HW4 Solutions
STAT431 Spr19

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Contents

1 Approximating Posterior Variance

1.1 Integrations needed

The integrals we need are :

$$\mathbb{E}\left[(\pi - \mu_\pi)^2\right] = \frac{1}{C}\int_0^1 (\pi - \mu_\pi)^2 L(Y \mid \pi, n)P(\pi)d\pi \tag{1}$$

where

$$C = \int_0^1 L(Y \mid \pi, n)P(\pi)d\pi \tag{2}$$

$L(Y \mid \pi, n)$ denotes the Binomial likelihood, while $P(\pi)$ denotes the Jeffrey's prior for π.

1.2 Approximate posterior variance

```
### Problem 1: Population Proportion: Numeric Integration ###

n <- 70

y <- 12


### Define likelihood and prior (up to proportionality)

like <- function(pi) pi^y * (1-pi)^(n-y)

prior <- function(pi) dbeta(pi, 0.5, 0.5)


### Compute posterior expectation of pi

numerator <- integrate(function(pi) pi * prior(pi) * like(pi), 0, 1)

denominator <- integrate(function(pi) prior(pi) * like(pi), 0, 1)

posterior_mean = numerator$value / denominator$value
posterior_mean



#### Posterior variance
numerator <- integrate(function(pi)
  (pi - posterior_mean)^2 * prior(pi) * like(pi), 0, 1)

posterior_var = numerator$value / denominator$value
posterior_var
```

Posterior variance = 0.002014729.

1.3 Analytic form of posterior variance

```
#### 1 C Analytic form of posterior variance
alpha = 0.5 + y
beta = 0.5 + n - y

posterior_var_analytic = (alpha*beta) / ((alpha+beta)^2 * (alpha+beta+1))
posterior_var_analytic
```

The analytic posterior variance is 0.002014729, the same value as the approximate posterior variance.

2 Difference in Means/Variances of MPGE for two classes of cars

```
##### 2
library(dplyr)
car_dat =
  "Hyundai Ioniq Electric, 136, Small Cars
Volkswagen e-Golf, 119, Small Cars
Honda Clarity EV, 114, Small Cars
BMW i3, 113, Small Cars
BMW i3s, 113, Small Cars
Nissan Leaf, 112, Small Cars
Fiat 500e, 112, Small Cars
smart EQ fortwo (coupe), 108, Small Cars
smart EQ fortwo (convertible), 102, Small Cars
Hyundai Kona Electric, 120, Sport Utility Vehicles (SUVs)
Tesla Model X 75D, 93, Sport Utility Vehicles (SUVs)
Tesla Model X 100D, 87, Sport Utility Vehicles (SUVs)
Tesla Model X P100D, 85, Sport Utility Vehicles (SUVs)
Jaguar I-Pace, 76, Sport Utility Vehicles (SUVs)" %>%
  read.csv(text=., header=F, stringsAsFactors = F, sep=",", strip.white = T)

colnames(car_dat) = c("Vehicle", "MPGE", "Class")
```

2.1 Sample means/standard deviations

```
##### 2 A
car_summaries = car_dat %>%
  group_by(Class) %>%
  summarise(mean = mean(MPGE) %>% round(3),
            sd = sd(MPGE) %>% round(3),
            n = n())
library(stargazer)
car_summaries %>% stargazer(summary = F)
```

Table 1:

	Class	mean	sd	n
1	Small Cars	114.333	9.341	9
2	Sport Utility Vehicles (SUVs)	92.2	16.694	5

2.2 95% Credible interval for $\mu_1 - \mu_2$

```
quantile(mu1s - mu2s, c(0.025, 0.975)) %>% round(3)
```

The posterior 95% equal-tailed credible interval for $\mu_1 - \mu_2$ is equal to **(0.080, 44.036)**. The means between the two classes of cars appears to differ as the 95% credible interval for the difference in means does not cover 0.

2.3 Approximate posterior probability that $\mu_1 < \mu_2$

```
#### 2 C
# approx. posterior probability that mu1 < mu2
mean(mu1s < mu2s)
```

The approximate posterior probability that $\mu_1 < \mu_2$ is equal to **0.02399**.

2.4 Frequentist Welch's two-sample t-test for testing $H_0 : \mu_1 < \mu_2$

```
##### 2 D
### For comparison: Frequentist Welch Approximate 95% CI
t.test(car_dat %>% subset(Class == "Small Cars", select=MPGE),
       car_dat %>% subset(Class != "Small Cars", select=MPGE),
       alternative='greater',
       var.equal = F)

t.test(car_dat %>% subset(Class == "Small Cars", select=MPGE),
       car_dat %>% subset(Class != "Small Cars", select=MPGE),
       alternative='greater',
       var.equal = T)
```

For the test with unequal variances, **p-value = 0.01879**. For the test with equal variances, **p-value = 0.00362**. Both frequentist p-values are smaller than the equivalent Bayesian probability for $\mu_1 < \mu_2$, as the Bayesian version includes additional uncertainty in the prior distributions of the parameters.

2.5 95% credible interval for $\frac{\sigma_1^2}{\sigma_2^2}$

```
#### 2 E
### Does sigma1^2 really exceed sigma2^2?
quantile(sigma1.2s / sigma2.2s, c(0.025, 0.975)) %>% round(3)
```

The 95% equal-tailed credible interval for the ratio $\frac{\sigma_1^2}{\sigma_2^2}$ is **(0.035, 1.579)**. As this interval covers 1, it does not appear that the variances in MPGE differs between the two classes of cars.

3 GRADUATE ONLY - Bayesian analysis with a single data point

$$P(Y \mid \nu+1, \beta) = \frac{\beta^{\nu+1}}{\Gamma(\nu+1)} Y^{\nu} e^{-\beta Y} \tag{3}$$

3.1 Full conditional of β

The prior for β is given by $P(\beta) \propto e^{-\beta},\ \beta > 0$. The posterior distribution for β is then proportional to

$$\begin{align} P(Y \mid \nu, \beta) P(\beta) &\propto \frac{\beta^{\nu+1}}{\Gamma(\nu+1)} Y^{\nu} e^{-\beta Y} e^{-\beta} \tag{4} \\ &\propto \frac{\beta^{\nu+1}}{C_{\nu}} e^{-\beta(Y+1)} \tag{5} \end{align}$$

which is the kernel of a Gamma distribution. Then $\beta \mid \nu, Y \sim \text{Gamma}\,(\nu+2, Y+1)$.

3.2 Full conditional of ν

The prior for ν is given by $P(\nu) \propto \frac{1}{2^{\nu}},\ \nu = 0, 1, 2, \ldots$. The posterior distribution for ν is then proportional to

$$\begin{align} P(Y \mid \nu, \beta) P(\nu) &\propto \frac{\beta^{\nu+1}}{\Gamma(\nu+1)} Y^{\nu} e^{-\beta Y} \frac{1}{2^{\nu}} \tag{6} \\ &\propto \frac{\beta^{\nu+1}}{\Gamma(\nu+1)} \left(\frac{Y}{2}\right)^{\nu} \tag{7} \\ &\propto \frac{\beta \left(\frac{\beta Y}{2}\right)^{\nu}}{\Gamma(\nu+1)} \tag{8} \end{align}$$

which is the kernel of a Poisson distribution. Then $\nu \mid \beta, Y \sim \text{Poisson}\left(\lambda = \frac{\beta Y}{2}\right)$.

3.3 Approximation of $P(\nu = 0 \mid Y = 4)$

We approximate $P(\nu = 0 \mid Y = 4)$ by using Gibbs sampling to draw posterior samples of $(\nu, \beta) \mid Y$.

```
Y = 4

nu_vec = beta_vec = rep(0, 100000)
nu_vec[1] = 0
beta_vec[1] = 1
for (i in 2:length(beta_vec)) {
```

```
  nu_vec[i] = rpois(1, beta_vec[i-1]*Y/2)
  beta_vec[i] = rgamma(1, nu_vec[i-1]+2, rate=Y+1)
}

mean(nu_vec[seq(length(nu_vec)/2, length(nu_vec))]==0)
```

Discarding the first half of the samples as burn-in, we obtain $P(\nu = 0 \mid Y = 4) \approx \mathbf{0.3629127}$.