MTH

H.M. Tahsin Sheikh
22201243

Partial Derivatives

Ex:01 $f(x,y) = x^2 + xy + y^2$, then using analytical definition find $f_x(-1,1)$, $f_y(2,5)$ and $f_{xy}(-1,1)$.

Solⁿ: Given that,

$$f(x,y) = x^2 + xy + y^2 \quad \text{——— (i)}$$

Then by analytical defination of partial derivatives,

$$\frac{df}{dx} \text{ or } f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\{(x+h)^2 + (x+h)y + y^2\} - \{x^2 + xy + y^2\}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2hx + h^2 + \cancel{xy} + hy + \cancel{y^2} - \cancel{x^2} - \cancel{xy} - \cancel{y^2}}{h}$$

$$= \lim_{h\to 0} \frac{2hx + h^2 + hy}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x + h + y)}{\cancel{h}}$$

$$= \lim_{h\to 0} 2x + h + y = 2x + 0 + y = 2x + y$$

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$\therefore f_x(x,y) = 2x + y$ ——— (ii)

Now, $f_x(-1,1) = 2(-1)+1$

$= -2+1 = -1$

$\frac{df}{dy}$ or $f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$

$= \lim_{k\to 0} \frac{\{x^2 + x(y+k) + (y+k)^2\} - (x^2 + xy + y^2)}{k}$

$= \lim_{k\to 0} \frac{\cancel{x^2} + \cancel{xy} + kx + \cancel{y^2} + 2yk + k^2 - \cancel{x^2} - \cancel{xy} - \cancel{y^2}}{k}$

$= \lim_{k\to 0} \frac{kx + 2yk + k^2}{k}$

$= \lim_{k\to 0} \frac{\cancel{k}(x+2y+k)}{\cancel{k}}$

$= \lim_{k\to 0} x + 2y + k$

$\therefore f_y(x,y) = x + 2y$ ——— (iii)

$f_y(2,5) = 2 + 2\times 5$

$= 2 + 10$

$= 12$

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$$\frac{d f_x}{dy} \text{ or } f_{xy}(x,y) = \lim_{k\to 0} \frac{f_x(x,y+k) - f_x(x,y)}{k}$$

$$= \lim_{k\to 0} \frac{(2x+y+k)-(2x+y)}{k} \quad [\because f_x = 2x+y]$$

$$= \lim_{k\to 0} \frac{\cancel{2x}+\cancel{y}+k-\cancel{2x}-\cancel{y}}{k}$$

$$= \lim_{k\to 0} \frac{k}{k}$$

$$= \lim_{k\to 0} 1 \quad f_{xy}(x,y) = 1$$

$$f_{xy}(-1,1) = 1$$

<u>H.W</u> If, $f(x,y) = x^2 + 3xy + 2y^3$, then $f_{xx}(1,2)$ and $f_{xy}(1,2)$.

<u>Soln:</u> Given that,

$$f(x,y) = x^2 + 3xy + 2y^3 \quad \text{——— (i)}$$

$$f_x = \frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\{(x+h)^2 + 3(x+h)y + 2y^3\} - \{x^2 + 3xy + 2y^3\}}{h}$$

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$$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2+\cancel{3xy}+3hy+\cancel{2y^3}-\cancel{x^2}-\cancel{3xy}-\cancel{2y^3}}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2+3hy}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+3y+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 2x+3y+h$$

$$= 2x+3y+0 = 2x+3y$$

$$\text{fun or } \frac{d}{dx}(fx) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{2(x+h)+3y-2x-3y}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2x}+2h+\cancel{3y}-\cancel{2x}-\cancel{3y}}{h}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}}$$

$$= \lim_{h\to 0} 2 = 2$$

$\text{fun}(x,y) = 2$

$\text{fun}(1,2) = 2$

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$$\frac{df_x}{dy} \text{ or } f_{xy}(x,y) = \frac{d}{dy}(f_x)$$

$$= \lim_{K\to 0} \frac{f_x(x, y+K) - f_x(x,y)}{K}$$

$$= \lim_{K\to 0} \frac{2x+3(y+K) - 2x+3y}{K} \quad [f_x = 2x+3y]$$

$$= \lim_{K\to 0} \frac{2x+3y+3K-2x+3y}{K}$$

$$= \lim_{K\to 0} \frac{3K}{K}$$

$$= \lim_{K\to 0} 3 = 3$$

$\therefore f_{xy}(x,y) = 3$

Now $f_{xy}(1,2) = 3$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3-y^3}{x^2+y^2} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{(my)^3-y^3}{(my)^2+y^2}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{m^3y^3-y^3}{m^2y^2+y^2}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{y^3(m^3-1)}{y^2(m^2+1)}$$

$$= \lim_{x\to 0} \frac{y(m^3-1)}{m^2+1} = \frac{0\times(m^3-1)}{(m^2+1)} = 0$$

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Differentiability (2 variables)

Ex: 01

Show that,
$$f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & ; \text{ when } (x,y) \neq (0,0) \\ 0 & ; \text{ when } (x,y) = (0,0) \end{cases}$$

Soln: This continuous; possess partial derivatives at (0,0) but not differentiable at (0,0)

Soln: For, simultaneous limit,

$y = mx$ where, $m = \pm 1, \pm 2 \cdots$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3-y^3}{x^2+y^2}$$

$x^2+y^2=0$
$\Rightarrow x^2 = -y^2$
$\Rightarrow x = \sqrt{-y^2}$
$\Rightarrow x = iy \Rightarrow x = my$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{(my)^3-y^3}{(my)^2+y^2}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{y^3(m^3-1)}{y^2(m^2+1)}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{y(m^3-1)}{(m^2+1)} = \frac{0\times(m^3-1)}{(m^2+1)} = 0$$

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$\therefore \lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^3-y^3}{x^2+y^2} = 0$, which is unique

Hence,

$\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does exist.

Now, we have, $f(x,y) = \dfrac{x^3-y^3}{x^2+y^2}$; $(x,y)\neq(0,0)$

$f(x,y) = 0$; $(x,y) = (0,0)$

$f(0,0) = 0$

Thus, $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y) = f(0,0) = 0$

The function of $f(x,y)$ is continuous at $(0,0)$.

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2) Partial Derivative at (0,0)

By analytical definition at partial derivatives,

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{h^3 - 0^3}{h^2+0^2} - 0\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{h^3}{h^2}\right]$$

$$= \lim_{h\to 0} 1$$

$$f_x(0,0) = 1$$

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again,

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x, y+k) - f(x,y)}{k}$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0, 0+k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f(0, k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{1}{k} \left[\frac{0^3 - k^3}{0^2+k^2} - 0\right]$$

$$= \lim_{k\to 0} \frac{1}{k} \left[\frac{-k^3}{k^2}\right]$$

$$f_y(0,0) = \lim_{k\to 0} -1 = -1$$

3) Differentiability at (0,0):

By analytical definition of differentiability

$$L.H.L = \lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(0,0) - h f_x(0,0) - k f_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}} \Big[f(0+h, 0+k) - f(0,0) - h f_x(0,0) - k f_y(0,0)\Big]$$

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$$= \lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}} \left[f(h,k) - 0 - h\cdot 1 - k(-1) \right]$$

$$= \lim_{(h,k)\to(0,0)} \left[\frac{h^3-k^3}{h^2+k^2} - 0 - h + k \right] \cdot \frac{1}{\sqrt{h^2+k^2}}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{1}{\sqrt{h^2+k^2}} \left[\frac{h^3-k^3-h(h^2+k^2)+k(h^2+k^2)}{h^2+k^2} \right]$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{1}{\sqrt{h^2+k^2}} \cdot \frac{\cancel{h^3}-\cancel{k^3}-\cancel{h^3}-\cancel{hk^2}+kh^2+\cancel{k^3}}{h^2+k^2}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{1}{\sqrt{h^2+k^2}} \cdot \frac{kh^2-hk^2}{h^2+k^2}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{1}{\sqrt{h^2+k^2}} \cdot \frac{hk(h-k)}{h^2+k^2}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{hk(h-k)}{(h^2+k^2)^{3/2}} \quad \text{——— (1)}$$

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Now, let, $h = mk$

Then,

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{hk(h-k)}{(h^2+k^2)^{3/2}} =$$

$h^2+k^2=0$
$\Rightarrow h^2 = -k^2$
$\Rightarrow h = \sqrt{-k^2}$
$\Rightarrow h = ik$
$\Rightarrow h = mk$

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{mk \cdot k(mk-k)}{(m^2k^2+k^2)^{3/2}}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{mk^2 \cdot k(m-1)}{\left[k^2(m^2+1)^{3/2}\right]}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{mk^3(m-1)}{k^3(m^2+1)^{3/2}}$$

$$= \frac{m(m-1)}{(m^2+1)^{3/2}}$$

$\therefore \lim_{\substack{h\to 0\\ k\to 0}} \frac{hk(h-k)}{(h^2+k^2)^{3/2}} = \frac{m(m-1)}{(m^2+1)^{3/2}}$; which is not unique

Hence, $\lim_{\substack{h\to 0\\ k\to 0}} \frac{hk(h-k)}{(h^2+k^2)^{3/2}}$ does not exist.

The function $f(x,y)$ is not differentiable at $(0,0)$

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H.W

(1) If the function $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{\sqrt{x^2+y^2}} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

Then examine differentiability of $f(x,y)$ at $(0,0)$

Soln: By analytical definition of partial derivative,

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{h\cdot 0(h^2-0^2)}{\sqrt{h^2+0^2}} - 0\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{0}{\sqrt{h^2+0^2}}\right]$$

$$= \lim_{h\to 0} \frac{1}{h} \times 0$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

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$$f_y(x,y) = \lim_{k\to 0} \frac{f(x, y+k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f(0, 0+k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{1}{k}\left[\frac{0\cdot k\,(0^2-k^2)}{\sqrt{0^2+k^2}} - 0\right]$$

$$= \lim_{k\to 0} \frac{1}{k}\left[\frac{0}{k}\right]$$

$$= \lim_{k\to 0} 0 = 0$$

By analytical definition of differentiability,

$$L.H.L = \lim_{\substack{h\to 0\\ k\to 0}} \frac{f(a+h, b+k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{f(0+h, 0+k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{1}{\sqrt{h^2+k^2}}\left[\frac{hk(h^2-k^2)}{\sqrt{h^2+k^2}} - 0 - h\times 0 - k\times 0\right]$$

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$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{1}{\sqrt{h^2+k^2}} \times \frac{hk(h^2-k^2)}{\sqrt{h^2+k^2}}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{hk(h^2-k^2)}{h^2+k^2} \quad \text{——— } (1)$$

Let $k = mh$

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{hk(h^2-k^2)}{h^2+k^2} =$$

$h^2+k^2=0$

$\Rightarrow k^2 = -h^2$

$\Rightarrow k = \sqrt{-h^2}$

$\therefore k = mh$

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{h\cdot mh\cdot(h^2-m^2h^2)}{h^2+m^2h^2}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{mh^2(h^2-m^2h^2)}{h^2+m^2h^2}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{mh^2\cdot h^2(1-m^2)}{h^2(1+m^2)}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{mh^2(1-m^2)}{(1+m^2)}$$

$$= \frac{m\times 0(1-m^2)}{(1+m^2)} = 0$$; which is unique.

Hence, $\lim_{\substack{h\to 0\\ k\to 0}} \frac{hk(h^2-k^2)}{\sqrt{h^2+k^2}}$ does exist.

$f(x,y)$ differentiable at $(0,0)$

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(ii) Let, $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

Then show both $f_x(0,0)$ and $f_y(0,0)$ exist but $f(x,y)$ is discontinuous at $(0,0)$

<u>Solⁿ:</u> $f_x(x,y) = \lim_{h\to 0} \dfrac{f(x+h,y)-f(x,y)}{h}$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{h\cdot 0}{h^2+0^2} - 0\right]$$

$$= \lim_{h\to 0} \frac{1}{h} \times \frac{0}{h^2} = \lim_{h\to 0} 0 = 0$$

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}$$

$$= \lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{1}{k}\left[\frac{0\cdot k}{0^2+k^2} - 0\right]$$

$$= \lim_{k\to 0} \frac{1}{k} \times \frac{0}{k^2} = \lim_{k\to 0} 0 = 0$$

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Here, $f_x(0,0)$ and $f_y(0,0)$ both exist.

Let, $y = mx$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x\cdot mx}{x^2+m^2x^2}$$

$x^2+y^2=0$
$\Rightarrow y^2=-x^2$
$\Rightarrow y=\sqrt{-x^2}$
$\Rightarrow y=ix$
$\Rightarrow y=mx$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{mx^2}{x^2(1+m^2)}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{m}{1+m^2}$$

$= \frac{m}{1+m^2}$, which is not unique

Since, m has multiples values the limit is not unique.

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2} \text{ does not exist}$$

As, simultaneous limit of $f(x,y)$ does not exist at $(0,0)$. The function is discontinuous.

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(iii) Let, $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

Show that $f_y(x,0) = x$; $f_x(0,y) = -y$ and

$f_{xy}(0,0) \neq f_{yx}(0,0)$

Solⁿ: $f_y(x,y) = \lim\limits_{k \to 0} \dfrac{f(x, y+k) - f(x,y)}{k}$

$f_y(x,0) = \lim\limits_{k \to 0} \dfrac{f_y(x, 0+k) - f_y(x,0)}{k}$

$= \lim\limits_{k \to 0} \dfrac{f_y(x,k) - f_y(x,0)}{k}$

$= \lim\limits_{k \to 0} \dfrac{1}{k} \left[\dfrac{x \cdot k(x^2-k^2)}{x^2+k^2} - 0 \right]$

$= \lim\limits_{k \to 0} \dfrac{1}{\cancel{k}} \times \dfrac{\cancel{k}x(x^2-k^2)}{x^2+k^2}$

$= \lim\limits_{k \to 0} \dfrac{x(x^2-k^2)}{x^2+k^2}$

$\cancel{= \lim\limits_{k \to 0}} = \dfrac{x(x^2-0)}{x^2+0}$

$= \dfrac{x^3}{x^2}$

$= x$ $\quad \therefore f(x,0) = x$

(Showed)

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_x(0,y) = \lim_{h\to 0} \frac{f_x(0+h,y) - f_x(0,y)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[f_x(h,y) - f_x(0,y)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{hy(h^2-y^2)}{h^2+y^2} - 0\right]$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}} \times \frac{\cancel{h}y(h^2-y^2)}{h^2+y^2}$$

$$= \lim_{h\to 0} \frac{y(h^2-y^2)}{h^2+y^2}$$

$$= \frac{y(0^2-y^2)}{0^2+y^2} = \frac{-y^3}{y^2} = -y$$

$\therefore f_x(0,y) = -y$ (Showed)

$\therefore$ L.H.L $f_{xy}(x,y) = \frac{d}{dy}(f_x)$

$$= \lim_{k\to 0} \frac{f_x(x,y+k) - f_x(x,y)}{k}$$

$$f_x(0,0) = \lim_{k\to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} \quad [f_x = k]$$

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$$= \lim_{K \to 0} \frac{-K-0}{K}$$

$$= \lim_{K \to 0} \frac{-K}{K} = \lim_{K \to 0} -1 = -1$$

R.H.L, $f_{yx}(x,y) = \frac{d}{dx}(f_y)$

$$= \lim_{h \to 0} \frac{f_y(x+h,y) - f_y(x,y)}{h}$$

Again,

$$f_{yx}(0,0) = \lim_{h \to 0} \frac{1}{h}\left[f_y(0+h,0) - f_y(0,0)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[f_y(h,0) - f_y(0,0)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}(h-0) \qquad [f_y = h]$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

$$\therefore f_{xy}(0,0) \neq f_{yx}(0,0)$$

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Chain rule

Ex:01 Suppose that, $z = f(u,v)$ where ~~u=e~~

$u = e^x \cos y$, $v = e^x \sin y$

Show that, (i) $\frac{dz}{dx} = u\frac{df}{du} + v\frac{df}{dv}$

(ii) $\frac{dz}{dy} = -v\frac{df}{du} + u\frac{df}{dv}$

(i)

$$\frac{dz}{dx} = \frac{df}{du}\cdot\frac{du}{dx} + \frac{df}{dv}\cdot\frac{dv}{dx}$$

$$= \frac{df}{du}\cdot\frac{d}{dx}(e^x\cos y) + \frac{df}{dv}\cdot\frac{d}{dx}(e^x\sin y)$$

$$= e^x\cos y\cdot\frac{df}{du} + e^x\sin y\,\frac{df}{dv}$$

$$= u\frac{df}{du} + v\cdot\frac{df}{dv}$$ [showed]

(ii)

$$\frac{dz}{dy} = \frac{df}{dy}\cdot\frac{du}{dy} + \frac{df}{dv}\cdot\frac{dv}{dy}$$

$$= \frac{df}{du}\cdot\frac{d}{dy}(e^x\cos y) + \frac{df}{dv}\cdot\frac{d}{dy}(e^x\sin y)$$

$$= -e^x\sin y\,\frac{df}{du} + e^x\cos y\,\frac{df}{dv}$$

$$= -v\frac{df}{du} + u\frac{df}{dv}$$

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Ex: 02

Suppose that, $u = f(x-y, y-z, z-x)$

Show that, $U_x + U_y + U_z = 0$

Solⁿ: Let,
$$p = x-y$$
$$q = y-z$$
$$r = r-x$$
$$\therefore u = f(p(x,y), q(y,z), r(z,x))$$

$$U_x = \frac{du}{dx} = \frac{df}{dp}\cdot\frac{dp}{dx} + \frac{df}{dr}\cdot\frac{dr}{dx}$$
$$= \frac{df}{dp}\cdot\frac{d}{dx}(x-y) + \frac{df}{dr}\cdot\frac{d}{dx}(z-x)$$
$$= \frac{df}{dp} - \frac{df}{dr}$$

$$U_y = \frac{du}{dy} = \frac{df}{dp}\cdot\frac{dp}{dy} + \frac{df}{dq}\cdot\frac{dq}{dy}$$
$$= \frac{df}{dp}\cdot\frac{d}{dy}(x-y) + \frac{df}{dq}\cdot\frac{d}{dy}(y-z)$$
$$= -\frac{df}{dp} + \frac{df}{dq}$$

$$U_z = \frac{df}{dq}\cdot\frac{dq}{dz} + \frac{df}{dr}\cdot\frac{dr}{dz}$$
$$= \frac{df}{dq}\cdot\frac{d}{dz}(y-z) + \frac{df}{dr}\cdot\frac{d}{dz}(z-x)$$
$$= -\frac{df}{dq} + \frac{df}{dr}$$

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$\therefore$ L.H.S $= U_x + U_y + U_z$

$$= \cancel{\frac{df}{dp}} - \cancel{\frac{df}{dr}} - \cancel{\frac{df}{dp}} + \cancel{\frac{df}{dq}} - \cancel{\frac{df}{dq}} + \cancel{\frac{df}{dr}}$$

$= 0$

$=$ R.H.S (Showed)

Homogeneous function & Eulers

$f(x,y) = 3x^5 + 4x^4y + 9x^3y^2 + y^5 \rightarrow$ Homogenous function of degree $5 = n$

$f(x,y) = 3x^4 + 9x^2y + y^4 + 9 \rightarrow$ non-homogenous.

If $u = u(x,y)$ is a homogeneous function of degree n.

(i) $x\frac{du}{dx} + y\frac{du}{dy} = nu$

(ii) $xU_x + yU_y = nu$

$u = U(x,y,z)$

$\Rightarrow xU_x + yU_y + zU_z = nU$

Ex: 01 Verify Euler's theorem for,

$$u = \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}$$

Solⁿ: We have,

$$u = x^2y^{-1} + y^2z^{-1} + z^2x^{-1}$$

Here, u is a homogeneous function of x, y, z of degree, $n = 1$

We know, $xU_x + yU_y + zU_z = u$

$\therefore$ L.H.S $= xU_x + yU_y + zU_z$

$$= x\frac{d}{dx}\left(\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}\right) + y\frac{d}{dy}\left(\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}\right) + z\frac{d}{dz}\left(\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}\right)$$

$$= x\left(\frac{2x}{y} - \frac{z^2}{x^2}\right) + y\left(-\frac{x^2}{y^2} + \frac{2y}{z}\right) + z\left(-\frac{y^2}{z^2} + \frac{2z}{x}\right)$$

$$= \frac{2x^2}{y} - \frac{z^2}{x} - \frac{x^2}{y} + \frac{2y^2}{z} - \frac{y^2}{z} + \frac{2z^2}{x}$$

$$= \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}$$

$$= u$$

$$= \text{R.H.S}$$ (Showed)

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H.W (Chain)

(i) If, $w = f\left(\frac{y-x}{xy}, \frac{z-y}{yz}\right)$ then show that, $x^2 \frac{dw}{dx} + y^2 \frac{dw}{dy} + z^2 \frac{dw}{dz} = 0$

Soln: Let, $p = \frac{y-x}{xy}$

$$q = \frac{z-y}{yz}$$

$$\therefore w = f\left(p\left(\frac{y-x}{xy}\right), q\left(\frac{z-y}{yz}\right)\right)$$

$$\frac{dw}{dx} = \frac{df}{dp} \cdot \frac{dp}{dx}$$

$$= \frac{df}{dp} \cdot \frac{d}{dx}\left(\frac{y-x}{xy}\right)$$

$$= -\frac{1}{x^2} \frac{df}{dp}$$

$$\frac{dw}{dy} = \frac{df}{dp} \cdot \frac{dp}{dy} + \frac{df}{dq} \cdot \frac{dq}{dy}$$

$$= \frac{df}{dp} \cdot \frac{d}{dy}\left(\frac{y-x}{xy}\right) + \frac{df}{dq} \cdot \frac{d}{dy}\left(\frac{z-y}{yz}\right)$$

$$= \frac{1}{y^2} \frac{df}{dp} - \frac{1}{y^2} \frac{df}{dq}$$

$$\frac{dw}{dz} = \frac{df}{dq} \cdot \frac{dq}{dz} = \frac{df}{dq} \cdot \frac{d}{dz}\left(\frac{z-y}{yz}\right)$$

$$= \frac{1}{z^2} \frac{df}{dq}$$

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$\therefore$ L.H.S $= x^2 \frac{dw}{dx} + y^2 \frac{dw}{dy} + z^2 \frac{dw}{dz}$

$= -\cancel{x^2} \cdot \frac{1}{\cancel{x^2}} \cdot \frac{df}{dp} + \cancel{y^2} \cdot \frac{1}{\cancel{y^2}} \cdot \frac{df}{dp} - \cancel{y^2} \cdot \frac{1}{\cancel{y^2}} \frac{df}{dq}$

$+ \cancel{z^2} \cdot \frac{1}{\cancel{z^2}} \cdot \frac{df}{dq}$

$= -\frac{df}{dp} + \frac{df}{dp} - \frac{df}{dq} + \frac{df}{dq}$

$= 0 =$ R.H.S (showed)

(2) If, $u = f(x^2 + 2yz, y^2 + 2zx)$ then shown that, $(y^2 - zx)\frac{du}{dx} + (x^2 - yz)\frac{du}{dy} + (z^2 - xy)\frac{du}{dz} = 0$

<u>Solⁿ:</u> Let, $p = x^2 + 2yz$

$q = y^2 + 2zx$

$\therefore u = f(p(x^2 + 2yz), q(y^2 + 2zx))$

$\frac{du}{dx} = \frac{df}{dp} \cdot \frac{dp}{dx} + \frac{df}{dq} \cdot \frac{dq}{dx}$

$= \frac{df}{dp} \cdot \frac{d}{dx}(x^2 + 2yz) + \frac{df}{dq} \cdot \frac{d}{dx}(y^2 + 2zx)$

$= 2x \frac{df}{dp} + 2z \frac{df}{dq}$

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$$\frac{du}{dy} = \frac{df}{dp}\cdot\frac{dp}{dy} + \frac{df}{dq}\cdot\frac{dq}{dy}$$

$$= \frac{df}{dp}\cdot\frac{d}{dy}(x^2+2yz) + \frac{df}{dq}\cdot\frac{d}{dy}(y^2+2zx)$$

$$= 2z\frac{df}{dp} + 2y\frac{df}{dq}$$

$$\frac{du}{dz} = \frac{df}{dp}\cdot\frac{dp}{dz} + \frac{df}{dq}\cdot\frac{dq}{dz}$$

$$= \frac{df}{dp}\cdot\frac{d}{dz}(x^2+2yz) + \frac{df}{dq}\cdot\frac{d}{dz}(y^2+2zx)$$

$$= 2y\frac{df}{dp} + 2x\frac{df}{dq}$$

$$\therefore L.H.S = (y^2-zx)\frac{du}{dx} + (x^2-yz)\frac{du}{dy} + (z^2-xy)\frac{du}{dz}$$

$$= (y^2-zx)\cdot\left(2x\frac{df}{dp} + 2z\frac{df}{dq}\right) +$$

$$(x^2-yz)\cdot\left(2z\frac{df}{dp} + 2y\frac{df}{dq}\right) +$$

$$(z^2-xy)\cdot\left(2y\frac{df}{dp} + 2x\frac{df}{dq}\right)$$

$$= 2xy^2\frac{df}{dp} - 2x^2z\frac{df}{dp} + 2zy^2\frac{df}{dq} - 2xz^2\frac{df}{dq}$$

$$+ 2zx^2\frac{df}{dp} - 2yz^2\frac{df}{dp} + 2yx^2\frac{df}{dq} - 2xy^2\frac{df}{dq}$$

$$+ 2yz^2\frac{df}{dp} - 2xy^2\frac{df}{dp} + 2xz^2\frac{df}{dq} - 2yx^2\frac{df}{dq}$$

$$= 0 = R.H.S$$

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③ If, $z = z(x, y)$ and $x = e^{u} + e^{-v}$;

$y = e^{-u} - e^{v}$ then show that,

$$\frac{dz}{du} - \frac{dz}{dv} = x\frac{dz}{dx} - y\frac{dz}{dy}$$

Soln: $\frac{dz}{du} = \frac{dz}{dx} \cdot \frac{dx}{du} + \frac{dz}{dy} \cdot \frac{dy}{du}$

$$= \frac{dz}{dx} \cdot \frac{d}{du}(e^{u} + e^{-v}) + \frac{dz}{dy} \cdot \frac{d}{du}(e^{-u} - e^{v})$$

$$= \frac{dz}{dx}(e^{u}) + \frac{d}{du}(-e^{-u})$$

$$= e^{u}\frac{dz}{dx} - e^{-u}\frac{d}{du}$$

$$\frac{dz}{dv} = \frac{dz}{dx} \cdot \frac{dx}{dv} + \frac{dz}{dy} \cdot \frac{dy}{dv}$$

$$= \frac{dz}{dx} \cdot \frac{d}{dv}(e^{u} + e^{-v}) + \frac{dz}{dy} \cdot \frac{d}{dv}(e^{-u} - e^{v})$$

$$= \frac{dz}{dx}(-e^{-v}) + \frac{dz}{dy}(-e^{v})$$

$$= e^{-v} \cdot \frac{dz}{dx} - e^{v} \cdot \frac{dz}{dy}$$

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$$L.H.S = \frac{dz}{du} - \frac{dz}{dv}$$

$$= e^{u}\frac{dz}{dx} - e^{-u}\frac{d}{du} + e^{-v}\frac{dz}{dx} - e^{v}\frac{dz}{dy}$$

$$= e^{u}\frac{dz}{dx} + e^{-v}\frac{dz}{dx} - e^{-u}\frac{dz}{dy} + e^{v}\frac{dz}{dy}$$

$$= \left(e^{u}\frac{dz}{dx} + e^{-v}\frac{dz}{dx}\right) - \left(e^{-u}\frac{dz}{dy} - e^{v}\frac{dz}{dy}\right)$$

$$= (e^{u} + e^{-v})\frac{dz}{dx} - (e^{-u} - e^{v})\frac{dz}{dy}$$

$$= x\frac{dz}{dx} - y\frac{dz}{dy}$$

$= R.H.S$ (Showed)

<u>H.W</u> (Homogeneous)

① Verify Euler's theorem for $u(x,y,z) = x^3 + y^3 + z^3$

<u>Solⁿ:</u> Here, u is a function of x, y, z of degree $= 3$

We have,

$$xU_x + yU_y + zU_z = 3u$$

$$L.H.S = xU_x + yU_y + zU_z$$

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$= x\frac{d}{dx}(x^3+y^3+z^3) + y\cdot\frac{d}{dy}(x^3+y^3+z^3) +$

$z\frac{d}{dz}(x^3+y^3+z^3)$

$= x\cdot 3x^2 + y\cdot 3y^2 + z\cdot 3z^2$

$= 3x^3 + 3y^3 + 3z^3$

$= 3(x^3+y^3+z^3) = 3u$ = R.H.S (showed)

(2) verify Eulers for $u(x,y,z) = x^2+y^2+z^2$

<u>Solⁿ:</u> Here,

u is a homogeneous function of x,y,z of degree = 2

we have, $xU_x + yU_y + zU_z = 2u$

$= x\cdot\frac{d}{dx}(x^2+y^2+z^2) + y\frac{d}{dy}(x^2+y^2+z^2) +$

$z\frac{d}{dz}(x^2+y^2+z^2)$

$= x\cdot 2x + y\cdot 2y + z\cdot 2z$

$= 2x^2 + 2y^2 + 2z^2$

$= 2(x^2+y^2+z^2)$

$= 2u$

= R.H.S (Showed)

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③ Verify Euler's for $u(x,y,z) = \frac{xy^2}{z} + \frac{yz^2}{x} + \frac{zx^2}{y}$

Soln: We have,

$$u = xy^2z^{-1} + yz^2x^{-1} + zx^2y^{-1}$$

Here, u is a homogeneous function of x,y,z of degree = 2.

We have,

$$xU_x + yU_y + zU_z = 2u$$

$$L.H.S = xU_x + yU_y + zU_z$$

$$= x\frac{d}{dx}\left(\frac{xy^2}{z} + \frac{yz^2}{x} + \frac{zx^2}{y}\right) + y\frac{d}{dy}\left(\frac{xy^2}{z} + \frac{yz^2}{x} + \frac{zx^2}{y}\right) + z\frac{d}{dz}\left(\frac{xy^2}{z} + \frac{yz^2}{x} + \frac{zx^2}{y}\right)$$

$$= x\left(\frac{y^2}{z} - \frac{yz^2}{x^2} + \frac{2xz}{y}\right) + y\left(\frac{2yx}{z} + \frac{z^2}{x} - \frac{zx^2}{y^2}\right) + z\left(-\frac{xy^2}{z^2} + \frac{2zy}{x} + \frac{x^2}{y}\right)$$

$$= \cancel{\frac{xy^2}{z}} - \cancel{\frac{yz^2}{x}} + \frac{2x^2z}{y} + \frac{2xy^2}{z} + \cancel{\frac{yz^2}{x}} - \cancel{\frac{zx^2}{y}} - \cancel{\frac{xy^2}{z}} + \frac{2z^2y}{x} + \cancel{\frac{zx^2}{y}}$$

$$= 2\left(\frac{x^2z}{y} + \frac{xy^2}{z} + \frac{yz^2}{x}\right) = 2u = R.H.S$$

(Showed)

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Maximum and Minimum

Q:1 Find maximum and minimum value of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

Sol^n: We have,

$$\frac{df}{dx} = 3x^2 - 3$$

$$\frac{df}{dy} = 3y^2 - 12$$

$$r = \frac{d^2f}{dx^2} = \frac{df}{dx} \cdot \frac{d}{dx}$$

$$= \frac{d}{dx}(3x^2 - 3)$$

$$= 6x$$

$$t = \frac{d^2f}{dy^2} = \frac{df}{dy} \cdot \frac{d}{dy} = \frac{d}{dy}(3y^2 - 12) = 6y$$

$$s = \frac{d^2f}{dx\,dy} = \frac{df}{dx} \cdot \frac{d}{dy} = \frac{d}{dy}(6x) = 0$$

For critical points

$$\frac{df}{dx} = 0$$

$$\Rightarrow 3x^2 - 3 = 0$$

$$\Rightarrow 3x^2 = 3$$

$$\therefore x = \pm 1$$

$$\frac{df}{dy} = 0$$

$$\Rightarrow 3y^2 - 12 = 0$$

$$\Rightarrow 3y^2 = 12$$

$$\Rightarrow y = \pm 2$$

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We have critical points $(1,2)$, $(1,-2)$, $(-1,2)$, $(-1,-2)$

At $(1,2)$; $r = 6x = 6\times1 = 6>0$; $t = 6y = 6\cdot2 = 12>0$; $rt-s^2 = 6\times12-0 = 72>0$

$\therefore f(x,y)$ has ~~max~~ minimum at $(1,2)$

$\therefore f(1,2) = 1+8-3-24+20 = 2$

At $(1,-2)$, $r = 6x = 6\times1 = 6>0$; $t = 6y = 6\times(-2) = -12<0$; $rt-s^2 = 6\times-12-0 = -72<0$

$\therefore f(x,y)$ has no minimum maximum value at $(1,-2)$

Hence, $(1,-2)$ is saddle point.

At $(-1,-2)$ $r = 6x = -6<0$; $t = 6y = 6\times-2 = -12<0$; $rt-s^2 = rt-s^2 = -6\times12-0 = 72>0$

$\therefore f(x,y)$ has maximum ~~or minimum~~ at $(-1,-2)$

Hence $f(-1,2) = -1-8+3+24+20 = 38$

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<u>At $(-1,2)$</u>

$r = 6x$
$= 6(-1)$
$= -6 < 0$

$t = 6y$
$\Rightarrow t = 6 \times 2$
$= 12 > 0$

$rt - s^2$
$= -6 \times 12 - 0$
$= -72 < 0$

$f(x,y)$ has no maximum or minimum value at $(-1,2)$. So, $(-1,2)$ is saddle point.

<u>H.W</u> (i) $f(x,y) = x^2 - xy + y^2 + 3x - 2y + 1$

<u>Solⁿ:</u> we have,

$$\frac{df}{dx} = 2x - y + 3 \qquad \frac{df}{dy} = -x + 2y - 2$$

$$r = \frac{d^2f}{dx^2} = \frac{df}{dx} \cdot \frac{d}{dx}$$
$$= \frac{d}{dx}(2x - y + 3)$$
$$= 2$$

$$t = \frac{d^2f}{dy^2}$$
$$= \frac{d}{dy} \cdot \frac{df}{dy}$$
$$= \frac{d}{dy}(-x + 2y - 2)$$
$$= 2$$

$$s = \frac{d^2f}{dx\,dy}$$
$$= \frac{df}{dx} \cdot \frac{d}{dy}$$
$$\cancel{= \frac{df}{dx} \cdot (2)}$$
$$= \frac{d}{dy}(2x - y + 3)$$
$$= -1$$

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For critical point,

$$\frac{df}{dx} = 0 \qquad \frac{df}{dy} = 0$$

$$\Rightarrow 2x - y + 3 = 0 \qquad \Rightarrow -x + 2y - 2 = 0$$

Thus we have critical points $(4/3, -1/3)$

At $(4/3, -1/3)$

$r = 2 > 0$ $\qquad t = 2 > 0$ $\qquad rt - s^2$

$= 4 - (-1)^2$

$= 3 > 0$

$f(x,y)$ has minimum at $(4/3, -1/3)$

$$f(4/3, -1/3) = (4/3)^2 - 4/3 \cdot (-1/3) + (-1/3)^2 + 3(-1/3) - 2(1/3) + 1$$

$$= 3$$

(ii) $f(x,y) = x^2 - 3xy + y^2 + 13x - 12y + 13$

$$\frac{df}{dx} = 2x - 3y + 13 \qquad \frac{df}{dy} = -3y + 2y - 12$$

$$r = \frac{d^2f}{dx^2} = \frac{df}{dx} \cdot \frac{d}{dx} \qquad t = \frac{d^2f}{dy^2} = \frac{d}{dy}\left(\frac{df}{dy}\right)$$

$$= \frac{d}{dx}(2x - 3y + 13) \qquad = \frac{d}{dy}(-3y + 2y - 12)$$

$$= 2 \qquad = -3 + 2$$

$$= -1$$

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$$s = \frac{d^2f}{dxdy} = \frac{df}{dx} \cdot \frac{d}{dy} = \frac{d}{dy}(2x - 3y + 13)$$

$$= -3$$

for critical point,

$$\frac{df}{dx} = 0 \qquad \frac{df}{dy} = 0$$

$$\Rightarrow 2x - 3y + 13 = 0 \qquad \Rightarrow -3y + 2y - 12 = 0$$

Thus, we have $(2, -3)$

At, $(2, -3)$

$$r = x, \quad t = -1 < 0 \qquad rt - s^2$$

$$= 2 > 0 \qquad = 2 \times (-1) - (-3)^2$$

$$= -2 + 9$$

$$= 7 > 0$$

$f(x,y)$ has no maximum or minimum value at $(2, -3)$.

Hence, $(2, -3)$ has a saddle point.

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Lagrange

Ex:01 Use lagrange multiplies to find three real numbers whose sum is 12 and the sum of whose square is minimum.

Sol^n: Suppose,

The three real number x, y, z

$$x+y+z=12$$

$$\Rightarrow g(x,y,z) = x+y+z-12 \quad \text{--- (i)}$$

We have to minimize,

$$f(x,y,z) = x^2+y^2+z^2$$

For extremum,

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow \left(\hat{i}\frac{df}{dx} + \hat{j}\frac{df}{dy} + \hat{k}\frac{df}{dz}\right) = \lambda\left[\hat{i}\frac{dg}{dx} + \hat{j}\frac{dg}{dy} + \hat{k}\frac{dg}{dz}\right]$$

$$\Rightarrow 2x\hat{i} + 2y\hat{i} + 2z\hat{k} = \lambda\hat{i} + \lambda\hat{j} + \lambda\hat{k}$$

$$\Rightarrow 2x = \lambda \quad 2y = \lambda \quad 2z = \lambda$$

$$\Rightarrow 2x = 2y = 2z$$

$$\Rightarrow x = y = z$$

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From, (i)

$x+x+x=12$

$\Rightarrow 3x=12 \quad \therefore x=4;\ y=4;\ z=4$

<u>Ex:02</u> For a rectangle whose perimeter is 20cm use the lagrangers multiplier method to find the dimensions that will maximize area.

<u>Soln:</u> Find x, y;

maximize, $xy = f(x,y)$

$\Rightarrow 2(x+y) = 20$

$\Rightarrow x+y=10 \quad \Rightarrow x+y-10=0$ —— (i)

We have to maximize,

$f(x,y) = xy$

For extreme,

$\nabla f = \lambda \nabla g$

$\Rightarrow \left(\hat{i}\frac{df}{dx} + \hat{j}\frac{df}{dy}\right) = \lambda\left[\hat{i}\frac{dg}{dx} + \hat{j}\frac{dg}{dy}\right]$

$\Rightarrow y\hat{i} + x\hat{j} = \lambda\hat{i} + \lambda\hat{j}$

$\Rightarrow y\hat{i} = \lambda\hat{i} \quad ; \quad x\hat{j} = \lambda\hat{j} \quad \therefore x=y$

$\Rightarrow y = \lambda \qquad \Rightarrow x = \lambda$

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From (i) $x+x-10=0$

$\Rightarrow 2x = 10$

$\therefore x = 5 \; ; y = 5$

<u>Ex: 03</u> Find the points on the circle $x^2+y^2=80$ which are closest and furthest from the Point $(1,2)$

<u>Soln</u> Suppose,

(x,y) be the points on the circle

$x^2+y^2 = 80$ ——— (i)

$\Rightarrow x^2+y^2-80=0$

$g(x,y) = x^2+y^2-80$ ——— (ii)

Now, the distance between (x,y) and $(1,2)$,

$d = \sqrt{(x-1)^2+(y-2)^2}$

$d^2 = (x-1)^2+(y-2)^2$

We have to maximize or minimize,

$f(x,y) = (x-1)^2+(y-2)^2$ ——— (iii)

For extremum,

$\nabla f = \lambda \nabla g$

$\Rightarrow \left(\hat{i}\frac{df}{dx} + \hat{j}\frac{df}{dy}\right) = \lambda\left[\hat{i}\frac{df}{dx} + \hat{j}\frac{df}{dy}\right]$

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$\Rightarrow 2\hat{i}(x-1) + 2\hat{j}(y-2) = 2x\lambda\hat{i} + 2y\lambda\hat{j}$

$\Rightarrow 2(x-1) = 2x\lambda \;,\; 2(y-2) = 2y\lambda$

$\Rightarrow \lambda = \frac{x-1}{x} \;;\; \lambda = \frac{y-2}{y}$

$\therefore \frac{x-1}{x} = \frac{y-2}{y}$

$\Rightarrow xy - y = xy - 2x \Rightarrow y = 2x$

(i) implies that,

$x^2 + y^2 = 80$

$\Rightarrow x^2 + 4x^2 = 80$

$\Rightarrow 5x^2 = 80$

$\Rightarrow x = \pm 4$

When, $x = 4 \;;\; y = 8 \Rightarrow (x,y) = (4,8)$

$x = -4, y = -8 \Rightarrow (x,y) = (-4,-8)$

at $(x,y) = (4,8)$

From (iii) $\Rightarrow f = (4-1)^2 + (8-2)^2$

$= 45$

At $(x,y) = (-4,-8)$

From (iii) $\Rightarrow f = (-4-1)^2 + (-8-2)^2$

$= 125$ (Ans)

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Multiple Integral

Ex:01 Prove that, $\int_0^{\ln 2}\int_0^1 xye^{xy^2}\,dy\,dx = \frac{1}{2}[1-\ln 2]$

Solⁿ: $L.H.S = \int_0^{\ln 2}\int_0^1 xye^{xy^2}\,dy\,dx$

$= \int_0^{\ln 2} x\left[\int_0^1 ye^{xy^2}\,dy\right]dx$

$= \int_0^{\ln 2} x\left[\frac{1}{2}\int_0^1 e^{xy^2}\,2y\,dy\right]dx$

$= \int_0^{\ln 2} x\left[\frac{1}{2}\int_0^1 e^{xy^2}\,dy^2\right]dx$

$= \frac{1}{2}\int_0^{\ln 2} x\left[\frac{e^{xy^2}}{x}\right]_0^1 dx$

$= \frac{1}{2}\int_0^{\ln 2} x\left[\frac{e^x}{x} - \frac{1}{x}\right]dx$

$= \frac{1}{2}\int_0^{\ln 2} x \times \frac{1}{x}(e^x - 1)\,dx$

$= \frac{1}{2}\left[e^x - x\right]_0^{\ln 2}$

$= \frac{1}{2}\left[(e^{\ln 2} - \ln 2) - (e^0 - 0)\right]$

$= \frac{1}{2}\left[1 - \ln 2\right]$

$= R.H.S$ (Showed)

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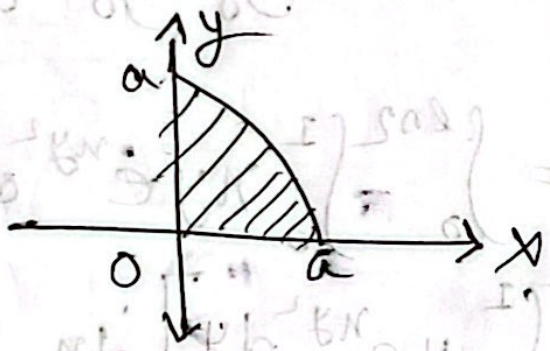
Ex: 03 $\iint_R xy\, dy\, dx$, where R is the quadrant of the circle $x^2+y^2=a^2$, where $x \geq 0$ and $y \geq 0$

$\int x^n dx = \frac{x^{n+1}}{n+1}$

Soln:



From the figure the region R, x varies from $x=0$ to $x=a$ and y varies from $y=0$ to $y=\sqrt{a^2-x^2}$.

Hence, we can write,

$$\iint_R xy\, dy\, dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx$$

$$= \int_0^a x \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{x}{2}\left[a^2-x^2\right] dx$$

$$= \frac{1}{2}\int_0^a (a^2x - x^3)\, dx$$

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$$= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\left(\frac{a^4}{2} - \frac{a^4}{4}\right) - (0-0)\right]$$

$$= 1/2 \times \frac{2a^4 - a^4}{4}$$

$$= 1/2 \times \frac{a^4}{4}$$

$$= \frac{a^4}{8} \quad \underline{\text{Ans:}}$$

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Jacobian

Let, $u = u(x, y, z)$

$v = v(x, y, z)$

$w = w(x, y, z)$

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)}$$

$$= \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

$\therefore du\, dv\, dw = |J|\, dx\, dy\, dz$

<u>Ex: 01</u> By transforming to polar co-ordinate show that,

$$\int_0^\infty \int_0^\infty dx\, dy = \int_0^\infty \int_0^{\pi/2} r\, d\theta\, dr$$

<u>Soln:</u> We know,

In polar co-ordinate system,

$x(r, \theta) = x = r\cos\theta$

$y(r, \theta) = y = r\sin\theta$

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$$dx\,dy = |J|\,dr\,d\theta \quad \text{——— (1)}$$

$$J = \frac{d(x,y)}{d(r,\theta)} = \begin{vmatrix} \partial x/\partial\theta & \partial x/\partial r \\ \partial y/\partial\theta & \partial y/\partial r \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(1) = r$$

Implicit (i) we get,

$$dx\,dy = |J|\,dr\,d\theta$$

$$\Rightarrow dx\,dy = r\,dr\,d\theta \quad \text{——— (ii)}$$

$$\Rightarrow \iint dx\,dy = \iint r\,dr\,d\theta \quad \text{——— (iii)}$$

When,

$x = 0$; $r\cos\theta = 0 \Rightarrow r = 0$ or $\cos\theta = 0$, ~~$r=0$~~

$r = 0$ or $\theta = \pi/2$

$x = \infty$; $r\cos\theta = \infty \Rightarrow r = \infty$ $[\cos\theta \neq \infty]$

When,

$x = 0$; $r\sin\theta = 0 \Rightarrow r = 0$ or $\theta = 0°$

$x = \infty$; $r\sin\theta = \infty \Rightarrow r = \infty$ $[\sin\theta \neq \infty]$

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Implies in (iii) $\Rightarrow$

$$\int_0^\infty \int_0^\infty dx\,dy = \int_0^{\pi/2} \int_0^\infty r\,dr\,d\theta$$

$$\therefore \int_0^\infty \int_0^\infty dx\,dy = \int_0^\infty \int_0^{\pi/2} r\,d\theta\,dr$$ [Showed]

* **Ex: 02** Transform $\iiint dx\,dy\,dz$ into spherical polar co-ordinate system.

Sol^n: We know,

$$x = r\cos\phi \sin\theta = x(r, \theta, \phi)$$

$$y = r\sin\phi \sin\theta = y(r, \theta, \phi)$$

$$z = r\cos\theta = z(r, \theta, \phi)$$

We know,

$$J = \frac{d(x, y, z)}{d(r, \theta, \phi)}$$

$$= \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta & \partial x/\partial \phi \\ \partial y/\partial r & \partial y/\partial \theta & \partial y/\partial \phi \\ \partial z/\partial r & \partial z/\partial \theta & \partial z/\partial \phi \end{vmatrix}$$

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$$= \begin{vmatrix} \cos\phi \sin\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\phi & r\cos\phi\sin\theta \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\phi\sin\theta\,(0 + r^2\sin^2\theta\cos\phi) -$$
$$r\cos\phi\cos\theta\,(0 - r\cos\theta\cos\phi\sin\theta) +$$
$$(-r\sin\phi\sin\theta)(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi)$$

$$= r^2\left[\sin^2\theta\,(\cos^2\theta + \sin^2\theta) + \sin\theta\cos^2\theta \cdot (\cos^2\theta + \sin^2\theta)\right]$$

$$= r^2\sin\theta$$

So, $dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$

$$= r^2\sin\theta\,dr\,d\theta\,d\phi$$

Hence,

$$\iiint dx\,dy\,dz = \iiint r^2\sin\theta\,dr\,d\theta\,d\phi$$

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H.W

Ex:03 By using transformation $x+y=u$, $y=uv$ then show that,

$$\int_0^1\int_0^{1-x} e^{\frac{y}{y+x}}\,dy\,dz = \frac{1}{2}(e-1)$$

Soln: Given, $x+y=u$

$\Rightarrow x = u-y$

$\Rightarrow x(u,v) = u - uv = u(1-v)$

$y(u,v) = y = uv$

the Jacobian law,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$= u - uv + uv$

$= u$

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$dx\,dy = |J|\,du\,dv$

$\Rightarrow dx\,dy = u\,du\,dv$

$\Rightarrow \int_0^1 \int_0^{1-x} e^{y/y+x}\,dy\,dx =$

$\iint_R u\,e^{uv/v}\,dv\,du$

$= \iint_R u\,e^{v}\,dv\,du$

When $x=0$ then, $0 = u(1-v)$ then $u=0$, $v=1$

$y=0$ then, $u=0$, $v=0$

$y=1-x$ then, $u=1$, $v=1$

$x=1$ then $u=1$, $v=1$

$\therefore \int_0^1 \int_0^{1-x} e^{y/y+x}\,dy\,dx$

$= \int_0^1 \int_0^1 u\,e^{v}\,dv\,du$

$\therefore \int_0^1 \int_0^1 u\,e^{v}\,dv\,du$

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L.H.S =

$$\int_0^1 u\,[e^u]_0^1\,du$$

$$= \int_0^1 u\,[e^1 - e^0]\,du$$

$$= \int_0^1 (e^1 - 1)\,u\,du$$

$$= (e-1)\left[\frac{u^2}{2}\right]_0^1 = (e-1)\left[\frac{1}{2} - 0\right]$$

$$= \frac{e-1}{2}$$

= R.H.S [Showed]

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