$$

$$= \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda-0 & 0-0 & 0+2 \\ 0-1 & \lambda-2 & 0-1 \\ 0-1 & 0-0 & \lambda-3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{vmatrix} = 0$$

$$\Rightarrow \lambda(\lambda-2)(\lambda-3) + 2(\lambda-2) = 0$$

$$\Rightarrow \lambda-2 \{\lambda(\lambda-3)+2\} = 0$$

$$\Rightarrow \lambda-2 (\lambda^2-2\lambda-\lambda+2) = 0$$

$$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-1) = 0$$

$\lambda = 1, 2, 2$

so, eigen values are 1 and 2.

Let, the eigan vector is $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

So, $[\lambda I - A] . X = 0$

$$= \begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \text{ --- } (1)$$

Put $\lambda = 1$ in (1)

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Equations are :

$x + 2z = 0$

$-x - y - z = 0 \Rightarrow x + y + z = 0$

$-x - 2z = 0 \Rightarrow x + 2z = 0$

Let, $z = t$

$x = -2t$

$y = t.$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is an eigan ~~value~~ vector for $\lambda = 1$.

Again, putting $\lambda = 2$ in (1)

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Equations are:

$2x + 2z = 0$

$-2x - 2z = 0$

$-x - z = 0 \Rightarrow x + z = 0$

Let, $y = t$

$z = s$

$x = -s$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix}$$

$$= t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are eigan ~~valu~~ vectors for $\lambda = 2$.

Problems:2 Find the eigan value and vectors of the matrix

1. $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

For eigan value, λ

Characteristic equation $\Rightarrow$ ~~$\lambda A - I$~~ $[\lambda I - A] = 0$

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = 0$$

Again putting 6 in (1),

$$\begin{bmatrix} 2 & 0 & -1 \\ 2 & 5 & 0 \\ 2 & 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Equations are :

$2x - 2z = 0$

$2x + 5y = 0$

$2x + 5z = 0$

Let, $z = t$

$\therefore 2x - 2t = 0$

$\Rightarrow x = 2t/2$

$\Rightarrow x = t$

$2t + 5y = 0$

$\Rightarrow 5y = -2t$

$\Rightarrow y = -\frac{2}{5} t$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ -2/5\, t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2/5 \\ 1 \end{bmatrix}$$ $\Rightarrow$ eigan vector for $\lambda = 6$

Problem : 3

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Characteristics equation $\Rightarrow [\lambda I - A] = 0$

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} \lambda - 0 & 0-1 & 0-1 \\ 0-1 & \lambda - 0 & 0-1 \\ 0-1 & 0-1 & \lambda - 0 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix} = 0$$

$$\Rightarrow \lambda(\lambda^2 - 1) + 1(-\lambda - 1) - 1(-\lambda - 1) = 0$$

$$\Rightarrow \lambda(\lambda^2 - 1) - \lambda - 1 + \lambda + 1 = 0$$

$\lambda = 0, 1$

So, eigan values are $0, 1$.

Let, the eigan value is, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$[\lambda I - A] . X = 0$

$$\Rightarrow \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \text{--- (1)}$$

putting 0 in (1),

$$\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-y - z = 0$ | $y + z = 0$

$-x - z = 0$ | $x + z = 0$

$-x - y = 0$ | $x + y = 0$

Let, $y = t$

$z = -t$

$x = t$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} l \\ l \\ -l \end{bmatrix} = l \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \text{eigan vector for } \lambda = 0.$$

Again, putting 1 in(1).

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x - y - z = 0 \Rightarrow x - y - z = 0$$
$$-x + y - z = 0 \Rightarrow x - y + z = 0$$
$$-x - y + z = 0 \Rightarrow x + y - z = 0$$

Let, $x = t$