Unruh Effect

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Abstract

In the project, we will formulate Unruh Effect.This project is just an effort to learn these topics in detail and hence I claim little or no originality. A large portion of this draft is taken from studying [1, 2, 3] The insight given by the Unruh effect is that a non inertial observer will see particles in the Minkowski vacuum. In this paper we will discuss the inertial observer, describe the Rindler (accelerating) observer in a special coordinate and show that the modes in the general solutions are related by a Bogoliubov transformation. Finally we show that, the expectation value of the number operator for the accelerated observer gives the blackbody spectrum with the temperature.

Contents

1 Introduction

The Unruh effect is a conclusion that an observer moving with uniform acceleration through the Minkowski vacuum observes a thermal spectrum of particles. This is called the Unruh Effect. Hawking effect[1]. Unruh effect shows how different set of observers will describe the same state in very different ways. By showing that a seemingly inert vacuuum has a thermal state, Unruh effect reveals thermal nature of vacuum in quantum field theory.

2 Inertial Observers

As we know all inertial observers agree on the notion of the vacuum. However when $\vec{v}$ is changing, we will be talking about a non inertial reference frame.

$$ds^2 = dt^2 - d\vec{x}^2 \tag{1}$$

Both the stationary observer and the **boosted** observer will agree on the notion of the vacuum. The natural question to ask is what if the observer is not moving with constant velocity but uniformly accelerating. For simplicity we will ignore the y and z coordinates. The Minkowski space would look like the following:

$$ds^2 = dt^2 - dx^2 \tag{2}$$

Relativistically, one could write this as

$$x^2 = t^2 + \frac{1}{\alpha^2} \tag{3}$$

[1]Hawking effect is a thermal radiation with a black body spectrum emitted by black holes.The radiation process reduces the mass of the black hole and also known as black hole evaporation. Unruh effect is conceptually similar to Hawking effect while in the Hawking effect we need curve spacetime coordinates to solve the wave equations.

After this we could parametrize x and t

$$x(s) = \frac{1}{\alpha} Cosh(\alpha s) \tag{4}$$

$$t(s) = \frac{1}{\alpha} Sinh(\alpha s) \tag{5}$$

We define the velocity as,

$$v^\mu = (cosh(\alpha s), sinh(\alpha s)) \tag{6}$$

and acceleration as,

$$a^\mu = a(sinh(\alpha s), cosh(\alpha s)) \tag{7}$$

Note,

$$v^\mu v_\mu = 1 \tag{8}$$

$$\alpha_\mu \alpha^\mu = -\alpha^2 \tag{9}$$

So the trajectory here is hyperbolic which is asymptotic at $x = \pm t$. Physically this means that the observer reaches the speed of light in the inifinite past or the infinite future.

2.1 Light Cone Coordinate

Let us introduce a special coordinate, commonly known as the **light cone coordinates**. Here the x_- and x_+ are light like and the other are spatial.

$$x_- = x - t \tag{10}$$

$$x_+ = x + t \tag{11}$$

That divides the spacetime into four distinct regions.The line element in these coordinates can be written as the following:

$$ds^2 = -dx_+ dx_- \tag{12}$$

3 Accelerating observer and the Rindler's coordinate

We will introduce relativistic polar coordinates in the Minkowski space where proper time axes parametrizes the hyperbola and the other coordinate is the straight line through the origin.

$$x = \frac{1}{\alpha} \rho \ cosh(\alpha\eta) \tag{13}$$

$$t = \frac{1}{\alpha} \rho \ sinh(\alpha\eta) \tag{14}$$

where $\rho > 0$ and η can take any value. In this coordinate the metric would look like the following:

$$ds^2 = -\frac{1}{\alpha^2} d\rho^2 + \rho^2 d\eta^2 \tag{15}$$

This space in region I is known as **Rindler's space** which is shown in the figure below:

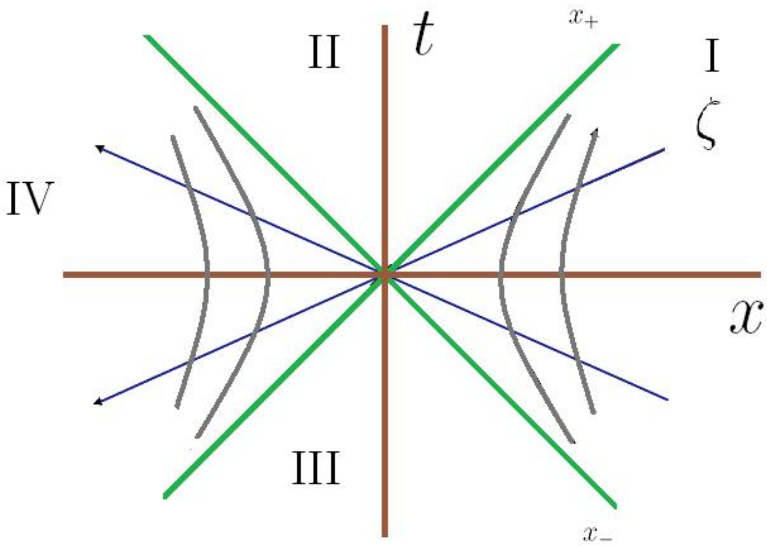

Figure 1: **Minkowski space in Rindler coordinates:** The light cones separate the region in four distinct parts. The part I region is accessible to the accelerated observer in +x direction. We use coordinate (η, ζ) in region I. The horizons represent boundaries of past and future as observed by the accelerated observer.

The ρ curves are given by the hyperbolic equation.

$$x^2 - t^2 = \frac{\rho^2}{\alpha^2} \tag{16}$$

and $\frac{t}{x} = tanh(\alpha\eta)$ For simplicity we will define a quantity

$$\zeta = \frac{1}{\alpha} ln\rho \tag{17}$$

Now the metric for the Rindler space with the above defined coordinates take the following form:

$$\boxed{ds^2 = \left\{ \left(\mathrm{d}\eta^2 - \mathrm{d}\zeta^2 \right) e^{2\alpha\zeta} \right\} \tag{18}}$$

One could write the metric in terms of light cone coordinate

$$ds^2 = -e^{(\alpha\zeta_+ + \alpha\zeta_-)} d\zeta_+ \, d\zeta_- \tag{19}$$

4 Wave Equations

We will consider a massless Klein Gordon equation:

$$\Box\phi = 0 \tag{20}$$

Here the positive energy plane wave solutions:

$$u_k(x,t) = \frac{1}{\sqrt{2\omega}} e^{-i(\omega t - kx)} \tag{21}$$

with and the negative energy solutions are given by u_k^*. As usual we will have a normalization factor $\frac{1}{\sqrt{2\omega}}$ When $k > 0$ we have the right moving wave,

$$u_k(x,t) = \frac{1}{\sqrt{2\omega}} e^{i\omega x_-} \tag{22}$$

This is the analytical function of x_- and is bounded in the upper half x_- plane:

$$|e^{i\omega x_-}| = e^{-\omega B x_-} \leq 1 \tag{23}$$

and when $k < 0$ we have the left moving wave

$$u_k(x,t) = \frac{1}{\sqrt{2\omega}} e^{-i\omega x_+} \tag{24}$$

Analytical function of x_+ and is bounded in lower half of the x_+ plane

$$|e^{i\omega x_+}| = e^{\omega B x_+} \leq 1 \tag{25}$$

We will come back to this in the following section: Now we can write the general solution of the above KG wave equation:

$$\phi = \int \frac{dk}{2\pi} (a_k u_k + a_k^\dagger u_k^*) \tag{26}$$

After we quantize this, the modes a_k and $a_k^\dagger$ becomes the creation and annihilation operators. Now we can define the vacuum,

$$a_k|0>= 0 \tag{27}$$

One could do the same thing for the Rindler accelerating observer.Just like one could write down the Klein Gordon equation in terms of (x,t) we could write down the wave equation for the Rindler observer:

$$\partial_\eta^2 \phi - \partial_\zeta^2 \phi = 0 \tag{28}$$

One crucial difference to note in the case of the Rindler observer is the fact that this is just the one fourth of the total Minkowski space described by η, ζ So we play the same game to obtain the following The right moving ($k > 0$) waves in the positive energy solutions are given by:

$$\bar{u}_k = \frac{1}{\sqrt{2\omega}} e^{i\omega \zeta_-} \tag{29}$$

The left moving wave ($k < 0$) are given by

$$\bar{u}_k = \frac{1}{\sqrt{2\omega}} e^{-i\omega \zeta_+} \tag{30}$$

Now one can analytically continue these plane waves in terms of light cone coordinates which will lead us to two sets of modes [2]. In region I $(+,+)$,

$$\bar{u}_k^1 = \frac{1}{\sqrt{2\omega}} e^{-(i\omega\eta - k\zeta)} \tag{31}$$

whereas in the region IV, $\bar{u}_k^2 = 0$ On the other hand, in region IV

$$\bar{u}_k^2 = \frac{1}{\sqrt{2\omega}} e^{(i\omega\eta - k\zeta)} \tag{32}$$

while in region I, $\bar{u}_k^1 = 0$ One can expand the general solution of the Rindler's wave equation:

$$\boxed{\phi = \int \frac{dk}{2\pi} (\beta_k^1 \overline{u}_k^1 + \beta_k^2 \overline{u}_k^2 + \beta_k^{1\dagger} \overline{u}_k^{1*} + \beta_k^{2\dagger} \overline{u}_k^{2*})} \tag{33}$$

We can define the vacuum as the following:

$$\beta_k^r |\Omega >= 0 \tag{34}$$

Now it is clear that we need to find a relation between the Rindler modes β and inertial modes a. It is non trivial to find the transformation and we will use the argument by Unruh to get the Bogoliubov transformation between these modes which will simply things for us.

5 The inertial modes revisted

In the right moving waves, the solution in region I is extended in region II as well

$$\overline{u}_k^{(1)} = \frac{1}{\sqrt{2\omega}} (\alpha x_-)^{i\omega/\alpha} \tag{35}$$

$$\overline{u}_k^{(1)} = 0 \tag{36}$$

Also we could region IV solution to region III and take the conjugate:

$$\overline{u}_k^{(2)} = 0 \tag{37}$$

$$\overline{u}_k^{(2)*} = \frac{1}{\sqrt{2\omega}} (\alpha x_-)^{i\omega/\alpha} \tag{38}$$

In the above solution we need to clarify $(-)^{i\omega/\alpha}$. The function in the upper half plane will be **analytic** if we impose a **branch cut** in the lower half x_- of the complex plane to avoid the discontinuity. As $x_- < 0$ in region IV, its argument must be π

$$x_- = (-x_-) e^{i\pi} \tag{39}$$

[2] There is no solution that can be analytically extended to region IV in the above picture. So we have to reverse t and x to get the complete set of solutions

$$-x_- = x_- e^{-i\pi} \tag{40}$$

Same thing happen in the region III. Consequently we will have the followsing:

$$\overline{u}_k^{(2)*} = 0 \tag{41}$$

$$\overline{u}_k^{(2)*} = \frac{1}{\sqrt{2\omega}} e^{-\omega\pi/\alpha} (\alpha x_-)^{i\omega/\alpha} \tag{42}$$

Lets rewrite the solution more generally,

$$\psi_k^1 = \overline{u}_k^1 + e^{-\omega\pi/\alpha} \overline{u}_k^{(2)*} = \frac{1}{\sqrt{2\omega}} (\alpha x_-)^{i\omega/\alpha} \tag{43}$$

in the entire four regions in the Minkowski space. Note that this function is analytic in the upper half of the plane We could write it,

$$|(\alpha x_-)^{i\omega/\alpha}| = e^{-(\omega/\alpha) arg x_-} \leq 1 \tag{44}$$

as $0 \leq arg x_- \leq \pi$ in the upper half plane as we placed the branch cut in the lower half plane. This property is the same for $k > 0$ modes in the Minkowski space as we showed in the case of right moving positive energy waves. So we could expand ψ_k in terms of u_k forming a set of positive energy solutions of the Minksowki wave equations.In the exact way we can also get the

$$\psi_k^2 = \overline{u}_k^2 + e^{-\omega\pi/\alpha} \overline{u}_k^{(1)*} = \frac{1}{\sqrt{2\omega}} (\alpha x_-)^{i\omega/\alpha} \tag{45}$$

These form the a set of positive energy solution of the Minkowski wave equation $k < 0$ of the Minkowski wave equation This is the **complete Minkowski Space**. So with this the general solution can be written as:

$$\boxed{\phi = \int \frac{dk}{2\pi} (\chi_k^1 \psi_k^1 + \chi_k^2 \psi_k^2 + \chi_k^{1\dagger} \psi_k^{1*} + \chi_k^{2\dagger} \psi_k^{2*})} \tag{46}$$

where

$$\psi_k^1 = \overline{u}_k^1 + e^{-\omega\pi/\alpha} u_k^{(2)*} = \frac{1}{2\omega} (\alpha x_-)^{i\omega/\alpha} \tag{47}$$

and

$$\psi_k^2 = \overline{u}_k^2 + e^{\omega\pi/\alpha} u_k^{(1)*} = \frac{1}{2\omega} (\alpha x_+)^{-i\omega/\alpha} \tag{48}$$

The vacuum of the inertial observer can be written as

$$\chi_k^r |0> = 0, \qquad\qquad r = 1, 2 \tag{49}$$

which is **equivalent** to $a_k|0> = 0$

6 Relation between Rindler's mode and inertial mode

We want to relate these modes with the accelerated Rindler's model

$$\beta_k^1 = \chi_k^1 + e^{-\pi\omega/\alpha}\chi_k^{2\dagger} \tag{50}$$

$$\beta_k^2 = \chi_k^2 + e^{-\pi\omega/\alpha}\chi_k^{1\dagger} \tag{51}$$

We can properly normalize the χ modes using the commutation relations of $[\beta_k^r, \beta_{k'}^s] = \delta^{rs}2\pi\delta(k-k')$ This leads to the commutation relation:

$$[\chi_k^r, \chi_{k'}^{s\dagger}] = \delta^{rs}2\pi\delta(k-k') \tag{52}$$

Now this finally will lead us to to Rindler's modes written in terms of creation and annhilation of the inertial modes. We define:

$$c_k^r = e^{-\pi\omega/2\alpha}\sqrt{2Sinh(\pi\omega/2\alpha)}\chi_k^r \tag{53}$$

Now the transformation we obtain is known as **Bogolioubov transformation**.

$$\beta_k^1 = \frac{1}{\sqrt{2Sinh(\pi\omega/\alpha)}}\{e^{\frac{\pi\omega}{2\alpha}}c_k^1 + e^{-\frac{\pi\omega}{2\alpha}}c^{2}{}_k^{\dagger}\} \tag{54}$$

$$\beta_k^2 = \frac{1}{\sqrt{2Sinh(\pi\omega/\alpha)}}\{e^{\frac{\pi\omega}{2\alpha}}c_k^2 + e^{-\frac{\pi\omega}{2\alpha}}c^{1}{}_k^{\dagger}\} \tag{55}$$

The number operator for the Rindler observer is given by the following. Note that $\beta^{2\dagger}$ is not accessible to the Rindler observer.

$$N_k = \beta^{1}{}_k^{\dagger}\beta_k^1 \tag{56}$$

This leads to the expectation value of the number operator turns out to be the following:

$$<0|N_K|0>= \frac{e^{\pi\omega/\alpha}}{2Sinh(\pi\omega/\alpha)} <0|c_k^2c^{2}{}_k^{\dagger}|0> \tag{57}$$

$$<0|N_K|0>= \frac{1}{e^{2\pi\omega/\alpha}-1}2\pi\delta(0) \tag{58}$$

which is a blackbody spectrum with the following temperature.[3]

$$\boxed{T = \frac{\hbar\alpha}{2\pi ck_B}} \tag{59}$$

Interestingly, the Rindler's observer is going through some some hot radiation in the vacuum.

[3] The delta function is an artifact of plane wave basis modes. If we had constructed normalized wave packet then we would have obtained a finite result with the identical spectrum.

7 Conclusion

So the insight given by the Unruh effect is that a non inertial observer will see particles in the Minkowski vacuum. On the other hand the Minkowski space will see no effect in the same state. By showing that the expectation value of the number operator for the accelerated observer turns out to be the blackbody spectrum with the temperature we have shown glimpse of deep relation between quantum field theory and thermal nature of vacuum. Note that for the Minkowski observer, the expectation value of the energy momentum tensor $T_{\mu\nu} = 0$. So if $T_{\mu\nu}$ is zero how can one detect particle. The argument is subtle. In order to detect a particle one needs to carry a detector of some sort and in the above paradigm the detector moves with a constant acceleration. This results in work being done in the detector in order to ensure that it keeps accelerating. So from the reference of Minkowski observer Rindler detector emits and absorb the particle. So the inertial observer will see the emission of a particle as detector detects the particle and also feel a radiation reaction force.

References

[1] G. Siopsis, *Quantum Field Theory lecture notes, http://aesop.phys.utk.edu/qft/*

[2] R. Wald, *Quantum Field Theory in Curved spacetime and Black Hole Thermodynamics*

[3] S. Caroll, *Spacetime and Geometry: An Introduction to General Relativity*

[4] T.A Jacobson, *Introductory lecture on Black Hole Thermodynamics, www.fys.ruu.nl/ wwwthe/lectures/itfuu-0196.ps*