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Spontaneously Broken Continuous Symmetries in Hyperbolic (or Open) de Sitter Spacetime

BHARAT RATRA

Joseph Henry Laboratories, Department of Physics, Princeton University, Princeton, N. J. 08544

ABSTRACT

The functional Schrödinger approach is used to study scalar field theory in hyperbolic (or open) de Sitter spacetime. While on intermediate length scales (small compared to the spatial curvature length scale) the massless minimally coupled scalar field two-point correlation function does have a term that varies logarithmically with scale, as in flat and closed de Sitter spacetime, the spatial curvature tames the infrared behaviour of this correlation function at larger scales in the open model. As a result, and contrary to what happens in flat and closed de Sitter spacetime, spontaneously broken continuous symmetries are not restored in open de Sitter spacetime (with more than one spatial dimension).

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1. Introduction and Summary

Data from observational cosmology suggests that it might not be entirely unreasonable to assume that the mass density of the universe is less than the critical Einstein-de Sitter density, and that the spatial hypersurfaces of the FLRW model that describes the zeroth-order homogeneous universe might be open and have constant negative spatial curvature, [1]. To generate the primordial energy density perturbations, that might be responsible for the observed large-scale cosmological structure, in such an open universe, one might be tempted to consider quantum-mechanical fluctuations in a very early epoch during which the energy density of the universe was dominated by a constant cosmological constant and negative spatial curvature, [2]. This homogeneous and isotropic background solution of Einstein's equations might be called hyperbolic de Sitter spacetime.

Quantum-mechanical zero point fluctuations of a scalar field in the hyperbolic de Sitter epoch would generate energy density perturbations which might be responsible for the observed large-scale structure of the present universe. The main purpose of this paper is to study these quantum-mechanical scalar field fluctuations in hyperbolic de Sitter spacetime. We are particularly interested in deriving an expression for the scalar field two-point correlation function, and choose to use the functional Schrödinger approach developed in Ref. [3] to accomplish this. (See Refs. [4,5] for extensions of the functional Schrödinger approach.)

In Ref. [3] we found that the logarithmic infrared divergence of massless minimally coupled scalar field theory in spatially flat or spatially closed de Sitter spacetime led to the restoration of spontaneously broken continuous symmetries, in any number of spacetime dimensions, in these models. That is, we found that the infrared behaviour of massless minimally coupled scalar field theory in these models was very similar to that of massless scalar field theory in 1+1 dimensional Minkowski spacetime. (See Ref. [6] for a discussion of massless scalar field theory in 1+1 dimensional Minkowski spacetime, and Refs. [3,7] for massless minimally coupled scalar field theory and symmetry breaking in flat and closed de Sitter spacetime.)

Interestingly, we find that massless minimally coupled scalar field theory in hyperbolic de Sitter spacetime has a very different infrared behaviour. Although on intermediate scales (where spatial curvature is unimportant) the scalar field two-point correlation function does have a spatially dependent logarithmic correction to the standard ultraviolet divergence (as do the correlation functions in flat and closed de Sitter spacetime), on large scales the correlation function is very strongly damped (and shows no sign of the logarithmic infrared divergence of the correlation function in the spatially flat and spatially closed models).

This behaviour has a number of interesting implications. First of all, because of the intermediate scale logarithm, the late time power spectrum of energy density perturbations in this model will have the standard scale-invariant form, [8], characteristic of spatially flat de Sitter inflation, [4,9,10], on scales small compared to the curvature of the spatial hypersurfaces, [2,11], while on scales comparable to the spatial hypersurface curvature the late time power spectrum will break away from the scale-invariant form, [2,11].

It is possible that spontaneously broken continuous symmetries in hyperbolic de Sitter spacetime (with more than one spatial dimension) will display a scale-dependent behaviour. It is known that in the limit in which the spatial radius of curvature is infinite, i.e., when the model is flat, spontaneously broken continuous symmetries are restored, [3,7]. In the open model, on intermediate scales, we find a term in the correlation function with a similar logarithmic scaling — the consequences of this term needs to examined in more detail, and we hope to return to this issue at some point — while on sufficiently large scales the correlation function is strongly damped, and there is no symmetry restoration.

The large-scale behaviour of the massless minimally coupled scalar field correlation function in hyperbolic de Sitter spacetime is somewhat similar to that in anti-de Sitter spacetime (the first reference in [7]) and to that in a constant negative curvature Euclidean signature space, [12] (except in the 1+1 dimensional case). The reason for this is presumably the similar geometry of the spatial hypersurfaces.

In spatially flat de Sitter spacetime each polarization of the graviton behaves like a minimally coupled massless scalar field. As a result the graviton two-point correlation function is also logarithmically infrared divergent, [13]. Whether or not a similar relation between the graviton and the massless minimally coupled scalar field holds in hyperbolic de Sitter spacetime, it does not seem unreasonable to suspect that the spatial geometry will ensure that the graviton correlation function in hyperbolic de Sitter spacetime will be very well behaved on large scales.

In Sec. 2 we review the $n+1$ dimensional hyperbolic de Sitter spacetime solution and record the scalar spatial harmonics (a technical issue is relegated to Appendix A). In Sec. 3 we study the massless conformally coupled scalar field model in this spacetime, using the functional Schrödinger approach, and in Sec. 4 we examine the massive minimally coupled case. Technical details and a derivation of the 'non-equal' time Green's function for the conformally coupled and massless minimally coupled cases in 3+1 dimensions are given in Appendices B and C. In Sec. 5 we comment on spontaneously broken continuous symmetries in hyperbolic de Sitter spacetime.

2. Technical Preliminaries

Hyperbolic de Sitter spacetime is the negative spatial curvature solution of Einstein's equations when the stress-energy tensor is dominated by a cosmological constant term. The spatial hypersurfaces are open, and the $n+1$ dimensional spacetime line element is

$$ds^2 = dt^2 - h^{-2}\sinh^2(ht)\left[d\theta_n^2 + \sinh^2(\theta_n)d\Omega_{n-1}^2\right], \tag{2.1}$$

where, for the expanding solution $0 \leq t < \infty$, the 'radial' coordinate $0 \leq \theta_n < \infty$, the usual line element on the $n-1$ dimensional sphere is denoted by $d\Omega_{n-1}^2$, and the constant

$$h^2 = \frac{8\pi G}{3}\rho_b, \tag{2.2}$$

where G is Newton's gravitational constant and ρ_b the energy density of the cosmological constant.

Conformal time $\tilde{t}$ is defined by $d\tilde{t} = dt/a(t)$ [where $a(t) = \sinh(ht)/h$ is the FLRW scale factor] and is related to t through

$$\tilde{t} = \ln\left[\tanh\left(\frac{ht}{2}\right)\right]; \tag{2.3}$$

$-\infty \leq \tilde{t} < 0$ for the expanding solution. With this variable the line element is

$$ds^2 = h^{-2}\mathrm{csch}^2(\tilde{t})\left[d\tilde{t}^2 - \left\{d\theta_n^2 + \sinh^2(\theta_n)d\Omega_{n-1}^2\right\}\right]. \tag{2.4}$$

Hyperbolic de Sitter spacetime is not conformally flat, except in $1+1$ dimensions.

Hyperbolic de Sitter spacetime, with the line element (2.1), may be viewed as an $n+1$ dimensional 'hyperboloid' in an $n+2$ dimensional space. The embedding space coordinates are

$$\begin{aligned}
z_0 &= h^{-1}\cosh(ht) \\
z_1 &= h^{-1}\sinh(ht)\cosh(\theta_n) \\
z_2 &= h^{-1}\sinh(ht)\sinh(\theta_n)\cos(\theta_{n-1}) \\
&\cdots \\
z_i &= h^{-1}\sinh(ht)\sinh(\theta_n)\sin(\theta_{n-1})\cdots\sin(\theta_{n+2-i})\cos(\theta_{n+1-i}) \\
&\cdots \\
z_{n+1} &= h^{-1}\sinh(ht)\sinh(\theta_n)\sin(\theta_{n-1})\cdots\sin(\theta_2)\sin(\theta_1),
\end{aligned} \tag{2.5}$$

and the $n+2$ dimensional line element

$$ds^2 = -dz_0^2 + dz_1^2 - dz_2^2 \cdots - dz_{n+1}^2 \tag{2.6}$$

reproduces eqn. (2.1).

The square of the distance between two points, $(t, \theta_n, \Omega_{n-1})$ and $(t', \theta'_n, \Omega'_{n-1})$, in hyperbolic de Sitter spacetime is

$$\sigma^2 = -\frac{2}{h^2}\sinh(ht)\sinh(ht')\left[\frac{1-\cosh(ht)\cosh(ht')}{\sinh(ht)\sinh(ht')} + \cosh(\hat{\gamma}_n)\right], \tag{2.7}$$

where

$$\cosh(\hat{\gamma}_n) = \cosh(\theta_n)\cosh(\theta'_n) - \sinh(\theta_n)\sinh(\theta'_n)\cos(\gamma_{n-1}), \tag{2.8}$$

where γ_{n-1} is the usual angle between the two points Ω_{n-1} $(= \theta_{n-1}, \theta_{n-2}, \cdots \theta_1)$ and Ω'_{n-1} on the $n-1$ dimensional unit sphere, given, for example, for the 2 dimensional unit sphere by

$$\cos(\gamma_2) = \cos(\theta_2)\cos(\theta'_2) + \sin(\theta_2)\sin(\theta'_2)\cos(\theta_1 - \theta'_1). \tag{2.9}$$

From eqn. (2.7), the square of the distance between two points on the same spatial hypersurface, $(t, \theta_n, \Omega_{n-1})$ and $(t, \theta'_n, \Omega'_{n-1})$, is

$$\sigma^2 = 2a^2(t)[1 - \cosh(\hat{\gamma}_n)]. \tag{2.10}$$

The n dimensional spatial Laplacian in the metric of eqn. (2.1) is

$$L^2_{(n)} = \frac{1}{\sinh^{n-1}(\theta_n)}\frac{\partial}{\partial\theta_n}\left(\sinh^{n-1}(\theta_n)\frac{\partial}{\partial\theta_n}\right) + \frac{1}{\sinh^2(\theta_n)}\mathcal{L}^2_{(n-1)}, \tag{2.11}$$

where $\mathcal{L}^2_{(n-1)}$ is the usual Laplacian on the $n-1$ dimensional unit sphere (Appendix B of Ref. [3]). The scalar eigenfunctions Z_{AW} of $L^2_{(n)}$ obey, [14],

$$L^2_{(n)}Z_{AW}(\hat{\Omega}_n) = -\left[A^2 + \left(\frac{n-1}{2}\right)^2\right]Z_{AW}(\hat{\Omega}_n), \tag{2.12}$$

where $\hat{\Omega}_n = (\theta_n, \Omega_{n-1})$, $A > 0$ and is continuous, and the $n-2$ 'magnetic' integral indices $B, C, \cdots$ are denoted by W $(B = 0, 1, 2, \cdots, C\epsilon[-B, B], D\epsilon[-C, C], \cdots)$.

The orthonormal eigenfunctions are, [14],

$$Z_{ABC\cdots}(\hat{\Omega}_n) = \frac{\Gamma\left(iA+B+\frac{n-1}{2}\right)}{\Gamma\left(iA\right)}[\sinh(\theta_n)]^{(2-n)/2}P_{iA-1/2}^{-B+(2-n)/2}(\cosh(\theta_n))\,Y_{BC\cdots}(\Omega_{n-1}), \tag{2.13}$$

where $Y_{BC\cdots}$ is the standard $n-1$ dimensional spherical harmonic (Appendix B of Ref. [3]), Γ is the gamma function, and P_ν^μ is the associated Legendre function of the first kind (Chap. 3 of Ref. [15] or Chap. 8 of Ref. [16]).

The orthonormality relation is

$$\int_0^\infty d\theta_n \sinh^{n-1}(\theta_n) \int_{S^{n-1}} d\Omega_{n-1} Z_{AW}(\hat{\Omega}_n)\left[Z_{A'W'}(\hat{\Omega}_n)\right]^* = \delta(A-A')\delta_{W,W'}. \tag{2.14}$$

where S^{n-1} is the $n-1$ dimensional unit sphere, $\delta(A-A')$ is a Dirac delta function, and $\delta_{W,W'}$ is a product of $n-1$ Kronecker deltas.

The addition theorem is (see Appendix A for a derivation),

$$P_{iA-1/2}^{(2-n)/2}(\cosh(\hat{\gamma}_n)) = (2\pi)^{n/2}\left|\frac{\Gamma(iA)}{\Gamma\left(iA+\frac{n-1}{2}\right)}\right|^2[\sinh(\hat{\gamma}_n)]^{(n-2)/2}\sum_W Z_{AW}(\hat{\Omega}_n)\left[Z_{AW}(\hat{\Omega}'_n)\right]^*, \tag{2.15}$$

where $\hat{\gamma}_n$, the angle between $\hat{\Omega}_n$ and $\hat{\Omega}'_n$, is given by eqn. (2.8).

3. The Conformally Coupled Scalar Field

Hyperbolic de Sitter spacetime is not conformally flat, so rescaling the Minkowski spacetime massless scalar field two-point correlation function does not give the corresponding hyperbolic de Sitter spacetime massless, conformally coupled, scalar field two-point correlation function. Nevertheless, the massless conformally coupled case is still significantly simpler than the general case, and in this section we use it to review the methods developed in Ref. [3]. We shall solve the functional Schrödinger equation for the wavefunction of the massless conformally coupled scalar field in hyperbolic de Sitter spacetime, and then use the wavefunction to derive the scalar field two-point correlation function.

The action for a massless conformally coupled complex scalar field ϕ is

$$S = \int dt\, d^n x \mathcal{L}(x) = \int dt\, d^n x \sqrt{|g|} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi + \frac{1}{2} \xi R |\phi|^2 \right], \tag{3.1}$$

where $\mathcal{L}(x)$ is the Lagrangian density, the metric tensor $g_{\mu\nu} = \mathrm{diag}(1, -a^2 H_{ij})$ is given in eqn. (2.1), $\xi = (n-1)/(4n)$, and the Ricci scalar $R = -n(n+1)h^2$. Integrating by parts, both spatially and temporally, and dropping surface terms, and introducing the dimensionless field $\chi = a^{(n-1)/2}\phi$ and conformal time $\tilde{t}$ [eqn. (2.3)], we find

$$S = \int d\tilde{t}\, d^n x \sqrt{|H|} \left[\frac{1}{2} |\dot{\chi}|^2 + \frac{1}{2} \chi^* L^2_{(n)} \chi + \frac{1}{8} (1-n)^2 |\chi|^2 \right], \tag{3.2}$$

where a dot denotes a derivative with respect to $\tilde{t}$, $L^2_{(n)}$ is the spatial Laplacian [eqn. (2.11)], and H is the determinant of the metric tensor on the spatial hypersurface.

Transforming to spatial momentum space,

$$\chi(\hat{\Omega}_n, \tilde{t}) = \int_0^\infty dA \sum_W \chi(A, W, \tilde{t}) Z_{AW}(\hat{\Omega}_n), \tag{3.3}$$

and using eqns. (2.12) and (2.14), we find

$$S = \int d\tilde{t} \int_0^\infty dA \sum_W \left[\frac{1}{2} \left|\dot{\chi}(A, W, \tilde{t})\right|^2 - \frac{A^2}{2} \left|\chi(A, W, \tilde{t})\right|^2 \right]. \tag{3.4}$$

Treating the real and imaginary parts of $\chi (= \chi_R + i\chi_I)$ as independent real variables, denoted generically as χ, we see that they are governed by the Hamiltonian density

$$\widetilde{\mathcal{H}}_{AW} = \frac{1}{2} p^2 + \frac{1}{2} A^2 \chi^2, \tag{3.5}$$

where p is the canonical momentum conjugate to χ.

With eqn. (3.5), the functional Schrödinger equation

$$\widetilde{\mathcal{H}}_{AW} \Psi_{AW}[\chi, \tilde{t}] = i \frac{\partial}{\partial \tilde{t}} \Psi_{AW}[\chi, \tilde{t}], \tag{3.6}$$

for the wavefunction Ψ_{AW}, reduces to

$$\left[-i \frac{\partial}{\partial \tilde{t}} - \frac{1}{2} \frac{\partial^2}{\partial \chi^2} + \frac{1}{2} A^2 \chi^2 \right] \Psi_{AW} = 0. \tag{3.7}$$

Following Ref. [3], we look for solutions of the form

$$\Psi_{AW}[\chi,\tilde{t}] = g(\tilde{t})\exp\left[-\frac{1}{2}f(\tilde{t})\chi^2\right]. \tag{3.8}$$

Requiring that the coefficients of χ^0 and χ^2 vanish, we find that eqn. (3.7) reduces to

$$-i\frac{\dot{g}}{g} + \frac{1}{2}f = 0, \tag{3.9}$$

$$i\dot{f} - f^2 + A^2 = 0. \tag{3.10}$$

The solution of eqn. (3.10) is

$$f(\tilde{t}) = A\left(\frac{c_2 e^{iA\tilde{t}} - c_1 e^{-iA\tilde{t}}}{c_2 e^{iA\tilde{t}} + c_1 e^{-iA\tilde{t}}}\right), \tag{3.11}$$

where c_1 and c_2 are A dependent constants of integration. The function g may be adjusted to ensure that the wavefunction is normalized, [3].

The expectation value of the Hamiltonian density in the state described by eqn. (3.8) is

$$\begin{aligned} E_{AW} \equiv \langle 0_{AW}|\mathcal{H}_{AW}|0_{AW}\rangle &= \left\langle 0_{AW}\left|i\frac{\partial}{\partial t}\right|0_{AW}\right\rangle \\ = \frac{i}{a}\left\langle 0_{AW}\left|\frac{\partial}{\partial\tilde{t}}\right|0_{AW}\right\rangle &= \frac{i}{a}\left\langle 0_{AW}\left|\left\{-\frac{1}{2}\dot{f}\chi^2 - \frac{i}{2}f\right\}\right|0_{AW}\right\rangle, \end{aligned} \tag{3.12}$$

where the second step uses the functional Schrödinger equation, the third changes to conformal time, and the fourth uses eqns. (3.8) and (3.9). It may be shown that, with an appropriate choice of $g(\tilde{t})$,

$$\langle 0_{AW}|0_{AW}\rangle = 1, \tag{3.13}$$

$$\langle 0_{AW}|\chi^2|0_{AW}\rangle = \frac{1}{f+f^*}. \tag{3.14}$$

Using eqns. (3.10) and (3.12) – (3.14) we find

$$E_{AW} = \frac{1}{2a}\left(\frac{A^2 + ff^*}{f+f^*}\right), \tag{3.15}$$

and from eqn. (3.11),

$$E_{AW} = \frac{A}{2a}\left(\frac{|c_2|^2 + |c_1|^2}{|c_2|^2 - |c_1|^2}\right). \tag{3.16}$$

The initial condition for the ground state is, [3,4,9],

$$\lim_{a\to 0} E_{AW} = \frac{A}{2a}, \tag{3.17}$$

so, from eqn. (3.16), we must have

$$c_1 = 0, \tag{3.18}$$

and eqns. (3.11), (3.14) and (3.18) give the equal-time Green's function in momentum space

$$\langle 0_{AW} | \chi^2(A, W, \tilde{t}) | 0_{AW} \rangle = \frac{1}{2A}. \tag{3.19}$$

(A derivation of the 'non-equal' times Green's function is given in Appendix B.)

The two-point correlation function is

$$\begin{aligned}
\left\langle 0 \left| \phi(\hat{\Omega}_n, t)\phi(\hat{\Omega}'_n, t) \right| 0 \right\rangle &= \frac{1}{a^{n-1}} \left\langle 0 \left| \chi(\hat{\Omega}_n, t)\chi(\hat{\Omega}'_n, t) \right| 0 \right\rangle \\
&= \frac{1}{2[a(t)]^{n-1}} \int_0^\infty \frac{dA}{A} \sum_W Z_{AW}(\hat{\Omega}_n) \left[Z_{AW}(\hat{\Omega}'_n) \right]^* \\
&= \frac{[\sinh(\hat{\gamma}_n)]^{(2-n)/2}}{2(2\pi)^{n/2}[a(t)]^{n-1}} \int_0^\infty \frac{dA}{A} \left| \frac{\Gamma\left(iA + \frac{n-1}{2}\right)}{\Gamma(iA)} \right|^2 P^{(2-n)/2}_{iA-1/2}\left(\cosh(\hat{\gamma}_n)\right),
\end{aligned} \tag{3.20}$$

where we have used the addition theorem (2.15) in the last step.

We have not yet been able to perform the integral in eqn. (3.20) for even n. For odd n we find, from eqn. 3.15.1(4) on p. 175, eqn. 3.15.2(30) on p. 178 and eqn. 3.6.1(12) on p. 150 of Ref. [15],

$$[\sinh(\hat{\gamma}_n)]^{(2-n)/2} P^{(2-n)/2}_{iA-1/2}(\cosh(\hat{\gamma}_n)) = \sqrt{\frac{2}{\pi}} \frac{\Gamma\left(iA - \frac{n-3}{2}\right)}{\Gamma\left(iA + \frac{n-1}{2}\right)} \left[\frac{d}{d\left[\cosh(\hat{\gamma}_n)\right]} \right]^{(n-1)/2} \frac{\cos(A\hat{\gamma}_n)}{iA}, \tag{3.21}$$

and from eqns. (6.1.17) and (6.1.29) of Ref. [16],

$$\Gamma\left(-iA + \frac{n-1}{2}\right)\Gamma\left(iA - \frac{n-3}{2}\right) = \frac{-\pi}{\sin\left[\pi\left(iA - \frac{n-1}{2}\right)\right]}, \tag{3.22}$$

$$|\Gamma(iA)|^2 = \frac{\pi}{A\sinh(\pi A)}, \tag{3.23}$$

and eqn. (3.20) becomes, for odd n,

$$\left\langle 0 \left| \phi(\hat{\Omega}_n, t)\phi(\hat{\Omega}'_n, t) \right| 0 \right\rangle = \frac{(-1)^{(n-1)/2}}{(2\pi)^{(n+1)/2}[a(t)]^{n-1}} \left[\frac{d}{d\left[\cosh(\hat{\gamma}_n)\right]} \right]^{(n-1)/2} \int_0^\infty \frac{dA}{A} \cos(A\hat{\gamma}_n). \quad (3.24)$$

The integral in eqn. (3.24) is straightforwardly evaluated. When $n = 1$ one recovers the standard $1+1$ dimensional infrared divergent logarithmic correlation. When $n = 3$ we find

$$\left\langle 0 \left| \phi(\hat{\Omega}_3, t)\phi(\hat{\Omega}'_3, t) \right| 0 \right\rangle = \frac{1}{4\pi^2 a^2(t) \sinh(\hat{\gamma}_3)\hat{\gamma}_3}, \quad (3.25)$$

where $\hat{\gamma}_3$ is the angle between $\hat{\Omega}_3$ and $\hat{\Omega}'_3$, eqn. (2.8). When $n = 5$ we find

$$\left\langle 0 \left| \phi(\hat{\Omega}_5, t)\phi(\hat{\Omega}'_5, t) \right| 0 \right\rangle = \frac{1}{8\pi^3 a^4(t)} \left[\frac{\sinh(\hat{\gamma}_5) + \hat{\gamma}_5 \cosh(\hat{\gamma}_5)}{\sinh^3(\hat{\gamma}_5)\hat{\gamma}_5{}^2} \right]. \quad (3.26)$$

On small scales ($\hat{\gamma}_n \to 0$) these equations reduce to

$$\left\langle 0 \left| \phi(\hat{\Omega}_3, t)\phi(\hat{\Omega}'_3, t) \right| 0 \right\rangle = \frac{1}{4\pi^2 a^2(t) \hat{\gamma}_3{}^2} + \cdots, \quad (3.27)$$

$$\left\langle 0 \left| \phi(\hat{\Omega}_5, t)\phi(\hat{\Omega}'_5, t) \right| 0 \right\rangle = \frac{1}{4\pi^3 a^4(t) \hat{\gamma}_5{}^4} + \cdots, \quad (3.28)$$

which is the correct normalization, and is consistent with the expected $(-\sigma^2)^{-1}$ and $(-\sigma^2)^{-2}$ [eqn. (2.10)] dependence, since $-\sigma^2 = a^2\hat{\gamma}_n{}^2 + \cdots$ in this limit. On large scales ($\hat{\gamma}_n \to \infty$) we find

$$\left\langle 0 \left| \phi(\hat{\Omega}_3, t)\phi(\hat{\Omega}'_3, t) \right| 0 \right\rangle = \frac{1}{2\pi^2 a^2(t)} \frac{e^{-\hat{\gamma}_3}}{\hat{\gamma}_3} + \cdots, \quad (3.29)$$

$$\left\langle 0 \left| \phi(\hat{\Omega}_5, t)\phi(\hat{\Omega}'_5, t) \right| 0 \right\rangle = \frac{1}{2\pi^3 a^4(t)} \frac{e^{-2\hat{\gamma}_5}}{\hat{\gamma}_5} + \cdots, \quad (3.30)$$

which drop off even faster than $(-\sigma^2)^{-1}$ and $(-\sigma^2)^{-2}$ (in this limit $-\sigma^2 = a^2 e^{\hat{\gamma}_n} + \cdots$). This is partly because spatial curvature provides an additional length scale, since hyperbolic de Sitter spacetime is not conformally flat.

4. The Minimally Coupled Scalar Field

In this section we generalize the analysis of the previous section to the case of the massive minimally coupled scalar field in hyperbolic de Sitter spacetime.

The action for a minimally coupled complex scalar field ϕ of mass m is

$$S = \int dt\, d^n x \mathcal{L}(x) = \int dt\, d^n x \sqrt{|g|} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi - \frac{1}{2} m^2 |\phi|^2 \right]. \tag{4.1}$$

Repeating the steps in eqns. (3.2) – (3.4) we find that the real and imaginary parts of the conformally rescaled field χ, generically denoted by χ, are governed by the momentum space Hamiltonian density

$$\widetilde{\mathcal{H}}_{AW} = \frac{1}{2} p^2 + \frac{1}{2} \left[A^2 + \left(\frac{m^2}{h^2} - \frac{n^2 - 1}{4} \right) \operatorname{csch}^2(\tilde{t}) \right] \chi^2, \tag{4.2}$$

where p is the canonical momentum conjugate to χ. With the ansatz (3.8) we find that the Schrödinger equation (3.6) reduces to eqn. (3.9) and

$$i\dot{f} - f^2 + A^2 + \left(\frac{1}{4} - \nu^2 \right) \operatorname{csch}^2(\tilde{t}) = 0, \tag{4.3}$$

where we have defined

$$\nu = \left(\frac{n^2}{4} - \frac{m^2}{h^2} \right)^{1/2}. \tag{4.4}$$

The solution of eqn. (4.3) is

$$f(\tilde{t}) = -i \left[\left(\nu - \frac{1}{2} \right) z - \left\{ A^2 + \left(\nu - \frac{1}{2} \right)^2 \right\} \frac{c_1 R^{(1)iA}_{\nu-3/2}(z) + c_2 R^{(2)iA}_{\nu-3/2}(z)}{c_1 R^{(1)iA}_{\nu-1/2}(z) + c_2 R^{(2)iA}_{\nu-1/2}(z)} \right], \tag{4.5}$$

where c_1 and c_2 are A dependent constants of integration,

$$z = -\coth(\tilde{t}) = \sqrt{1 + h^2 a^2(t)}, \tag{4.6}$$

and we have defined the functions

$$R^{(1)iA}_{\nu-1/2}(z) = \Gamma(-iA + \nu + 1/2)\, P^{iA}_{\nu-1/2}(z), \tag{4.7}$$

$$R^{(2)iA}_{\nu-1/2}(z) = \Gamma(-iA+\nu+1/2)\left[P^{iA}_{\nu-1/2}(z) - i\frac{2}{\pi}\sinh(\pi A)\hat{Q}^{iA}_{\nu-1/2}(z)\right], \tag{4.8}$$

$$\hat{Q}^{\mu}_{\nu}(z) = e^{-i\pi\mu}Q^{\mu}_{\nu}(z), \tag{4.9}$$

where P^{μ}_{ν} and Q^{μ}_{ν} are associated Legendre functions of the first and second kind. It may be shown that

$$\left[R^{\binom{1}{2}iA}_{\nu-1/2}(z)\right]^* = R^{\binom{2}{1}iA}_{\nu-1/2}(z), \tag{4.10}$$

and the functions $R^{\binom{1}{2}iA}_{\nu-3/2}(z)$ in eqn. (4.5) are derived from those in eqns. (4.7) and (4.8) by replacing ν with $\nu-1$.

Adjusting the function $g(\tilde{t})$ [eqn. (3.8)] to normalize the wavefunction, and following the derivation of eqns. (3.12) – (3.14), we find that the expectation value of the Hamiltonian density in the state described by eqns. (3.8) and (4.5) is

$$E_{AW} = \frac{1}{2a(f+f^*)}\left[A^2 + \left(\frac{1}{4}-\nu^2\right)\mathrm{csch}^2(\tilde{t}) + ff^*\right]. \tag{4.11}$$

As is fairly evident, when eqn. (4.5) is used to evaluate eqn. (4.11), the expression for E_{AW} is somewhat lengthy so we record here only the limit of this result

$$\lim_{a\to 0}E_{AW} = \frac{A}{2a}\left(\frac{|c_2|^2+|c_1|^2}{|c_2|^2-|c_1|^2}\right); \tag{4.12}$$

comparing to the ground state initial condition of Refs. [3,4,9], eqn. (3.17), we must require

$$c_1 = 0. \tag{4.13}$$

Using eqns. (3.14), (4.5) and (4.13) we find that the equal-time Green's function in momentum space is

$$\langle 0_{AW}|\chi^2(A,W,\tilde{t})|0_{AW}\rangle = \frac{\pi}{2\sinh(\pi A)}P^{iA}_{\nu-1/2}\left(-\coth(\tilde{t})\right)\left[P^{iA}_{\nu-1/2}\left(-\coth(\tilde{t})\right)\right]^*. \tag{4.14}$$

Comparing the Lagrangians (3.1) and (4.1) we see that the massless conformal case is equivalent to the minimally coupled case with a mass given by

$$m^2 = \frac{1}{4}(n^2 - 1)h^2, \tag{4.15}$$

or, from eqn. (4.4), $\nu = 1/2$. It may be shown that in this case eqn. (4.14) reduces to eqn. (3.19), as expected.

The massless minimally coupled case in $3+1$ dimensions corresponds to $\nu = 3/2$, and since

$$P_1^{iA}\left(-\coth(\tilde{t})\right) = -\left[\frac{iA + \coth(\tilde{t})}{\Gamma(2 - iA)}\right] e^{-iA\tilde{t}}, \tag{4.16}$$

eqn. (4.14) reduces to

$$\langle 0_{AW}|\chi^2(A, W, \tilde{t})|0_{AW}\rangle = \frac{1}{2A}\left[1 + \frac{h^2 a^2(t)}{1 + A^2}\right], \tag{4.17}$$

which, on rescaling by the appropriate power of a to get $\langle 0_{AW}|\phi^2(A, W, t)|0_{AW}\rangle$, is consistent with eqns. (33) and (34) of Ref. [17] and eqn. (4.27) of Ref. [11]. We note that at late times the right hand side of eqn. (4.17) is $\propto a^2$, as is the corresponding two-point function in exponentially expanding flat de Sitter spacetime (eqn. (5.18) of Ref. [3]), however, the dependence on spatial momentum in the long wavelength limit are quite different. (A derivation of the 'non-equal' times Green's function is given in Appendix C.)

From eqn. (4.14) we find the position space two-point correlation function:

$$\left\langle 0\left|\phi(\hat{\Omega}_n, t)\phi(\hat{\Omega}'_n, t)\right|0\right\rangle = \frac{[2\pi\sinh(\hat{\gamma}_n)]^{(2-n)/2}}{4[a(t)]^{n-1}}$$
$$\times \int_0^\infty \frac{dA}{\sinh(\pi A)}\left|\frac{\Gamma\left(iA + \frac{n-1}{2}\right)}{\Gamma(iA)}\right|^2 P^{iA}_{\nu-1/2}(z)\left[P^{iA}_{\nu-1/2}(z)\right]^* P^{(2-n)/2}_{iA-1/2}(\cosh(\hat{\gamma}_n)), \tag{4.18}$$

where z is given in eqn. (4.6). We have not yet been able to perform the integral in eqn. (4.18) for even n. For odd n eqns. (3.21) – (3.23) may be used to simplify eqn. (4.18) to

$$\left\langle 0\left|\phi(\hat{\Omega}_n, t)\phi(\hat{\Omega}'_n, t)\right|0\right\rangle = \frac{1}{2[-2\pi a^2(t)]^{(n-1)/2}}\left[\frac{d}{d\left[\cosh(\hat{\gamma}_n)\right]}\right]^{(n-1)/2}$$

$$\times \int_0^\infty \frac{dA}{\sinh(\pi A)} P^{iA}_{\nu-1/2}(z) \left[P^{iA}_{\nu-1/2}(z)\right]^* \cos(A\hat{\gamma}_n). \qquad (4.19)$$

We have not yet been able to compute the general integral in eqn. (4.19). However, if we set $m = 0$ we find, for instance, when $n = 3$,

$$\left\langle 0 \left| \phi(\hat{\Omega}_3, t)\phi(\hat{\Omega}'_3, t) \right| 0 \right\rangle = \frac{1}{4\pi^2 a^2(t) \sinh(\hat{\gamma}_3)} \left[\frac{1}{\hat{\gamma}_3} + \frac{1}{2} h^2 a^2(t) \left\{ e^{\hat{\gamma}_3} E_1(\hat{\gamma}_3) + e^{-\hat{\gamma}_3} Ei(\hat{\gamma}_3) \right\}\right], \qquad (4.20)$$

where $\hat{\gamma}_3$ is the angle between $\hat{\Omega}_3$ and $\hat{\Omega}'_3$, eqn. (2.8), and E_1 and Ei are exponential integrals, Ch. 5 of Ref. [16].

On small scales ($\hat{\gamma}_3 \to 0$) this reduces to

$$\left\langle 0 \left| \phi(\hat{\Omega}_3, t)\phi(\hat{\Omega}'_3, t) \right| 0 \right\rangle = \frac{1}{4\pi^2} \left[\frac{1}{a^2(t)\hat{\gamma}_3{}^2} - h^2 \ln\left(\hat{\gamma}_3 e^{\gamma - 1}\right) - \frac{1}{6a^2(t)} + \cdots \right], \qquad (4.21)$$

where γ is the Euler-Mascheroni constant. The leading term is consistent with eqn. (3.27), as expected. On these small scales spatial curvature is unimportant, and it is not unexpected that the subleading spatially dependent term,

$$-\frac{h^2}{4\pi^2} \ln(\hat{\gamma}_3), \qquad (4.22)$$

agrees with the leading large-scale term for a massless minimally coupled scalar field in $3+1$ dimensional exponentially expanding spatially flat (or closed) de Sitter spacetime, eqn. ($E6$) of Ref. [3] (where we have ignored the zero mode on the n sphere). This is the remnant of the logarithmic infrared divergence of massless minimally coupled scalar field theory in spatially flat (or closed) de Sitter spacetime. [One important difference is that the scale in the logarithm in eqn. ($E6$) of Ref. [3] is set by the Hubble radius, while in eqn. (4.21) here the scale is set by the spatial radius of curvature (which we have set to unity).]

The interesting new effect is that on scales comparable to the spatial curvature scale the logarithmic infrared divergence of spatially flat (or closed) de Sitter spacetime has been tamed. On large scales ($\hat{\gamma}_3 \to \infty$) we find that eqn. (4.20) reduces to

$$\left\langle 0 \left| \phi(\hat{\Omega}_3, t)\phi(\hat{\Omega}'_3, t) \right| 0 \right\rangle = \frac{\left[1 + h^2 a^2(t)\right]}{2\pi^2 a^2(t)} \frac{e^{-\hat{\gamma}_3}}{\hat{\gamma}_3} + \cdots, \tag{4.23}$$

which tends to a time-independent constant at late times, unlike eqn. (3.29).

5. Spontaneously Broken Continuous Symmetries

A discussion of spontaneous symmetry breaking in the $O(2)$ model (a complex scalar field Φ with a standard Φ^4 potential) in de Sitter spacetime is given in Sec. X of Ref. [3] so we can be brief here. Parameterizing the spin-wave excitations along the circle of minima $|\Phi| = \rho$ by $\Phi(x) = \rho\exp[i\theta(x)]$ and defining the free massless field $\phi(x) \equiv \rho\theta(x)$, the correlation function

$$\left\langle \Phi(\vec{x})\Phi^*(\vec{x}') \right\rangle \propto \exp\left[\frac{\langle\phi(\vec{x})\phi(\vec{x}')\rangle}{\rho^2}\right] \tag{5.1}$$

is a diagnostic of symmetry breaking. If this correlation function asymptotically approaches zero for large physical separation the theory is in the symmetric phase; if it asymptotes to a positive constant the theory is in a broken phase.

In spatially flat and spatially closed de Sitter spacetime we found that this correlation function asymptotically approached zero and so spontaneously broken continuous symmetries were restored, [3]. In $3+1$ dimensional hyperbolic de Sitter spacetime we have, from eqn. (4.23),

$$\lim_{\hat{\gamma}_3 \to \infty} \left\langle \phi(\hat{\Omega}_3)\phi(\hat{\Omega}'_3) \right\rangle = 0, \tag{5.2}$$

so the correlation function in eqn. (5.1) asymptotically approaches a positive constant and the symmetry is broken.

It would be of some interest to examine what effect the logarithm in eqn. (4.21) has on symmetry breaking. [In the open de Sitter model the curvature scale can be very much larger than the Hubble scale, so eqn. (4.21) could be valid on fairly large scales, however, the logarithm is most significant on scales comparable to the curvature scale, when eqn. (4.21) is presumably invalid.]

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Appendix A: Addition Theorem (2.15)

We generalize the analysis of Appendix B of Ref. [3] to prove eqn. (2.15).

From symmetry we have

$$P^{(2-n)/2}_{iA-1/2}(\cosh(\hat{\gamma}_n)) = c_3 \sum_W Z_{AW}(\hat{\Omega}_n)\left[Z_{AW}(\hat{\Omega}'_n)\right]^* \tag{A1}$$

where $\hat{\gamma}_n$, the angle between $\hat{\Omega}_n$ and $\hat{\Omega}'_n$, is given by eqn. (2.8), and c_3 is a coefficient which we must determine.

The right hand side of eqn. $(A1)$ is invariant under 'rotations', so we can rotate $\hat{\Omega}'_n$ to the pole at $\theta'_n = 0$. Using eqn. $(B7)$ of Ref. [3]

$$Y_0(\Omega'_{n-1}) = \left[\frac{\Gamma(n/2)}{2\pi^{n/2}}\right]^{1/2}, \tag{A2}$$

and eqn. 3.9.2(3) on p. 163 of Ref. [15],

$$\lim_{\theta'_n\to 0}\left[\sinh(\theta'_n)\right]^{(2-n)/2} P^{(2-n)/2}_{iA-1/2}\left(\cosh(\theta'_n)\right) = \frac{2^{(2-n)/2}}{\Gamma(n/2)}, \tag{A3}$$

we have, from eqn. (2.13),

$$Z_{AW}(\hat{\Omega}'_n = \text{Pole}) = \delta_{W,0}\frac{2^{(1-n)/2}}{\pi^{n/4}}\frac{\Gamma\left(iA + \frac{n-1}{2}\right)}{\Gamma^{1/2}(n/2)\,\Gamma(iA)}, \tag{A4}$$

where $\delta_{W,0}$ is a product of $n-1$ Kronecker deltas. At $\theta'_n = 0$ we have, from eqn. (2.8),

$$\cosh(\hat{\gamma}_n) = \cosh(\theta_n). \qquad (A5)$$

Using eqns. (2.13), ($A1$), ($A2$), ($A4$) and ($A5$), we find

$$c_3 = (2\pi)^{n/2} \left[\sinh(\theta_n)\right]^{(n-2)/2} \left| \frac{\Gamma(iA)}{\Gamma\left(iA + \frac{n-1}{2}\right)} \right|^2, \qquad (A6)$$

and, since $\hat{\gamma}_n = \theta_n$ at $\theta'_n = 0$, eqn. (2.15) follows.

Appendix B: 'Non-Equal' Time Green's Function for Conformally Coupled Massless Scalar Field

Since the Hamiltonian density in eqn. (3.5) coincides with that in eqn. (3.10) of Ref. [3] we may use the results of the analysis of eqns. (3.24) - (3.26) of Ref. [3] to derive the generalization of eqn. (3.19) to 'non-equal' times:

$$\left\langle 0_{AW} \middle| T\chi(A, W, \tilde{t})\chi(A, W, \tilde{t}') \middle| 0_{AW} \right\rangle = \frac{1}{2A} e^{-iA(\tilde{t}-\tilde{t}')}. \qquad (B1)$$

Repeating the analysis of eqns. (3.20) - (3.24), we find, for odd n, with the $\epsilon > 0$ prescription,

$$\left\langle 0 \left| T\phi(\hat{\Omega}_n, t)\phi(\hat{\Omega}'_n, t') \right| 0 \right\rangle$$
$$= \frac{(-1)^{(n-1)/2}}{(2\pi)^{(n+1)/2}\left[a(t)a(t')\right]^{(n-1)/2}} \left[\frac{d}{d\left[\cosh(\hat{\gamma}_n)\right]}\right]^{(n-1)/2} \int_0^\infty \frac{dA}{A} \cosh(A\hat{\gamma}_n) e^{-A(\epsilon + i\widetilde{T})}, \qquad (B2)$$

where $\widetilde{T} = \tilde{t} - \tilde{t}'$, and is related to t and t' through eqn. (2.3).

The integral is straightforwardly evaluated, and for $n = 3$ we find

$$\left\langle 0 \left| T\phi(\hat{\Omega}_3, t)\phi(\hat{\Omega}'_3, t') \right| 0 \right\rangle = \frac{1}{4\pi^2 a(t)a(t')} \frac{\hat{\gamma}_3}{\sinh(\hat{\gamma}_3)} \frac{1}{\hat{\gamma}_3{}^2 - (\widetilde{T} - i\epsilon)^2}. \qquad (B3)$$

Eqn. (3.25) follows from eqn. ($B3$), and eqn. (3.26) may be derived in a similar manner.

Appendix C: 'Non-Equal' Time Green's Function for the Massless Minimally Coupled Scalar Field in 3 + 1 Dimensions

To determine the Green's function for 'non-equal' times in the Schrödinger approach we need the propagator of the functional Schrödinger equation, [18,3]. For the case at hand the Hamiltonian density, eqn. (4.2), is time dependent so this is not a straightforward exercise. For our derivation we shall, instead, use the solution of the wave-equation, eqns. (4.17), (4.23) and (4.25) of Ref. [11]. We find, for the $3+1$ dimensional massless minimally coupled case

$$\langle 0_{AW}|T\phi(A,t)\phi(A,t')|0_{AW}\rangle = \frac{e^{-iA(\tilde{t}-\tilde{t}')}}{2A(A^2+1)a(t)a(t')} \tag{C1}$$
$$\times \left[\sqrt{[1+h^2a^2(t)][1+h^2a^2(t')]} - iA\left\{\sqrt{1+h^2a^2(t')} - \sqrt{1+h^2a^2(t)}\right\} + A^2\right].$$

In the limit when $t = t'$, and with the appropriate rescaling, this expression reduces to eqn. (4.17).

Defining

$$C_1 = \sqrt{[1+h^2a^2(t)][1+h^2a^2(t')]} - 1, \tag{C2}$$

$$C_2 = \sqrt{1+h^2a^2(t')} - \sqrt{1+h^2a^2(t)}, \tag{C3}$$

and repeating the analysis that lead to eqn. (4.19) (specializing to $n = 3$) we have, with the $\epsilon > 0$ prescription,

$$\left\langle 0 \left| T\phi(\hat{\Omega}_3,t)\phi(\hat{\Omega}'_3,t')\right| 0\right\rangle = \frac{-i}{8\pi^2 a(t)a(t')\sinh(\hat{\gamma}_3)} \tag{C4}$$
$$\times \int_0^\infty dA\left[1+\frac{1}{1+A^2}\left\{C_1 + C_2\frac{\partial}{\partial\widetilde{T}}\right\}\right]\left[e^{-A(-i\hat{\gamma}_3+i\widetilde{T}+\epsilon)} - e^{-A(i\hat{\gamma}_3+i\widetilde{T}+\epsilon)}\right],$$

where $\widetilde{T} = \tilde{t} - \tilde{t}'$.

The integral of the part of the integrand that is proportional to $1/(1+A^2)$ is related to the exponential integral, Ch. 5 of Ref. [16]. Similar integrals are evaluated in Appendix A of Ref. [11]; here we only record the result of evaluating $(C4)$. Defining the functions

$$f_{\pm}(z) = e^{z}E_1(z) \pm e^{-z}E_1(-z), \tag{C5}$$

where E_1 is the exponential integral, and defining

$$y_{\pm} = \pm\hat{\gamma}_3 - \widetilde{T} + i\epsilon, \tag{C6}$$

we find that eqn. $(C4)$ becomes

$$\begin{aligned}\left\langle 0 \left| T\phi(\hat{\Omega}_3, t)\phi(\hat{\Omega}'_3, t') \right| 0 \right\rangle = & \frac{1}{4\pi^2 a(t)a(t')\sinh(\hat{\gamma}_3)} \\ \times & \left[\frac{\hat{\gamma}_3}{\hat{\gamma}_3{}^2 - (\widetilde{T} - i\epsilon)^2} + \frac{C_1}{4}\left\{f_-(y_+) - f_-(y_-)\right\} - \frac{C_2}{4}\left\{f_+(y_+) - f_+(y_-)\right\} \right].\end{aligned} \tag{C7}$$

Eqn. (4.20) follows from this equation in the limit when $\widetilde{T} = 0$.

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