HW2 Solutions
STAT 431 Spring 2019

February 2019

Contents

1 Bayesian inference on Binomial π

We observe $X = 6$ successes in our sample of size $N = 70$, where we assume $X \sim \text{Binomial}(N = 70, \pi)$, where the success probability π is unknown. The likelihood of this model is given by

$$P(X \mid \pi, N) = \binom{N}{X} \pi^X (1 - \pi)^{N-X} \tag{1}$$

1.1 A - Posterior densities

1.1.1 I - Uniform prior

$$
\begin{aligned}
P(\pi \mid X) &\propto P(X \mid \pi, N)P(\pi) && (2)\\
&= P(X \mid \pi, N) && (3)\\
&= \binom{N}{X}\pi^X(1-\pi)^{N-X} && (4)\\
&= \binom{70}{6}\pi^6(1-\pi)^{64} && (5)
\end{aligned}
$$

The posterior distribution for π (under a uniform prior) is $P(\pi \mid X) =$ Beta$(X+1,\, N-X+1) =$ **Beta(7, 65)**. The full density is then:

$$
P(\pi \mid X) = \frac{\Gamma(72)}{\Gamma(7)\Gamma(65)}\pi^6(1-\pi)^{64}, \qquad 0 < \pi < 1 \tag{6}
$$

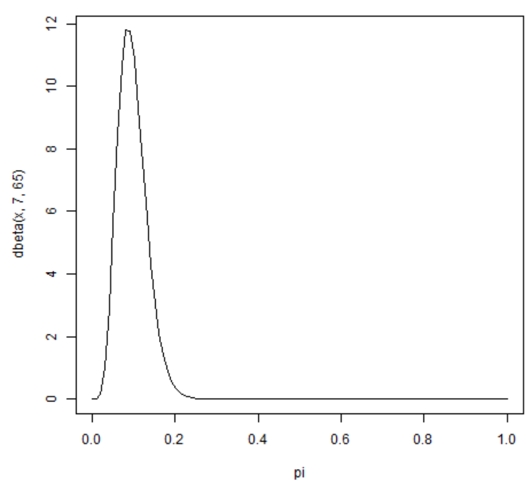

Figure 1: Posterior distribution of π under a uniform prior for π.

1.1.2 II - Informative Beta prior

A priori $\pi \sim$ Beta$(100, 100)$. Then the form of the posterior is given as follows

$$
\begin{align}
P(\pi \mid X) &\propto P(X \mid \pi, N) P(\pi) \tag{7}\\
&= \pi^X (1-\pi)^{N-X} \left(\pi^{99}(1-\pi)^{99}\right) \tag{8}\\
&= \pi^{X+99}(1-\pi)^{N-X+99} \tag{9}\\
&= \pi^{105}(1-\pi)^{163} \tag{10}
\end{align}
$$

The posterior distribution for π is $P(\pi \mid X) = \text{Beta}(X+100,\, N-X+100) =$ **Beta(106, 164)**. The full density is:

$$
P(\pi \mid X) = \frac{\Gamma(270)}{\Gamma(106)\Gamma(164)} \pi^{105}(1-\pi)^{163}, \qquad 0 < \pi < 1 \tag{11}
$$

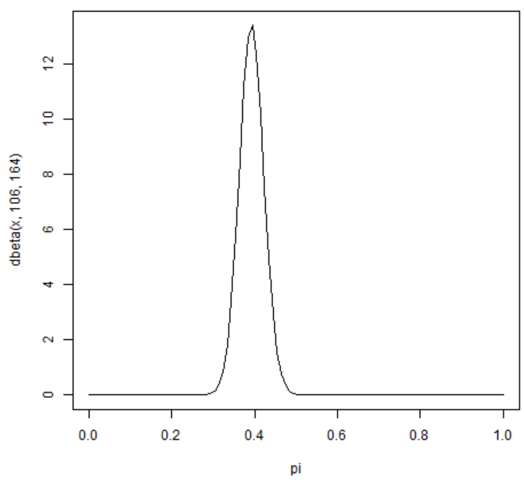

Figure 2: Posterior distribution of π under a Beta$(100, 100)$ prior for π.

1.2 B - Posterior mean and standard deviations

1.2.1 I - Uniform prior

The posterior mean is $\frac{X+1}{X+1+N-X+1} = \frac{X+1}{N+2} = \frac{7}{72} =$ **0.09722**, and the posterior standard deviation is $\sqrt{\frac{(X+1)*(N-X+1)}{(X+1+N-X+1)^2(X+1+N-X+1+1)}} = \sqrt{\frac{455}{72^2*73}} =$ **0.03467**.

1.2.2 II - Informative Beta prior

The posterior mean is **0.3925**, and the posterior standard deviation is **0.02966**.

1.3 C - 95% Equal-tailed credible intervals

1.3.1 I - Uniform prior

We obtain our 95% posterior credible interval for π under a uniform prior for π using the following R commands:

```
qbeta(c(0.025, 0.975), 7, 65) %>% round(3)
```

Our credible interval is **(0.041, 0.175)**.

1.3.2 II - Informative Beta prior

Our credible interval is **(0.335, 0.451)**.

1.4 D - Posterior inference

1.4.1 I - Uniform prior

Under a uniform prior for π,

$$P(\pi \geq 0.2 \mid X, N) = \mathbf{0.007} \tag{12}$$
$$P(\pi < 0.2 \mid X, N) = \mathbf{0.993} \tag{13}$$

Where we found the first probability using the following R command:

```
(1-pbeta(0.2, 7, 65)) %>% round(3)
```

1.4.2 II - Informative Beta prior

Under a Beta$(100, 100)$ prior for π,

$$P(\pi \geq 0.2 \mid X, N) = \mathbf{1} \tag{14}$$
$$P(\pi < 0.2 \mid X, N) = \mathbf{0} \tag{15}$$

2 Cowles, Problem 5.3

Prompt: If $\phi = g(\pi) = \text{logit}(\pi)$, show that a uniform prior on π induces the following density for ϕ:

$$P_\phi(\phi) = \frac{e^\phi}{(1 + e^\phi)^2} \tag{16}$$

Solution:

$$P_\phi(\phi) = P_\pi(g^{-1}(\phi)) \left|\frac{d\pi}{d\phi}\right| \tag{17}$$

$$= P_\pi\left(\frac{e^\phi}{1+e^\phi}\right)\left(\frac{d}{d\phi}\frac{e^\phi}{1+e^\phi}\right) \tag{18}$$

$$= P_\pi(\pi)\,\frac{(e^\phi)'(1+e^\phi) - (e^\phi)(1+e^\phi)'}{(1+e^\phi)^2} \tag{19}$$

$$= \frac{(e^\phi)(1+e^\phi) - (e^\phi)(e^\phi)}{(1+e^\phi)^2} \tag{20}$$

$$= \frac{e^\phi}{(1+e^\phi)^2} \tag{21}$$

In line 15 we apply the quotient rule. Going from line 15 to line 16, $P_\pi(\pi) = 1$ as P_π is the uniform distribution.

3 GRADUTE SECTION ONLY

Continuing Problem 1, suppose that we encounter $N = 20$ new individuals, who we assume are a random sample from a population of "people like us".

3.1 Frequentist inference

The frequentist estimate for π is simply what we observe, i.e. $\hat{\pi} = \frac{6}{70} = 0.0857$. The probability that *at least* 4 individuals have played Euchre before $P(X \geq 4 \mid \pi, N = 20)$ can be equivalently written as $1 - P(X < 4 \mid \pi, N = 20)$.

We evaluate this in R using the following command:

```
1 - pbinom(3, 20, 6/70)
```

and we find that $P(\tilde{X} \geq 4 \mid \hat{\pi}, \tilde{N} = 20) = \mathbf{0.08639}$.

3.2 Posterior predictive probabilities

Using the *LearnBayes* package in R, with the following commands,

```
library(LearnBayes)
1 - (pbetap(c(7, 65), 20, 0:3) %>% sum)
```

We find that $= P(\tilde{X} \geq 4 \mid X, \tilde{N} = 20) = \mathbf{0.1459}$.