Partial Derivatives

Definition:- Let $f(x,y)$ be the functions of two variables x and y. Then the partial derivative of f with respect to x, defined as its ordinary differential coefficients with respect to x by treating the other variable y as constant. It is denoted by $\frac{\partial f}{\partial x}$ or f_x.

$$\therefore \frac{\partial f}{\partial x} \text{ or } f_x = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h};$$

provided that the limit exists.

Similarly, let $f(x,y)$ be the functions of two variables x and y. Then the partial derivative of f with respect to y defined as its ordinary coefficients with respect to y by treating the other variable x as constant. It is denoted by $\frac{\partial f}{\partial y}$ or f_y

$$\therefore \frac{\partial f}{\partial y} \text{ or } f_y = \lim_{K \to 0} \frac{f(x, y+K) - f(x,y)}{K}$$

provided the limit exists.

Example-01:-

~~If~~ ~~$f(x,y)$~~

If $f(x,y) = x^2 + xy + y^2$, then using analytical definition find $f_x(-1,1)$, $f_y(2,5)$ and $f_{xy}(-1,1)$

Solution:-

Given that,

$$f(x,y) = x^2 + xy + y^2 \quad \text{——} \quad (1)$$

Then by analytical definition of partial derivatives.

$$\frac{\partial f}{\partial x} \text{ or } f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\{(x+h)^2 + (x+h)y + y^2\} - \{x^2 + xy + y^2\}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2hx + h^2 + \cancel{xy} + hy + \cancel{y^2} - \cancel{x^2} - \cancel{xy} - \cancel{y^2}}{h}$$

$$= \lim_{h\to 0} \frac{2hx + h^2 + hy}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x + h + y)}{\cancel{h}}$$

$$= \lim_{h\to 0} 2x + h + y$$

$$= 2x + 0 + y$$

$$= 2x + y$$

$\therefore f_x(x,y) = 2x+y$ ——— (ii)

Now, $f_x(-1,1) = 2(-1)+1$

$= -2+1$

$= -1$

$$\frac{\partial f}{\partial y} \text{ or } f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

$$= \lim_{k\to 0} \frac{\{x^2+x(y+k)+(y+k)^2\} - (x^2+xy+y^2)}{k}$$

$$= \lim_{k\to 0} \frac{\cancel{x^2}+\cancel{xy}+kx+\cancel{y^2}+2yk+k^2-\cancel{x^2}-\cancel{xy}-\cancel{y^2}}{k}$$

$$= \lim_{k\to 0} \frac{kx+2yk+k^2}{k}$$

$$= \lim_{k\to 0} \frac{\cancel{k}(x+2y+k)}{\cancel{k}}$$

$$= \lim_{k\to 0} x+2y+k$$

$\therefore f_y(x,y) = x+2y$ ——— (iii)

$f_y(2,5) = 2+2\times 5$

$= 2+10$

$= 12$

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again,

$$\frac{\partial f_x}{\partial y} \text{ or } f_{xy}(x,y) = \lim_{k\to 0} \frac{f_x(x, y+k) - f_x(x,y)}{k}$$

$$= \lim_{k\to 0} \frac{(2x+y+k) - (2x+y)}{k} \quad [\because f_x = 2x+y]$$

$$= \lim_{k\to 0} \frac{\cancel{2x}+\cancel{y}+k-\cancel{2x}-\cancel{y}}{k}$$

$$= \cancel{2x+y} \lim_{k\to 0} \frac{\cancel{k}}{\cancel{k}}$$

$$= \lim_{k\to 0} 1$$

$$f_{xy}(x,y) = \underline{1}$$

$$f_{xy}(-1,1) = 1$$

$$f_x(x,y)\ \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y)\ \frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(f_x) = \lim_{k\to 0} \frac{f_x(x,y+k) - f_x(x,y)}{k}$$

$$f_{yx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(f_y) = \lim_{h\to 0} \frac{f_y(x+h,y) - f_y(x,y)}{h}$$

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(f_x) = \lim_{h\to 0} \frac{f_x(x+h,y) - f_x(x,y)}{h}$$

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* If $f(x,y) = x^2 + 3xy + 2y^3$, then $f_{xx}(1,2)$ and $f_{xy}(1,2)$

Given that,

$$f(x,y) = x^2+3xy+2y^3 \quad \text{——} \ (1)$$

Then by analytical definition of partial derivatives

$$\frac{\partial f_x}{\partial x} \text{ or } f_{xx}(x,y) = \lim_{k\to 0} \frac{f_x(x,y+k) - f_x(x,y)}{k}$$

$$= \lim_{k\to 0} \frac{\{x^2+3x(y+k)+2(y+k)^3\} - (x^2+3xy+2y^3}{k}$$

$$= \lim_{k\to 0} x^2+3xy+3kx+2(y^3+3yk+$$

~~$f_{xx} = \frac{\partial f_x}{\partial x}$~~

$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\{(x+h)^2 + 3(x+h)y + 2y^3\} - \{x^2+3xy+2y^3\}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2+\cancel{3xy}+3hy+\cancel{2y^3}-\cancel{x^2}-\cancel{3xy}-\cancel{2y^3}}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2+3hy}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+3y+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 2x+3y+h$$

$$= 2x+3y+0 \quad \text{①}$$

$$= 2x+3y$$

$$\therefore f_{xx} \text{ or } \frac{\partial}{\partial x}(f_x) = \lim_{h\to 0} \frac{f_x(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{2(x+h)+3y-2x-3y}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2x}+2h+\cancel{3y}-\cancel{2x}-\cancel{3y}}{h}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}}$$

$$= \lim_{h\to 0} 2$$

$$\therefore f_{xx}(x,y) = 2$$

$$f_{xx}(1,2) = 2 \quad \text{(Ans)}$$

$\therefore$ ~~f_{xx}~~, ~~$f_{xx}(x,y) = 2x+3y$ ①~~

~~Now,~~

~~$f_{xx}(1,2) = 2\cdot 1 + 3\cdot 2$~~

~~$= 2+6$~~

~~$= 8$~~

$$\frac{\partial f_x}{\partial y} \text{ or } f_{xy}(x,y) = \frac{\partial}{\partial y}(f_x)$$

$$= \lim_{k\to 0} \frac{f_x(x, y+k) - f_x(x,y)}{k}$$

$$\cancel{= \lim_{k\to 0} \frac{2x+3(y+k) - 2x+3y}{k}} \quad \cancel{[f_x =}$$

$$\cancel{= \lim_{k\to 0} x^2+3x(y+k)+2(y+k)^2}$$

$$= \lim_{k\to 0} \frac{2x+3(y+k) - 2x+3y}{k} \quad [f_x = 2x+3y]$$

$$= \lim_{k\to 0} \frac{\cancel{2x}+\cancel{3y}+3k-\cancel{2x}+\cancel{3y}}{k}$$

$$= \lim_{k\to 0} \frac{3\cancel{k}}{\cancel{k}}$$

$$= \lim_{k\to 0} 3$$

$$= 3$$

$\therefore f_{xy}(x,y) = 3$

Now,

$$f_{xy}(1,2) = 3$$

(Ans)

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3-y^3}{x^2+y^2} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{(my)^3-y^3}{(my)^2+y^2}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{m^3y^3-y^3}{m^2y^2+y^2}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{y^3(m^3-1)}{y^2(m^2+1)}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{y(m^3-1)}{(m^2+1)}$$

$$= \frac{0\times(m^3-1)}{(m^2+1)}$$

$$= 0$$

$\therefore \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3-y^3}{x^2+y^2} = 0$; which is unique

Hence,

$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does exist

Now, we have, $f(x,y) = \frac{x^3-y^3}{x^2+y^2}$ for $(x,y)\neq(0,0)$

$f(x,y) = 0$ for $(x,y)=(0,0)$

$f(0,0) = 0$

Thus,

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = f(0,0) = 0$$

Thus, The function of $f(x,y)$ is continuous at $(0,0)$

Differentiability of a function of two variables

Definition:

The function $f(x,y)$ is said to be differentiable at the point (a,b) if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k)-f(a,b)-hf_x(a,b)-kf_y(a,b)}{\sqrt{h^2+k^2}}=0$$

Example-01:

Show that the function

$$f(x,y)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2} ; & \text{when } (x,y)\neq(0,0)\\ 0 ; & \text{when } (x,y)=(0,0)\end{cases}$$

is continuous, possess partial derivatives at $(0,0)$ but not differentiable at $(0,0)$

Solution:

For simultaneous limit

Suppose that, $y=mx$ where $m=\pm1,\pm2\ldots$

Then,

$$\lim_{\substack{x\to0\\y\to0}} \frac{x^3-y^3}{x^2+y^2}$$ along $x=my$ is

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$x^2+y^2=0$

$x^2=-y^2$

$x=\sqrt{-y^2}$

$x=iy$

$x=my$

2. Partial Derivative at (0,0):

By analytical definition of partial derivatives,

$$f_x(x,y) = \lim_{h\to0}\frac{f(x+h, y) - f(x,y)}{h}$$

$$f_x(0,0) = \lim_{h\to0}\frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h\to0}\frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to0}\frac{1}{h}\left[\frac{h^3-0^3}{h^2+0^2} - 0\right]$$

$$= \lim_{h\to0}\frac{1}{\cancel{h}}\left[\frac{\cancel{h^3}}{\cancel{h^2}}\right]$$

$$= \lim_{h\to0} 1$$

$\therefore f_x(0,0) = 1$

again,

$$f_y(x,y) = \lim_{k\to0}\frac{f(x, y+k) - f(x,y)}{k}$$

$$f_y(0,0) = \lim_{k\to0}\frac{f(0, 0+k) - f(0,0)}{k}$$

$$= \lim_{k\to0}\frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{1}{k}\left[\frac{0^3 - k^3}{0^2+k^2} - 0\right]$$

$$= \lim_{k \to 0} \frac{1}{k}\left[\frac{-k^{\cancel{3}}\,k}{k^2}\right]$$

$$= \lim_{k \to 0} -1$$

$\therefore f_y(0,0) = -1$

3. Differentiability at (0,0) :-

By analytical definition of differentiability

$$L.H.L = \lim_{(h,k)\to(0,0)} \frac{f(a+h,\, b+k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}}\left[f(0+h,\, 0+k) - f(0,0) - h\cdot \underbrace{1}_{f_x(0,0)} - k\underbrace{(-1)}_{f_y(0,0)}\right] \quad [a,b=0]$$

$$= \lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}}\left[f(h,k) - \underbrace{f(0,0)}_{0} - h\cdot 1 - k\cdot(-1)\right]$$

$$= \lim_{h,k\to(0,0)} \frac{1}{\sqrt{h^2+k^2}}\left[\frac{h^3-k^3}{h^2+k^2} - 0 - h + k\right]$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{1}{\sqrt{h^2+k^2}}\left[\frac{h^3-k^3 - h(h^2+k^2) + k(h^2+k^2)}{h^2+k^2}\right]$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{1}{\sqrt{h^2+k^2}} \cdot \frac{\cancel{h^3} - \cancel{k^3} - \cancel{h^3} - hk^2 + kh^2 + \cancel{k^3}}{h^2+k^2}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{1}{\sqrt{h^2+k^2}} \cdot \frac{\cancel{kh^2 - hk^2}}{\cancel{h^2+k^2}} \qquad \frac{kh^2 - hk^2}{h^2+k^2}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{1}{\sqrt{h^2+k^2}} \cdot \frac{\cancel{hk(k-h)}}{\cancel{(h^2+k^2)}} \qquad \frac{hk(h-k)}{h^2+k^2}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{hk(h-k)}{(h^2+k^2)^{3/2}} \quad \text{———} \quad (1)$$

Now, let $h = mk$

Then,

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{hk(h-k)}{(h^2+k^2)^{3/2}} = \lim_{\substack{h\to 0\\ k\to 0}} \frac{mk\cdot k(mk-k)}{(m^2k^2+k^2)^{3/2}}$$

Now,

$h^2+k^2 = 0$

$h^2 = -k^2$

$h = \sqrt{-k^2}$

$h = ik$

$h = mk$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{m\cancel{k}^{k^2}\cdot k(m-1)}{\left[k^3(m^2+1)^{3/2}\right]}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{mk^3(m-1)}{k^3(m^2+1)^{3/2}}$$

$$= \frac{m(m-1)}{(m^2+1)^{3/2}}$$

$$\therefore \lim_{\substack{h\to 0\\ k\to 0}} \frac{hk(h-k)}{(h^2+k^2)^{3/2}} = \frac{m(m-1)}{(m^2+1)^{3/2}};$$ which is not unique

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Hence, $\lim_{\substack{h\to 0\\ k\to 0}} \frac{hk(h-k)}{(h^2+k^2)^{3/2}}$ does not exist

Hence, the function $f(x,y)$ is not differentiable at $(0,0)$

H.W

① If the function, $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{\sqrt{x^2+y^2}} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$

Then examine the differentiability of $f(x,y)$ at $(0,0)$

Solution:-

By analytical definition of ~~differentiability~~ Partial derivative,

$$\cancel{L.H.L = \lim_{\substack{h\to 0\\ k\to 0}} f(a+h, b+k) - f(0,0) - hf_x(0,0)}$$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{h.0(h^2-0^2)}{\sqrt{h^2+0^2}} - 0\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{0}{\sqrt{h^2+0^2}}\right]$$

$$= \lim_{h\to 0} \frac{1}{h} \times 0$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x, y+k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f(0, 0+k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f(0.k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{1}{k}\left[\frac{0.k\,(0^2-k^2)}{\sqrt{0^2+k^2}} - 0\right]$$

$$= \lim_{k\to 0} \frac{1}{k}\left[\frac{0}{k}\right]$$

$$= \lim_{k\to 0} 0$$

$$= 0$$

By analytical definition of differentiability,

$$L.H.L = \lim_{\substack{h\to 0\\ k\to 0}} \frac{f(a+h, b+k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{f(0+h,\ 0+k) - f(0,0) - h f_x(0,0) - k f_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{f(h,k) - f(0,0) - h f_x(0,0) - k f_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{1}{\sqrt{h^2+k^2}} \left[\frac{hk(h^2-k^2)}{\sqrt{h^2+k^2}} - 0 - h\times 0 - k\times 0\right]$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{1}{\sqrt{h^2+k^2}} \times \frac{hk(h^2-k^2)}{\sqrt{h^2+k^2}}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{hk(h^2-k^2)}{h^2+k^2} \text{ ——— } (1)$$

let, $k = mh$

Then,

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{hk(h^2-k^2)}{h^2+k^2} = \lim_{\substack{h\to 0\\ k\to 0}} \frac{h.mh(h^2-m^2h^2)}{h^2+m^2h^2}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{mh^2(h^2-m^2h^2)}{h^2+m^2h^2}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{mh^2.\cancel{h^2}(1-m^2)}{\cancel{h^2}(1+m^2)}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{mh^2(1-m^2)}{(1+m^2)}$$

Now,

$h^2+k^2=0$

$\Rightarrow k^2 = -h^2$

$\Rightarrow k = \sqrt{-h^2}$

$\Rightarrow k = ih$

$\Rightarrow k = mh$

$$= \lim_{\substack{h\to 0\\ k\to 0}} = \frac{m\times 0^2(1-m^2)}{(1+m^2)}$$

$$= \frac{0}{1+m^2}$$

$= 0$; which is unique and equal

Hence, $\lim_{\substack{h\to 0\\ k\to 0}} \frac{hk(h^2-k^2)}{\sqrt{h^2+k^2}}$ does exist.

Hence, the function $f(x,y)$ is defferentiable at $(0,0)$

(11) let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} ; & (x,y \neq 0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$

Then show that both $f_x(0,0)$ and $f_y(0,0)$ both exist but $f(x,y)$ is discontinuous at $(0,0)$

Solution:-

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h.0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{h.0}{h^2+0^2} - 0\right]$$

$$= \lim_{h \to 0} \frac{1}{h} \times \frac{0}{h^2}$$

$$= \lim_{h \to 0} 0$$

$$= 0$$

$$f_y(x,y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x,y)}{k}$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0, 0+k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{f(0.k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{1}{k}\left[\frac{0.k}{0^2+k^2} - 0\right]$$

$$= \lim_{k \to 0} \frac{1}{k} \times \frac{0}{k^2}$$

$$= \lim_{k \to 0} 0$$

$$= 0$$

Here, $f_x(0,0)$ and $f_y(0,0)$ both exist

let, $y = mx$

Now,
$x^2+y^2=0$
$\Rightarrow y^2 = -x^2$
$\Rightarrow y = \sqrt{-x^2}$
$\Rightarrow y = ix$
$\Rightarrow y = mx$

Then,

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x.mx}{x^2+m^2x^2}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{mx^2}{x^2(1+m^2)}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{m}{1+m^2}$$

$$= \frac{m}{1+m^2}$$; which is not unique

Since, m has multiples values, hence the limit is not unique.

Hence, $\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2}$ does not exist

As, simultaneous limit of $f(x,y)$ does not exist at $(0,0)$, hence, the function is discontinuous at $(0,0)$

(iii) let $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & ; \quad (x,y) \neq (0,0) \\ 0 & ; \quad (x,y) = (0,0) \end{cases}$

show that. $f_y(x,0) = x$; $f_x(0,y) = -y$ and $f_{xy}(0,0) \neq f_{yx}(0,0)$

Solution:-

$$f_y(x,y) = \lim_{K\to 0} \frac{f(x, y+K) - f(x,y)}{K}$$

$$f_y(x,0) = \lim_{K\to 0} \frac{f_y(x, 0+K) - f_y(x,0)}{K}$$

$$= \lim_{K\to 0} \frac{f_y(x.K) - f_y(x,0)}{K}$$

$$= \lim_{K\to 0} \frac{1}{K} \left[\frac{x.K(x^2-K^2)}{x^2+K^2} - 0 \right]$$

$$= \lim_{K\to 0} \frac{1}{\cancel{K}} \times \frac{\cancel{K}x(x^2-K^2)}{x^2+K^2}$$

$$= \lim_{K\to 0} \frac{x(x^2-K^2)}{x^2+K^2}$$

$$= \frac{x(x^2-0)}{x^2+0}$$

$$= \frac{\cancel{x^3}}{\cancel{x^2}} x$$

$$= x$$

$\therefore f(x,0) = x$ (showed)

$$f_x(x,y) = \lim_{h \to 0} \frac{f_x(x+h, y) - f(x,y)}{h}$$

$$f_x(0,y) = \lim_{h \to 0} \frac{f_x(0+h, y) - f_x(0,y)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \; f_x(h, y) - f_x(0,y)$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{hy(h^2-y^2)}{h^2+y^2} - 0 \right]$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}} \times \frac{\cancel{h}y(h^2-y^2)}{h^2+y^2}$$

$$= \lim_{h \to 0} \frac{y(h^2-y^2)}{h^2+y^2}$$

$$= \frac{y(0^2-y^2)}{0^2+y^2}$$

$$= \frac{-y^3}{y^2} \quad -y$$

$$= -y$$

$\therefore f_x(0,y) = -y$ (showed)

∴ L.H.L

$$f_{xy}(x,y) = \frac{\partial}{\partial y}(f_x)$$

$$f_x(x,y) = \lim_{k\to 0} \frac{f_x(x,y+k) - f_x(x,y)}{k}$$

$$f_x(0,0) = \lim_{k\to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{1}{k}\left[\frac{0.k(0^2-k^2)}{0^2+k^2} - 0\right] = \lim_{k\to 0} \frac{-k-0}{k} \quad (f_x = -k)$$

$$= \lim_{k\to 0} \frac{-k}{k}$$

$$= \lim_{k\to 0} -1$$

$$= -1$$

$$= \lim_{k\to 0} \frac{1}{k} \times \frac{0}{k^2}$$

$$= \lim_{k\to 0} 0$$

$$= 0$$

∴ R.H.L

$$f_{yx}(x,y) = \frac{\partial}{\partial x}(f_y)$$

$$f_{yx}(x,y) = \lim_{h\to 0} \frac{f_y(x+h,y) - f_y(x,y)}{h}$$

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{1}{h}\, f_y(0+h,0) - f_y(0,0)$$

$$= \lim_{h\to 0} \frac{1}{h}\, f_y(h,0) - f_y(0,0)$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{h.0(h^2-0^2)}{h^2+0^2} - 0\right] = \lim_{h\to 0} \frac{1}{h} h - 0 \quad (f_y = h)$$

$$= \lim_{h\to 0} \frac{1}{h} \times \frac{0}{h^2} \qquad = \lim_{h\to 0} \frac{h}{h} 1$$

$$= \lim_{h\to 0} 0 \qquad = \lim_{h\to 0} 1$$

$$= 1$$

$$= 0$$

$\therefore$ ~~$f_{xy}(0,0) = f_{yx}(0,0)$~~

$\therefore f_{xy}(0,0) \neq f_{yx}(0,0)$

Chain rule

Suppose that, $z = f\left(u(x,y),\ v(x,y)\right)$

Then,

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

Example:-

Suppose that $z = f(u,v)$, where, $u = e^x\cos y$, $v = e^x\sin y$

Show that (i) $\frac{\partial z}{\partial x} = u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v}$

(ii) $\frac{\partial z}{\partial y} = -v\frac{\partial f}{\partial u} + u\frac{\partial f}{\partial v}$

Solve:-

(i) $$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial f}{\partial u}\cdot\frac{\partial}{\partial x}(e^x\cos y) + \frac{\partial f}{\partial v}\cdot\frac{\partial}{\partial x}(e^x\sin y)$$

$$= e^x\cos y\cdot\frac{\partial f}{\partial u} + e^x\sin y\ \frac{\partial f}{\partial v}$$

$$= u\cdot\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v}$$

(ii) $\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$

$= \frac{\partial f}{\partial u} \frac{\partial}{\partial y} (e^x \cos y) + \frac{\partial f}{\partial v} \frac{\partial}{\partial y} (e^x \sin y)$

$= -e^x \sin y \frac{\partial f}{\partial u} + e^x \cos y \frac{\partial f}{\partial v}$

$= -v \frac{\partial f}{\partial u} + u \frac{\partial f}{\partial v}$

Example-02:-

Suppose that, $u = f(x-y, y-z, z-x)$

show that, $U_x + U_y + U_z = 0$ (i)

Solution:-

Let $p = x-y$, $q = y-z$, $r = z-x$

$\therefore u = f(p(x,y), q(y,z), r(z,x))$

$U_x = \frac{\partial u}{\partial x} = \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x}$

$= \frac{\partial f}{\partial p} \cdot \frac{\partial}{\partial x} (x-y) + \frac{\partial f}{\partial r} \cdot \frac{\partial}{\partial x} (z-x)$

$= \frac{\partial f}{\partial p} \cdot - \frac{\partial f}{\partial r}$

$$U_y = \frac{\partial u}{\partial y} = \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial y}$$

$$= \frac{\partial f}{\partial p} \cdot \frac{\partial}{\partial y}(x-y) + \frac{\partial f}{\partial q} \cdot \frac{\partial}{\partial y}(y-z)$$

$$= -\frac{\partial f}{\partial p} + \frac{\partial f}{\partial q}$$

$$U_z = \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial z}$$

$$= \frac{\partial f}{\partial q} \cdot \frac{\partial}{\partial z}(y-z) + \frac{\partial f}{\partial r} \cdot \frac{\partial}{\partial z}(z-x)$$

$$= -\frac{\partial f}{\partial q} + \frac{\partial f}{\partial r}$$

$$\therefore L.H.S = U_x + U_y + U_z$$

$$= \cancel{\frac{\partial f}{\partial p}} - \cancel{\frac{\partial f}{\partial r}} - \cancel{\frac{\partial f}{\partial p}} + \cancel{\frac{\partial f}{\partial q}} - \cancel{\frac{\partial f}{\partial q}} + \cancel{\frac{\partial f}{\partial r}}$$

$$= 0$$

$$= R.H.S$$

(showed)

Homogeneous function:-

$f(x,y) = 3x^5 + 4x^4y + 9x^3y^2 + y^5 \rightarrow$ Homogeneous function of degree $5 = n$

$f(x,y) = 3x^4 + 9x^2y + y^4 + 9 \rightarrow$ non-homogeneous

Euler's theorem:-

Q. If $u = u(x,y)$ is a homogeneous function of degree n

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

$xU_x + yU_y = nu$

If $u = U(x,y,z)$

$xU_x + yU_y + zU_z = nU$

* Verify Euler's theorem for $u = \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}$

Solve:-

we have,

$$u = x^2y^{-1} + y^2z^{-1} + z^2x^{-1}$$

Here, u is a homogeneous function of x,y,z of degree $n = 1$

we have to show that,

$$xU_x + yU_y + zU_z = u$$

$\therefore$ L.H.S $= xU_x + yU_y + zU_z$

$$= x\frac{\partial}{\partial x}\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)+y\frac{\partial}{\partial y}\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)+ z\frac{\partial}{\partial z}\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)$$

$$= x\left(\frac{2x}{y} - \frac{z^2}{x^2}\right)+y\left(-\frac{x^2}{y^2}+\frac{2y}{z}\right)+z\left(-\frac{y^2}{z^2}+\frac{2z}{x}\right)$$

$$= \frac{2x^2}{y} - \frac{z^2}{x} - \frac{x^2}{y} + \frac{2y^2}{z} - \frac{y^2}{z} + \frac{2z^2}{x}$$

$$= \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}$$

$= u$

$=$ R.H.S (showed)

HW on chain Rule:-

① If $w = f\left(\frac{y-x}{xy}, \frac{z-y}{yz}\right)$ then shown that

$$x^2\frac{\partial w}{\partial x} + y^2\frac{\partial w}{\partial y} + z^2\frac{\partial w}{\partial z} = 0$$

Solution:-

let $p = \frac{y-x}{xy}$, $q = \frac{z-y}{yz}$

$\therefore w = f\left(p\left(\frac{y-x}{xy}\right), q\left(\frac{z-y}{yz}\right)\right)$

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial x}$$

$$= \frac{\partial f}{\partial p} \cdot \frac{\partial}{\partial x}\left(\frac{y-x}{xy}\right)$$

$$= \cancel{\frac{\partial f}{\partial p}} - \frac{1}{x^2} \cdot \frac{\partial f}{\partial p}$$

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial y}$$

$$= \frac{\partial f}{\partial p} \frac{\partial}{\partial y}\left(\frac{y-x}{xy}\right) + \frac{\partial f}{\partial q} \cdot \frac{\partial}{\partial y}\left(\frac{z-y}{yz}\right)$$

$$= \cancel{\frac{\partial f}{\partial p} +}$$

$$= \frac{1}{y^2} \frac{\partial f}{\partial p} - \frac{1}{y^2} \frac{\partial f}{\partial q}$$

$$\frac{\partial w}{\partial z} = \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial z}$$

$$= \frac{\partial f}{\partial q} \cdot \frac{\partial}{\partial z}\left(\frac{z-y}{yz}\right)$$

$$= \frac{1}{z^2} \frac{\partial f}{\partial q}$$

$\therefore$ L.H.S $= x^2 \frac{\partial w}{\partial x} + y^2 \frac{\partial w}{\partial y} + z^2 \frac{\partial w}{\partial z}$

$= -\cancel{x^2} \cdot \frac{1}{\cancel{x^2}} \frac{\partial f}{\partial p} + \cancel{y^2} \frac{1}{\cancel{y^2}} \frac{\partial f}{\partial p} - \cancel{y^2} \cdot \frac{1}{\cancel{y^2}} \frac{\partial f}{\partial q} + \cancel{z^2} \cdot \frac{1}{\cancel{z^2}} \frac{\partial f}{\partial q}$

$= -\frac{\partial f}{\partial p} + \frac{\partial f}{\partial p} - \frac{\partial f}{\partial q} + \frac{\partial f}{\partial q}$

$= 0$

$=$ R.H.S (showed)

② If $u = f(x^2 + 2yz,\ y^2 + 2zx)$, then shown that ~~$(y^2 - zx)\frac{\partial u}{\partial x}$~~

$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$

Solution:-

Let $p = x^2 + 2yz$, $q = y^2 + 2zx$

$\therefore u = f\left(p(x^2 + 2yz),\ q(y^2 + 2zx)\right)$

$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial x}$

$= \frac{\partial f}{\partial p} \cdot \frac{\partial}{\partial x}(x^2 + 2yz) + \frac{\partial f}{\partial q} \cdot \frac{\partial}{\partial x}(y^2 + 2zx)$

$= 2x \frac{\partial f}{\partial p} + 2z \frac{\partial f}{\partial q}$

$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial y}$

$= \frac{\partial f}{\partial p} \cdot \frac{\partial}{\partial y}(x^2 + 2yz) + \frac{\partial f}{\partial q} \cdot \frac{\partial}{\partial y}(y^2 + 2zx)$

$$= 2z\frac{\partial f}{\partial p} + 2y\frac{\partial f}{\partial q}$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial z}$$

$$= \frac{\partial f}{\partial p}\cdot\frac{\partial}{\partial z}(x^2+2yz) + \frac{\partial f}{\partial q}\cdot\frac{\partial}{\partial z}(y^2+2zx)$$

$$= 2y\frac{\partial f}{\partial p} + 2x\frac{\partial f}{\partial q}$$

$\therefore$ L.H.S $= (y^2-zx)\frac{\partial u}{\partial x} + (x^2-yz)\frac{\partial u}{\partial y} + (z^2-xy)\frac{\partial u}{\partial z}$

$$= (y^2-zx)\cdot\left(2x\frac{\partial f}{\partial p} + 2z\frac{\partial f}{\partial q}\right) + (x^2-yz)\left(2z\frac{\partial f}{\partial p} + 2y\frac{\partial f}{\partial q}\right) + (z^2-xy)\cdot\left(2y\frac{\partial f}{\partial p} + 2x\frac{\partial f}{\partial q}\right)$$

$$= \cancel{2xy^2\frac{\partial f}{\partial p}} - \cancel{2x^2z\frac{\partial f}{\partial p}} + 2zy^2\frac{\partial f}{\partial q} - 2xz^2\frac{\partial f}{\partial q} + \cancel{2zx^2\frac{\partial f}{\partial p}}$$
$$- \cancel{2yz^2\frac{\partial f}{\partial p}} + 2yx^2\frac{\partial f}{\partial q} - \cancel{2zy^2\frac{\partial f}{\partial q}} + \cancel{2yz^2\frac{\partial f}{\partial p}} - \cancel{2xy^2\frac{\partial f}{\partial p}}$$
$$+ \cancel{2xz^2\frac{\partial f}{\partial q}} - 2yx^2\frac{\partial f}{\partial q}$$

$= 0$

$=$ R.H.S

(showed)

③ If $z = z(x, y)$ and $x = e^u + e^{-v}$; $y = e^{-u} - e^v$, then show that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Solution:-

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}\cdot\frac{\partial}{\partial u}\left(e^u + e^{-v}\right) + \frac{\partial z}{\partial y}\cdot\frac{\partial}{\partial u}\left(e^{-u} - e^v\right)$$

$$= \frac{\partial z}{\partial x}\left(e^u\right) + \frac{\partial}{\partial u}\left(-e^{-u}\right)$$

$$= e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial}{\partial u}$$

$$\therefore \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}\cdot\frac{\partial}{\partial v}\left(e^u + e^{-v}\right) + \frac{\partial z}{\partial y}\cdot\frac{\partial}{\partial v}\left(e^{-u} - e^v\right)$$

$$= \frac{\partial z}{\partial x}\left(-e^{-v}\right) + \frac{\partial z}{\partial y}\left(-e^v\right)$$

$$= -e^{-v}\frac{\partial z}{\partial x} - e^v\frac{\partial z}{\partial y}$$

$$L.H.S = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$= e^u \frac{dz}{dx} - e^{-u} \frac{\partial}{\partial u} + e^{-v} \frac{\partial z}{\partial x} - e^{v} \frac{\partial z}{\partial y}$$

$$= e^u \frac{dz}{dx} + e^{-v} \frac{dz}{dx} - e^{-u} \frac{dz}{dy} + e^{v} \frac{dz}{dy}$$

$$= \left(e^u \frac{dz}{dx} + e^{-v} \frac{dz}{dx}\right) - \left(e^{-u} \frac{dz}{dy} - e^{v} \frac{dz}{dy}\right)$$

$$= \left(e^u + e^{-v}\right) \frac{dz}{dx} - \left(e^{-u} - e^{v}\right) \frac{dz}{dy}$$

$$= x \frac{dz}{dx} - y \frac{dz}{dy}$$

$= R.H.S$

$\therefore$ L.H.S = R.H.S (showed)

H.W on Homogeneous function:-

① Verify Euler's theorem for $u(x,y,z) = x^3+y^3+z^3$

Solution:-

Here, u is a homogeneous function of x, y, z of degree = 3

we have to show that,

$$xU_x + yU_y + zU_z = 3u$$

$\therefore$ L.H.S $= xU_x + yU_y + zU_z$

$$= x\frac{\partial}{\partial x}(x^3+y^3+z^3) + y\cdot\frac{\partial}{\partial y}(x^3+y^3+z^3) + z\cdot\frac{\partial}{\partial z}(x^3+y^3+z^3)$$

$$= x\cdot 3x^2 + y\cdot 3y^2 + z\cdot 3z^2$$

$$= 3x^3 + 3y^3 + 3z^3$$

$$= 3(x^3+y^3+z^3)$$

$$= 3u$$

$=$ R.H.S (showed)

② Verify Euler's theorem for $u(x,y,z) = x^2+y^2+z^2$

Solution:-

Here, u is a homogeneous function of x,y,z of degree $=2$

we have to show that,

$$xU_x + yU_y + zU_z = 2u$$

$\therefore L.H.S = xU_x + yU_y + zU_z$

$= x \cdot \frac{\partial}{\partial x}(x^2+y^2+z^2) + y \cdot \frac{\partial}{\partial y}(x^2+y^2+z^2) + z \cdot \frac{\partial}{\partial z}(x^2+y^2+z^2)$

$= x \cdot 2x + y \cdot 2y + z \cdot 2z$

$= 2x^2 + 2y^2 + 2z^2$

$= 2(x^2+y^2+z^2)$

$= 2u$

$= R.H.S$ (showed)

③ Verify Euler's theorem for $u(x,y,z) = \frac{xy^2}{z} + \frac{yz^2}{x} + \frac{zx^2}{y}$

Solution:-

we have,

$$u = xy^2z^{-1} + yz^2x^{-1} + zx^2y^{-1}$$

Here, u is a homogeneous function of x,y,z of degree $=2$

we have to show that,

$$xU_x + yU_y + zU_z = 2u$$

$\therefore$ L.H.S $= xU_x + yU_y + zU_z$

$$= x\frac{\partial}{\partial x}\left(\frac{xy^2}{z} + \frac{yz^2}{x} + \frac{zx^2}{y}\right) + y\frac{\partial}{\partial y}\left(\frac{xy^2}{z} + \frac{yz^2}{x} + \frac{zx^2}{y}\right) + z\frac{\partial}{\partial z}\left(\frac{xy^2}{z} + \frac{yz^2}{x} + \frac{zx^2}{y}\right)$$

$$\cancel{= x\left(2x - \frac{yz^2}{x^2} + 2z\right) + y\left(2y\right.}$$

$$= x\left(\frac{y^2}{z} - \frac{yz^2}{x^2} + \frac{2xz}{y}\right) + y\left(\frac{2yx}{z} + \frac{z^2}{x} - \frac{zx^2}{y^2}\right) + z\left(-\frac{xy^2}{z^2} + \frac{2zy}{x} + \frac{x^2}{y}\right)$$

$$= \frac{xy^2}{z} - \frac{xyz^2}{x^2 x} + \frac{2x^2z}{y} + \frac{2xy^2}{z} + \frac{yz^2}{x} - \frac{x^2yz}{y^2y} - \frac{xy^2z}{z^2z} + \frac{2z^2y}{x} + \frac{zx^2}{y}$$

$$= \cancel{\frac{xy^2}{z}} - \cancel{\frac{yz^2}{x}} + \frac{2x^2z}{y} + \frac{2xy^2}{z} + \cancel{\frac{yz^2}{x}} - \cancel{\frac{zx^2}{y}} - \cancel{\frac{xy^2}{z}} + \frac{2z^2y}{x} + \cancel{\frac{zx^2}{y}}$$

$$= \frac{2x^2z}{y} + \frac{2xy^2}{z} + \frac{2z^2y}{x}$$

$$= 2\left(\frac{x^2z}{y} + \frac{xy^2}{z} + \frac{yz^2}{x}\right)$$

$= 2u$

$=$ R.H.S

(showed)

Maximum and Minimum of a function:-

Suppose that, $f(x,y)$ be the given function

Step-01:- Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

Step-02:- For critical point,

Consider, $\frac{\partial f}{\partial x} = 0$ ——— (i)

$\frac{\partial f}{\partial y} = 0$ ——— (ii)

After solving (i) and (ii), you will find some values of (x,y). Suppose that the point is $(x,y) = (a,b)$

Step-03:-

Now put $(x,y) = (a,b)$ at r, s and t

(i) If $r<0$, $t<0$ and $rt-s^2>0$, then $f(x,y)$ has maximum value at $(x,y)=(a,b)$.
and maximum value is $f(a,b)$

(ii) If $r>0$, $t>0$, and $rt-s^2>0$, then $f(x,y)$ has minimum value at $(x,y) = (a,b)$ and the minimum value is $f(a,b)$

(iii) If any of the conditions violets conditions of (i) and (ii) then $f(x,y)$ has neither maximum nor minimum at (a,b) and we say that (a,b) is a saddle point.

Q:- Find maximum and minimum value of the function
$$f(x,y) = x^3 + y^3 - 3x - 12y + 20$$

Solve:-

We have,

$$\frac{\partial f}{\partial x} = 3x^2 - 3 \qquad \frac{\partial f}{\partial y} = 3y^2 - 12$$

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial x} \cdot \frac{\partial}{\partial x}$$

$$= \frac{\partial}{\partial x}(3x^2 - 3)$$

$$= 6x$$

$$\cancel{s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f}{\partial x} \cdot \frac{\partial}{\partial y}}$$

$$\cancel{= \frac{\partial}{\partial y}(3y^2 - 12)}$$

$$\cancel{= 6y}$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial y} \cdot \frac{\partial}{\partial y}$$

$$= \frac{\partial}{\partial y}(3y^2 - 12)$$

$$= 6y$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f}{\partial x} \cdot \frac{\partial}{\partial y}$$

$$= \frac{\partial}{\partial y}(6x)$$

$$= 0$$

For critical point,

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

$\Rightarrow 3x^2 - 3 = 0$

$\Rightarrow 3y^2 - 12 = 0$

$\Rightarrow 3x^2 = 3$

$\Rightarrow 3y^2 = 12$

$\Rightarrow x = \pm 1$

$\Rightarrow y = \pm 2$

Thus, we have critical points $(1,2)$, $(1,-2)$, $(-1,2)$, $(-1,-2)$

At $(1,2)$

$r = 6x$, $t = 6y$, $rt - s^2 = 6 \times 12 - 0$

$= 6 \times 1$ $\qquad = 6 \times 2$ $\qquad = 72 > 0$

$= 6 > 0$ $\qquad = 12 > 0$

$\therefore f(x,y)$ has minimum at $(1,2)$

$\therefore$ Minimum value $f(1,2) = 1 + 8 - 3 - 24 + 20$

$= 2$

At $(1,2)$

$r = 6x$ $\qquad t = 6y$ $\qquad rt - s^2 = 6 \times -12 - 0$

$= 6 \times 1$ $\qquad = 6 \times -2$ $\qquad = -72 < 0$

$= 6 > 0$ $\qquad = -12 < 0$

$\therefore f(x,y)$ has no maximum or minimum value at $(1,-2)$

Hence, $(1,-2)$ is saddle point

At $(-1,2)$

$r = 6x$ $\quad$ $t = 6y$ $\quad$ $rt - s^2 = -6\times 12 - 0$

$= 6\times -1$ $\quad$ $= 6\times 2$ $\quad$ $= -72 < 0$

$= -6 < 0$ $\quad$ $= 12 > 0$

$\therefore f(x,y)$ has no maximum or minimum value at $(-1,2)$

Hence, $(-1,2)$ is saddle point

At $(-1,-2)$

$r = 6x$ $\quad$ $t = 6y$ $\quad$ $rt - s^2 = rt - s^2$

$= -6 < 0$ $\quad$ $= 6\times -12$ $\quad$ $= -6\times -12 - 0$

$= -12 < 0$ $\quad$ $= 72 > 0$

$\therefore f(x,y)$ has maximum at $(-1,-2)$

$\therefore$ Maximum value $f(-1,-2) = -1 - 8 + 3 + 24 + 20$

$= 38$

(Ans)

H.W

① $f(x,y) = x^2 - xy + y^2 + 3x - 2y + 1$

Solution:-

we have,

$$\frac{\partial f}{\partial x} = 2x - y + 3 \qquad \frac{\partial f}{\partial y} = -x + 2y - 2$$

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial x} \cdot \frac{\partial}{\partial x} \qquad t = \frac{\partial^2 f}{\partial y^2}$$

$$= \frac{\partial}{\partial x}(2x - y + 3) \qquad = \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial y}$$

$$= 2 \qquad = \frac{\partial}{\partial y}(-x + 2y - 2)$$

$$t = 2$$

$$s = \frac{\partial^2 f}{\partial x \partial y}$$

$$= \frac{\partial f}{\partial x} \cdot \frac{\partial}{\partial y}$$

$$= \frac{\partial}{\partial y}(2)$$

$$= 0$$

For critical point,

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow 2x - y + 3 = 0 \qquad \Rightarrow -x + 2y - 2 = 0$$

Thus, we have critical points $\left(\frac{4}{3}, -\frac{1}{3}\right)$

At $\left(\frac{4}{3}, -\frac{1}{3}\right)$

$r = 2 > 0$ $\qquad t = 2 > 0$ $\qquad rt - s^2$

$= 2 \times 2 - 0$

$= 4 > 0$

$f(x,y)$ has minimum at $\left(\frac{4}{3}, -\frac{1}{3}\right)$

$\therefore$ minimum value $f\left(\frac{4}{3}, -\frac{1}{3}\right) \cancel{= \left(\frac{4}{3}\right)^2 - \frac{4}{3}\cdot\left(-\frac{1}{3}\right)}$

$$= \left(\frac{4}{3}\right)^2 - \frac{4}{3}\cdot\left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right)^2 + 3\cdot\left(-\frac{1}{3}\right) - 2\cdot\left(-\frac{1}{3}\right) + 1$$

$= 3$

$f(x,y) = x^2 - 3xy + y^2 + 13x - 12y + 13$

$\frac{\partial f}{\partial x} = 2x - 3y + 13$ $\qquad \frac{\partial f}{\partial y} = -3y + 2y - 12$

$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial x} \cdot \frac{\partial}{\partial x}$ $\qquad t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$

$= \frac{\partial}{\partial x}(2x - 3y + 13)$ $\qquad = \frac{\partial}{\partial y}(-3y + 2y - 12)$

$= x$ $\qquad = -3 + 2$

$\qquad = -1$

$s = \frac{\partial^2 f}{\partial x \partial y}$

$= \frac{\partial f}{\partial x} \cdot \frac{\partial}{\partial y}$

$= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$= \frac{\partial}{\partial y}(x) = 0$

For critical point,

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

$\Rightarrow 2x - 3y + 13 = 0 \qquad \Rightarrow -3y + 2y - 12 = 0$

Thus, we have critical points $(2, -3)$

At $(2, -3)$

$r = x \qquad t = -1 < 0 \qquad rt - s^2$

$= 2 > 0 \qquad\qquad = 2 \times (-1) - 0$

$= -2 < 0$

$f(x, y)$ has no maximum or minimum value at $(2, -3)$

Hence, $(2, -3)$ has a saddle point.

Use lagrange multiplaies to find three real numbers whose sum is 12 and the sum of whose squares is minimum.

Solve:-

Suppose that the three real numbers are x, y, z

Given,

$$x+y+z=12$$

$$\Rightarrow x+y+z-12=0 \quad \text{--- (1)}$$

$$\therefore g(x,y,z) = x+y+z-12$$

We have to minimize

$$f(x,y,z) = x^2+y^2+z^2$$

For extremum,

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right) = \lambda\left[\hat{i}\frac{\partial g}{\partial x} + \hat{j}\frac{\partial g}{\partial y} + \hat{k}\frac{\partial g}{\partial z}\right]$$

$$\vec{\nabla} f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$\Rightarrow 2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda\hat{i} + \lambda\hat{j} + \lambda\hat{k}$$

$$\Rightarrow 2x = \lambda; \quad 2y = \lambda; \quad 2z = \lambda$$

$$\Rightarrow 2x = 2y = 2z$$

$$\Rightarrow x = y = z$$

From ①

$$x+x+x=12$$

$\Rightarrow 3x=12$

$\Rightarrow x=4$

$\therefore x=4,\ y=4,\ z=4$

⊞ For a rectangle whose perimeter is 20cm. Use the lagrange's multiplier method to find the ~~dimet~~ dimensions that will maximize area.

Solution:-

Find x, y

maximize,

$$xy = f(x,y)$$

$\Rightarrow 2(x+y)=20$

$\Rightarrow x+y=10$

$\Rightarrow x+y-10=0$ ——— ①

We have to maximize,

$$f(x,y) = xy$$

For extreme,

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y}\right) = \lambda\left[\hat{i}\frac{\partial g}{\partial x} + \hat{j}\frac{\partial g}{\partial y}\right]$$

$$\cancel{\Rightarrow 2x\hat{i} + 2y\hat{j} = \lambda\hat{i} + \lambda\hat{j}}$$

$$\cancel{\Rightarrow 2x = \lambda \; ; \; 2y = \lambda}$$

$$\cancel{\Rightarrow x = y}$$

~~From ①~~

$$\cancel{x + x = 10}$$

$$\cancel{\Rightarrow 2x = 10}$$

$$\cancel{\Rightarrow x}$$

$$\Rightarrow y\hat{i} + x\hat{j} = \lambda\hat{i} + \lambda\hat{j}$$

$$\Rightarrow y\hat{i} = \lambda\hat{i} \; ; \; x\hat{j} = \lambda\hat{j}$$

$$\Rightarrow y = \lambda \; ; \; x = \lambda$$

$$\therefore x = y$$

From ①

$$x + x - 10 = 0$$

$$\Rightarrow x + x = 10$$

$$\Rightarrow 2x = 10$$

$$\Rightarrow x = 5$$

$$\therefore x = 5, \; y = 5$$

Find the points on the circle $x^2+y^2=80$ which are closest and furthest from the point $(1,2)$.

Solution:-

Suppose that, (x,y) be the points on the circle

$$x^2+y^2=80 \text{ ——— (i)}$$

$$x^2+y^2-80=0$$

$$g(x,y) = x^2+y^2-80 \text{ ——— (ii)}$$

Now, the distance between (x,y) and $(1,2)$ is

$$d=\sqrt{(x-1)^2+(y-2)^2}$$

$$d^2 = (x-1)^2+(y-2)^2$$

We have to maximize or minimize

$$f(x,y) = (x-1)^2+(y-2)^2 \text{ ——— (iii)}$$

For extremum,

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow \left(\hat{i}\frac{\partial f}{\partial x}+\hat{j}\frac{\partial f}{\partial y}\right) = \lambda\left[\hat{i}\frac{\partial f}{\partial x}+\hat{j}\frac{\partial f}{\partial y}\right]$$

$$\Rightarrow \hat{i}\,2(x-1) + 2\hat{j}\,(y-2) = 2x\lambda\hat{i} + 2y\lambda\hat{j}$$

$$\Rightarrow 2(x-1) = 2x\lambda \;;\; 2(y-2) = 2y\lambda$$

$$\Rightarrow \lambda = \frac{x-1}{x} \;;\; \lambda = \frac{y-2}{y}$$

$$\therefore \frac{x-1}{x} = \frac{y-2}{y}$$

$\cancel{\Rightarrow y = 2x}$

$$\Rightarrow xy - y = xy - 2x$$

$$\Rightarrow y = 2x$$

(1) implies that,

$$x^2+y^2 = 80$$

$$\Rightarrow x^2+4x^2 = 80$$

$$\Rightarrow 5x^2 = 80$$

$$\Rightarrow x^2 = 16$$

$$\Rightarrow x = \pm 4$$

when,

$x = 4,\; y = 8 \Rightarrow (x,y) = (4,8)$

$x = -4 \quad y = -8 \Rightarrow (x,y) = (-4,-8)$

at $(x,y) = (4,8)$

(3) $\Rightarrow f = (4-1)^2 + (8+12)^2$

$= 45$

at $(x,y) = (-4,-8)$

③ $\Rightarrow f = (-4-1)^2 + (-8-2)^2$

$= 125$

(Ans)

Multiple Integral:-

$$\int_c^d \int_a^b f(x,y)\, dx\, dy$$

prove that, $\int_0^{\ln 2} \int_0^1 xye^{xy^2}\, dy\, dx = \frac{1}{2}[1 - \ln 2]$

Solution:-

$$LHS = \int_0^{\ln 2} \int_0^1 xye^{xy^2}\, dy\, dx$$

$$= \int_0^{\ln 2} x \left[\int_0^1 ye^{xy^2}\, dy \right] dx$$

$$= \int_0^{\ln 2} x \left[\frac{1}{2} \int_0^1 e^{xy^2}\, 2y\, dy \right] dx$$

$$= \int_0^{\ln 2} x \left[\frac{1}{2} \int_0^1 e^{xy^2} \, dy^2 \right] dx$$

$$= \frac{1}{2} \int_0^{\ln 2} x \left[\frac{e^{xy^2}}{x} \right]_0^1 dx$$

$$= \frac{1}{2} \int_0^{\ln 2} x \left[\frac{e^x}{x} - \frac{1}{x} \right] dx$$

$$= \frac{1}{2} \int_0^{\ln 2} \cancel{x} \times \frac{1}{\cancel{x}} \left(e^x - 1 \right) dx$$

$$= \frac{1}{2} \left[e^x - x \right]_0^{\ln 2}$$

$$= \frac{1}{2} \left[\left(e^{\ln 2} - \ln 2 \right) - \left(e^0 - 0 \right) \right]$$

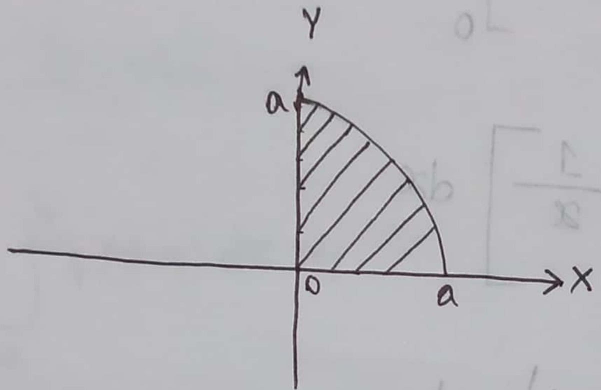
$$= \frac{1}{2} \left[2 - \ln 2 - 1 \right]$$

$$= \frac{1}{2} \left[1 - \ln 2 \right]$$

$= R.H.S$ (showed)

Evaluate $\iint_R xy\,dy\,dx$, where R is the quadrent of the circle $x^2+y^2=a^2$, where $x \geq 0$ and $y \geq 0$

Solve:-



$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

From figure, In the region R, x varies from $x=0$ to $x=a$ and y varies from $y=0$ to $y=\sqrt{a^2-x^2}$

Hence, we can write,

$$\iint_R xy\,dy\,dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$= \int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx^2$$

$$= \int_0^a \frac{x}{2}\left[a^2 - x^2\right] dx$$

$$= \frac{1}{2}\int_0^a \left(a^2x - x^3\right) dx$$

$$= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\left(\frac{a^4}{2} - \frac{a^4}{4}\right) - (0-0)\right]$$

$$= \frac{1}{2} \times \frac{a^4}{4}$$

$$= \frac{a^4}{8}$$