1. Consider the relation x = 2y + 1 between real numbers x, y. Is it reflexive? Is it symmetric? Is it transitive?

   Solution: Denote the given relation by R, so that
   xRy ↔ x = 2y + 1.

   This relation is not reflexive, because the condition 1R1 does not hold.
   This relation is not symmetric, because the condition 1R3 holds, but the condition 3R1 doesn't.
   This relation is not transitive, because the conditions 1R3 and 3R7 hold, but the condition 1R7 doesn't.

2. What is the total number of binary relations on the set {1, . . ., 10}? How many of them are reflexive?

   Solution: A binary relation is an arbitrary subset of the set {1, . . ., 10}× {1, . . ., 10}. So the total number of binary relations is 2^{100}. Such a subset is a reflexive relation if it contains 10 pairs of the form (n, n). So the number of reflexive relations is 2^{90}
   $[2^{n^2-n}]$

let's say we have a set with two elements $A = \{0, 1\}$
So Cartesian product is $C = A \times A = \{(0,0), (0,1), (1,0), (1,1)\}$
Then we should write down all subsets of this product to get all binary relations. Right?

1. $\emptyset$
2. $\{(0,0)\}$
3. $\{(0,1)\}$
4. $\{(1,0)\}$
5. $\{(1,1)\}$
6. $\{(0,0),(0,1)\}$
7. $\{(0,0),(1,0)\}$
8. $\{(0,0),(1,1)\}$
9. $\{(0,1),(1,0)\}$
10. $\{(0,1),(1,1)\}$
11. $\{(1,0),(1,1)\}$
12. $\{(0,0),(0,1),(1,0)\}$
13. $\{(0,0),(0,1),(1,1)\}$
14. $\{(0,0),(1,0),(1,1)\}$
15. $\{(0,1),(1,0),(1,1)\}$
16. $\{(0,0),(0,1),(1,0),(1,1)\}$

We know that there are only 4 reflexive binary relations.
those are: 8, 13, 14, 16
because they satisfied reflexive property: for all $x \in A, \rightarrow (x, x) \in R$.

3. Let R be the relation on N defined by x + 3y = 12, i.e. R = {(x, y)| x + 3y = 12}. Write R as a set of ordered pairs.
   Ans:
   {(9, 1), (6, 2), (3, 3)};

4. Each of the following defines a relation on the positive integers N:
   i. "x is greater than y."
   ii. "xy is the square of an integer."
   iii. x + y = 10
   iv. x + 4y = 10.

   Determine which of the relations are: (a) reflexive; (b) symmetric; (c) antisymmetric; (d) transitive.

   Ans:
   (a)Reflexive: ii;
   (b) Symmetric: (ii) and (iii);
   (c) Antisymmetric: (i) and (iv);
   (d) transitive: all except (iii).

5. For each of the following relations between positive integers m, n, determine whether it is a partial order:
   i. m / n
   ii. m / n^2
   iii. m^2 / n

(a) $m|n$.

Solution: This relation is reflexive (every number evenly divides itself), anti-symmetric (if m divides n and n divides m then $m = n$), and transitive (if k divides m and m divides n then k divides n). Consequently this is a partial order. But it is not total: for example, 2 doesn't divide 3 and 3 doesn't divide 2.

(b) $m|n^2$.

Solution: This relation is not anti-symmetric; for instance, 2 divides 4^2 and 4 divides 2^2. So it is not a partial order and hence not a total order.

(c) $m^2|n$.

Solution: This relation is not reflexive; for instance, 2^2 doesn't divide 2. So it is not a partial order and hence not a total order.

6. Determine whether the relation R, $m = -n$ on the set of integers is (a) reflexive, (b) irreflexive, (c) symmetric, (d) transitive.

   Solution.
   This relation R is not reflexive, because 1R1 is not true. It is not irreflexive, because 0R0 is true. It is symmetric, because the condition $m = -n$ is equivalent to $n = -m$. It is not transitive, because the conditions $1R{-1}$ and $-1R1$ hold, but the condition 1R1 doesn't.

7. Consider the following three relations on the integers:
   $R = \{(x,y) | x \cdot y >= 0\}$
   $S = \{(x,y) | \gcd(x,y) = 1\}$
   $T = \{(x,y) | x - y < 1\}$

   For each entry of the following matrix, write YES if the relation labeling its column has the property labeling its row. Else write NO.

	Reflexive	Symmetric	Antisymmetric	Transitive
R	YES	YES	NO	NO
S	NO	YES	NO	NO
T	YES	NO	YES	YES

8. For each "NO" answer for Question no. 7, give a counter-example to prove that the given relation does not have the given property.

	refl	*sym*	*anti − sym*	*trans*
$R = \{(x,y) \mid x \cdot y \geq 0\}$	Y	Y	N	N
$S = \{(x,y) \mid \gcd(x,y) = 1\}$	N	Y	N	N
$T = \{(x,y) \mid x - y < 1\}$	Y	N	Y	Y

SOLUTION

R not anti-sym: $4 \cdot 2 \geq 0 \wedge 2 \cdot 4 \geq 0$, but $2 \neq 4$

R not trans: $-4 \cdot 0 \geq 0 \wedge 0 \cdot 2 \geq 0$, but $-4 \cdot 2 < 0$

S not refl: $\gcd(5,5) = 5 \neq 1$

S not anti-sym: $\gcd(4,5) = 1 \wedge \gcd(5,4) = 1$, but $5 \neq 4$

S not trans: $\gcd(4,5) = 1 \wedge \gcd(5,6) = 1$, but $\gcd(4,6) \neq 1$

T not sym: $3 - 5 < 1$, but $5 - 3 > 1$

9. For each of the following relations, state whether they fulfill each of the 4 main properties - reflexive, symmetric, antisymmetric, transitive. Briefly substantiate each of your answers.
(a) The coprime relation on Z. (Recall that a, b ∈ Z are coprime if and only if gcd(a, b) = 1.)
(b) Divisibility on Z.

Solution

(a) It's definitely not reflexive, as no integer is coprime with itself except -1 and 1. It is symmetric because $\gcd(a,b) = \gcd(b,a)$, so $\gcd(a,b) = 1$ iff $\gcd(b,a) = 1$. Not antisymmetric — *every* coprime pair, such as (5,7) and (7,5), will show this. Not transitive — $\gcd(5,7) = 1$, $\gcd(7,10) = 1$, but $\gcd(5,10) \neq 1$.

(b) It's reflexive since any integer divides itself. Not symmetric, for example $2 \mid 4$ but $4 \nmid 2$. It not antisymmetric on $\mathbb{Z}$, since $a \mid -a$ and $-a \mid a$, although it would be antisymmetric if restricted to $\mathbb{N}$. It is transitive — if $a \mid b$ then $b = ka$ for some $k \in \mathbb{Z}$, and if $b \mid c$ then $c = lb$ for some $l \in \mathbb{Z}$, thus $c = (lk)a$ and $(lk) \in \mathbb{Z}$ so $a \mid c$.

10. Let R be the relation defined below. Determine which properties, reflexive, irreflexive, symmetric, antisymmetric, transitive, the relation satisfies. Prove each answer.

i. R is the relation on {a, b, c}, R = {(a, b),(b, a),(b, b),(c, c)}

Solution:

R is the relation on $\{a, b, c\}$, $R = \{(a, b), (b, a), (b, b), (c, c)\}$

reflexive: No. (a, a) is not in R.

irreflexive: No. (b, b) is in R.

symmetric: Yes. For each pair $(x, y) \in R$ you can check that the pair $(y, x) \in R$.

antisymmetric: No. $(a, b), (b, a) \in R$ and $a \neq b$.

transitive: No. $(a, b), (b, a) \in R$, but $(a, a) \notin R$.

11. Consider the following five relations on the set A = {1, 2, 3, 4}:

R1 = {(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)}

R2 = {(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

R3 = {(1, 3), (2, 1)}

R4 = ∅, the empty relation

R5 = A × A, the universal relation

Determine which of the relations are reflexive, symmetric, antisymmetric, transitive.

Solution:

Since A contains the four elements 1, 2, 3, and 4, a relation R on A is reflexive if it contains the four pairs (1, 1), (2, 2), (3, 3), and (4, 4).

Thus only R2 and the universal relation R5 = A × A are reflexive.

Note that

R1, R3, and R4 are not reflexive since, for example, (2, 2) does not belong to any of them.

R1 is not symmetric since (1, 2) ∈ R1 but (2, 1) ∉ R1. R3 is not symmetric since (1, 3) ∈ R3 but (3, 1) ∉ R3. The other relations are symmetric.

R_2 is not antisymmetric since $(1, 2)$ and $(2, 1)$ belong to R_2, but $1 \neq 2$. Similarly, the universal relation R_3 is not antisymmetric. All the other relations are antisymmetric.

The relation R_3 is not transitive since $(2, 1), (1, 3) \in R_3$ but $(2, 3) \notin R_3$. All the other relations are transitive.

12. Consider the following five relations:

(1) Relation ≤ (less than or equal) on the set Z of integers.

(2) Set inclusion ⊆ on a collection C of sets.

(3) Relation ⊥ (perpendicular) on the set L of lines in the plane.

(4) Relation (parallel) on the set L of lines in the plane.

(5) Relation | of divisibility on the set N of positive integers. (Recall x |y if there exists z such that xz = y.)
Determine which of the relations are reflexive, symmetric, antisymmetric, transitive.

Solution:
The relation (3) is not reflexive since no line is perpendicular to itself. Also (4) is not reflexive since no line is parallel to itself. The other relations are reflexive; that is, $x \le x$ for every $x \in Z$, $A \subseteq A$ for any set $A \in C$, and $n|n$ for every positive integer $n \in N$.

The relation $\perp$ is symmetric since if line a is perpendicular to line b then b is perpendicular to a. Also, $\parallel$ is symmetric since if line a is parallel to line b then b is parallel to line a. The other relations are not symmetric. For example:

$$3 \le 4 \text{ but } 4 \not\le 3; \quad \{1, 2\} \subseteq \{1, 2, 3\} \text{ but } \{1, 2, 3\} \not\subseteq \{1, 2\}; \quad \text{and} \quad 2 \mid 6 \text{ but } 6 \nmid 2.$$

The relation $\le$ is antisymmetric since whenever $a \le b$ and $b \le a$ then $a = b$. Set inclusion $\subseteq$ is antisymmetric since whenever $A \subseteq B$ and $B \subseteq A$ then $A = B$. Also, divisibility on $\mathbf{N}$ is antisymmetric since whenever $m \mid n$ and $n \mid m$ then $m = n$. (Note that divisibility on $\mathbf{Z}$ is not antisymmetric since $3 \mid -3$ and $-3 \mid 3$ but $3 \ne -3$.) The relations $\perp$ and $\parallel$ are not antisymmetric.

The relations $\le$, $\subseteq$, and $\mid$ are transitive, but certainly not $\perp$. Also, since no line is parallel to itself, we can have $a \parallel b$ and $b \parallel a$, but $a \parallel a$. Thus $\parallel$ is not transitive. (We note that the relation "is parallel or equal to" is a transitive relation on the set L of lines in the plane.)

13. Consider the Z of integers and an integer m > 1. We say that x is congruent to y modulo m, written as $x \equiv y \pmod m$
if $x - y$ is divisible by m. Show that this defines an equivalence relation on Z. (We must show that the relation is reflexive, symmetric, and transitive.)

Ans:

We must show that the relation is reflexive, symmetric, and transitive.

(i) For any x in Z we have $x \equiv x \pmod m$ because $x - x = 0$ is divisible by m. Hence the relation is reflexive.

(ii) Suppose $x \equiv y \pmod m$, so $x - y$ is divisible by m. Then $-(x - y) = y - x$ is also divisible by m, so $y \equiv x \pmod m$. Thus the relation is symmetric.

(iii) Now suppose $x \equiv y \pmod m$ and $y \equiv z \pmod m$, so $x - y$ and $y - z$ are each divisible by m. Then the sum

$$(x - y) + (y - z) = x - z$$

is also divisible by m; hence $x \equiv z \pmod m$. Thus the relation is transitive.

Accordingly, the relation of congruence modulo m on $\mathbf{Z}$ is an equivalence relation.