Coordinate systems in De Sitter spacetime

Bachelor thesis

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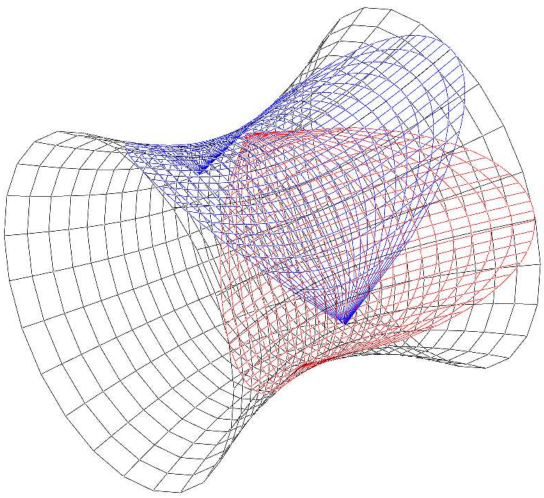

Abstract

The De Sitter metric is a solution for the Einstein equation with positive cosmological constant, modelling an expanding universe. The De Sitter metric has a coordinate singularity corresponding to an event horizon. The physical properties of this horizon are studied. The Klein-Gordon equation is generalized for curved spacetime, and solved in various coordinate systems in De Sitter space.

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Cover illustration

Projection of two-dimensional De Sitter spacetime embedded in Minkowski space. Three coordinate systems are used: global coordinates, static coordinates, and static coordinates rotated in the (x_1, x_2)-plane.

Contents

Preface

For ages, people have studied the motion of objects. These objects could be close to home, like marbles on a table, or far away, like planets and stars. They found out that objects could exert forces on each other on contact, like marbles bumping into one another. Stellar matters seemed more puzzling, because there seemed to be nothing actually making contact with the stars.

Isaac Newton solved this by his law of gravitation. Newton stated that there could be a gravitational force acting on a distance. To many, this seemed a preposterous idea, but Newton's law proved to be very successful. The motion of stars and planets could be predicted accurately and, even on earth, the gravitational law could be tested, e.g. by Cavendish' experiment.

Newton, however, did not state *what* the gravitational force is. In his own words, Newton did not "frame hypotheses." The origins of the gravitational force did not become clear until Einstein's theory of General Relativity of 1915. In Einstein's hypothesis, mass curves spacetime. A free-falling particle follows a geodesic in spacetime. In flat space, this geodesic is a straight line, and we conclude that the particle is at rest. In a curved spacetime, the geodesic is curved. We conclude that something 'exerts a force' on the particle, and we call this phenomenon gravity.

To me, curvature of spacetime is a very intuitive explanation of what a force is. This is the reason why I wanted to study General Relatvity for my bachelor thesis. Furthermore, GR is a subject that is somewhat neglected in the bachelor curriculum in Nijmegen. Although it is one of the most successful theories we have about the fundaments of physics and the universe, only one course is dedicated to this field. That is a pity, for GR is a most elegant theory.

The quantitative description of mass curving spacetime is given by the Einstein equation. This is a nonlinear second order differential equation, and therefore very difficult to solve. Only under severe restrictions on the symmetry of spacetime, the Einstein equation has been solved in a closed form. The De Sitter solution is one of these solutions. As we shall see, De Sitter spacetime is a model for an empty expanding spacetime.

De Sitter space has an interesting property: the metric has a degeneracy at a fixed distance from the origin of the coordinate system. This means that there must be some kind of horizon; this horizon depends on the coordinate sytem's origin. How does this singularity transform under a coordinate transformation? This will be the first part of this thesis.

The second part is about quantum mechanics. Quantum mechanics has proved to be another most successful theory. However, QM does not incorporate curved spacetimes. My goal for the second part was to see how one could do quantum mechanics in De Sitter space.

The reader is expected to be familiar with some concepts of GR, like tensor calculus, covariant derivatives et cetera. The reader should also know non-relativistic quantum mechanics, as well as the basics of relativistic QM. Some of these ideas are reviewed in chapter 1.

Chapter 1

Introduction

In this chapter, I will review some concepts of general relativity. We shall derive the Einstein field equations and the geodesic equations from the principle of mimimal action. Then, we shall take a look at the De Sitter solution to the Einstein equation. We then review the Klein-Gordon equation, and its generalization to curved spacetimes.

1.1 The Einstein field equations

The following derivation of the Einstein field equation is based on [1]. The Lagrange formalism is based on the principle of minimal action. From variational calculus, it is known that for this minimum, the action is stationary. This is represented by

$$\delta S = 0 \tag{1.1}$$

We define the action for general relativity to be

$$S_G = \frac{1}{2\kappa}\int_{\mathcal{M}} \mathcal{L}(g_{\mu\nu})\sqrt{-g}\ \mathrm{d}^4x \tag{1.2}$$

This integral goes over the entire spacetime $\mathcal{M}$; $\mathcal{L}$ is called the Lagrangian density. $g_{\mu\nu}$ is the metric tensor; g is the determinant of this tensor. The constant κ is chosen such that the weak field limit gives Newtonian gravity.

Now, we propose the following Lagrangian density:

$$\mathcal{L}(g_{\mu\nu}) = R - 2\Lambda \tag{1.3}$$

In this equation, R is the Ricci scalar. This is an object obtained by contracting the Riemann tensor two times. Λ is called the cosmological constant; as we will see later, this is a measure for the expansion of the universe.

To show that the Einstein field equations follow from this Lagrangian, we choose a point P such that a small variation of the metric and its derivative are $\delta g_{\mu\nu} = \delta g_{\mu\nu,\lambda} = 0$. Here, the comma notation is used to denote the partial derivative to the λ-th coordinate. Next, we rewrite the action in terms of the Ricci tensor:

$$S_G = \frac{1}{2\kappa}\int_{\mathcal{M}} \left(R_{\mu\nu}g^{\mu\nu}\sqrt{-g} - 2\Lambda\sqrt{-g}\right)\ \mathrm{d}^4x \tag{1.4}$$

This gives for the variation of the action:

$$\delta S_G = \frac{1}{2\kappa}\int_{\mathcal{M}} \Big(\delta\left(R_{\mu\nu}\right)g^{\mu\nu}\sqrt{-g} + R_{\mu\nu}\delta\left(g^{\mu\nu}\sqrt{-g}\right) - 2\Lambda\delta\sqrt{-g}\Big)\ \mathrm{d}^4x \tag{1.5}$$

According to the local flatness theorem, there exists a coordinate system such that in $\mathcal{V}$, the Christoffel symbols vanish; the Ricci tensor is then given by

$$R_{\alpha\beta}|_{\mathcal{V}} = \Gamma^{\rho}{}_{\beta\alpha,\rho} - \Gamma^{\rho}{}_{\rho\alpha,\beta} + \Gamma^{\rho}{}_{\rho\lambda}\Gamma^{\lambda}{}_{\beta\alpha} - \Gamma^{\rho}{}_{\beta\alpha}\Gamma^{\lambda}{}_{\rho\alpha}\Big|_{\mathcal{V}} = \Gamma^{\rho}{}_{\beta\alpha,\rho} - \Gamma^{\rho}{}_{\rho\alpha,\beta} \tag{1.6}$$

Symmetry in the bottom indices of the Christoffel symbols then gives

$$R_{\alpha\beta} = \Gamma^{\rho}{}_{\beta\alpha,\rho} - \Gamma^{\rho}{}_{\alpha\rho,\beta} \tag{1.7}$$

Taking the variation commutes with taking partial derivatives, so we have

$$\delta R_{\alpha\beta} = \delta\left(\Gamma^{\rho}{}_{\beta\alpha}\right)_{,\rho} - \delta\left(\Gamma^{\rho}{}_{\alpha\rho}\right)_{,\beta} \tag{1.8}$$

As said, local flatness implies $g_{\mu\nu,\lambda} = 0$, so the metric can be pulled inside the partial derivative:

$$g^{\mu\nu}\delta R_{\mu\nu} = \left(g^{\mu\nu}\delta\Gamma^{\lambda}{}_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^{\nu}{}_{\mu\nu}\right)_{,\lambda} \tag{1.9}$$

Define the vector $\mathbf{A}$ by $A^{\lambda} \equiv g^{\mu\nu}\delta\Gamma^{\lambda}{}_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^{\nu}{}_{\mu\nu}$, so we can write

$$g^{\mu\nu}\delta R_{\mu\nu} = A^{\lambda}{}_{,\lambda} \tag{1.10}$$

This is a divergence, so we may use Gauss' integral theorem says $\int_{\mathcal{V}} A^{\lambda}{}_{,\lambda}\ \mathrm{d}V = \oint_{\mathcal{S}} \mathbf{A}\cdot\mathbf{n}\ \mathrm{d}S$. This means the integral depends only on the behaviour on the boundary of $\mathcal{V}$. Furthermore, we demanded that all variations of derivatives of the metric vanished, so that $\Gamma^{\lambda}{}_{\mu\nu} = 0$. This means that the integral over the boundary is zero, and therefore the integral over the volume vanishes. Thus, in equation (1.5), the first term vanishes.

Next, we calculate the last term in equation (1.5). Taking the variation of the determinant gives

$$\delta\sqrt{-g} = \frac{\partial\sqrt{-g}}{\partial g_{\alpha\beta}}\delta g_{\alpha\beta} = \frac{1}{2\sqrt{-g}}\frac{\partial g}{\partial g_{\alpha\beta}}\delta g_{\alpha\beta} \tag{1.11}$$

For the derivative of the determinant, we use a theorem in matrix theory [2]:

$$\frac{\partial \det A}{\partial A_{ij}} = \det A\left(A^{-1}\right)_{ji} \tag{1.12}$$

which gives

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta} \tag{1.13}$$

With the product rule for differentiation and the equation above, the second term in equation (1.5) is written as

$$\delta\left(g^{\mu\nu}\sqrt{-g}\right) = \sqrt{-g}\left(\delta g^{\mu\nu} + \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\delta g_{\alpha\beta}\right) \tag{1.14}$$

Using the product rule, we obtain by varying $\delta\left(g^{\mu\nu}g_{\mu\nu}\right) = \delta\left(\delta^{\mu}_{\nu}\right)$

$$\delta g_{\alpha\beta} = -g_{\alpha\mu}g_{\nu\beta}\delta g^{\mu\nu} \tag{1.15}$$

Inserting equations (1.13) and (1.14) in (1.5) and using the above identities gives

$$\delta S_G = \frac{1}{2\kappa}\int\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right)\sqrt{-g}\delta g^{\mu\nu}\ \mathrm{d}^4x \tag{1.16}$$

This integral should vanish for *any* variation $\delta g^{\mu\nu}$. This can only be when the integrand is zero, giving

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0 \tag{1.17}$$

These are the Einstein equations in vacuum. For a non-vacuum spacetime, it can be shown that the Einstein equations take the form of

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = -\frac{8\pi G}{c^4}T_{\alpha\beta} \tag{1.18}$$

where $T_{\alpha\beta}$ is the stress-energy tensor.

The indices in the equation above can be contracted to give

$$R - \frac{1}{2} \cdot 4R + 4\Lambda = -\frac{8\pi G}{c^4}T \Rightarrow R = \frac{8\pi G}{c^4}T + 4\Lambda \tag{1.19}$$

where T is the contraction of the stress-energy tensor. Use this identity to rewrite equation (1.18):

$$R_{\alpha\beta} - \Lambda g_{\alpha\beta} = -\frac{8\pi G}{c^4}\left(T_{\alpha\beta} + Tg_{\alpha\beta}\right) \tag{1.20}$$

This is called the trace-reversed Einstein equation.

1.2 The geodesic equations

Particles under the influence of a gravitational field are free particles. Their equation of motion is the geodesic equation; in this section, the geodesic equation is derived. The derivation is based on [3].

In classical dynamics, the Lagrangian of a free particle is given by

$$\mathcal{L} = \frac{1}{2}M\left\|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right\|^2 \tag{1.21}$$

The relativistic analogon of this equation gives the Lagrangian of a free particle in a gravitational field:

$$\mathcal{L} = \frac{1}{2}Mg_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \tag{1.22}$$

In order to calculate the path of the particle, choose a parameterization $x^\mu \to x^\mu(s)$. The Euler-Lagrange equations are then given by

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) - \frac{\partial\mathcal{L}}{\partial x^\mu} = 0 \tag{1.23}$$

First, we calculate the derivatives:

$$\frac{\partial\mathcal{L}}{\partial x^\beta} = \frac{1}{2}g_{\mu\nu,\beta}\dot{x}^\mu\dot{x}^\nu; \qquad \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\beta}\right) = g_{\beta\nu}\ddot{x}^\nu + \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\partial g_{\beta\nu}}{\partial x^\mu}\dot{x}^\nu = g_{\beta\nu}\ddot{x}^\nu + \frac{\partial g_{\beta\nu}}{\partial x^\mu}\dot{x}^\mu\dot{x}^\nu$$

This can be put in equation (1.22) to obtain

$$\left(g_{\beta\nu}\ddot{x}^\nu + g_{\beta\nu,\mu}\dot{x}^\mu\dot{x}^\nu\right) - \frac{1}{2}g_{\mu\nu,\beta}\dot{x}^\mu\dot{x}^\nu \tag{1.24}$$

By rewriting $g_{\beta\nu,\mu}\dot{x}^\mu\dot{x}^\nu$, we can express this as

$$0 = g_{\beta\nu}\ddot{x}^\nu + \frac{1}{2}\left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right)\dot{x}^\mu\dot{x}^\nu \tag{1.25}$$

This can be multiplied by $g^{\mu\nu}$ to give

$$\ddot{x}^\alpha + \frac{1}{2}g^{\alpha\beta}\left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right)\dot{x}^\mu\dot{x}^\nu = \ddot{x}^\alpha + \Gamma^\alpha{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \tag{1.26}$$

This is the geodesic equation, the equation of motion for a free particle in curved spacetime.

1.3 De Sitter space

Now we have seen how to derive the Einstein equations, it is time to find solutions. The Einstein equations are non-linear second order differential equations; therefore, it is notoriously difficult to find solutions in closed form. However, with a few additional requirements, a solution can be found.

First, we neglect any mass terms, so we obtain an empty universe. Therefore, we take the stress-energy tensor and the contracted stress-energy tensor in equation (1.20) to be zero. The Einstein equation then becomes

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{1.27}$$

Static coordinates

In this section, we proceed as in [4]. We are looking for a static metric with spherical symmetry. This means it is of the form

$$\mathrm{d}s^2 = f(r)\mathrm{d}t^2 - g(r)\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2 \mathrm{d}\phi^2\right) \tag{1.28}$$

Here, $f(r)$ and $g(r)$ are real functions of r. We take the signature of the metric $(+ - - -)$. This is a matter of convention; in this thesis, I have used both signatures. It is therefore necessary to specify in each calculation the signature used.

The signature of the metric specifies that $f(r)$ and $g(r) \geq 0$; this means we can write the metric as

$$\mathrm{d}s^2 = \mathrm{e}^{A(r)}\mathrm{d}t^2 - \mathrm{e}^{B(r)}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2 \mathrm{d}\phi^2\right) \tag{1.29}$$

The components of the Ricci tensor are calculated to be

$$\begin{aligned} R_{tt} =& -\mathrm{e}^{A-B}\left(\frac{1}{2}A'' - \frac{1}{4}A'B' + \frac{1}{4}\left(A'\right)^2 + \frac{A'}{r}\right) \\ R_{rr} =& \frac{1}{2}A'' - \frac{1}{4}A'B' + \frac{1}{4}\left(A'\right)^2 - \frac{B'}{r} \\ R_{\theta\theta} =& \mathrm{e}^{-B}\left(1 + \frac{1}{2}r\left(A' - B'\right)\right) - 1 \\ R_{\phi\phi} =& R_{\theta\theta}\sin^2\theta \end{aligned} \tag{1.30}$$

where $A' = \frac{\mathrm{d}A}{\mathrm{d}r}$ and $B' = \frac{\mathrm{d}B}{\mathrm{d}r}$. All other components are equal to zero. Setting these components equal to $\Lambda g_{\mu\nu}$ gives for the (tt) and (rr) components:

$$\begin{aligned} \Lambda\mathrm{e}^{A} =& -\mathrm{e}^{A-B}\left(\frac{1}{2}A'' - \frac{1}{4}A'B' + \frac{1}{4}(A')^2 + \frac{A'}{r}\right) \\ -\Lambda\mathrm{e}^{B} =& \frac{1}{2}A'' - \frac{1}{4}A'B' + \frac{1}{4}(A')^2 - \frac{B'}{r} \end{aligned} \tag{1.31}$$

A little calculation shows that from this follows

$$A' = -B' \rightarrow A = -B \tag{1.32}$$

The integration constant is taken zero here; otherwise, it would result in a factor in the $\mathrm{d}t^2$ terms. By a rescaling of t, this factor could be absorbed anyway. The (tt) equation now gives

$$\mathrm{e}^{A}\left(1 + rA'\right) = 1 - \Lambda r^2 \tag{1.33}$$

Substitute $\alpha \equiv \mathrm{e}^{A(r)}$. This gives the differential equation

$$\alpha + r\alpha' = 1 - \Lambda r^2 \tag{1.34}$$

This can be written as

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\alpha) = \frac{\mathrm{d}}{\mathrm{d}r}\left(r - \frac{\Lambda}{3}r^3\right) \tag{1.35}$$

giving

$$r\alpha = r - \frac{\Lambda}{3}r^3 + M \tag{1.36}$$

where M is the integration constant. Putting this in the metric (1.29) we obtain the De Sitter-Schwarzschild metric:

$$\mathrm{d}s^2 = \left(1 - \frac{M}{r} - \frac{1}{3}\Lambda r^2\right)\mathrm{d}t^2 - \frac{1}{1 - \frac{M}{r} - \frac{1}{3}\Lambda r^2}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \tag{1.37}$$

The parameter M corresponds to a spherical mass in the origin. For $\Lambda = 0$, we obtain the Schwarzschild solution to the Einstein equation; this metric models the curvature of spacetime under the influence of a mass M in the origin of the coordinate system.

For $M = 0$, this metric reduces to the De Sitter metric:

$$\mathrm{d}s^2 = \left(1 - \frac{1}{3}\Lambda r^2\right)\mathrm{d}t^2 - \frac{1}{1 - \frac{1}{3}\Lambda r^2}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \tag{1.38}$$

The coordinates in this metric are also called static, because they do not depend explicitly on the time coordinate t.

The De Sitter horizon

First, we rewrite the De Sitter metric using the definition $\ell \equiv \sqrt{\frac{3}{\Lambda}}$:

$$\mathrm{d}s^2 = \left(1 - \frac{r^2}{\ell^2}\right)\mathrm{d}t^2 - \frac{1}{1 - \frac{r^2}{\ell^2}}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \tag{1.39}$$

At $r = \ell$, the coefficient in front of $\mathrm{d}t^2$ vanishes, whereas the coefficient of $\mathrm{d}r^2$ diverges. This means the metric has a degeneracy at $r = \ell$; this degeneracy corresponds to some kind of horizon. It is peculiar that this horizon depends on the choice of origin. One may choose two coordinate patches of De Sitter spacetime with different origins. These coordinate systems have different horizons.

In the overlapping region of the two coordinate patches, one can calculate the coordinate transformation. What is the coordinate transformation in this region? And what happens to the De Sitter horizon? These questions are discussed in chapter 2.

Another way to study the horizon, is to calculate the geodesics. The properties of the geodesics may tell us something about whether the horizon is a property of De Sitter space, or is a consequence of the choice of coordinates. This is done in chapter 3.

Flat slicing coordinates

The Sitter space can be parameterized by various coordinate systems, just as one can choose various coordinate systems for flat space. Another choice for coordinates in De Sitter space can be obtained by choosing a metric of the form

$$\mathrm{d}s_F^2 = \mathrm{d}\tau^2 - g(\tau)\mathrm{d}\rho^2 - g(\tau)\rho^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \tag{1.40}$$

Calculating the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, we get

$$G_{\mu\nu} = \begin{pmatrix} \frac{3}{4}\frac{\dot{g}(\tau)^2}{g(\tau)^2} & & & \\ & -\ddot{g}(\tau) + \frac{1}{4}\frac{\dot{g}(\tau)^2}{g(\tau)} & & \\ & & \left(-\ddot{g}(\tau) + \frac{1}{4}\frac{\dot{g}(\tau)^2}{g(\tau)}\right)\rho^2 & \\ & & & \left(-\ddot{g}(\tau) + \frac{1}{4}\frac{\dot{g}(\tau)^2}{g(t)}\right)\rho^2 \sin^2\theta \end{pmatrix} \tag{1.41}$$

From the (00) component in the Einstein equation, one derives

$$\dot{g}(\tau) = \sqrt{-\frac{4}{3}\Lambda} g(\tau) \Rightarrow g(\tau) = A\mathrm{e}^{\sqrt{-\frac{4}{3}\Lambda}\tau} \tag{1.42}$$

One can check that this expression for $g(\tau)$ is consistent for the other components of the Einstein equation. The minus sign in the square root can be absorbed in the cosmological constant Λ. By a rescaling of the radial coordinate ρ, the integration constant A can be taken equal to one. Thus, we can write the metric in the form

$$\mathrm{d}s_F^2 = \mathrm{d}\tau^2 - \mathrm{e}^{2\tau/\ell}\left(\mathrm{d}\rho^2 + \rho^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right) \tag{1.43}$$

The coordinates corresponding to this metric are called flat slicing coordinates.

The latter part of the metric can be seen as the spatial part of spacetime. As time τ progresses, distances in the spatial part increase by virtue of the exponential $\mathrm{e}^{2\tau/\ell}$. This metric is therefore a model for an expanding universe.

Embedding in Minkowski space

De Sitter space can be embedded in five-dimensional Minkowski space as the hypersurface given by [5]

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = \ell^2 \tag{1.44}$$

where we have used Cartesian coordinates in Minkowski space. De Sitter space is therefore a hyperboloid. De Sitter coordinates can be retrieved using the parameterization [6]:

$$\begin{aligned} x_0 &= -\sqrt{\ell^2 - r^2}\sinh\left(\frac{t}{\ell}\right) & x_1 &= -\sqrt{\ell^2 - r^2}\cosh\left(\frac{t}{\ell}\right) \\ x_2 &= r\cos\theta & x_3 &= r\sin\theta\cos\phi \\ x_4 &= r\sin\theta\sin\phi & & \end{aligned} \tag{1.45}$$

Note that $x_2 \ldots x_4$ are a parameterization of a sphere with radius r. The De Sitter metric can be retrieved by calculating the induced metric:

$$\begin{aligned} \mathrm{d}x_0 = \mathrm{d}\left(-\sqrt{\ell^2 - r^2}\sinh\left(\frac{t}{\ell}\right)\right) &= -\frac{1}{\ell}\sqrt{\ell^2 - r^2}\cosh\left(\frac{t}{\ell}\right)\mathrm{d}t + \frac{r}{\sqrt{\ell^2 - r^2}}\sinh\left(\frac{t}{\ell}\right)\mathrm{d}r \\ \mathrm{d}x_1 = \mathrm{d}\left(-\sqrt{\ell^2 - r^2}\cosh\left(\frac{t}{\ell}\right)\right) &= -\frac{1}{\ell}\sqrt{\ell^2 - r^2}\sinh\left(\frac{t}{\ell}\right)\mathrm{d}t + \frac{r}{\sqrt{\ell^2 - r^2}}\cosh\left(\frac{t}{\ell}\right)\mathrm{d}r \\ \mathrm{d}x_i = \omega^{(i)}\mathrm{d}r &+ r\mathrm{d}\omega^{(i)} \quad (i = 2 \ldots 4) \end{aligned} \tag{1.46}$$

where $\omega^{(i)}$ is a parameterization for the 2-sphere. Putting this in the Minkowski metric

$$\mathrm{d}s_M^2 = -\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2 \tag{1.47}$$

one obtains the De Sitter metric

$$\mathrm{d}s^2 = -\left(1 - \frac{r^2}{\ell^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{r^2}{\ell^2}} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \tag{1.48}$$

Notice that the signature of this metric is $(-+++)$, as opposed to the metric derived in section 1.3.

Global coordinates

The De Sitter hyperboloid can be parameterized as follows [6]:

$$x_0 = \ell \sinh\left(\frac{\tau}{\ell}\right) \qquad x_j = \ell\cosh\left(\frac{\tau}{\ell}\right)\omega^{(j)} \qquad (j = 1 \ldots 4) \tag{1.49}$$

where $\omega^{(j)}$ denotes a parameterization of a unit 3-sphere. Calculating the metric of this parameterization gives

$$\mathrm{d}s_g^2 = -\mathrm{d}\tau^2 + \ell^2\cosh^2\left(\frac{\tau}{\ell}\right)\mathrm{d}\Omega_3^2 \tag{1.50}$$

with $\mathrm{d}\Omega_3^2$ the metric of the unit 3-sphere. Again, we have used the $(-+++)$ sign convention. For reasons that become clear in the next chapter, these coordinates are called global coordinates.

1.4 The Klein-Gordon equation

The next topic which we discuss is quantum mechanics in curved spacetime. Quantum mechanics in Minkowski space is well-known, and the basis for the Standard Model. In curved spacetime, gravitational effects have to be taken into account.

Special relativistic quantum mechanics is governed by two equations: the Klein-Gordon equation for spin-0 particles and the Dirac equation for spin-$\frac{1}{2}$ particles. The Klein-Gordon equation is derived from the relativistic energy-momentum equation for a free particle [7]:

$$p^\nu p_\nu - M^2c^2 = 0 \tag{1.51}$$

The quantum mechanical version is obtained by replacing the classical quantities by operators:

$$p_\nu \to i\hbar\partial_\nu \tag{1.52}$$

where ∂_ν denotes the partial derivative with respect to x^ν. Letting these operators act on a wavefunction ψ, we get the Klein-Gordon equation:

$$-\hbar^2\partial^\nu\partial_\nu\psi - M^2c^2\psi = 0 \tag{1.53}$$

which can be rewritten as

$$\partial^\nu\partial_\nu\psi + \mu^2\psi = 0 \tag{1.54}$$

where we have substituted $\mu \equiv \frac{Mc}{\hbar}$.

Although this equation has been very successful in predicting the behaviour of particles in flat space, it is not valid in curved spacetime. The reason for this is that the partial derivatives are not invariant under a coordinate transformation. In order to obtain this invariance, we apply the comma-goes-to-semicolon rule: replace each partial derivative by the covariant derivative. This generalizes the Klein-Gordon equation to curved spacetime:

$$\nabla^\nu\nabla_\nu\psi + \mu^2\psi = 0 \tag{1.55}$$

These covariant derivatives can be written in terms of the Christoffel symbols. The equation then becomes

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\psi - g^{\alpha\beta}\Gamma^\lambda{}_{\alpha\beta}\partial_\lambda\psi + \mu^2\psi = 0 \tag{1.56}$$

What is the solution for De Sitter space? In chapter 4, we will solve this equation for various coordinate systems.

Chapter 2

Coordinate transformations

As we have seen in the last chapter, the De Sitter metric has a horizon at $r = \ell$. This singularity is dependent on the choice of origin. If we choose two coordinate systems with different origins, we should have two different horizons. If the origins are near each other, the patches described by the systems may overlap. What is the transformation between the coordinate systems? And what happens to the De Sitter horizon? These questions will be discussed here.

First, we will derive coordinate transformations between the static coordinate system and other types of coordinates. We know that flat slicing coordinates and global coordinates have no degeneracies. Next, we will investigate the De Sitter group in order to generate coordinate systems with a translated origin. We will then see how the De Sitter horizon behaves under these coordinate transformations.

2.1 Transformations to non-singular coordinates

The De Sitter hyperboloid can be parameterized in various coordinate systems. Not all of these systems have a coordinate singularity like static coordinates. In this section, we investigate the behaviour of the De Sitter horizon when observed in coordinate systems without singularity.

From static to global coordinates

Recall from chapter 1 that the De Sitter hyperboloid can be parameterized in static and global coordinates by

$$\begin{aligned}
x_0 &= -\sqrt{\ell^2 - r^2}\sinh\left(\frac{t}{\ell}\right) = \ell\sinh\left(\frac{\tau}{\ell}\right)\\
x_1 &= -\sqrt{\ell^2 - r^2}\cosh\left(\frac{t}{\ell}\right) = \ell\cosh\left(\frac{\tau}{\ell}\right)\cos\theta_1\\
x_2 &= r\cos\theta = \ell\cosh\left(\frac{\tau}{\ell}\right)\sin\theta_1\cos\theta_2\\
x_3 &= r\sin\theta\cos\phi = \ell\cosh\left(\frac{\tau}{\ell}\right)\sin\theta_1\sin\theta_2\cos\theta_3\\
x_4 &= r\sin\theta\sin\phi = \ell\cosh\left(\frac{\tau}{\ell}\right)\sin\theta_1\sin\theta_2\sin\theta_3
\end{aligned} \tag{2.1}$$

where the metric has signature $(- + + +)$. We have used coordinates (t, r, θ, ϕ) for static coordinates and $(\tau, \theta_1, \theta_2, \theta_3)$ for global coordinates. Note that we have parameterized the 3-sphere in the global system with polar coordinates.

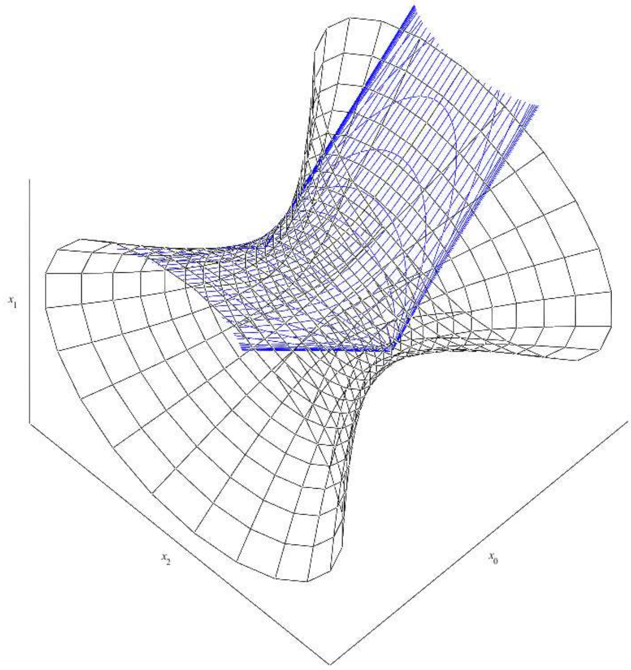

Figure 2.1: Parameterization of the De Sitter hyperboloid in global (black) and static (blue) coordinates. The x_3 and x_4 axes are suppressed. Static coordinates span one quarter of De Sitter space; global coordinates describe the entire hyperboloid.

If the x_3 and x_4 axes are suppressed, this parameterization can be drawn. An image of such a projection is given in figure 2.1.

Global coordinates are a parameterization for the entire hyperboloid. This is the reason to call these coordinates global. Static coordinates, however, describe only a quarter. The coordinate singularity is visible in figure 2.1 as the sharp points on the edges of the blue area.

The appearance of the coordinate singularity in the plot as a point is in accordance with the discarding of two dimensions. In four dimensions, a point corresponds to a 2-sphere. This was to be expected, based on the shape of the metric.

We can now calculate the coordinate transformation between global and static coordinates. One immediately identifies the 2-sphere with two dimensions in the 3-sphere, so that

$$\theta_2 = \theta; \qquad \theta_3 = \phi \tag{2.2}$$

This reduces equations (2.1) to

$$\begin{gathered} -\sqrt{1-\frac{r^2}{\ell^2}}\sinh\left(\frac{t}{\ell}\right) = \ell\sinh\left(\frac{\tau}{\ell}\right); \quad -\sqrt{1-\frac{r^2}{\ell^2}}\cosh\left(\frac{t}{\ell}\right) = \cosh\left(\frac{\tau}{\ell}\right)\cos\theta_1; \\ \frac{r}{\ell} = \cosh\left(\frac{\tau}{\ell}\right)\sin\theta_1 \end{gathered} \tag{2.3}$$

The transformation for τ can be obtained by inverting the first equation. θ_1 is calculated by dividing the second by the third. This gives the following coordinate transformation:

$$\tau = \ell \sinh^{-1}\left[-\sqrt{1-\frac{r^2}{\ell^2}}\sinh\left(\frac{t}{\ell}\right)\right] \qquad \theta_1 = \tan^{-1}\left[-\frac{r}{\sqrt{\ell^2-r^2}}\frac{1}{\cosh\left(\frac{t}{\ell}\right)}\right] \tag{2.4}$$

Conversely, one can calculate the coordinate transformation to static coordinates. The t coordinate is obtained by dividing the first equation in (2.3) by the second. The inverse coordinate transformation therefore is:

$$t = \ell \tanh^{-1}\left(\frac{\tanh\left(\frac{\tau}{\ell}\right)}{\cos\theta_1}\right) \qquad r = \ell\cosh\left(\frac{\tau}{\ell}\right)\sin\theta_1 \tag{2.5}$$

Let us see what happens at the De Sitter horizon $r = \ell$. Evaluating equation (2.4) at this point, we observe

$$\tau(r=\ell) = 0 \qquad \lim_{r\to\ell}\theta_1(r) = -\frac{\pi}{2} \tag{2.6}$$

Note that the transformation fails to exist for $r = \ell$; however, the limit $r \to \ell$ exists. The failing of the transformation is due to the singularity in the coordinate system. From the existence of the limit, however, we conclude that the coordinate singularity is a regular point on the hyperboloid.

From static to flat slicing coordinates

We now derive the coordinate transformations from static to flat slicing coordinates. Recall that the metrics are given by

$$\begin{aligned} \mathrm{d}s^2 &= \left(1-\frac{r^2}{\ell^2}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1-\frac{r^2}{\ell^2}} - r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right) \\ \mathrm{d}s_F^2 &= \mathrm{d}\tau^2 - \mathrm{e}^{2\tau/\ell}\left(\mathrm{d}\rho^2+\rho^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)\right) \end{aligned} \tag{2.7}$$

where we have used the signature $(+---)$. We immediately identify the angular part in the metrics, that is, θ and ϕ can identified in both coordinate systems. This gives the equality

$$r^2 = \mathrm{e}^{2\tau/\ell}\rho^2 \Rightarrow r = \rho\mathrm{e}^{\tau/\ell} \tag{2.8}$$

This gives the differential

$$\mathrm{d}r^2 = \frac{\rho^2}{\ell^2}\mathrm{e}^{2\tau/\ell}\mathrm{d}\tau^2 + \mathrm{e}^{2\tau/\ell}\mathrm{d}\rho^2 + \frac{2\rho}{\ell}\mathrm{e}^{2\tau/\ell}\mathrm{d}\tau\mathrm{d}\rho \tag{2.9}$$

Furthermore, we have

$$\left(1-\frac{r^2}{\ell^2}\right)\mathrm{d}t^2 = \left(1-\frac{\rho^2}{\ell^2}\mathrm{e}^{\tau/\ell}\right)\left(\left(\frac{\partial t}{\partial\tau}\right)^2\mathrm{d}\tau^2 + \left(\frac{\partial t}{\partial\rho}\right)^2\mathrm{d}\rho^2 + 2\frac{\partial t}{\partial\tau}\frac{\partial t}{\partial\rho}\mathrm{d}\tau\mathrm{d}\rho\right) \tag{2.10}$$

We can now write the r and t part of the static metric in terms of τ and ρ:

$$\begin{aligned} \left(1-\frac{r^2}{\ell^2}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1-\frac{r^2}{\ell^2}} &= \left(1-\frac{\rho^2}{\ell^2}\mathrm{e}^{2\tau/\ell}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2\mathrm{d}\tau^2 \\ &\quad - \frac{\rho^2}{\ell^2}\mathrm{e}^{2\tau/\ell}\frac{\mathrm{d}\tau^2}{1-\frac{\rho^2}{\ell^2}\mathrm{e}^{2\tau/\ell}} + O(\mathrm{d}\tau,\mathrm{d}\rho,\mathrm{d}\rho^2) \end{aligned} \tag{2.11}$$

where we have used the notation $O(\mathrm{d}\tau, \mathrm{d}\rho, \mathrm{d}\rho^2)$ for terms linear in $\mathrm{d}\tau$, $\mathrm{d}\rho$ and $\mathrm{d}\rho^2$. From the flat slicing metric, it follows that the terms linear in $\mathrm{d}\tau^2$ must be equal to one, giving the differential equation

$$\left(1 - \frac{\rho^2}{\ell^2}\mathrm{e}^{2\tau/\ell}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \frac{\rho^2}{\ell^2}\mathrm{e}^{2\tau/\ell}\frac{1}{1 - \frac{\rho^2}{\ell^2}\mathrm{e}^{2\tau/\ell}} = 1 \tag{2.12}$$

which can be rewritten as

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{1 - \frac{\rho^2}{\ell^2}\mathrm{e}^{2\tau/\ell}} \tag{2.13}$$

To solve this equation, integrate to obtain

$$t = \tau - \frac{\ell}{2}\ln\left(-\ell^2 + \rho^2\mathrm{e}^{2\tau/\ell}\right) \tag{2.14}$$

We thus have the coordinate transformation

$$r = \rho\mathrm{e}^{\tau/\ell} \qquad\qquad t = \tau - \frac{\ell}{2}\ln\left(-\ell^2 + \rho^2\mathrm{e}^{2\tau/\ell}\right) \tag{2.15}$$

2.2 Transformations of the static metric

We now turn to transformation of the static metric. We have seen that the horizon is dependent of the choice of origin. When we choose different origins, the horizons may overlap. The goal of this section is to calculate a transformation between these coordinate systems.

The coordinate transformation we are looking for, is basically a translation. However, as De Sitter space is curved, this translation is not trivial. What we do know, is that the metric of both coordinate systems is the De Sitter metric. We are thus looking for transformations that leave the metric invariant.

This group of transformations is called the De Sitter group. In the embedding in 5-dimensional Minkowski space, this group is the same as the Lorentz group $\mathrm{SO}(1,4)$ [8].

We can see this when we notice that De Sitter space was given by the points in Minkowski space satisfying

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = \ell^2 \tag{2.16}$$

where we have used the $(- + + + +)$ signature. We observe that this is actually a sphere in the 5-dimensional Minkowski metric

$$\eta_{MN}x^M x^N = \ell^2 \tag{2.17}$$

where we have used the indices M and N to emphasize that we are summing from 0 to 4. Therefore, transformations that leave the Minkowski metric invariant, leave the De Sitter metric invariant.

From special relativity, we know that the Minkowski group consists of boosts and rotations. In the next sections, we study the effects of these transformations on De Sitter space.

Rotations of the sphere

As we have seen in equation (1.45), the last three coordinates are parameterized as $\mathbb{R}^3$ in spherical coordinates. The elements of the symmetry group of this space are rotations;

this subgroup is generated by:

$$
T_{\mathrm{rot}}^{(1)} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \cos\alpha & -\sin\alpha & \\ & & \sin\alpha & \cos\alpha & \\ & & & & 1 \end{pmatrix} \quad T_{\mathrm{rot}}^{(2)} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \cos\alpha & -\sin\alpha \\ & & & \sin\alpha & \cos\alpha \end{pmatrix}
$$
$$
T_{\mathrm{rot}}^{(3)} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \cos\alpha & & -\sin\alpha \\ & & & 1 & \\ & & \sin\alpha & & \cos\alpha \end{pmatrix} \tag{2.18}
$$

The effect of this transformation on De Sitter space is a rotation of the angular part of the metric; were the spatial part written in Cartesian coordinates, the rotation would redefine the x, y and z-axes. There would be no changes in the origin of the coordinate system, nor shifts of the horizon.

Time translations

Because the static metric does not depend on time t, it must be invariant under a time translation. Apparently, a time translation is a boost in the (x_0, x_1)-plane:

$$
T_{t\ \mathrm{trans}} = \begin{pmatrix} \cosh(\beta/\ell) & -\sinh(\beta/\ell) & & & \\ -\sinh(\beta/\ell) & \cosh(\beta/\ell) & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \tag{2.19}
$$

This matrix gives rise to transformed coordinates:

$$
\begin{pmatrix} x_0' \\ x_1' \end{pmatrix} = \begin{pmatrix} \cosh(\beta/\ell) & -\sinh(\beta/\ell) \\ -\sinh(\beta/\ell) & \cosh(\beta/\ell) \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \tag{2.20}
$$

x_2, x_3 and x_4 are kept the same. One can check that this transformation gives the static metric again:

$$
\begin{aligned}
\mathrm{d}x_0' =& \frac{r}{\sqrt{\ell^2 - r^2}} \left(\sinh\left(\frac{t}{\ell}\right) \cosh(\beta/\ell) - \cosh\left(\frac{t}{\ell}\right) \sinh(\beta/\ell) \right) \mathrm{d}r \\
&+ \sqrt{1 - \frac{r^2}{\ell^2}} \left(-\cosh\left(\frac{t}{\ell}\right) \cosh(\beta/\ell) + \sinh\left(\frac{t}{\ell}\right) \sinh(\beta/\ell) \right) \mathrm{d}t \\
\mathrm{d}x_1' =& \frac{r}{\sqrt{\ell^2 - r^2}} \left(-\sinh\left(\frac{t}{\ell}\right) \sinh(\beta/\ell) + \cosh\left(\frac{t}{\ell}\right) \cosh(\beta/\ell) \right) \mathrm{d}r \\
&+ \sqrt{1 - \frac{r^2}{\ell^2}} \left(\cosh\left(\frac{t}{\ell}\right) \sinh(\beta/\ell) - \sinh\left(\frac{t}{\ell}\right) \cosh(\beta/\ell) \right) \mathrm{d}t \\
\mathrm{d}x_i' =& \omega^{(i)} \mathrm{d}r + r \mathrm{d}\omega^{(i)}
\end{aligned} \tag{2.21}
$$

where $i = 2, 3, 4$ and $\omega^{(i)}$ is a parameterization of the 2-sphere. Using the identities $\cosh^2\left(\frac{t}{\ell}\right) - \sinh^2\left(\frac{t}{\ell}\right) = 1$ and $\cos^2(\beta/\ell) + \sin^2(\beta/\ell) = 1$, one obtains the static metric again.

Using this transformation, we can write down equations for the transformed coordinates in terms of the original:

$$\begin{aligned} x'_0 = -\sqrt{\ell^2 - r'^2}\sinh\left(\frac{t'}{\ell}\right) &= -\sqrt{\ell^2 - r^2}\sinh\left(\frac{t}{\ell}\right)\cosh(\beta/\ell) \\ &\quad + \sqrt{\ell^2 - r^2}\cosh\left(\frac{t}{\ell}\right)\sinh(\beta/\ell) \\ x'_1 = -\sqrt{\ell^2 - r'^2}\cosh\left(\frac{t'}{\ell}\right) &= \sqrt{\ell^2 - r^2}\sinh\left(\frac{t}{\ell}\right)\sinh(\beta/\ell) \\ &\quad - \sqrt{\ell^2 - r^2}\cosh\left(\frac{t}{\ell}\right)\cosh(\beta/\ell) \end{aligned} \tag{2.22}$$

From the other three axes, we conclude that $r' = r$, $\theta' = \theta$, and $\phi' = \phi$. The transformation for t' is obtained by dividing the first by the second line in the equation above. The result is:

$$t' = \ell\tanh^{-1}\left(\frac{\tanh(t/\ell) - \tanh(\beta/\ell)}{1 - \tanh(t/\ell)\tanh(\beta/\ell)}\right) = \ell\tanh^{-1}\left(\tanh\left(\frac{t-\beta}{\ell}\right)\right) = t - \beta \tag{2.23}$$

We see that this boost gives indeed a time translation. We can plot this translated coordinates when we suppress the x_3 and x_4 axes. This plot is given in figure 2.2.

In the plot, we see that the transformed coordinates describe the same patch of the

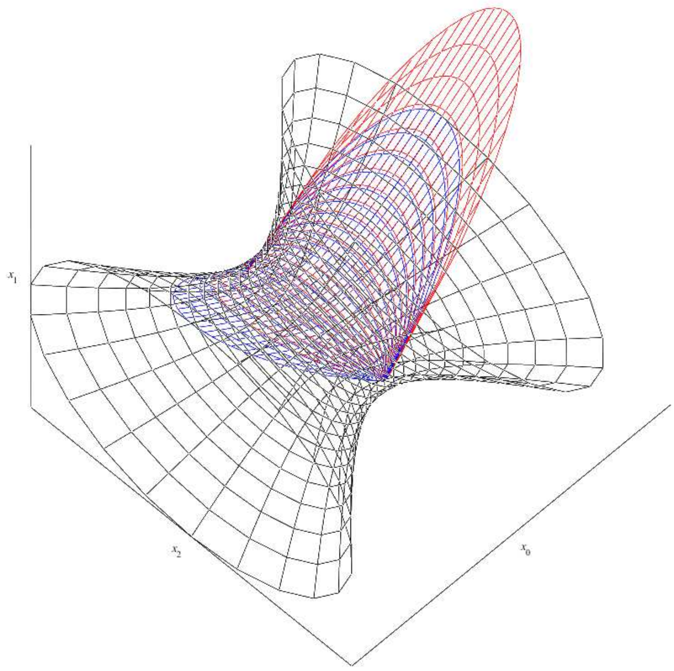

Figure 2.2: Static coordinates translated in time. Blue coordinates are the standard static coordinates, translated coordinates are shown in red. I have used $\ell = 1$; The boost parameter β/ℓ used in this plot was -0.4. Both static coordinate systems are drawn with $t \in [-1.5 \ldots 1.5]$. One sees that the red coordinates stick out on one side; this is due to the time translation.

hyperboloid. The transformation also leaves the singularity invariant, for the r coordinate is unchanged.

Rotations on the hyperboloid

Another transformation is a rotation along the hyperboloid. These rotations are transformations which involve the x_1 coordinate. The subgroup is generated by the following transformations

$$
T_{\text{rot}}^{(4)} = \begin{pmatrix} 1 &&&& \\ & \cos\alpha & -\sin\alpha && \\ & \sin\alpha & \cos\alpha && \\ &&& 1 & \\ &&&& 1 \end{pmatrix} \quad T_{\text{rot}}^{(5)} = \begin{pmatrix} 1 &&&& \\ & \cos\alpha && -\sin\alpha & \\ && 1 && \\ & \sin\alpha && \cos\alpha & \\ &&&& 1 \end{pmatrix}
$$
$$
T_{\text{rot}}^{(6)} = \begin{pmatrix} 1 &&&& \\ & \cos\alpha &&& -\sin\alpha \\ && 1 && \\ &&& 1 & \\ & \sin\alpha &&& \cos\alpha \end{pmatrix} \tag{2.24}
$$

The coordinate transformation can be calculated from the following equations:

$$
\begin{aligned}
x_0' &= -\sqrt{\ell^2 - r'^2}\sinh\left(\frac{t'}{\ell}\right) = -\sqrt{\ell^2 - r^2}\sinh\left(\frac{t}{\ell}\right) \\
x_1' &= -\sqrt{\ell^2 - r'^2}\cosh\left(\frac{t'}{\ell}\right) = -\sqrt{\ell^2 - r^2}\cosh\left(\frac{t}{\ell}\right)\cos\alpha - r\omega^{(i)}\sin\alpha \\
x_i' &= r'\omega^{(i)\prime} = \sqrt{\ell^2 - r^2}\cosh\left(\frac{t}{\ell}\right)\sin\alpha + r\omega^{(i)}\cos\alpha \\
& x_j' = r'\omega^{(j)\prime} = r\omega^{(j)} \quad x_k' = r'\omega^{(k)\prime} = r\omega^{(k)}
\end{aligned} \tag{2.25}
$$

with (i, j, k) a permutation of $\{2, 3, 4\}$ and $\omega^{(i,j,k)}$ a parameterization for the 2-sphere. This means we have the identities

$$
(\omega^{(i)})^2 + (\omega^{(j)})^2 + (\omega^{(k)})^2 = 1 \qquad (\omega^{(i)\prime})^2 + (\omega^{(j)\prime})^2 + (\omega^{(k)\prime})^2 = 1 \tag{2.26}
$$

Using these identities, we can verify that this transformation indeed preserves the metric; one obtains the differentials

$$
\begin{aligned}
\mathrm{d}x_0' =& \frac{r}{\sqrt{\ell^2 - r^2}}\sinh\left(\frac{t}{\ell}\right)\mathrm{d}r - \sqrt{1 - \frac{r^2}{\ell^2}}\cosh\left(\frac{t}{\ell}\right)\mathrm{d}t \\
\mathrm{d}x_1' =& \left(\frac{r}{\sqrt{\ell^2 - r^2}}\cosh\left(\frac{t}{\ell}\right)\cos\alpha - \omega^{(i)}\sin\alpha\right)\mathrm{d}r \\
& - \sqrt{1 - \frac{r^2}{\ell^2}}\sinh\left(\frac{t}{\ell}\right)\cos\alpha\mathrm{d}t - r\sin\alpha\mathrm{d}\omega^{(i)} \\
\mathrm{d}x_i' =& \left(\frac{r}{\sqrt{\ell^2 - r^2}}\cosh\left(\frac{t}{\ell}\right)\sin\alpha + \omega^{(i)}\cos\alpha\right)\mathrm{d}r \\
& - \sqrt{1 - \frac{r^2}{\ell^2}}\sinh\left(\frac{t}{\ell}\right)\sin\alpha\mathrm{d}t + r\sin\alpha\mathrm{d}\omega^{(i)} \\
\mathrm{d}x_j' =& r\mathrm{d}\omega^{(j)} + \omega^{(j)}\mathrm{d}r \quad \mathrm{d}x_k'; = r\mathrm{d}\omega^{(k)} + \omega^{(k)}\mathrm{d}r
\end{aligned} \tag{2.27}
$$

Squaring and putting these line elements in the Minkowski metric $\mathrm{d}s^2 = -\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2$ shows that this transformation does preserve the metric.

We can derive the coordinate transformation for r' by squaring and adding the last three equations in (2.25). This gives

$$r'^2 = (\ell^2 - r^2)\cosh^2\left(\frac{t}{\ell}\right)\sin^2\alpha + r^2(\omega^{(i)})^2\cos^2\alpha \\ + 2r\omega^{(i)}\sqrt{\ell^2 - r^2}\cosh\left(\frac{t}{\ell}\right)\sin\alpha\cos\alpha + r^2((\omega^{(j)})^2 + (\omega^{(k)})^2) \quad (2.28)$$

Rewriting all angular parts in terms of $\omega^{(i)}$ gives

$$r'^2 = (\ell^2 - r^2)\cosh^2\left(\frac{t}{\ell}\right)\sin^2\alpha + r^2(\omega^{(i)})^2\cos^2\alpha \\ + 2r\omega^{(i)}\sqrt{\ell^2 - r^2}\cosh\left(\frac{t}{\ell}\right)\sin\alpha\cos\alpha + r^2(1 - (\omega^{(i)})^2) \quad (2.29)$$

which can be reduced to

$$r'^2 = (\ell^2 - r^2)\cosh^2\left(\frac{t}{\ell}\right)\sin^2\alpha + r^2\left(1 - (\omega^{(i)})^2\sin^2\alpha\right) \\ + 2r\omega^{(i)}\sqrt{\ell^2 - r^2}\cosh\left(\frac{t}{\ell}\right)\sin\alpha\cos\alpha \quad (2.30)$$

The transformation for t' can be obtained by dividing the first two equations in (2.25). This gives

$$t' = \ell\tanh^{-1}\left[\frac{\sqrt{\ell^2 - r^2}\sinh\left(\frac{t}{\ell}\right)}{\sqrt{\ell^2 - r^2}\cosh\left(\frac{t}{\ell}\right)\cos\alpha + r\omega^{(i)}\sin\alpha}\right] \quad (2.31)$$

Let us see what effect this transformation has on the origin in the new coordinate system. The origin in the unprimed coordinate system has become

$$r'(r = 0) = \ell\sin\alpha\cosh\left(\frac{t}{\ell}\right); \qquad t'(r = 0) = \ell\tanh^{-1}\left[\frac{\tanh\left(\frac{t}{\ell}\right)}{\cos\alpha}\right] \quad (2.32)$$

Writing $r'(r = 0)$ in terms of t', we get

$$r'(r = 0) = \frac{\ell\sin\alpha}{\sqrt{1 - \cos^2\alpha\tanh^2\left(\frac{t'}{\ell}\right)}} \quad (2.33)$$

This means that the origin in transformed coordinates is now moving in time. The singularity appears in rotated coordinates as

$$r'(r = \ell) = \pm\ell^2\sqrt{1 - (\omega^{(i)})^2\sin^2\alpha} \qquad t'(r = \ell) = 0 \quad (2.34)$$

The effect of these transformation becomes clear when we draw a plot. This is done in figure 2.3. Because we suppressed two dimensions, the shown De Sitter space has a zero-dimensional spherical part. This means that $\omega^{(i)} = \pm 1$. The singularity in unprimed coordinates therefore has the primed coordinates

$$r'(r = \ell) = r\cos\alpha \qquad t'(r = \ell) = 0 \quad (2.35)$$

Another thing we note, is that the coordinate patch described by primed coordinates is different from the original. This means that a new part of space has become visible. Furthermore, the unprimed singularity is now somewhere in the middle of the primed coordinate patch; it is not in any way a singular point for the observer in primed coordinates.

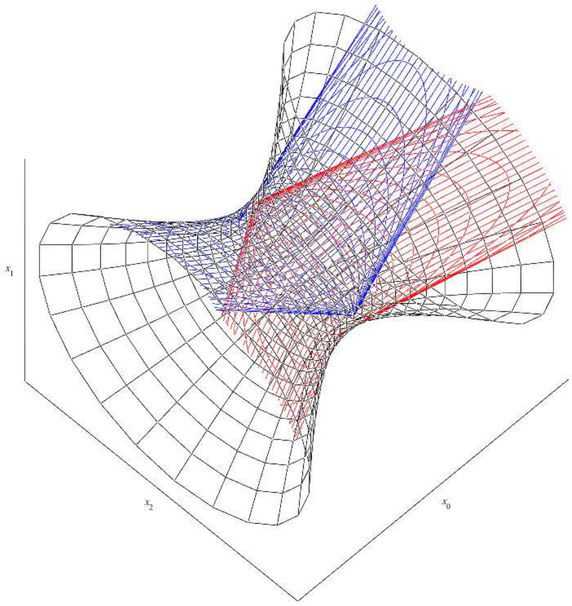

Figure 2.3: Static coordinates rotated on the hyperboloid. Blue coordinates are the standard static coordinates, translated coordinates are shown in red. The value of ℓ is $\ell = 1$; the angle of rotation is $\pi/4$.

Boosts

The last class of transformation we will study are the Lorentz boosts. The subgroup is generated by the matrices

$$T^{(1)}_{\text{boost}} = \begin{pmatrix} \cosh\beta & & -\sinh\beta & & \\ & 1 & & & \\ -\sinh\beta & & \cosh\beta & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \quad T^{(2)}_{\text{boost}} = \begin{pmatrix} \cosh\beta & & & -\sinh\beta & \\ & 1 & & & \\ & & 1 & & \\ -\sinh\beta & & & \cosh\beta & \\ & & & & 1 \end{pmatrix}$$

$$T^{(3)}_{\text{boost}} = \begin{pmatrix} \cosh\beta & & & & -\sinh\beta \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ -\sinh\beta & & & & \cosh\beta \end{pmatrix} \tag{2.36}$$

The parameterization is very similar to that of rotated coordinates; observe the following equations:

$$\begin{aligned} x_0' &= -\sqrt{\ell^2 - r'^2}\sinh\left(\frac{t'}{\ell}\right) = -\sqrt{\ell^2 - r^2}\sinh\left(\frac{t}{\ell}\right)\cosh\beta - r\omega^{(i)}\sinh\beta \\ x_1' &= -\sqrt{\ell^2 - r'^2}\cosh\left(\frac{t'}{\ell}\right) = -\sqrt{\ell^2 - r^2}\cosh\left(\frac{t}{\ell}\right) \\ x_i' &= r'\omega^{(i)\prime} = \sqrt{\ell^2 - r^2}\sinh\left(\frac{t}{\ell}\right)\sinh\beta + r\omega^{(i)}\cosh\beta \\ & \qquad x_j' = r'\omega^{(j)\prime} = r\omega^{(j)} \quad x_k' = r'\omega^{(k)\prime} = r\omega^{(k)} \end{aligned} \tag{2.37}$$

We can check that this transformation preserves the metric by calculating the following line elements:

$$
\begin{aligned}
\mathrm{d}x'_0 =& \left(\frac{r}{\sqrt{\ell^2-r^2}}\sinh\left(\frac{t}{\ell}\right)\cosh\beta - \omega^{(i)}\sinh\beta\right)\mathrm{d}r \\
&- \sqrt{1-\frac{r^2}{\ell^2}}\cosh\left(\frac{t}{\ell}\right)\cosh\beta\mathrm{d}t - r\sinh\beta\mathrm{d}\omega^{(i)} \\
\mathrm{d}x'_1 =& \frac{r}{\sqrt{\ell^2-r^2}}\cosh\left(\frac{t}{\ell}\right)\mathrm{d}r - \sqrt{1-\frac{r^2}{\ell^2}}\sinh\left(\frac{t}{\ell}\right)\mathrm{d}t \\
\mathrm{d}x'_i =& \left(-\frac{r}{\sqrt{\ell^2-r^2}}\sinh\left(\frac{t}{\ell}\right)\sinh\beta + \omega^{(i)}\cosh\beta\right)\mathrm{d}r \\
& \sqrt{1-\frac{r^2}{\ell^2}}\cosh\left(\frac{t}{\ell}\right)\sinh\beta\mathrm{d}t - r\cosh\beta\mathrm{d}\omega^{(i)} \\
\mathrm{d}x'_j =& r\mathrm{d}\omega^{(j)} + \omega^{(j)}\mathrm{d}r \quad \mathrm{d}x'_k; = r\mathrm{d}\omega^{(k)} + \omega^{(k)}\mathrm{d}r
\end{aligned}
\tag{2.38}
$$

Just like with rotated coordinates, squaring and putting these expressions in the Minkowski metric gives the De Sitter metric.

The coordinate transformation is calculated by adding the last three equations in (2.37):

$$
\begin{aligned}
r'^2 =& (\ell^2-r^2)\sinh^2\left(\frac{t}{\ell}\right)\sinh^2\beta + r^2(\omega^{(i)})^2\cosh^2\beta \\
&+ 2r\omega^{(i)}\sqrt{\ell^2-r^2}\sinh\left(\frac{t}{\ell}\right)\sinh\beta\cosh\beta + r^2\left((\omega^{(j)})^2+(\omega^{(k)})^2\right)
\end{aligned}
\tag{2.39}
$$

This can be rewritten as

$$
\begin{aligned}
r'^2 =& (\ell^2-r^2)\sinh^2\left(\frac{t}{\ell}\right)\sinh^2\beta + r^2\left(1+(\omega^{(i)})^2\sinh^2\beta\right) \\
&+ 2r\omega^{(i)}\sqrt{\ell^2-r^2}\sinh\left(\frac{t}{\ell}\right)\sinh\beta\cosh\beta
\end{aligned}
\tag{2.40}
$$

The transformation for t' is obtained by dividing the first by the second line in (2.37):

$$
t' = \ell\tanh^{-1}\left[\frac{\sqrt{\ell^2-r^2}\sinh\left(\frac{t}{\ell}\right)\cosh\beta + r\omega^{(i)}\sinh\beta}{\sqrt{\ell^2-r^2}\cosh\left(\frac{t}{\ell}\right)}\right] \tag{2.41}
$$

From these transformations, we conclude that the origin is not at rest in transformed coordinates. This becomes clear when we evaluate the primed coordinates at $r=0$:

$$
r'(r=0) = \ell\sinh\beta\sinh\left(\frac{t}{\ell}\right); \qquad t'(r=0) = \ell\tanh^{-1}\left[\cosh\beta\tanh\left(\frac{t}{\ell}\right)\right] \tag{2.42}
$$

Writing $r'(r=0)$ in terms of t', this becomes

$$
r'(r=0) = \frac{\ell\tanh\beta\tanh\left(\frac{t'}{\ell}\right)}{\sqrt{1-\frac{\tanh^2\left(\frac{t'}{\ell}\right)}{\cosh^2\beta}}} \tag{2.43}
$$

The singularity, however, remains at rest. When we evaluate the primed coordinates at $r=\ell$, we obtain

$$
r'(r=\ell) = \pm\ell\sqrt{1+(\omega^{(i)})^2\sinh^2\beta}; \qquad t'(r=\ell) = \ell\tanh^{-1}\left[\frac{\ell\omega^{(i)}\sinh\beta}{0}\right] \tag{2.44}
$$

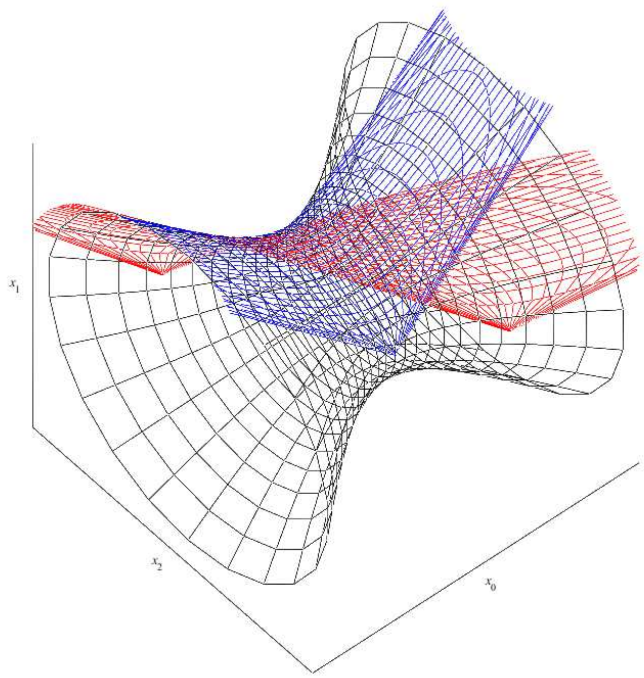

Figure 2.4: Boosted static coordinates. Unboosted coordinates are given in blue, boosted coordinates are shown in red. The used boost parameter β is 1.

This expression is undefined; it shows that the singularity is moved outside the coordinate patch.

We can visualize this by plotting the hyperboloid in two dimensions. This plot is given in figure 2.4. We clearly see that the singularity in blue coordinates is outside the coordinate patch of the red coordinate system. Furthermore, a new part of spacetime has become visible to an observer in the red coordinate system. The origin, which is the middle line in blue coordinates, appears in red coordinates as moving in time.

2.3 A translation of the origin

We have now seen all transformations in the De Sitter group. This group, generated by the ten matrices from the last sections, are all transformations that preserve the static metric. We conclude that many of them change the origin of the coordinate system and the singularity at $r = \ell$.

We also conclude that it is not possible to define a coordinate transformation which simply translate the coordinate system in space; apart from time translated coordinates, there are no transformed coordinates in which the origin appears at rest. The best we can do is define a coordinate system which the origin is at $t' = T, r' = R$ only at a certain moment. Let us calculate such a translation.

First, we need to define our transformation parameters. Say that at $t = 0$, the origin appears in transformed coordinates at $t' = T, r' = R$. Equation (2.42) shows that a boost does not change the origin at $t = 0$; we can therefore choose the boost parameter to be zero. From equation (2.32), however, we know that a rotation does change the origin at

$t = 0$. The new origin is at

$$r'(r = \ell, t = 0) = \ell \sin\alpha \qquad\qquad t'(r = 0, t = 0) = 0 \tag{2.45}$$

So, if we take $\alpha = \sin^{-1}\left(\frac{R}{\ell}\right)$ the origin is moved to $r' = R$. The time coordinate of the origin is shifted by a time translation; see equation (2.23). If we take the translation parameter $\beta = -T$, the origin appears at $t' = T$.

We see that this transformation only holds for $R < \ell$; otherwise, the inverse sine is undefined. This was actually to be expected as for $R > \ell$, the transformed coordinate patch is moved entirely beyond the horizon of untransformed coordinates.

Thus, if we first rotate and then translate the coordinates, we obtain a coordinate system with the original origin at $(t'' = T, r'' = R)$ at $t = 0$. We use single primed coordinates for the system which is only rotated and double primed coordinates for the system which is both rotated and translated. We can embed the double primed coordinates in 5-dimensional Minkowski space using

$$\begin{pmatrix} x_0'' \\ x_1'' \\ x_2'' \\ x_3'' \\ x_4'' \end{pmatrix} = \begin{pmatrix} \cosh\left(\frac{\beta}{\ell}\right) & -\sinh\left(\frac{\beta}{\ell}\right) & & & \\ -\sinh\left(\frac{\beta}{\ell}\right) & \cosh\left(\frac{\beta}{\ell}\right) & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & \cos\alpha & -\sin\alpha & & \\ & \sin\alpha & \cos\alpha & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{2.46}$$

Combining the transformations derived in the last sections, we obtain the coordinate transformation

$$\begin{aligned} t'' = t' - \beta = \ell \tanh^{-1}&\left[\frac{\sqrt{\ell^2 - r^2}\sinh\left(\frac{t}{\ell}\right)}{\sqrt{\ell^2 - r^2}\cosh\left(\frac{t}{\ell}\right)\cos\alpha + r\cos\theta\sin\alpha}\right] - \beta \\ (r'')^2 = r'^2 =&(\ell^2 - r^2)\cosh^2\left(\frac{t}{\ell}\right)\sin^2\alpha + r^2\left(1 - \cos^2\theta\sin^2\alpha\right) \\ &+ 2r\cos\theta\sqrt{\ell^2 - r^2}\cosh\left(\frac{t}{\ell}\right)\sin\alpha\cos\alpha \end{aligned} \tag{2.47}$$

Note that we have used $\omega^{(i)} = \cos\theta$. Putting in the rotation parameters gives

$$\begin{aligned} t'' = t' - \beta = \ell \tanh^{-1}&\left[\frac{\sqrt{\ell^2 - r^2}\sinh\left(\frac{t}{\ell}\right)}{\sqrt{\ell^2 - r^2}\cosh\left(\frac{t}{\ell}\right)\sqrt{1 - \frac{R^2}{\ell^2}} + r\cos\theta\frac{R}{\ell}}\right] + T \\ (r'')^2 = r'^2 =&(\ell^2 - r^2)\cosh^2\left(\frac{t}{\ell}\right)\frac{R^2}{\ell^2} + r^2\left(1 - \frac{R^2}{\ell^2}\cos^2\theta\right) \\ &+ 2r\cos\theta\sqrt{\ell^2 - r^2}\cosh\left(\frac{t}{\ell}\right)\frac{R}{\ell}\sqrt{1 - \frac{R^2}{\ell^2}} \end{aligned} \tag{2.48}$$

Evaluating these transformations at $(t = 0, r = 0)$ shows that the origin in these coordinates is indeed at (T, R). The horizon is given by $r = \ell$; in transformed coordinates, this appears as

$$t''(r = \ell) = T \qquad\qquad r''(r = \ell) = \sqrt{\ell^2 - R^2\cos^2\theta} \tag{2.49}$$

We observe that the coordinate singularity in transformed coordinates is at a distance smaller than ℓ. This means that the singularity is a regular point in the transformed coordinates. The conclusion of this calculations is that the singularity is a *coordinate* singularity, rather than a *physical* singularity. There is no special point in De Sitter space at $r = \ell$; it is the coordinate system which has a singularity.

If the horizon were a physical singularity, we should have seen a singularity in the curvature of the space. However, the Gauss curvature is a constant:

$$R = \frac{12}{\ell^2}, \tag{2.50}$$

so the De Sitter horizon could never be a physical singularity.

Chapter 3

Geodesics

In this chapter we calculate the geodesics on De Sitter space. As we have seen in chapter 1, the geodesic equations can be written down as

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} + \Gamma^\mu{}_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \frac{\mathrm{d}x^\beta}{\mathrm{d}s} = 0 \tag{3.1}$$

Furthermore, a geodesic can always be parameterized by its proper time. This means we have the additional requirement

$$g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 1 \tag{3.2}$$

Take static coordinates (t, r, θ, ϕ). De Sitter space is maximally symmetric [8]; this means that the geodesics have constant angular part. To simplify the calculations, we take $\theta = \frac{\pi}{2}, \phi = 0$. This is without loss of generality, as one can always rotate the coordinate system such that the geodesic satisfies this.

Using Maple (see section A.4), I calculated the geodesic equations. They can be written as

$$\ddot{t} - \frac{2r\dot{t}\dot{r}}{\ell^2 - r^2} = 0; \quad \ddot{r} - \frac{(\ell^2 - r^2) r \dot{t}^2}{\ell^4} + \frac{r\dot{r}^2}{\ell^2 - r^2} = 0; \quad \left(1 - \frac{r^2}{\ell^2}\right)\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{r^2}{\ell^2}} = 1 \tag{3.3}$$

From the first equation in (3.3) follows:

$$(\ell^2 - r^2)\ddot{t} - 2r\dot{r}\dot{t} = \ell^2\ddot{t} - \left[r^2\dot{t}\right]^{\cdot} = 0 \tag{3.4}$$

This equation can be solved to give

$$\dot{t} = \frac{a\ell^2}{\ell^2 - r^2} \tag{3.5}$$

where $a \in \mathbb{R}$ is the integration term. This number is a free parameter.

The second equation in (3.3) can be rewritten as:

$$(\ell^2 - r^2)\ddot{r} - \frac{1}{\ell^4}(\ell^2 - r^2)^2 r\dot{t}^2 + r\dot{r}^2 = 0 \tag{3.6}$$

Inserting equation (3.5) gives:

$$(\ell^2 - r^2)\ddot{r} - a^2 r + r\dot{r}^2 = 0 \tag{3.7}$$

Solving this differential equation using Maple gives:

$$r(s) = \frac{1}{2\omega}\left[\omega^2 \mathrm{e}^{\pm\frac{s+\alpha}{\omega}} + (\ell^2 - a^2\omega^2)\mathrm{e}^{\mp\frac{s+\alpha}{\omega}}\right] \tag{3.8}$$

The parameters $\alpha, \omega \in \mathbb{C}; \omega \neq 0$ are free to be chosen. In fact, one parameter is determined by the third equation in (3.3). If we choose $\omega = \ell$, we can check that $r(s)$ satisfies the third equation exactly. This gives us the solution

$$r(s) = \frac{\ell}{2}\left[e^{\pm\frac{s+\alpha}{\ell}} + (1-a^2)e^{\mp\frac{s+\alpha}{\ell}}\right] = \ell\cosh\left(\frac{s+\alpha}{\ell}\right) - \frac{a^2\ell}{2}e^{\pm\frac{s+\alpha}{\ell}} \tag{3.9}$$

Of course, $r(s)$ should be a real function; this implies that α is real as well. In fact, the physical siginificance of α is the starting point of the geodesic; in the following, we take $\alpha = 0$. The meaning of the $\pm$-sign becomes also clear; this is simply the orientation of the geodesic. In the following, we choose the negative sign for the exponential.

Furthermore, $r(s)$ should be non-negative; this means that the exponential term must never exceed the hyperbolic cosine. This means we have a restriction on a; we should have $a \leq 1$.

The differential equation for $t(s)$ is now

$$\dot{t}(s) = \frac{a}{1-\left[\cosh(s/\ell) - \frac{a^2}{2}e^{-s/\ell}\right]^2} \tag{3.10}$$

Solving this differential equation with Maple gives the solutions

$$t(s) = \frac{\ell}{2}\ln\left[\frac{e^{2s/\ell} - 1 - a^2 + 2a}{e^{2s/\ell} - 1 - a^2 - 2a}\right] + t_0 \tag{3.11}$$

which can be rewritten as

$$t(s) = \ell\tanh^{-1}\left(\frac{e^{2s/\ell} - (a^2+1)}{2a}\right) + t_0 \tag{3.12}$$

What is the physical significance of this result? First of all, we make a plot of r as a function of proper time. This is given in figure 3.1. We note that the codomain of $r(s)$ is $(0, \infty)$. This means that all geodesics cross the De Sitter horizon. The proper time at which this happens is at

$$r(s_h) = \ell \Rightarrow s_h = \ell\ln(1 \pm a) \tag{3.13}$$

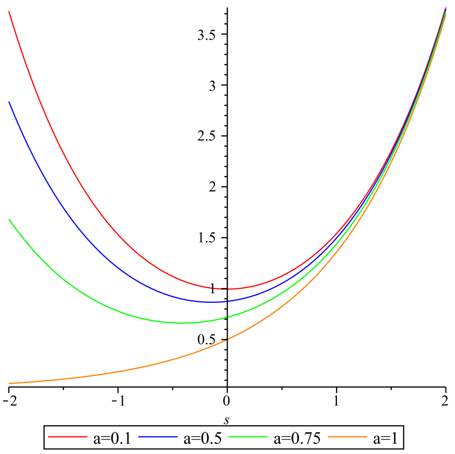

Figure 3.1: The function $r(s)$ for $\ell = 1$. One sees that the geodesics cross the horizon. The parameter a defines the minimal value of r.

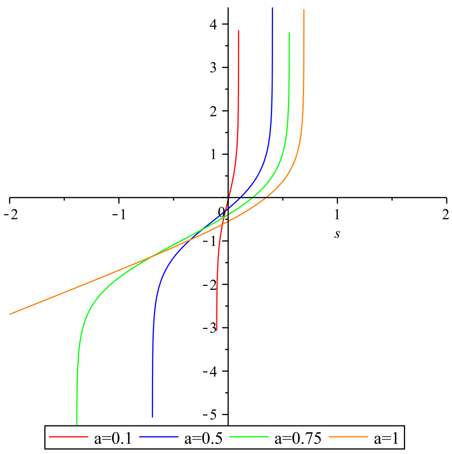

Figure 3.2: The function $t(s)$ for $\ell = 1$. The coordinate time blows up at finite values of proper time, determined by the parameter a.

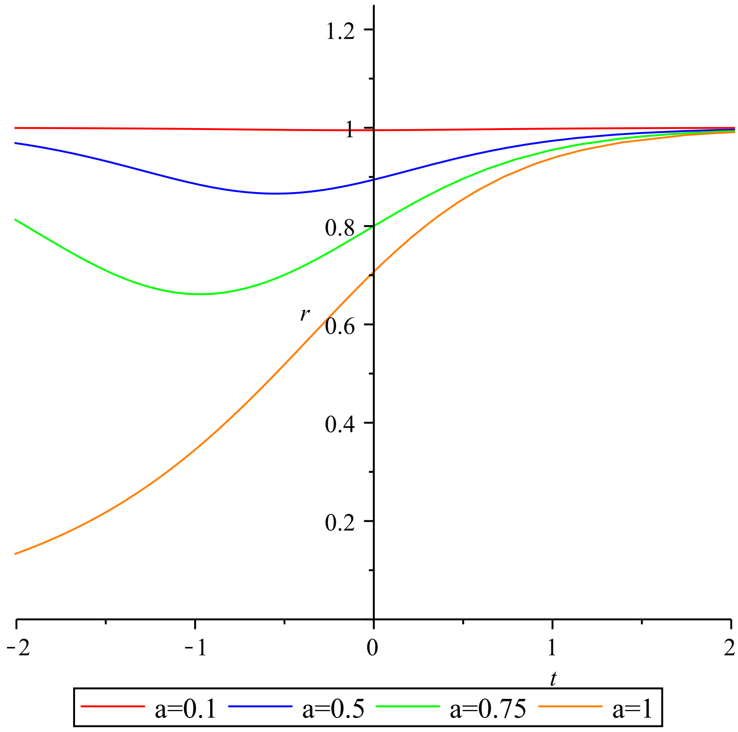

Figure 3.3: Parametric plot of $[t(s), r(s)]$. I have taken $\ell = 1$. The plot shows that no geodesics cross the De Sitter horizon in finite coordinate time. The parameter a determines the smallest value of r; only for $a = 1$, the geodesic comes arbitrarily close to the origin of the coordinate system.

If we plot the coordinate time, however, we see that coordinate time blows up at certain values. The function $t(s)$ is given in figure 3.2. We see that the proper time at which coordinate time blows up depends on a. This means that for an observer at rest, a clock traveling towards the horizon seems to slow down.

If we calculate the coordinate time corresponding to the crossing of the horizon, we obtain

$$t(s_h) = \ell \tanh^{-1}\left(\frac{(1 \pm a)^2 - (a^2 + 1)}{2a}\right) + t_0 = \ell \tanh^{-1}(\pm 1) + t_0 = \pm\infty \tag{3.14}$$

Therefore, the coordinate time for crossing the horizon is infinite. An observer in static coordinates can never see a free particle cross the horizon. This can be seen even clearer if we plot the geodesics as a parameter of s. This is done in figure 3.3. For $a < 1$, a particle seems to travel from the De Sitter horizon, reaches a nearest point, and then travels back towards the horizon. Nevertheless, it never reaches the horizon.

Note the special case $a = 1$. This geodesic travels from the origin $r = 0$ at $t = -\infty$ towards the horizon. Also in this case, the coordinate time to reach the horizon is infinite.

In the reference system of the particle though, the time to reach the horizon is finite. This is another argument to show that the De Sitter horizon is a coordinate singularity, instead of a physical singularity.

Chapter 4

The Klein-Gordon equation

In this chapter, we will solve the Klein-Gordon equation in static coordinates and in flat slicing coordinates. For flat slicing coordinates, we will check that in the special relativistic limit, we retrieve the classical solution.

In Minkowski space, a solution for the Klein-Gordon equation is given by plane waves, that is

$$\psi(\mathbf{x}) = \mathrm{e}^{-ik^\mu x_\mu} \tag{4.1}$$

where the wave vector $\mathbf{k}$ satisfies $k^\mu k_\mu = \frac{M^2c^2}{\hbar^2}$ [9]. In fact, the vector $\mathbf{k}$ is the energy-momentum vector:

$$k^\mu = \frac{p^\mu}{\hbar} \tag{4.2}$$

The wave function ψ is therefore an eigenfunction for the energy and momentum operators.

The plane wave character is particularly obvious when the Klein-Gordon equation is solved in Cartesian coordinates. However, both static and flat slicing coordinates are spherical. Therefore, we first turn our attention to solving the Klein-Gordon equation for flat space in spherical coordinates.

In this chapter, we use the metric with signature $(+ - - -)$.

4.1 Flat space

Recall from section 1.4 that the Klein-Gordon equation for general coordinate systems is given by

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\psi - g^{\alpha\beta}\Gamma^\lambda{}_{\alpha\beta}\partial_\lambda\psi + \mu^2\psi = 0 \tag{4.3}$$

where $\mu = \frac{Mc}{\hbar}$. For flat space in spherical coordinates, we have the line element

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2 \tag{4.4}$$

I have calculated the Christoffel symbols using Maple; one obtains then the partial differential equation

$$\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} - \frac{1}{r^2\sin\theta}\frac{\partial^2\psi}{\partial\phi^2} - \frac{2}{r}\frac{\partial\psi}{\partial r} - \frac{1}{r^2\tan\theta}\frac{\partial\psi}{\partial\theta} + \mu^2\psi = 0 \tag{4.5}$$

We solve this equation using separation of variables; using the Ansatz $\psi = T(t)\xi(r,\theta,\phi)$, we get

$$\frac{\partial T}{\partial t}\xi - \frac{\partial^2\xi}{\partial r^2}T - \frac{1}{r^2}\frac{\partial^2\xi}{\partial\theta^2}T - \frac{1}{r^2\sin\theta}\frac{\partial^2\xi}{\partial\phi^2}T - \frac{2}{r}\frac{\partial\xi}{\partial r}T - \frac{1}{r^2\tan\theta}\frac{\partial\xi}{\partial\theta}T + \mu^2 T\xi = 0 \tag{4.6}$$

Dividing the equation by $T\xi$ and shuffling the terms gives

$$-\frac{1}{\xi}\left(\frac{\partial^2\xi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\xi}{\partial\theta^2}+\frac{1}{r^2\sin\theta}\frac{\partial^2\xi}{\partial\phi^2}+\frac{2}{r}\frac{\partial\xi}{\partial r}+\frac{1}{r^2\tan\theta}\frac{\partial\xi}{\partial\theta}\right)+\mu^2=-\frac{1}{T}\frac{\partial T}{\partial t} \tag{4.7}$$

The left hand side of the equation depends on r, θ and ϕ, whereas the right hand side only depends on t. Therefore, both sides must be constant. This constant can take any value; for reasons that will become clear soon, we take this 'separation constant' $\frac{E^2}{\hbar^2}$. We then get the ordinary differential equation

$$\frac{\mathrm{d}T}{\mathrm{d}t}=-\frac{E^2}{\hbar^2}T \tag{4.8}$$

This can be solved to give the solution

$$T(t)=\mathrm{e}^{\pm itE/\hbar} \tag{4.9}$$

We see that the solution we have obtained is what we expected: a part of a plane wave. The frequency exactly coincides with the energy.

We now return to the spatial part of the equation; this is equal to

$$\frac{\partial^2\xi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\xi}{\partial\theta^2}+\frac{1}{r^2\sin\theta}\frac{\partial^2\xi}{\partial\phi^2}+\frac{2}{r}\frac{\partial\xi}{\partial r}+\frac{1}{r^2\tan\theta}\frac{\partial\xi}{\partial\theta}=-\left(\frac{E^2}{\hbar^2}-\mu^2\right)\xi \tag{4.10}$$

Note that $\frac{E^2}{\hbar^2}-\mu^2=\frac{p^2}{\hbar^2}$, where $p^2=p^jp_j$. Here, we have summed over $j=1\ldots3$. Again, we try separation of variables. This time, use the Ansatz $\xi=R(r)Y(\theta,\phi)$. Putting this in the equation gives

$$\frac{\partial^2R}{\partial r^2}Y+\frac{1}{r^2}\frac{\partial^2Y}{\partial\theta^2}R+\frac{1}{r^2\sin\theta}\frac{\partial^2Y}{\partial\phi^2}R+\frac{2}{r}\frac{\partial R}{\partial r}Y+\frac{1}{r^2\tan\theta}\frac{\partial Y}{\partial\theta}R=-\frac{p^2}{\hbar^2}RY \tag{4.11}$$

Dividing by $\frac{RY}{r^2}$ and sorting terms yields

$$\frac{1}{R}\left(r^2\frac{\partial^2R}{\partial r^2}+2r\frac{\partial R}{\partial r}\right)+\frac{p^2}{\hbar^2}r^2=-\frac{1}{Y}\left(\frac{\partial^2Y}{\partial\theta^2}+\frac{1}{\sin\theta}\frac{\partial^2Y}{\partial\phi^2}+\frac{1}{\tan\theta}\frac{\partial Y}{\partial\theta}\right) \tag{4.12}$$

We take the separation constant $l(l+1)$. The right hand side gives now the angular part of the Laplace equation [10]:

$$\frac{1}{Y}\left(\frac{\partial^2Y}{\partial\theta^2}+\frac{1}{\sin\theta}\frac{\partial^2Y}{\partial\phi^2}+\frac{1}{\tan\theta}\frac{\partial Y}{\partial\theta}\right)=-l(l+1) \tag{4.13}$$

Its solutions are spherical harmonics Y_l^m. Regularity conditions require that l is a non-negative integer, and $m=-l,-l+1,\ldots,l-1,l$. These functions are eigenfunctions both for the total angular momentum L and the projection of the angular momentum on the $z-$axis L_z.

The radial equation is now an ordinary differential equation:

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+2r\frac{\mathrm{d}R}{\mathrm{d}r}-\left[l(l+1)-\frac{p^2}{\hbar^2}r^2\right]R=0 \tag{4.14}$$

To solve this equation, we first use the Ansatz $R(r)=r^{-1/2}J(r)$. This gives the derivatives

$$\frac{\mathrm{d}R}{\mathrm{d}r}=r^{-1/2}\left(\frac{\mathrm{d}J}{\mathrm{d}r}-\frac{1}{2r}J\right)\qquad\frac{\mathrm{d}^2R}{\mathrm{d}r^2}=r^{-1/2}\left(\frac{3}{4r^2}J-\frac{1}{r}\frac{\mathrm{d}J}{\mathrm{d}r}+\frac{\mathrm{d}^2J}{\mathrm{d}r^2}\right) \tag{4.15}$$

Inserting this in equation (4.14) and dividing by $r^{-1/2}$, one obtains

$$r^2\left(\frac{3}{4r^2}J - \frac{1}{r}\frac{\mathrm{d}J}{\mathrm{d}r} + \frac{\mathrm{d}^2J}{\mathrm{d}r^2}\right) + 2r\left(\frac{\mathrm{d}J}{\mathrm{d}r} - \frac{1}{2r}J\right) - \left[l(l+1) - \frac{p^2}{\hbar^2}r^2\right]J = 0 \tag{4.16}$$

which simplifies to

$$r^2\frac{\mathrm{d}^2J}{\mathrm{d}r^2} + r\frac{\mathrm{d}J}{\mathrm{d}r} - \left[\left(l+\frac{1}{2}\right)^2 - \frac{p^2}{\hbar^2}r^2\right]J = 0 \tag{4.17}$$

Define the variable $\rho = \frac{p}{\hbar}r$. The derivatives for this variable are

$$\frac{\mathrm{d}J}{\mathrm{d}r} = \frac{p}{\hbar}\frac{\mathrm{d}J}{\mathrm{d}\rho} \qquad\qquad \frac{\mathrm{d}^2J}{\mathrm{d}r^2} = \frac{p^2}{\hbar^2}\frac{\mathrm{d}^2J}{\mathrm{d}\rho^2} \tag{4.18}$$

Inserting this in the differential equation gives

$$\rho^2\frac{\mathrm{d}^2J}{\mathrm{d}\rho^2} + \rho\frac{\mathrm{d}J}{\mathrm{d}\rho} - \left[\left(l+\frac{1}{2}\right)^2 - \rho^2\right]J = 0 \tag{4.19}$$

This is Bessel's differential equation. [11] The solution to this equation are Bessel functions; the total solution for $R(r)$ therefore is

$$R(r) = \frac{A}{\sqrt{r}}\left(J_{l+\frac{1}{2}}\left(\frac{p}{\hbar}r\right) \pm iY_{l+\frac{1}{2}}\left(\frac{p}{\hbar}r\right)\right) \tag{4.20}$$

where A is a complex normalisation constant. J_ν is called the Bessel function of the first kind, Y_ν is the Bessel function of the second kind. The linear combinations in the above equation are also called Hankel functions; an alternative way to write down the solution is

$$R(r) = \frac{A}{\sqrt{r}}H^{(1,2)}_{l+\frac{1}{2}}\left(\frac{p}{\hbar}r\right) \tag{4.21}$$

The total solutions of the Klein-Gordon equation can therefore be written as

$$\psi = \mathrm{e}^{i\frac{E}{\hbar}t}\frac{1}{\sqrt{r}}H_{l+\frac{1}{2}}\left(\frac{p}{\hbar}r\right)Y_l^m \tag{4.22}$$

4.2 Static coordinates

In this section, we try to solve the Klein-Gordon equation in static coordinates. We still use the metric with signature $(+---)$; The Klein-Gordon equation in these coordinates becomes

$$\frac{1}{1-\frac{r^2}{\ell^2}}\frac{\partial^2\psi}{\partial t^2} - \left(1-\frac{r^2}{\ell^2}\right)\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} \\ - \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \frac{2\ell^2-4r^2}{\ell^2 r}\frac{\partial\psi}{\partial r} - \frac{1}{r^2\tan\theta}\frac{\partial\psi}{\partial\theta} + \mu^2\psi = 0 \tag{4.23}$$

Separation of variables

Just like in flat space, we solve this equation with separation of variables. Use the Ansatz $\psi = T(t)\xi(r,\theta,\phi)$. Inserting this in the differential equation, dividing by $\frac{\psi}{1-\frac{r^2}{\ell^2}}$ and sorting the terms gives

$$\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = \frac{1-\frac{r^2}{\ell^2}}{\xi}\left\{\left(1-\frac{r^2}{\ell^2}\right)\frac{\partial^2\xi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\xi}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\xi}{\partial\phi^2} - \frac{2\ell^2-4r^2}{\ell^2 r}\frac{\partial\xi}{\partial r} + \frac{1}{r^2\tan\theta}\frac{\partial\xi}{\partial\theta} - \mu^2\xi\right\} \tag{4.24}$$

Call the separation constant $-\frac{E^2}{\hbar^2}$. This gives for the left hand side of the equation

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{E^2}{\hbar^2}T \Rightarrow T(t) = \mathrm{e}^{\pm it\frac{E}{\hbar}} \tag{4.25}$$

The right hand side of the equation becomes

$$\left(1-\frac{r^2}{\ell^2}\right)\frac{\partial^2\xi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\xi}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\xi}{\partial\phi^2} - \frac{2\ell^2-4r^2}{\ell^2 r}\frac{\partial\xi}{\partial r} + \frac{1}{r^2\tan\theta}\frac{\partial\xi}{\partial\theta} = \frac{-E^2/\hbar^2+\mu^2-\mu^2r^2/\ell^2}{1-\frac{r^2}{\ell^2}}\xi \tag{4.26}$$

We again use the identity $\frac{E^2}{\hbar^2}-\mu^2=\frac{p^2}{\hbar^2}$ to simplify the equation. Then, we use another Ansatz to separate r and the angular part of the equation: insert $\xi = R(r)Y(\theta,\phi)$, divide by $\frac{RY}{r^2}$ and sort the terms to obtain

$$\frac{1}{R}\left(\left(1-\frac{r^2}{\ell^2}\right)r^2\frac{\partial^2 R}{\partial r^2} - \frac{2\ell^2-4r^2}{\ell^2}r\frac{\partial R}{\partial r}\right) + \frac{r^2p^2/\hbar^2+\mu^2r^4/\ell^2}{1-\frac{r^2}{\ell^2}} = -\frac{1}{Y}\left(\frac{\partial^2 Y}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} + \frac{1}{\tan\theta}\frac{\partial Y}{\partial\theta}\right) \tag{4.27}$$

Define the separation constant to be $l(l+1)$; the right hand side is then the angular part of the Laplace equation. The angular part of the solution are therefore spherical harmonics:

$$Y(\theta,\phi) = Y_l^m \tag{4.28}$$

The left hand side is now an ordinary differential equation given by

$$\left(1-\frac{r^2}{\ell^2}\right)r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2} - \frac{2\ell^2-4r^2}{\ell^2}r\frac{\mathrm{d}R}{\mathrm{d}r} + \frac{r^2p^2/\hbar^2+\mu^2r^4/\ell^2}{1-\frac{r^2}{\ell^2}}R = l(l+1)R \tag{4.29}$$

Solution to the R equation

The equation for the R function is quite hard to solve analytically. Therefore, I will just state the result that can be obtained with Maple. A solution for R is given by

$$R(r) = r^{\frac{3}{2}+\gamma}\left(\ell^2-r^2\right)^{1-\frac{1}{2}\sqrt{4-\ell^2\frac{E^2}{\hbar^2}}}\,{}_2\mathrm{F}_1\left(\alpha,\beta;1+\gamma;\frac{r^2}{\ell^2}\right) \tag{4.30}$$

where ${}_2\mathrm{F}_1$ is the hypergeometric function. The parameters α,β,γ are given by

$$\begin{aligned}\alpha &= \frac{1}{4}\left[-2\sqrt{4-\ell^2\frac{E^2}{\hbar^2}}+2+2\gamma+\sqrt{25-4\ell^2\mu^2}\right]\\ \beta &= \frac{1}{4}\left[-2\sqrt{4-\ell^2\frac{E^2}{\hbar^2}}+2+2\gamma-\sqrt{25-4\ell^2\mu^2}\right]\\ \gamma &= \pm\sqrt{\frac{9}{4}+l(l+1)}\end{aligned} \tag{4.31}$$

The general solution is then given by a linear combination of the solution with a plus or a minus sign in γ.

Physical interpretation

We now have the complete solution to the Klein-Gordon equation in static coordinates. The solution appears to be an eigenfunction for the energy operator. Angular momentum is quantized, just like in the special relativistic limit.

The interpretation of the radial solution, however, is somewhat more vague. This is not helped by the tricky hypergeometric function. The function R is dependent not only on energy and mass, but also on angular momentum. We also know that the solution is valid only for $r \leq \ell$; at this point, the hypergeometric equation as a singularity. This is in accordance with the previous findings on the De Sitter horizon; no information can be obtained from beyond the horizon.

This makes the analysis of the Klein-Gordon solution rather difficult. Perhaps the solution would be better to understand in a coordinate system without a singularity. Therefore, let us see what the solution looks like in flat slicing coordinates.

4.3 Flat slicing coordinates

We take the metric in flat slicing coordinates, using the signature $(+---)$:

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 - \mathrm{e}^{2\tau/\ell}\left(\mathrm{d}\rho^2 + \rho^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\right) \tag{4.32}$$

Plugging this in the covariant Klein-Gordon equation (4.3), we get

$$\frac{\partial^2\psi}{\partial\tau^2} + \frac{3}{\ell}\frac{\partial\psi}{\partial\tau} - \mathrm{e}^{-2\tau/\ell}\left(\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \frac{2}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2\tan\theta}\frac{\partial\psi}{\partial\theta}\right) + \mu^2\psi = 0 \tag{4.33}$$

As with the previous Klein-Gordon equations, we solve this by separation of variables.

Separation of variables

First, we strip off the τ part by the Ansatz $\psi = T(\tau)\xi(\rho,\theta,\phi)$. Putting this in the differential equation, dividing by $T\xi$ and rearranging the terms gives

$$\frac{\mathrm{e}^{2\tau/\ell}}{T}\left(\frac{\partial^2 T}{\partial\tau^2} + \frac{3}{\ell}\frac{\partial T}{\partial\tau}\right) + \mu^2\mathrm{e}^{2\tau/\ell} = \frac{1}{\xi}\left(\frac{\partial^2\xi}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2\xi}{\partial\theta^2} + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2\xi}{\partial\phi^2} + \frac{2}{\rho}\frac{\partial\xi}{\partial\rho} + \frac{1}{\rho^2\tan\theta}\frac{\partial\xi}{\partial\theta}\right) \tag{4.34}$$

Taking the separation constant $-\frac{p^2}{\hbar^2}$, we obtain for the right hand side the partial differential equation

$$\frac{\partial^2\xi}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2\xi}{\partial\theta^2} + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2\xi}{\partial\phi^2} + \frac{2}{\rho}\frac{\partial\xi}{\partial\rho} + \frac{1}{\rho^2\tan\theta}\frac{\partial\xi}{\partial\theta} = -\frac{p^2}{\hbar^2}\xi \tag{4.35}$$

We notice that this equation is exactly the same as equation (4.10); its solutions must therefore be plane waves:

$$\xi = \mathrm{e}^{ik^j x_j} \qquad (j = 1\ldots 3) \tag{4.36}$$

The equation for T is given by

$$\frac{\mathrm{d}^2 T}{\mathrm{d}\tau^2} + \frac{3}{\ell}\frac{\mathrm{d}T}{\mathrm{d}\tau} + \left(\mu^2 + \frac{p^2}{\hbar^2}\mathrm{e}^{-2\tau/\ell}\right)T = 0 \tag{4.37}$$

Solution to the T equation

In order to solve the T equation, we use the Ansatz $T(\tau) = \mathrm{e}^{-\frac{3}{2}\tau\ell} J(\tau)$. This gives the derivatives

$$\frac{\mathrm{d}T}{\mathrm{d}\tau} = \left(-\frac{3}{2\ell}J + \frac{\mathrm{d}J}{\mathrm{d}\tau}\right)\mathrm{e}^{-\frac{3}{2}\tau/\ell} \qquad \frac{\mathrm{d}^2T}{\mathrm{d}\tau^2} = \left(\frac{\mathrm{d}^2J}{\mathrm{d}\tau^2} - \frac{3}{\ell}\frac{\mathrm{d}J}{\mathrm{d}\tau} + \frac{9}{4\ell^2}J\right)\mathrm{e}^{-\frac{3}{2}\tau/\ell} \tag{4.38}$$

Inserting this in the differential equation and dividing by $\mathrm{e}^{-\frac{3}{2}\tau/\ell}$, we obtain

$$\left(\frac{\mathrm{d}^2J}{\mathrm{d}\tau^2} - \frac{3}{\ell}\frac{\mathrm{d}J}{\mathrm{d}\tau} + \frac{9}{4\ell^2}J\right) + \frac{3}{\ell}\left(-\frac{3}{2\ell}J + \frac{\mathrm{d}J}{\mathrm{d}\tau}\right) + \left(\mu^2 + \frac{p^2}{\hbar^2}\mathrm{e}^{-2\tau/\ell}\right)J = 0 \tag{4.39}$$

which simplifies to

$$\frac{\mathrm{d}^2J}{\mathrm{d}\tau^2} + \left(-\frac{9}{4\ell^2} + \mu^2 + \frac{p^2}{\hbar^2}\mathrm{e}^{-2\tau/\ell}\right)J = 0 \tag{4.40}$$

Define the new variable $t = \frac{\ell p}{\hbar}\mathrm{e}^{-\tau/\ell}$. This has the derivatives

$$\frac{\mathrm{d}J}{\mathrm{d}\tau} = -\frac{t}{\ell}\frac{\mathrm{d}J}{\mathrm{d}t} \qquad \frac{\mathrm{d}^2J}{\mathrm{d}\tau^2} = \frac{t}{\ell^2}\frac{\mathrm{d}J}{\mathrm{d}t} + \frac{t^2}{\ell^2}\frac{\mathrm{d}^2J}{\mathrm{d}t^2} \tag{4.41}$$

When we plug this in the differential equation, we get

$$\frac{t^2}{\ell^2}\frac{\mathrm{d}^2J}{\mathrm{d}t^2} + \frac{t}{\ell^2}\frac{\mathrm{d}J}{\mathrm{d}t} + \left(-\frac{9}{4\ell^2} + \mu^2 + \frac{t^2}{\ell^2}\right)J = 0 \tag{4.42}$$

Multiplying by ℓ^2 brings this in the form of the Bessel equation:

$$t^2\frac{\mathrm{d}^2J}{\mathrm{d}t^2} + t\frac{\mathrm{d}J}{\mathrm{d}t} + \left(-\frac{9}{4} + \mu^2\ell^2 + t^2\right)J = 0 \tag{4.43}$$

The solutions are therefore Hankel functions, denoted by

$$T(\tau) = \mathrm{e}^{-\frac{3}{2}\tau/\ell}\left(J_{\sqrt{\frac{9}{4}-\mu^2\ell^2}}\left(\frac{\ell p}{\hbar}\mathrm{e}^{-\tau/\ell}\right) \pm iY_{\sqrt{\frac{9}{4}-\mu^2\ell^2}}\left(\frac{\ell p}{\hbar}\mathrm{e}^{-\tau/\ell}\right)\right) \tag{4.44}$$

Weak field limit

We have shown that the spatial part of the Klein-Gordon solution in flat slicing coordinates is the same as in flat space. Therefore, we only have to check that the T function produces the special relativistic result for $\Lambda \to 0$. This is the same as $\ell \to \infty$. I shall give three arguments to show that the weak field limit is retrieved.

First of all, we look at equation (4.37). As we set $\ell \to \infty$, the differential equation becomes

$$\frac{\mathrm{d}^2T}{\mathrm{d}\tau^2} + \left(\mu^2 + \frac{p^2}{\hbar^2}\right)T = \frac{\mathrm{d}^2T}{\mathrm{d}\tau^2} + \frac{E^2}{\hbar^2}T = 0 \tag{4.45}$$

The solution to this equation is indeed the equation for a complex exponential.

Secondly, we can plot the solution in the complex plane for several values of ℓ. This is done in figure 4.1. We can clearly see that for large ℓ, the flat slicing solution approaches the special relativistic function.

The last argument is a Taylor expansion. If the function T reduces to an exponential as $\ell \to \infty$, its logarithm should reduce to something linear in τ:

$$\ln T(\tau) \overset{?}{\sim} i\tau E/\hbar \tag{4.46}$$

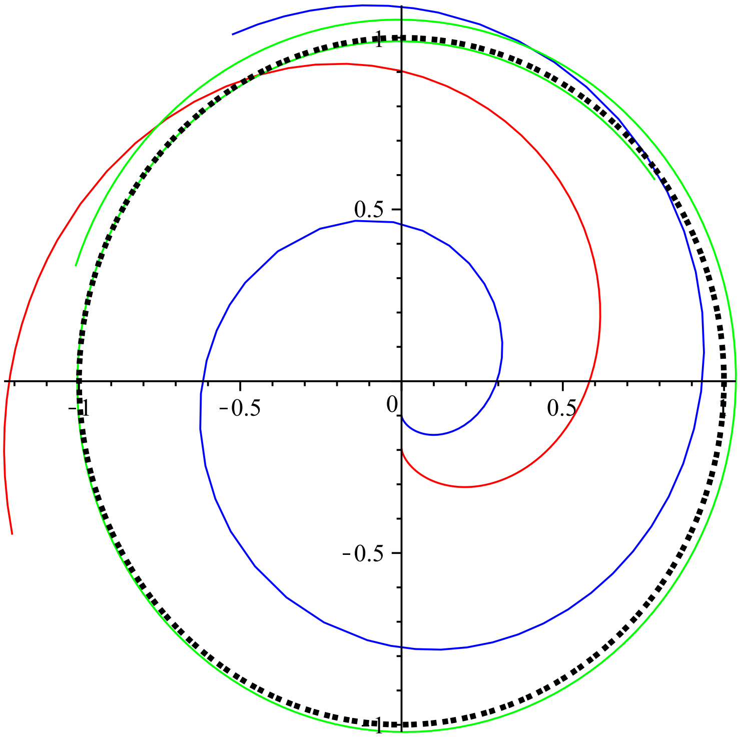

Figure 4.1: Complex plot of the T function for several values of ℓ. I have taken $c = \hbar = 1$, $M = 0$ and $p = 1$. The red plot shows $T(\tau)$ for $\ell = 5$ and $-1.3 \le \tau \le 100$. The blue plot shows $T(\tau)$ for $\ell = 5$ and $-1.3 \le \tau \le 100$. The green plot shows $T(\tau)$ for $\ell = 100$ and $-2\pi \le \tau \le 2$. The dotted plot is the solution in flat space for a particle with $M = 0$, $p = 1$. One sees that for $\ell \to \infty$, $T(\tau)$ approaches the dotted circle.

We expand this in the Taylor series around $z \equiv \frac{\tau}{\ell} = 0$; this gives

$$\ln T(\tau) = \ln\left(H_{\sqrt{\frac{9}{4}-\mu^2\ell^2}}\left(\frac{\ell p}{\hbar}\right)\right) - \frac{3}{2}z + \left[\ln\left(H_{\sqrt{\frac{9}{4}-\mu^2\ell^2}}\left(\frac{\ell p}{\hbar}\mathrm{e}^{-z}\right)\right)\right]'_{z=0} z + \left[\ln\left(H_{\sqrt{\frac{9}{4}-\mu^2\ell^2}}\left(\frac{\ell p}{\hbar}\mathrm{e}^{-z}\right)\right)\right]''_{z=0} \frac{z^2}{2} + O(z^3) \quad (4.47)$$

I have used H to denote both Hankel functions, that is, with both the plus and minus sign. The first term is a constant; this appears in the exponential as a constant factor. This can be absorbed in the normalisation constant. Therefore, this term is left out in the following calculations. Calculating the Taylor coefficients was done using Maple. Giving all the steps is a tedious and messy job, so I will skip some steps. The logarithm can be expanded as

$$\ln T(\tau) \sim \left(-\frac{3}{2} + \frac{\frac{\ell p}{\hbar} H_{\sqrt{\frac{9}{4}-\mu^2\ell^2}+1}\left(\frac{\ell p}{\hbar}\right)}{H_{\sqrt{\frac{9}{4}-\mu^2\ell^2}}\left(\frac{\ell p}{\hbar}\right)} - \frac{1}{2}\sqrt{9-4\mu^2\ell^2}\right) z + \left(-\frac{\ell^2}{2}\left(\frac{p^2}{\hbar^2}+\mu^2\right) - \frac{1}{2}\left(\frac{\frac{\ell p}{\hbar} H_{\sqrt{\frac{9}{4}-\mu^2\ell^2}+1}\left(\frac{\ell p}{\hbar}\right)}{H_{\sqrt{\frac{9}{4}-\mu^2\ell^2}}\left(\frac{\ell p}{\hbar}\right)} - \frac{1}{2}\sqrt{9-4\mu^2\ell^2}\right)^2\right) z^2 + O(z^3) \quad (4.48)$$

Now, set $\ell \to \infty$ to obtain

$$\ln T(\tau) \sim \left(\frac{\frac{\ell p}{\hbar} H_{i\frac{Mc}{\hbar}\ell+1}\left(\frac{\ell p}{\hbar}\right)}{H_{i\frac{Mc}{\hbar}\ell}\left(\frac{\ell p}{\hbar}\right)} - i\frac{Mc}{\hbar}\ell \right) z + \left(-\frac{\ell^2}{2}\left(\frac{E^2}{\hbar^2}\right) - \frac{1}{2}\left(\frac{\frac{\ell p}{\hbar} H_{i\frac{Mc}{\hbar}\ell+1}\left(\frac{\ell p}{\hbar}\right)}{H_{i\frac{Mc}{\hbar}\ell}\left(\frac{\ell p}{\hbar}\right)} - i\frac{Mc}{\hbar}\ell \right)^2 \right) z^2 + O(z^3) \quad (4.49)$$

Recall that $z = \frac{\tau}{\ell}$; this reduces the expansion to

$$\ln T(\tau) \sim \left(\frac{p}{\hbar} \frac{H_{i\frac{Mc}{\hbar}\ell+1}\left(\frac{\ell p}{\hbar}\right)}{H_{i\frac{Mc}{\hbar}\ell}\left(\frac{\ell p}{\hbar}\right)} - i\frac{Mc}{\hbar} \right) \tau + \left(-\frac{1}{2}\left(\frac{E^2}{\hbar^2}\right) - \frac{1}{2}\left(\frac{p}{\hbar} \frac{H_{i\frac{Mc}{\hbar}\ell+1}\left(\frac{\ell p}{\hbar}\right)}{H_{i\frac{Mc}{\hbar}\ell}\left(\frac{\ell p}{\hbar}\right)} - i\frac{Mc}{\hbar} \right)^2 \right) \tau^2 + O(z^3) \quad (4.50)$$

Define the parameter $\nu = i\frac{Mc}{\hbar}\ell$. Insert this in the expansion:

$$\ln T(\tau) \sim \left(\frac{p}{\hbar} \frac{H_{\nu+1}\left(\nu\frac{p}{iMc}\right)}{H_{\nu}\left(\nu\frac{p}{iMc}\right)} - i\frac{Mc}{\hbar} \right) \tau + \left(-\frac{1}{2}\left(\frac{E^2}{\hbar^2}\right) - \frac{1}{2}\left(\frac{p}{\hbar} \frac{H_{\nu+1}\left(\nu\frac{p}{iMc}\right)}{H_{\nu}\left(\nu\frac{p}{iMc}\right)} - i\frac{Mc}{\hbar} \right)^2 \right) \tau^2 + O(z^3) \quad (4.51)$$

From [12], we have the following limit for fractions of Hankel functions:

$$\lim_{n\to\infty} H_{n+1}(n\beta)/H_n(n\beta) = \frac{\beta}{1+\sqrt{1-\beta^2}} \quad (4.52)$$

The limit of the fractions therefore is:

$$\lim_{\nu\to\infty} \frac{H_{\nu+1}\left(\nu\frac{p}{iMc}\right)}{H_{\nu\ell}\left(\nu\frac{p}{iMc}\right)} = \frac{\frac{p}{iMc}}{1+\sqrt{1+\frac{p^2}{M^2c^2}}} = -\frac{ip}{Mc+E} \quad (4.53)$$

Inserting this into the expansion gives

$$\ln T(\tau) \sim \left(-\frac{p}{\hbar}\frac{ip}{Mc+E} - i\frac{Mc}{\hbar} \right) \tau + \left(-\frac{1}{2}\left(\frac{E^2}{\hbar^2}\right) - \frac{1}{2}\left(-\frac{p}{\hbar}\frac{ip}{Mc+E} - i\frac{Mc}{\hbar} \right)^2 \right) \tau^2 + O(z^3)$$

which can be rewritten as

$$\ln T(\tau) \sim -i\left(\frac{E}{\hbar}\right)\tau + O(z^3) \quad (4.54)$$

This is the expected result. It does not prove that the higher-order terms vanish as well, but it is, regarding the previous arguments, very likely. We can therefore state that flat slicing coordinates satisfy the special relativistic limit.

Physical interpretation

We have seen the solution of the Klein-Gordon equation in flat slicing coordinates and proved that it reproduces the special relativistic solution for $\ell \to \infty$. Now, we want to give a physical interpretation to the solution.

As said, we will only investigate the T solution, for the spatial part of the wave function is already the same as in flat space.

First of all, we check that the solution admits antimatter. Antimatter is a state for which the energy is negative. In the special relativistic case, the difference between matter and antimatter is the direction of rotation of the exponential in the complex plane. Matter rotates clockwise in time, whereas antimatter rotates anticlockwise. The solution of T provides this: T could contain two types of Hankel functions. When T contains a $H^{(1)}$ function, we have a clockwise rotating curve. For a $H^{(2)}$ function, the curve rotates anticlockwise. Figure 4.2 shows that this gives the desired result.

Already in chapter 1, we stated that flat slicing coordinates are a model for an expanding spacetime. We expect to see this in our wavefunctions; the absolute value of the wave function is a measure for the particle density. In an expanding spacetime, the absolute value of the wavefunction should therefore become smaller in time.

In figure 4.3, the absolute value is plotted. We can see that the particle density indeed

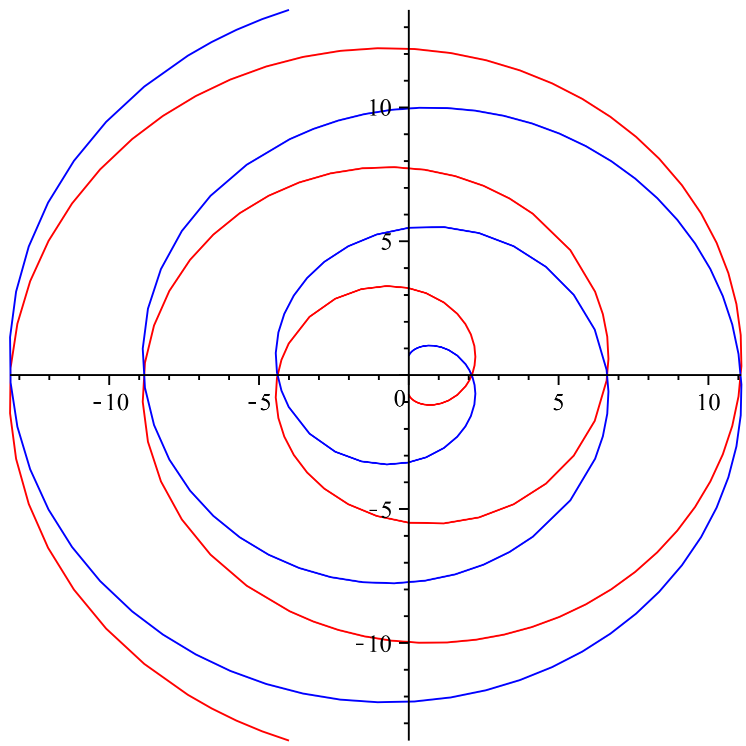

Figure 4.2: Complex plot of the T function containing $H^{(1)}$ and $H^{(2)}$. I have taken $c = \hbar = \ell = p = 1$, $M = 0$. The red plot shows the matter curve with a Hankel $H^{(1)}$ function. The antimatter curve is the blue plot with a Hankel $H^{(2)}$ function. The curves are plotted for $-3 \leq \tau \leq 100$. The curves both spiral inwards; this means the red plot rotates clockwise, the blue plot anticlockwise.

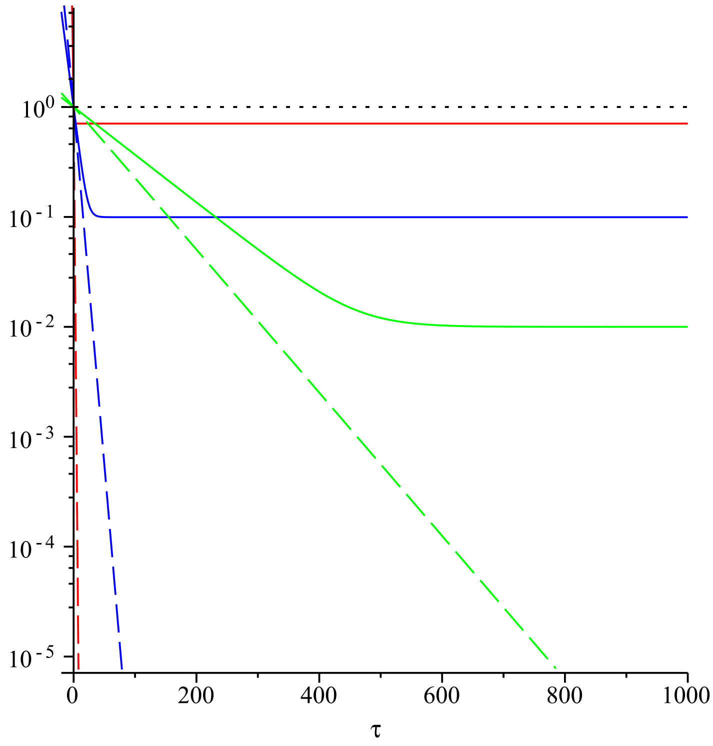

Figure 4.3: Logarithmic plot of the particle density $|T(\tau)|$. I have used the values $c = \hbar = p = 1$. The solid lines show the graph for $M = 0$; the dashed plots are for $M = 4$. The red plot is for $\ell = 1$; the blue for $\ell = 10$ and the green for $\ell = 100$. The dotted plot shows the special relativistic limit, which is a constant function. The wave functions are normalized so that their norm is 1 at $\tau = 0$.

decreases. For massive particles, the decrease is exponentially. This is the same rate at which the spacetime expands. In the massless case, however, the particle density reaches an equilibrium. This can also be shown using the asymptotic value of $T(\tau)$. For large τ, the argument of the Hankel function is small. It can thus be approximated by [13]:

$$H_{\sqrt{\frac{9}{4}-\mu^2\ell^2}}\left(\frac{\ell p}{\hbar}\mathrm{e}^{-\tau/\ell}\right) \sim Y_{\sqrt{\frac{9}{4}-\mu^2\ell^2}}\left(\frac{\ell p}{\hbar}\mathrm{e}^{-\tau/\ell}\right) \sim \left(\mathrm{e}^{-\tau/\ell}\right)^{-\sqrt{\frac{9}{4}-\mu^2\ell^2}} \tag{4.55}$$

The function $T(\tau)$ may be approximated for large τ by

$$T(\tau) \sim \mathrm{e}^{\left(-\frac{3}{2}+\sqrt{\frac{9}{4}-\mu^2\ell^2}\right)\tau/\ell} \tag{4.56}$$

This shows clearly that in the massless case, $T(\tau)$ approaches a constant. For massive particles, the exponent has negative real part. Therefore, in the massive case, the particle density decreases exponentially.

The particle density may be different for observers in other coordinate systems; in particular, the particle density for a static observer is different than the density for an accelerating observer. This is called the Unruh effect [14]. It will be interesting to see whether it is possible to show that the Unruh effect is visible for various coordinate systems in De Sitter space. However, I have not had the time to work this out yet.

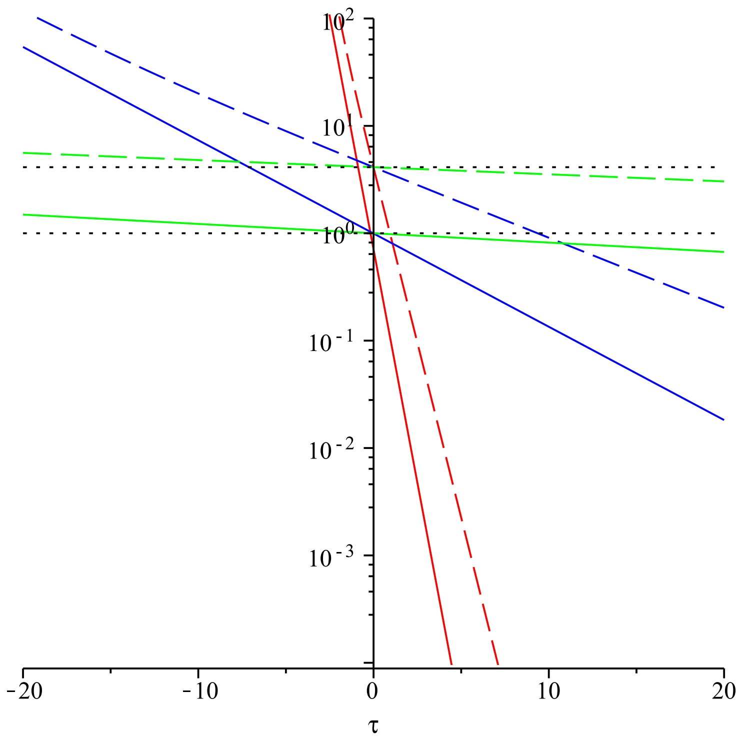

Figure 4.4: Logarithmic plot of $E \propto \left|\frac{dT}{d\tau}\right|$. I have used the values $c = \hbar = p = 1$. The solid lines show the graph for $M = 0$; the dashed plots are for $M = 4$. The red plot is for $\ell = 1$; the blue for $\ell = 10$ and the green for $\ell = 100$. The dotted plot shows the special relativistic limit, which is a constant function.

Another characteristic of an expanding space is redshift. This means that the wavelength or energy observed in the origin of the coordinate system is lower than the energy of the particle when it was created.

We can find this in our solution for T; we can calculate the velocity of the curve in the complex plane. That is, we have

$$|E| \propto \left|\frac{\mathrm{d}T}{\mathrm{d}\tau}\right| \tag{4.57}$$

This is plotted for various values of ℓ in figure 4.4. We clearly see that the energy in the coordinate past is lower than it is in the future. We observe that the energy decreases exponentially; at $t = 0$, the energy is the same as in the special relativistic limit. Furthermore, we notice that the energy decreases more slowly as ℓ becomes larger. It therefore approaches the special relativistic limit. A massive particle is redshifted more, for it couples stronger to the gravitational field.

Chapter 5

Conclusion

In this thesis, we have investigated some properties of De Sitter space and the De Sitter horizon. The first question we asked ourselves was whether the De Sitter horizon is a true, physical singularity, or a property of the static coordinate system.

It turned out to be the latter; we have seen that static coordinates could be transformed into coordinate systems without a singularity. In these coordinate systems, the singularity appeared as just a regular point.

We have shown that static coordinates themselves could be transformed by a boost of the origin; by this boost, the coordinate singularity was removed.

We then calculated geodesics in static coordinates that could cross the horizon. For a static observer in the origin, this would take an infinite amount of time, so we concluded that the De Sitter horizon is an event horizon, similar to the Schwarzschild horizon of a black hole. Particles are able to cross the horizon in a finite amount of proper time, but for the static observer, it is impossible to send information across the horizon.

We then turned our attention to quantum mechanics on a De Sitter background. We showed the solution of the Klein-Gordon equation in static coordinates and in flat slicing coordinates.

We showed the physical significance of the solution in flat slicing coordinates as a model for an expanding space. We demonstrated the existence of antimatter. Furthermore, we checked that we obtained the solution in flat space if the cosmological constant is zero. We showed that the expansion could be observed in the redshifted energy of the particle. For massive particles, the particle density would decrease exponentially, as was expected. For a massless particle, however, we uncovered the spontaneous creation of particles. This creation of particles from vacuum may be a manifestation of the Unruh effect; further investigation on this subject may be a problem for a follow-up research.

The Klein-Gordon equation governs the motion of spin-0 particles. The next step would be to generalize the field equation for spin-$\frac{1}{2}$ particles: the Dirac equation. The fields in this equation are spinorial. The Dirac equation is therefore a lot harder to generalize to curved spacetimes.

As we have seen, the unification of gravity and quantum mechanics is, to say the least, not straightforward. I think it will be a hot topic for many years to come,[1] and I am glad to have studied the basics.

[1] This is, in the opinion of some people, a slight understatement.

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Appendix A

Maple code

In this appendix, I will list the Maple code used in calculations and to generate pictures. The version used was Maple 15.

A.1 The procedure 'riemann'

The following code was written by Ronald Kleiss. This program calculates various tensorial objects from the metric, such as Christoffel symbols, the Ricci tensor and the Einstein tensor. In many calculations, I have used this program.

```
restart; with(linalg):
print("

This code contains three routines.

The procedure 'riemann' does the hard work, its options are self
    -explanatory. The coordinates MUST be called x[1],x[2],x
    [3],... and so on.

The procedure 'cartesianembed' computes the metric induced by an
     embedding of a 'dmnsS'-dimensional space with coordinates x
    [1],x[2],... using a list of embedding functions X, into a
    cartesian space with standard cartesian metric.

The code 'embed' embeds into a space with a more general metric
    'G'. This metric must have the same dimension as the list of
    embedding functions 'X'.

The option 'geodesics' in 'riemann' gives the geodesic equations
     ONLY if the metric 'M' is completely specified, i.e. it
    accepts no generic functions such as 'f(x[1])'.");

riemann:= proc(optie,gcov,info)
  local optlist,chopt,dmns,mu,nu,A,gcon,ro,chf,chs,si,al,
        rcf,be,rcs,rif,ris,gauss,eif,eis,itag,chsg,gcovg,
        geo;
```

```
optlist:={contravariantmetric, christoffel1, christoffel2,
   riemanntensor1, riemanntensor2, riccitensor1, riccitensor2,
   gausscurvature, einsteintensor1, einsteintensor2, geodesics};
chopt:=optie in optlist;
if evalb(chopt)=false then
  print("Error: option",optie," is invalid");
  print("Options are:");
  print(contravariantmetric," the contravariant form of the
     metric");
  print(christoffel1," Christoffel symbols of the first kind")
     ;
  print(christoffel2," Christoffel symbols of the second kind
     ");
  print(riemanntensor1," Riemann tensor in standard form");
  print(riemanntensor2," Riemann tensor in fully covariant
     form");
  print(riccitensor1," Ricci tensor in fully covariant form");
  print(riccitensor2," Ricci tensor in fully contravariant
     form");
  print(gausscurvature," Gauss curvature");
  print(einsteintensor1," Einstein tensor in fully covariant
     form");
  print(einsteintensor2," Einstein tensor in fully
     contravariant form");
  print(geodesics," Geodesic equation for arc length parameter
     ",s," plus kinematic equation. This option cannot be used
      if the metric involves unresolved functions of the
     coordinates.");
  return;
fi;
dmns:=nops(gcov);
for mu from 1 to dmns do
  if nops(gcov[mu]) <> dmns then
    print("Error: metric is not square");
    return;
  fi;
od;
for mu from 1 to dmns do
for nu from 1 to dmns do
  if gcov[mu][nu]<>gcov[nu][mu] then
    print("Error: metric is not symmetric");
    return;
  fi;
od;od;
if LinearAlgebra[Determinant](Matrix(gcov)) = 0 then
  print("Error: metric is singular");
  return;
fi;
A:=LinearAlgebra[MatrixInverse](Matrix(gcov));
```

```
for mu from 1 to dmns do
for nu from 1 to dmns do
  gcon[mu][nu]:=A[mu,nu];
od;od;
if info=1 then
  print("Matrix form of covariant metric:");
  print(Matrix(gcov));
  print("Matrix form of contravariant metric:");
  print(Matrix(A));
fi;
if optie=contravariantmetric then
  return [seq([seq(gcon[i1][i2],i2=1..dmns)],i1=1..dmns)];
fi;
for mu from 1 to dmns do
for nu from 1 to dmns do
for ro from 1 to dmns do
  chf[mu][nu][ro]:=1/2*(
    diff(gcov[mu][nu],x[ro]) +
    diff(gcov[mu][ro],x[nu]) -
    diff(gcov[nu][ro],x[mu]) );
od;od;od;
for mu from 1 to dmns do
for nu from 1 to dmns do
for ro from 1 to dmns do
  chs[mu][nu][ro]:=0;
  for si from 1 to dmns do
    chs[mu][nu][ro]:=chs[mu][nu][ro] +
      gcon[mu][si]*chf[si][nu][ro];
  od;
od;od;od;
if info=1 then
  print("nonzero Christoffel symbols of the first kind:");
itag:=0;
  for mu from 1 to dmns do
  for nu from 1 to dmns do
  for ro from 1 to dmns do
    A:=chf[mu][nu][ro];
    if A<>0 then
      itag:=1;
      print(Gamma[mu,nu,ro]=A);
    fi;
  od;od;od;
  if itag=0 then print(none); fi;
  print("nonzero Christoffel symbols of the second kind:");
  itag:=0;
  for mu from 1 to dmns do
  for nu from 1 to dmns do
  for ro from 1 to dmns do
  A:=chs[mu][nu][ro];
```

```
    if A<>0 then
      itag:=1;
      print((Gamma^([mu]))[nu,ro]=A);
    fi;
  od;od;od;
  if itag=0 then print(none); fi;
fi;
if optie=christoffel1 then
  return [seq([seq([seq(chf[i1][i2][i3],i3=1..dmns)],i2=1..
     dmns)],i1=1..dmns)];
fi;
if optie=christoffel2 then
  return [seq([seq([seq(chs[i1][i2][i3],i3=1..dmns)],i2=1..
     dmns)],i1=1..dmns)];
fi;
if optie=geodesics then
  for mu from 1 to dmns do
  for nu from 1 to dmns do
  for ro from 1 to dmns do
    chsg[mu][nu][ro]:=chs[mu][nu][ro];
    for si from 1 to dmns do
      chsg[mu][nu][ro]:=subs(x[si]=x[si](s),chsg[mu][nu][ro]);
    od;
  od;od;od;
  for mu from 1 to dmns do
  for nu from 1 to dmns do
    gcovg[mu][nu]:=gcov[mu][nu];
    for si from 1 to dmns do
      gcovg[mu][nu]:=subs(x[si]=x[si](s),gcovg[mu][nu]);
    od;
  od;od;
  if info=1 then print("Geodesic equations:") fi;
  for mu from 1 to dmns do
    geo[mu]:=diff(x[mu](s),s$2);
    for al from 1 to dmns do
    for be from 1 to dmns do
      geo[mu]:=geo[mu]
        + chsg[mu][al][be]*diff(x[al](s),s)*diff(x[be](s),s);
    od;od;
    geo[mu]:=geo[mu]=0;
    if info=1 then print(geo[mu]) fi;
  od;
  if info=1 then print("Kinematic equation:") fi;
  geo[dmns+1]:=0;
  for al from 1 to dmns do
  for be from 1 to dmns do
    geo[dmns+1]:=geo[dmns+1]
      + gcovg[al][be]*diff(x[al](s),s)*diff(x[be](s),s);
  od;od;
```

```
    geo[dmns+1]:=geo[dmns+1]=1;
    if info=1 then print(geo[dmns+1]) fi;
    return [seq(geo[i1],i1=1..dmns+1)];
fi;
for al from 1 to dmns do
for mu from 1 to dmns do
for nu from 1 to dmns do
for ro from 1 to dmns do
    rcf[al][mu][nu][ro]:=
      - diff(chs[al][mu][nu],x[ro])
      + diff(chs[al][mu][ro],x[nu]);
    for be from 1 to dmns do
      rcf[al][mu][nu][ro]:=rcf[al][mu][nu][ro]
        + chs[al][nu][be]*chs[be][mu][ro]
        - chs[al][ro][be]*chs[be][mu][nu];
    od;
od;od;od;od;
for al from 1 to dmns do
for mu from 1 to dmns do
for nu from 1 to dmns do
for ro from 1 to dmns do
    rcs[al][mu][nu][ro]:=0;
    for be from 1 to dmns do
      rcs[al][mu][nu][ro]:=rcs[al][mu][nu][ro]
        + gcov[al][be]*rcf[be][mu][nu][ro];
    od;
od;od;od;od;
if info=1 then
    print("nonzero elements of the standard form of the Riemann
        tensor:");
    itag:=0;
    for al from 1 to dmns do
    for mu from 1 to dmns do
    for nu from 1 to dmns do
    for ro from 1 to dmns do
      A:=rcf[al][mu][nu][ro];
      if A<>0 then
        itag:=1;
        print((R^[al])[mu,nu,ro]=A);
      fi;
    od;od;od;od;
    if itag=0 then print(none); fi;
    print("nonzero elements of the fully covariant form of the
        Riemann tensor:");
    itag:=0;
    for al from 1 to dmns do
    for mu from 1 to dmns do
    for nu from 1 to dmns do
    for ro from 1 to dmns do
```

```
    A:=rcs[al][mu][nu][ro];
    if A<>0 then
      itag:=1;
      print(R[al,mu,nu,ro]=A);
    fi;
  od;od;od;od;
  if itag=0 then print(none); fi;
fi;
if optie=riemanntensor1 then
  return [seq([seq([seq([seq(rcf[i1][i2][i3][i4],i4=1..dmns)],
      i3=1..dmns)],i2=1..dmns)],i1=1..dmns)];
fi;
if optie=riemanntensor2 then
  return [seq([seq([seq([seq(rcs[i1][i2][i3][i4],i4=1..dmns)],
      i3=1..dmns)],i2=1..dmns)],i1=1..dmns)];
fi;
for mu from 1 to dmns do
for nu from 1 to dmns do
  rif[mu][nu]:=0;
  for al from 1 to dmns do
    rif[mu][nu]:=expand(rif[mu][nu]
      + rcf[al][mu][al][nu]);
  od;
od;od;
for mu from 1 to dmns do
for nu from 1 to dmns do
  ris[mu][nu]:=0;
  for al from 1 to dmns do
  for be from 1 to dmns do
    ris[mu][nu]:=expand(ris[mu][nu] +
      gcon[mu][al]*gcon[nu][be]*rif[al][be]);
  od;od;
od;od;
if info=1 then
  print("nonzero elements of the fully covariant Ricci tensor
      :");
  itag:=0;
  for mu from 1 to dmns do
  for nu from 1 to dmns do
    A:=rif[mu][nu];
    if A<>0 then
      itag:=1;
      print(R[mu,nu]=A);
    fi;
  od;od;
  if itag=0 then print(none); fi;
  print("nonzero elements of the fully contravariant Ricci
      tensor:");
  itag:=0;
```

```
  for mu from 1 to dmns do
  for nu from 1 to dmns do
    A:=ris[mu][nu];
    if A<>0 then
      itag:=1;
      print(R^[mu,nu]=A);
    fi;
  od;od;
  if itag=0 then print(none); fi;
fi;
if optie=riccitensor1 then
  return [seq([seq(rif[i1][i2],i2=1..dmns)],i1=1..dmns)];
fi;
if optie=riccitensor2 then
  return [seq([seq(ris[i1][i2],i2=1..dmns)],i1=1..dmns)];
fi;
gauss:=0;
for mu from 1 to dmns do
for nu from 1 to dmns do
  gauss:=gauss + gcon[mu][nu]*rif[mu][nu];
od;od;
gauss:=expand(gauss);
if info=1 then
  print("Gauss curvature:");
  print(R=gauss);
fi;
if optie=gausscurvature then
  return gauss;
fi;
for mu from 1 to dmns do
for nu from 1 to dmns do
  eif[mu][nu]:=expand(rif[mu][nu]-gauss*gcov[mu][nu]/2);
od;od;
for mu from 1 to dmns do
for nu from 1 to dmns do
  eis[mu][nu]:=0;
  for al from 1 to dmns do
  for be from 1 to dmns do
    eis[mu][nu]:=expand(eis[mu][nu]+
      gcon[mu][al]*gcon[nu][be]*eif[al][be]);
  od;od;
od;od;
if info=1 then
  print("nonzero elements of the fully covariant Einstein
     tensor:");
  itag:=0;
  for mu from 1 to dmns do
  for nu from 1 to dmns do
    A:=eif[mu][nu];
```

```
      if A<>0 then
        itag:=1;
        print(E[mu,nu]=A);
      fi;
    od;od;
    if itag=0 then print(none); fi;
    print("nonzero elements of the fully contravariant Einstein
       tensor:");
    itag:=0;
    for mu from 1 to dmns do
    for nu from 1 to dmns do
      A:=eis[mu][nu];
      if A<>0 then
        itag:=1;
        print(E^[mu,nu]=A);
      fi;
    od;od;
    if itag=0 then print(none); fi;
  fi;
  if optie=einsteintensor1 then
    return [seq([seq(eif[i1][i2],i2=1..dmns)],i1=1..dmns)];
  fi;
  if optie=einsteintensor2 then
    return [seq([seq(eis[i1][i2],i2=1..dmns)],i1=1..dmns)];
  fi;
end proc:

embed:=proc(dmnsS,X,G)
  local dmnsT,A,B,gind,mu,nu;
  dmnsT:=nops(X);
  if dmnsT<dmnsS then
    print("Error: embedding space has too low dimension");
    return;
  fi;
  if nops(G)<>dmnsT then
    print("Error: embedding metric unacceptable");
    return;
  fi;
  for A from 1 to dmnsT do
    if nops(G[A])<>dmnsT then
    print("Error: embedding metric not square");
    return;
    fi;
  od;
  for A from 1 to dmnsT do
  for B from 1 to dmnsT do
    if G[A][B]<>G[B][A] then
    print("Error: embedding metric not symmetric");
    return;
```

```
    fi;
  od;od;
  for mu from 1 to dmnsS do
  for nu from mu to dmnsS do
    gind[mu][nu]:=0;
    for A from 1 to dmnsT do
    for B from 1 to dmnsT do
      gind[mu][nu]:=gind[mu][nu] +
      G[A][B]*diff(X[A],x[mu])*diff(X[B],x[nu]);
    od;od;
    gind[mu][nu]:=expand(gind[mu][nu]);
    gind[nu][mu]:=gind[mu][nu];
  od;od;
  return simplify([seq([seq(gind[i1][i2],i2=1..dmnsS)],i1=1..
     dmnsS)]);
end proc:

cartesianembed:= proc(dmnsS,X)
  local dmnsT,Gg,A,B,G;
  dmnsT:=nops(X);
  for A from 1 to dmnsT do
  for B from 1 to dmnsT do
    if A=B then
      Gg[A][B]:=1;
    else
      Gg[A][B]:=0;
    fi;
  od;od;
  G:=[seq([seq(Gg[i1][i2],i2=1..dmnsT)],i1=1..dmnsT)];
  return embed(dmnsS,X,G);
end proc:
"

  This code contains three routines.

  The procedure 'riemann' does the hard work, its options are

   self-explanatory. The coordinates MUST be called x[1],x[2],x
      [\

  3],... and so on.

  The procedure 'cartesianembed' computes the metric induced by

   an embedding of a 'dmnsS'-dimensional space with coordinates

   x[1],x[2],... using a list of embedding functions X, into a

   cartesian space with standard cartesian metric.
```

```
The code 'embed' embeds into a space with a more general
  metric 'G'. This metric must have the same dimension as the
  list of embedding functions 'X'.
The option 'geodesics' in 'riemann' gives the geodesic
  equations ONLY if the metric 'M' is completely specified,
  i.e. it accepts no generic functions such as 'f(x[1])'."
```

A.2 Flat slicing coordinates

In this section, the code for deriving the flat slicing metric is shown. The code uses the procedure 'riemann', as shown above.

```
> tau := x[1]; rho := x[2]; theta := x[3]; phi := x[4];
> M := [[1, 0, 0, 0], [0, -g(tau), 0, 0], [0, 0, -g(tau)*rho^2,
    0], [0, 0, 0, -g(tau)*rho^2*sin(theta)^2]];
> eqn := simplify(riemann(einsteintensor1, M, 0));
> eqns := simplify([eqn[1][1]+Lambda*M[1][1] = 0, eqn[2][2]+
    Lambda*M[2][2] = 0, eqn[3][3]+Lambda*M[3][3] = 0, eqn[4][4]+
    Lambda*M[4][4] = 0]);
> dsolve(eqns, g(tau));
```

A.3 Transformations of the static metric

This code creates the 3-dimensional plots of the De Sitter hyperboloid with various coordinate systems. The plot 'glob' shows the hyperboloid in global coordinates and is used as the background on which the static coordinate patches are drawn. The plot 'statplot' shows static coordinates without any transformation. 'rotstatplot' shows static coordinates rotated on the hyperboloid; 'tbooststatplot' shows coordinates translated in time; 'sbooststatplot' shows boosted coordinates. In each plot, I have taken $\ell = 1$. For the boundaries of the hyperboloid, I have chosen $\tau = \pm 1.5$ in global coordinates. For the boost parameter or rotation angle, I have chosen values $\pi/4$, 1 and -0.4, depending on the used transformation.

```
> with(plots);
> `&ell;` := 1; phi := 1; taumax := 1.5;
> glob := plot3d([`&ell;`*sinh(t/`&ell;`), `&ell;`*cosh(t/`&ell
    ;`)*cos(theta), `&ell;`*cosh(t/`&ell;`)*sin(theta)], t = -
    taumax .. taumax, theta = -Pi .. Pi, view = [-cosh(taumax) ..
     cosh(taumax), -cosh(taumax) .. cosh(taumax), -cosh(taumax)
     .. cosh(taumax)], style = wireframe, color = black, labels =
    ['x[0]', 'x[1]', 'x[2]'], axes = frame, orientation = [45,
    35, 300], tickmarks = [0, 0, 0], numpoints = 500);
> rot := array([[1, 0, 0], [0, cos(phi), -sin(phi)], [0, sin(phi
    ), cos(phi)]]);
```

```
>        statpar := array([[-`&ell;`*sqrt(1-r^2/`&ell;`^2)*sinh(t
    /`&ell;`)], [-`&ell;`*sqrt(1-r^2/`&ell;`^2)*cosh(t/`&ell;`)],
     [r]]);
> statplot := plot3d(statpar, t = -2.5 .. 2.5, r = -`&ell;` ..
    `&ell;`, color = blue, style = wireframe);
> rotstatpar := evalm(`&*`(rot, statpar));
> rotstatplot := plot3d(rotstatpar, t = -2.5 .. 2.5, r = -`&ell
    ;` .. `&ell;`, color = red, style = wireframe);
> display(glob, statplot, rotstatplot);
> tboost := array([[cosh(phi), -sinh(phi), 0], [-sinh(phi), cosh
    (phi), 0], [0, 0, 1]]);
> tbooststatpar := evalm(`&*`(tboost, statpar));
> tbooststatplot := plot3d(tbooststatpar, t = -1.5 .. 1.5, r =
    -`&ell;` .. `&ell;`, color = red, style = wireframe);
> display(glob, statplot, tbooststatplot);
> sboost := array([[cosh(phi), 0, -sinh(phi)], [0, 1, 0], [-sinh
    (phi), 0, cosh(phi)]]);
> sbooststatpar := evalm(`&*`(sboost, statpar));
> sbooststatplot := plot3d(sbooststatpar, t = -1.5 .. 1.5, r =
    -`&ell;` .. `&ell;`, color = red, style = wireframe);
> display(glob, statplot, sbooststatplot);
```

The De Sitter horizon was also argued to be a coordinate singularity because of constantness of the curvature; the Gauss curvature was calculated using the following code:

```
> t := x[1]; r := x[2]; theta := x[3]; phi := x[4];
> S := [[-1+r^2/l^2, 0, 0, 0], [0, 1/(1-r^2/l^2), 0, 0], [0, 0,
    r^2, 0], [0, 0, 0, r^2*sin(theta)^2]];
> G := simplify(riemann(gausscurvature, S, 0))
```

A.4 Geodesics

This piece of code calculates the geodesic equations in De Sitter coordinates. The used signature is $(+\,-\,-\,-)$. The additional requirement is that the angular part of the geodesic is constant; I have chosen $\theta = \frac{\pi}{2}, \phi = 0$. This simplifies the equations, so that an algebraic expression for the geodesics can be calculated. For the calculation of the geodesic equations, the procedure 'riemann' was used.

```
> t := x[1]; r := x[2]; theta := x[3]; phi := x[4];
> M := [[1-r^2/l^2, 0, 0, 0], [0, -1/(1-r^2/l^2), 0, 0], [0, 0,
    -r^2, 0], [0, 0, 0, -r^2*sin(theta)^2]];
> geo := riemann(geodesics, M, 0);
> x[4] := 0; x[3] := proc (s) options operator, arrow; (1/2)*Pi
    end proc;
```

The following code solves the differential equation equation (3.7). The option 'simplify' is used to simplify the result.

```
> dsolve((l^2-r(s)^2)*((D@@2)(r))(s)-a^2*r(s)+r(s)*(D(r))(s)^2 =
    0);
> sol1 := (1/2)*(_C1^2/((exp(s/_C1))^2*(exp(_C2/_C1))^2)-a^2*_C1
    ^2+l^2)*exp(s/_C1)*exp(_C2/_C1)/_C1;
```

```
> sol2 := (1/2)*((exp(s/_C1))^2*(exp(_C2/_C1))^2*_C1^2-a^2*_C1
    ^2+l^2)/(exp(s/_C1)*exp(_C2/_C1)*_C1);
> simplify(sol1);
> simplify(sol2);
```

The function $r(s)$ is then used to solve the equation for $t(s)$; this is done using the following code:

```
> r := proc (s) options operator, arrow; l*cosh(s/l)-(1/2)*a^2*l
    *exp(-s/l) end proc;
> eq := diff(t(s), s) = a*l^2/(l^2-r(s)^2);
> simplify(dsolve(eq, t(s)));
```

Last, the proper time at which the geodesic crosses the De Sitter horizon is calculated:

```
> solve(r(s) = l, s);
```

A.5 The Klein Gordon equation

In this section, the Maple code which I used for calculating the Klein-Gordon equation is given. To calculate Christoffel symbols, the procedure 'riemann' was used. The result of each piece of code is $g^{\alpha\beta}{\Gamma^\lambda}_{\alpha\beta}\partial_\lambda$.

The Klein-Gordon equation in flat space

First, the metric is defined. In the Klein-Gordon equation appears an inverse metric, so this has to be calculated as well. The Christoffel symbols are calculated. Then, the sum $g^{\alpha\beta}{\Gamma^\lambda}_{\alpha\beta}\partial_\lambda$ is calculated. In order to display the partial derivative operator, the variable $d[i]$ is used. As every metric used in this section is diagonalized, we only have to sum over the diagonal terms. The equation 'diffeqn' is the radial differential equation.

```
> t:=x[1];r:=x[2];theta:=x[3];phi:=x[4];
> flatmetric := [[1, 0, 0, 0], [0, -1, 0, 0], [0, 0, -r^2, 0],
    [0, 0, 0, -r^2*sin(theta)^2]];
> invflatmetric := inverse(flatmetric);
> flatChristoffel := riemann(christoffel2, flatmetric, 0);
> simplify(sum(invflatmetric[j, j]*(sum(flatChristoffel[i][j][j
    ]*d[i], i = 1 .. 4)), j = 1 .. 4));
> diffeqn := r^2*(diff(R(r), `$`(r, 2)))+2*r*(diff(R(r), r))-(-p
    ^2*r^2/`&hbar;`^2+l(l+1))*R(r) = 0;
> dsolve(diffeqn, R(r));
```

Static coordinates

I have used the following code to calculate the Klein-Gordon equation in static coordinates. The structure is the same as above; the names of the variables may be different.

```
> t:=x[1];r:=x[2];theta:=x[3];phi:=x[4];
> M:=[[(1-r^2/l^2),0,0,0],[0,-1/(1-r^2/l^2),0,0],[0,0,-r
    ^2,0],[0,0,0,-r^2*sin(theta)^2]];
> C := riemann(christoffel2, M, 0);
> invM := inverse(M);
```

```
> simplify(sum(invM[j, j]*(sum(C[i][j][j]*d[i], i = 1 .. 4)), j
    = 1 .. 4));
> eqn := (1-r^2/l^2)*r^2*(diff(R(r), `$`(r, 2)))-(2*l^2-4*r^2)*r
    *(diff(R(r), r))/l^2+(r^2*p^2/`&hbar;`^2+mu^2*r^4/l^2)*R(r)
    /(1-r^2/l^2) = s(s+1)*R(r);
> dsolve(eqn, R(r));
```

Flat slicing coordinates

This code calculates the Klein-Gordon equation in flat slicing coordinates. The equation 'eq2' is defined as the equation for $T(\tau)$, and then solved.

```
> tau := x[1]; theta1 := x[2]; theta2 := x[3]; theta3 := x[4];
> S := [[1, 0, 0, 0], [0, -exp(2*tau/l), 0, 0], [0, 0, -r^2*exp
    (2*tau/l), 0], [0, 0, 0, -exp(2*tau/l)*r^2*sin(theta)^2]];
> CS := riemann(christoffel2, S, 0);
> invS := inverse(S);
> simplify(sum(invS[j, j]*(sum(CS[i][j][j]*d[i], i = 1 .. 4)), j
    = 1 .. 4));
> eq2 := diff(T(tau), `$`(tau, 2))+3*(diff(T(tau), tau))/l+(mu
    ^2+p^2*exp(-2*tau/l)/`&hbar;`^2)*T(tau) = 0;
> dsolve(eq2, T(tau));
```

The next piece of code defines T in terms of $z = \frac{\tau}{\ell}$. It uses the definition of the Hankel function $H_\nu(x) = J_\nu(x) + iY_\nu(x)$, for otherwise, Maple could not calculate the third-order Taylor expansion of $\ln T(z)$.

```
> T := proc (z) options operator, arrow; exp(-(3/2)*z)*(BesselJ(
    sqrt(9/4-M^2*c^2*`&ell;`^2/`&hbar;`^2), p*`&ell;`*exp(-z)/`&
    hbar;`)+I*BesselY(sqrt(9/4-M^2*c^2*`&ell;`^2/`&hbar;`^2), p
    *`&ell;`*exp(-z)/`&hbar;`)) end proc;
> taylor(ln(T(z)), z = 0, 3);
```

Here, the code for generating the plots of $T(\tau)$ is given. First, the function $T(\tau)$ is defined with a normalization factor such that T has norm 1 at $\tau = 0$. The function T is defined with a Hankel $H^{(1)}$ function for matter, the function *antiT* is defined with a $H^{(2)}$ function to represent antimatter.

Then the plot in figure 4.1 is generated. Next up is figure 4.2, followed by figure 4.3 and figure 4.4.

```
> with(plots);
> c := 1; `&hbar;` := 1;
> T := proc (t) options operator, arrow; exp(-(3/2)*t/`&ell;`)*
    HankelH1(sqrt(9/4-M^2*c^2*`&ell;`^2/`&hbar;`^2), p*`&ell;`*
    exp(-t/`&ell;`)/`&hbar;`)/abs(HankelH1(sqrt(9/4-M^2*c^2*`&ell
    ;`^2/`&hbar;`^2), p*`&ell;`/`&hbar;`)) end proc;
> antiT := proc (t) options operator, arrow; exp(-(3/2)*t/`&ell
    ;`)*HankelH2(sqrt(9/4-M^2*c^2*`&ell;`^2/`&hbar;`^2), p*`&ell
    ;`*exp(-t/`&ell;`)/`&hbar;`)/abs(HankelH1(sqrt(9/4-M^2*c^2*`&
    ell;`^2/`&hbar;`^2), p*`&ell;`/`&hbar;`)) end proc;

> cpa := complexplot(eval(eval(eval(T(t), M = 0), `&ell;` = 5),
    p = 1), t = -1.3 .. 100, color = red, numpoints = 1000);
```

```
> cpb := complexplot(eval(eval(eval(T(t), M = 0), `&ell;` = 10),
    p = 1), t = -1.3 .. 100, color = blue);
> cpc := complexplot(eval(eval(eval(T(t), M = 0), `&ell;` = 100)
    , p = 1), t = -2*Pi .. 2, color = green);
> cpexp := complexplot(eval(eval(eval(exp(-I*sqrt(p^2+M^2)*t/`&
    hbar;`), M = 0), `&ell;` = 100), p = 1), t = -2*Pi .. 2*Pi,
    color = black, thickness = 3, linestyle = dot);
> display([cpa, cpb, cpc, cpexp]);

> matterplot := complexplot(eval(eval(eval(T(t), M = 0), `&ell;`
    = 1), p = 1), t = -3 .. 100, color = red, numpoints = 1000);
> antimatterplot := complexplot(eval(eval(eval(antiT(t), M = 0),
    `&ell;` = 1), p = 1), t = -3 .. 100, color = blue, numpoints
    = 1000);
> display(matterplot, antimatterplot);

> Ms := [0, 4]; ls := [1, 10, 100]; colors := [red, blue, green
    ]; linestyles := [solid, dash]; tmax := 1000; tmin := -20;
> ampplot := seq(seq(logplot(eval(eval(eval(abs(T(tau)), M = Ms[
    i]), `&ell;` = ls[j]), p = 1), tau = tmin .. tmax, view = 0
    .. 8, color = colors[j], numpoints = 400, linestyle =
    linestyles[i]), i = 1 .. 2), j = 1 .. 3);
> ampexpplot := seq(logplot(eval(eval(eval(abs(exp(-I*sqrt(p^2+M
    ^2)*tau/`&hbar;`)), M = Ms[i]), `&ell;` = 100), p = 1), tau =
    tmin .. tmax, color = black, linestyle = dot), i = 1 .. 2);
> display(ampplot, ampexpplot);

> energyplot := seq(seq(logplot(eval(eval(eval(abs(diff(T(tau),
    tau)), M = Ms[i]), `&ell;` = ls[j]), p = 1), tau = -20 .. 20,
    view = 0 .. 100, color = colors[j], numpoints = 400,
    linestyle = linestyles[i]), i = 1 .. 2), j = 1 .. 3);
> energyexpplot := seq(logplot(eval(eval(eval(sqrt(p^2+M^2)/`&
    hbar;`, M = Ms[i]), `&ell;` = 100), p = 1), tau = -20 .. 20,
    color = black, linestyle = dot), i = 1 .. 2);
> display(energyplot, energyexpplot);
```