An Empirical Comparison of Top-Lists Aggregation Methods

Ammar Eltigani and Tai Henrichs

The Problem

- Suppose there are **n** candidates and **N** voters.
- Define a **top-List** π_i to be the **strict** ordering of the candidates provided by voter *i*.
- Top-lists do not require a voter to rank all candidates. Instead
  - Each preference can rank up to n candidates $|\pi_i| \leq n$
- We assume unranked candidates are tied at the bottom of a top-list, i.e if *u* and *v* are not ranked by voter *i*, then $\pi_i(u) = \pi_i(v)$
- Define the **Generalized Kendall Tau Distance** as the traditional Kendall Tau Distance (number of pairwise disagreements between two ranking lists) that disregards ties. Formally,

$$K(\pi_i, \pi_j) = \sum_{u,v \in [n]} 1\{\, \pi_i(u) < \pi_i(v) \;\&\&\; \pi_j(v) < \pi_j(u) \}$$

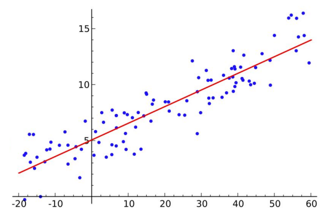
- **Problem:** Given N top-lists, output a ranking of all candidates that minimizes average generalized Kendall Tau Distance to each top-list

Ammar speaks

Note about how transformations from pi to tau are made
How total distance is computed for the whole dataset
Could mention top-k special-case

Why Top-List Aggregation?

- **Applications** to
  - Likelihood approximation
  - Web content aggregation
  - Social choice / ranking
- Our own **motivations** for choosing it
  - More realistic to have top-lists than full rankings
  - Relatively new, no known implementations
  - Fun!



- Tai Speaks
- Also condorcet consistent and ranks those preferred by the majority at the top, if there is such a majority

Why Approximations?

- **Complexity**
  - Top-List Aggregation is NP-Hard for 4 or more candidates
- An approximation algorithm is an efficient (poly-time usually) approximate solutions to NP-Hard optimization problems with provable guarantees on the distance returned
  - p-Approximation for some problem guarantees that $\begin{cases} \mathrm{OPT} \leq f(x) \leq \rho\mathrm{OPT}, & \text{if } \rho > 1; \\ \rho\mathrm{OPT} \leq f(x) \leq \mathrm{OPT}, & \text{if } \rho < 1. \end{cases}$
- Approximation algorithms we implemented*: **Borda+, Score-Then-Borda+, Score-Then-Adjust, FootRule+, RandomSort**

**these algorithms are mostly generalization from known full rank algorithms to the top-list problem (Mathieu & Simon 2020)*

Ammar Speaks

- Don't read off the algorithms

Example: Score-Then-Borda+*

Input: an instance (n, p) of TOP-AGG
Step 1, partition candidates into intervals:
$u \leftarrow$ uniformly random value on $[0, 1)$.
for all candidate $i \in [n]$ **do**
Compute $\boxed{Score_i \leftarrow p(\pi_i < \infty)}$.
Set $t \leftarrow \lfloor u - \ln(Score_i) \rfloor$ and put candidate i in interval E_t.
Step 2, solve the problem in each interval:
for all $t \in \mathbb{N} \cup \{\infty\}$ such that E_t is non-empty **do**
Order E_t sorting candidates i by average rank $\boxed{Rank_i \leftarrow \sum_{r=1}^{n} \frac{p(\pi_i = r)}{p(\pi_i < \infty)} \cdot r}$
Concatenate the ranking of E_0, ranking of E_1, ..., and ranking of E_∞.
Output resulting full-ranking.

Legend

- ★ E_i is the i^{th} partition
- ★ $p(\pi_i = r)$ is the probability that candidate i is placed in the r^{th} rank
- ★ $p(\pi_i < \infty)$ is the probability that candidate i is ranked

**randomized (8e+4) approximation, where e is the euler number*

Ammar Speaks
Define score and rank!
If score = 0, don't compute t. Automatically throw candidate into E_inf

- Intuition for the approximation
  - Borda+ sorts candidates from best to worst rank and is a non-constant approximation O(n)
  - Score-Then-Borda+ is a constant approximation
  - Kemeny cares about swaps, but it doesn't care about swap distance. For Kemeny, it only matters that a is ranked before b, not that a is 5 vs. 4 places better than b
  - A candidate has a greater score the more often they appear in top-lists - if a candidate c is unranked by voter v, then no matter how c is ranked by an algorithm, v does not care. So, this roughly prioritizes candidates that are generally cared about more, in that voters have preferences about them
  - But, even if two candidates are ranked with similar frequency, one may be more highly ranked very often
  - So, we sort partitions by ranks

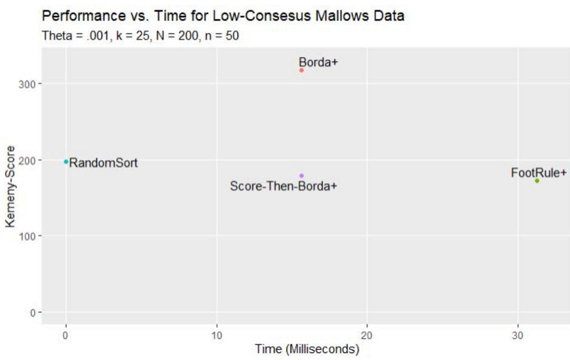
Example Input

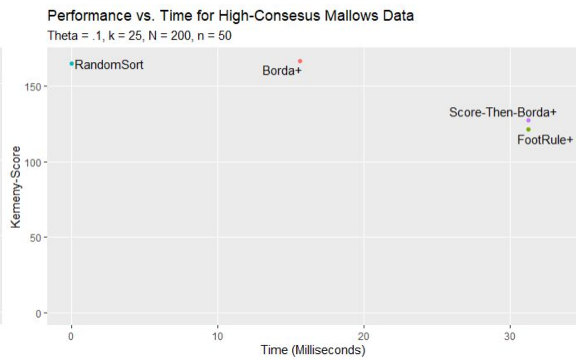
- Suppose the candidates are (1,2,3,4,5)
- Three voters submit the following top-lists: π_1 = (1,2,3), π_2 = (1,4,2), π_3=(2,1)

1. Suppose u ∈ [0,1) equals 0.5
2. Scores: $Score_1 = Score_2 = 3/3 = 1$, $Score_3 = Score_4 = ⅓$, $Score_5 = 0$
3. $\forall$ i ∈ {1,2,...,5} $t_i = \lfloor u - \ln(score_i) \rfloor$ if $score_i > 0$ else $t_i = \infty$
   a. $t_1 = t_2 = \lfloor u - \ln(score_1) \rfloor = \lfloor u - \ln(score_2) \rfloor = \lfloor u - \ln(1) \rfloor = \lfloor .5 - 0 \rfloor = 0$
   b. $t_3 = t_4 = \lfloor u - \ln(score_3) \rfloor = \lfloor u - \ln(score_4) \rfloor = \lfloor u - \ln(1/3) \rfloor = \lfloor .5 - (-1.09) \rfloor = \lfloor .5 + 1.09 \rfloor = 1$
   c. $t_5 = \infty$
4. $E_t = \{i : t_i = t\} \rightarrow E_0 = \{1,2\}$, $E_1 = \{3,4\}$, $E_\infty = \{5\}$
5. Average Ranks: $Rank_1 = 1 * ⅔ + 2 * ⅓ = 4/3$, $Rank_2 = ⅓ (2 + 3 + 1) = 2$, $Rank_3 = 1/1 * 3 = 3$, $Rank_4 = 1/1$, $Rank_5$=N/A
6. Sort each E_t by increasing ranks: $E_0 = \{1,2\}$, $E_1 = \{4,3\}$, $E_\infty = \{5\}$
7. Output full-rank: $\boldsymbol{\sigma}$ = (1,2,4,3,5)

- Tai speaks

Preliminary Results





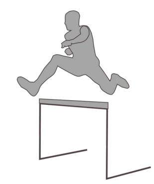
- Inspired by Ali and Meilă to consider data generated by the **Mallows Model** for varying levels of consensus
- Consensus increases with theta in the Mallows Model

Tai

- Mallows is a statistical model for ranks, has a ground truth ranking, which is the most common ranking. As theta grows, ranks that are very different from the ground truth ranking become rarer
- Ali and Meila found that performance and time varied with consensus level - intuitively, high-consensus regimes have fewer swaps between the provided lists, hence the better scores above when there is more consensus.

Experience thus far, hurdles, and what's next...

- Finding a library for generating synthetic data from a **Mallows Model*** for top-lists
- Learning **Integer Programming** for exact solutions because brute force was very slow
- Gave up on one algorithm that was a **PTAS** that would call multiple other PTASes as black boxes
  - Would have been very slow for reasonably small epsilon
  - Very hard to implement (even the PTAS co-designer advised against it!)
- Examine real-world data-sets from PrefLib
- Implement algorithms not considered in Simon and Mathieu
- Model **voting blocks** in synthetic data



*Fabien, Collas, and Irurozki Ekhine. Concentric mixtures of Mallows models for top-k rankings: sampling and identifiability.

We both speak, tai first

Define words in bold while presenting:

Voting Block consists of with preferences similar to each other, allowing us to consider cases where groups internally agree but disagree with other groups. Not considered in full-list aggregation literature that we could see.

An **integer programming** problem is a mathematical optimization or feasibility program in which some or all of the variables are restricted to be integers. In many settings the term refers to **integer linear programming** (ILP), in which the objective function and the constraints (other than the integer constraints) are linear.

Integer programming is NP-complete. In particular, the special case of 0-1 integer linear programming

A **PTAS** is an algorithm which takes an instance of an optimization problem and a parameter $\varepsilon > 0$ and, in polynomial time, produces a solution that is within a factor $1 + \varepsilon$ of being optimal

Resources and Papers

- Ailon, Nir. "Aggregation of partial rankings, p-ratings and top-m lists." Algorithmica 57.2 (2010): 284-300. **[Paper]**
- Ali, Alnur, and Marina Meilă. "Experiments with Kemeny ranking: What works when?." Mathematical Social Sciences 64.1 (2012): 28-40. **[Paper]**
- Fabien, Collas, and Irurozki Ekhine. "Concentric mixtures of Mallows models for top-k rankings: sampling and identifiability." arXiv preprint arXiv:2010.14260 (2020). **[Code]**
- Mathieu, Claire, and Simon Mauras. "How to aggregate Top-lists: Approximation algorithms via scores and average ranks." Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms. Society for Industrial and Applied Mathematics, 2020. **[Paper]**
- Nicholas Mattei and Toby Walsh. “PrefLib”: A Library of Preference Data, Springer. **[Dataset]**
- Schalekamp, Frans, and Anke van Zuylen. "Rank aggregation: Together we're strong." 2009 Proceedings of the Eleventh Workshop on Algorithm Engineering and Experiments (ALENEX). Society for Industrial and Applied Mathematics, 2009. **[Paper]**

- **Python Libraries:** numpy, scipy, and pulp.
- **R Libraries:** ggplot2