STAT 431 — Applied Bayesian Analysis — Spring 2019

Homework 3

Please submit your assignment *on paper*. Make sure your answers are completely justified and clear enough to read! Any computer code and output should be included.

1. The following are times (days) since last modification for 10 randomly-chosen English Wikipedia articles:

   481 144 93 446 69 170 383 63 79 181

   On the *log* scale, they are roughly normally distributed. Take *natural logarithms* of these values, then model the resulting (log-transformed) values as a *two-parameter* normal sample, with both mean μ and variance σ^2 unknown.

   For the following parts, use a normal-inverse gamma prior: For the conditional normal part, specify a mean of zero and an *equivalent prior sample size* of 1. For the inverse gamma part, let σ^2 have a (marginal) prior *mean* of 1 and a (marginal) prior *variance* of 1.

   (a) [3 pts] Determine the numerical values of all of the parameters of the normal-inverse gamma prior.

   (b) [2 pts] Determine the *marginal* posterior distribution of μ: Name it, and give the value(s) of its parameter(s).

   (c) [1 pt] Compute a posterior 95% equal-tailed credible interval for μ.

   (d) [2 pts] Determine the *marginal* posterior distribution of σ^2: Name it, and give the value(s) of its parameter(s).

   (e) [1 pt] Compute a posterior 95% equal-tailed credible interval for σ^2.

2. Consider the same data and data model as in the previous problem, but now use the *standard noninformative prior*

   $$p(\mu, \sigma^2) \propto \frac{1}{\sigma^2} \qquad (\sigma^2 > 0)$$

   in the following parts:

   (a) [2 pts] Determine the *marginal* posterior distribution of μ: Name it, and give the value(s) of its parameter(s).

   (b) [1 pt] Compute a posterior 95% equal-tailed credible interval for μ.

   (c) [2 pts] Determine the *marginal* posterior distribution of σ^2: Name it, and give the value(s) of its parameter(s).

   (d) [1 pt] Compute a posterior 95% equal-tailed credible interval for σ^2.

3. GRADUATE SECTION ONLY

   Let $L(\mu, \sigma^2; \boldsymbol{y})$ be the likelihood function for the two-parameter normal model from lecture, based on n independent observations.

(a) [3 pts] *Derive* a simplified expression for

$$I_{12}(\mu,\sigma^2) \;=\; -\,\mathrm{E}\left(\frac{\partial^2}{\partial\mu\,\partial\sigma^2}\ln L(\mu,\sigma^2;\boldsymbol{Y}) \;\middle|\; \mu,\sigma^2\right)$$

where the expectation is over the (random) data vector $\boldsymbol{Y}$, for fixed values of μ, σ^2.

(b) [2 pts] The *(Fisher) information matrix* for a two-parameter normal sample is defined as the 2×2 matrix

$$I(\mu,\sigma^2) \;=\; \begin{bmatrix} I_{11}(\mu,\sigma^2) & I_{12}(\mu,\sigma^2) \\ I_{21}(\mu,\sigma^2) & I_{22}(\mu,\sigma^2) \end{bmatrix}$$

where

$$I_{11}(\mu,\sigma^2) \;=\; -\,\mathrm{E}\left(\frac{\partial^2}{\partial\mu^2}\ln L(\mu,\sigma^2;\boldsymbol{Y}) \;\middle|\; \mu,\sigma^2\right) \;=\; \frac{n}{\sigma^2}$$

(which is the Fisher information for the μ-only model) and

$$I_{22}(\mu,\sigma^2) \;=\; -\,\mathrm{E}\left(\frac{\partial^2}{\partial(\sigma^2)^2}\ln L(\mu,\sigma^2;\boldsymbol{Y}) \;\middle|\; \mu,\sigma^2\right) \;=\; \frac{n}{2(\sigma^2)^2}$$

(which is the Fisher information for the σ^2-only model) and

$$I_{21}(\mu,\sigma^2) \;=\; I_{12}(\mu,\sigma^2).$$

The (general) Jeffreys prior is defined to be the *square root of the determinant* of the information matrix. Using part (a) and what is given above, derive the Jeffreys prior (up to proportionality) for the two-parameter normal model.