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FIELD QUANTIZATION ON THE SURFACE X^2 = CONSTANT

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Abstract: In analogy to non-relativistic quantum field theory where the Galilei invariant plane t = const. is used for a description of bound (and scattering) states we take the hyperboloid x^2 = const. which is invariant under homogeneous Lorentz transformations. We give the appropriate complete set of basis functions for the expansion of any scalar and spin $\frac{1}{2}$ field on the hyperboloid. The free Klein Gordon and Dirac fields are discussed in detail. In this formulation a multi-particle system can be described by a Lorentz covariant wave function.

1. Introduction

Relativistic quantum field theory has been most successful in applications in which the perturbation series could be used. Bound state problems on the other hand could so far not be treated with the same success. Hadron spectroscopy and experiments on the electromagnetic structure of hadrons have indicated, however, that a bound state description of hadrons is needed. In particular, the deep inelastic electron and neutrino experiments [1] suggest that hadrons are built up from point-like constituents [2]. More carefully expressed, these experiments seem to show that hadrons are composed of quanta of strongly interacting fields with local commutation relations. In this picture the spatial extension of the hadrons is due to the space time distribution of the constituents. Whether or not these fields interpolate particles (nucleons [3] or heavy quarks [4]), fictitious particles (unobservable quarks of low mass [5]) or physical scattering states (gnomes [6]) is so far undecided.

In this paper we suggest a formulation of relativistic quantum field theory which seems to us especially suited for bound state problems. In particular, we will pursue the analogy to the bound state description in non-relativistic field theory where the Schrödinger wave function describes the instantaneous distribution of the constituents in space. There, the commutator of two field operators is only independent of the interaction at equal times. Thus the state vectors are built using operators taken at the same time. In the Heisenberg representation a two particle state vector, for example, has the form

$$|A_\alpha\rangle = \int d^3x\, d^3y\, A_\alpha(x, y, t)\, \psi^\dagger(x, t)\, \psi^\dagger(y, t)\, |0\rangle \ , \tag{1.1}$$

where $\psi^+(x, t)$ is the interacting field operator which creates the constituents out of the vacuum. The wave function $A_\alpha(x, y, t)$ describes the size, charge distribution etc. of the state. The use of the hyperplane t = const. in eq. (1.1) leads to an important property which is independent of the dynamics: any wave function $A_\alpha(x, y, t)$ which is covariant with respect to homogeneous Galilei transformations ensures the covariance of the state $|A_\alpha\rangle$ and *vice versa.* This follows from the invariance of the plane t = const. with respect to these transformations. The dynamical problem on the other hand, is contained in an inhomogeneous part of the Galilei group, the time translation. The time independence of the state $|A_\alpha\rangle$ requires $A_\alpha(x, y, t)$ to obey the Schrödinger equation. Thus, the non-relativistic treatment contains a clear separation between kinematical and dynamical properties. This teaches us that in the relativistic theory a similar state vector description on planes t = const. is not appropriate. The hyperplane t = const. is not invariant under homogeneous Lorentz transformations. The correct analogy to the non-relativistic theory requires an invariant space-like surface. Thus, for the initial value problem, for the quantization and for a description of the state vectors the space-like hyperboloid x^2 = const. > 0 (with x_0 either in the backward or forward light cone) is more appropriate. In the non-relativistic limit (velocity of light $\to \infty$) this surface again approaches the hyperplane t = const. The advantages of this choice are the following:

(i) By homogeneous Lorentz transformations points on the hyperboloid remain on this surface and the commutation relations for two operators on the same surface will be manifestly covariant.

(ii) If a state is described by a multi-operator product acting on the vacuum with all points on the same surface x^2 = const. > 0 (in analogy to the non-relativistic form (1.1)) the corresponding wave function will be covariant by the requirement of covariance of the state vector.

(iii) The separation of an interacting field operator in two parts corresponding to formal annihilation and creation operators is invariant under the homogeneous Lorentz group.

Clearly, there are also disadvantages connected with the use of the invariant surface. The covariance with respect to the 3-space translations is now part of the dynamical problem which concerns the dependence on x^2. The translation covariance of a solution, even in the free field case, is not easily recognized in the non-Cartesian coordinates. Furthermore, the analogue of the Hamilton operator depends on x^2. It is therefore preferable to use directly the field equation instead of working with the Hamiltonian formalism. Nevertheless, we think that the advantages of Lorentz covariant wave functions outweigh the disadvantages.

In the literature a field quantization on light-like planes has been used extensively [7]. This procedure is practically equivalent to the infinite momentum limit and seems particularly suited to high energy processes. But also here, there is no clear separation between the effects of the homogeneous and inhomogeneous Lorentz group. The surfaces x^2 = const. > 0 where first discussed by Dirac [8]. An explicit quantization was performed by Fubini, Hanson, Jackiw [9]. These authors work in

Euclidean space and restrict themselves to scale invariant theories. In a very recent preprint, which we received after this work was completed, Sommerfield [10] treated the quantization on hyperboloids in Minkowski space for the case of the Klein-Gordon field.

In sect. 2 we present the solution of the free Klein-Gordon equation in the appropriate coordinates. Here we find the corresponding complete set of functions in which a field can be decomposed. In sect. 3 the free Dirac equation is treated in the same way. In sect. 4 we quantize the free fields and give the covariant commutation rules.

2. The Klein-Gordon field

In this section we study the solutions of the Klein-Gordon equation in hyperbolic coordinates. These solutions provide us with a complete set of functions on the hyperboloid x^2 = const., which may be used as a basis for the expansion of interacting fields. We introduce the coordinates ρ, ξ, θ, φ where θ, φ are the usual polar angles, and ρ, ξ are related to the Minkowski coordinates $(t, \boldsymbol{x})$ by the transformations

$$\rho = \sqrt{x^2} = \sqrt{t^2 - r^2}\,, \qquad \xi = \frac{t}{\sqrt{x^2}}\,, \tag{2.1}$$

where $r = |\boldsymbol{x}|$; or setting $\xi = \cosh u$ we have

$$t = \rho\,\xi = \rho\cosh u\,, \qquad r = \rho\sqrt{\xi^2 - 1} = \rho\sinh u\,.$$

The transformation (2.1) is a one-to-one mapping of the interior of the forward (backward) light cone onto the region $\xi \geqslant 1$, ($\xi \leqslant -1$) if we restrict ourselves to the positive branch of the square root. In what follows we shall concentrate on the solutions to the field equation in the forward light cone. These solutions may then be continued into the remaining space time regions. This is most easily accomplished by re-expressing ξ and ρ in terms of t and r.

2.1. *Complete set of solutions of the Klein-Gordon equation*

In the variables ρ, ξ, θ, φ, the Klein-Gordon operator takes the form

$$\square + m^2 = \frac{\partial^2}{\partial\rho^2} + \frac{3}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2}\left((\xi^2 - 1)\frac{\partial^2}{\partial\xi^2} + 3\xi\frac{\partial}{\partial\xi} - \frac{L^2}{\xi^2 - 1}\right) + m^2\,, \tag{2.2}$$

where $\boldsymbol{L}^2$ is the square of the angular momentum operator. Alternatively we may write (2.2) in the form

$$\square + m^2 = \frac{\partial^2}{\partial\rho^2} + \frac{3}{\rho}\frac{\partial}{\partial\rho} + \frac{F}{\rho^2} + m^2\,, \tag{2.3}$$

where F is the Casimir operator of the homogeneous Lorentz group in the spin zero case:

$$F = -\tfrac{1}{2} M_{\mu\nu} M^{\mu\nu} = N^2 - L^2 \; . \tag{2.4}$$

Here $M_{\mu\nu}$ are the generators of the group

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \; , \tag{2.5}$$

and

$$L = (M_{23}, M_{31}, M_{12}) \; , \quad N = (M_{01}, M_{02}, M_{03}) \; . \tag{2.6}$$

To solve the Klein-Gordon equation we look for solutions of the form

$$\phi_{\lambda lm}(\rho, \xi, \theta, \varphi) = h_\lambda(\rho) f_{\lambda l}(\xi)\, Y_l^m(\theta, \varphi) \; , \tag{2.7}$$

where

$$f_{\lambda lm}(\xi, \theta, \varphi) \equiv f_{\lambda l}(\xi)\, Y_l^m(\theta, \varphi) \tag{2.8}$$

are eigenfunctions of the Casimir operator

$$F = -(\xi^2 - 1) \frac{\partial^2}{\partial \xi^2} - 3\xi \frac{\partial}{\partial \xi} + \frac{L^2}{\xi^2 - 1} \; , \tag{2.9}$$

with eigenvalue $1 + \lambda^2$ (the reason for writing the eigenvalue in this form will become clear later), i.e.,

$$F f_{\lambda lm} = (1 + \lambda^2) f_{\lambda lm} \; . \tag{2.10}$$

This leads to the following differential equation for $f_{\lambda l}(\xi)$:

$$\left[(\xi^2 - 1) \frac{\partial^2}{\partial \xi^2} + 3\xi \frac{\partial}{\partial \xi} - \frac{l(l+1)}{\xi^2 - 1} + (1 + \lambda^2) \right] f_{\lambda l}(\xi) = 0 \; . \tag{2.11}$$

The corresponding differential equation for $h_\lambda(\rho)$ then reads:

$$\left[\frac{\partial^2}{\partial \rho^2} + \frac{3}{\rho} \frac{\partial}{\partial \rho} + m^2 + \frac{1 + \lambda^2}{\rho^2} \right] h_\lambda(\rho) = 0 \; . \tag{2.12}$$

In solving (2.11) we look for solutions which are regular at $\xi = 1$. They are of the form

$$f_{\lambda l}(\xi) = i^l \frac{(l + i\lambda)!}{(i\lambda - 1)!} \frac{1}{(\xi^2 - 1)^{\frac{1}{4}}} \mathcal{P}^{(l+\frac{1}{2})}_{\frac{1}{2} + i\lambda}(\xi) \; , \tag{2.13}$$

where $\mathcal{P}^\mu_\nu(\xi)$ are the Legendre functions of the first kind [11].

For $\xi \to 1$ and $\xi \to \infty$ the $f_{\lambda l}(\xi)$ behave as follows

$$f_{\lambda l}(\xi) \underset{\xi \to 1}{\sim} i^l 2^{\frac{1}{2}(l+\frac{1}{2})} \frac{(l + i\lambda)!}{(i\lambda - 1)!\,(l + \frac{1}{2})!} \; , \tag{2.14}$$

$$f_{\lambda l}(\xi) \underset{\xi\to\infty}{\sim} \frac{i^l}{\sqrt{\pi}} \frac{(l+i\lambda)!}{(i\lambda-1)!}\left[\frac{(i\lambda-1)!}{(i\lambda+l-1)!}\,\xi^{i\lambda-1} + \text{c.c.}\right]. \tag{2.15}$$

The multiplicative factor has been included for normalization reasons. For real values of λ the functions (2.8) with $f_{\lambda l}(\xi)$ given by (2.13) are normalizable (cf. eq. (2.17) below) and form the basis of an irreducible unitary representation of the homogeneous Lorentz group [12] characterized by the eigenvalue $1+\lambda^2$ of the Casimir operator F (cf. eq. (2.10)). In fact an explicit calculation shows that the generators L and N are hermitian in the space of the basis function (2.8):

$$L_3 f_{\lambda lm} = m f_{\lambda lm};\quad (L_1 \pm iL_2) f_{\lambda lm} = [(l\mp m)(l\pm m+1)]^{\frac12} f_{\lambda l m\pm 1}\,,$$

$$N_3 f_{\lambda lm} = -(l+1-i\lambda)\left[\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}\right]^{\frac12} f_{\lambda,l+1,m}$$

$$-(l+i\lambda)\left[\frac{(l+m)(l-m)}{(2l+1)(2l-1)}\right]^{\frac12} f_{\lambda,l-1,m}\,, \tag{2.16}$$

$$(N_1 \pm iN_2) f_{\lambda lm} = \pm(l+1-i\lambda)\left[\frac{(l\pm m+2)(l\pm m+1)}{(2l+1)(2l+3)}\right]^{\frac12} f_{\lambda,l+1,m\pm 1}$$

$$\mp(l+i\lambda)\left[\frac{(l\mp m-1)(l\mp m)}{(2l+1)(2l-1)}\right]^{\frac12} f_{\lambda,l-1,m\pm 1}\,.$$

The functions $f_{\lambda lm}$ form an orthogonal and complete basis

$$\int \mathrm{d}\Omega \int_1^\infty \mathrm{d}\xi \sqrt{\xi^2-1}\, f^*_{\lambda' l' m'}(\xi,\theta,\varphi) f_{\lambda lm}(\xi,\theta,\varphi) = \delta_{ll'}\,\delta_{mm'}\,\delta(\lambda'-\lambda),\quad \lambda,\lambda', \geqslant 0\,, \tag{2.17}$$

$$\sum_{l,m}\int_0^\infty \mathrm{d}\lambda\, f_{\lambda lm}(\xi',\theta',\varphi') f^*_{\lambda lm}(\xi,\theta,\varphi) = \frac{1}{\sqrt{\xi^2-1}}\,\delta(\xi'-\xi)\,\delta(\cos\theta'-\cos\theta)\,\delta(\varphi'-\varphi)\,. \tag{2.18}$$

Notice that only positive values of λ occur. This is because the real conical functions $\mathcal{P}^{-(l+\frac12)}_{-\frac12+i\lambda}(\xi)$ are invariant under the replacement $\lambda\to-\lambda$.

We now turn to the discussion of the differential equation (2.12). As two linearly independent solutions we take

$$h_\lambda(\rho) = \tfrac12\sqrt{\pi}\,\mathrm{e}^{\frac12\pi\lambda}\frac{1}{\rho}\,H^{(2)}_{i\lambda}(m\rho)\,, \tag{2.19}$$

and its complex conjugate

$$h^*_\lambda(\rho) = \tfrac12\sqrt{\pi}\,\mathrm{e}^{-\frac12\pi\lambda}\frac{1}{\rho}\,H^{(1)}_{i\lambda}(m\rho)\,. \tag{2.20}$$

The reason for choosing the Hankel functions $H^{(2)}_{i\lambda}$ and $H^{(1)}_{i\lambda}$ is that they possess a positive and negative Fourier spectrum respectively which will be important later when we discuss the quantization of the Klein-Gordon field. With the normalization factor chosen as in (2.19) the Wronskian becomes very simple:

$$h^*_\lambda(\rho)\overleftrightarrow{\frac{\partial}{\partial\rho}}h_\lambda(\rho) = -\frac{i}{\rho^3}\ . \tag{2.21}$$

The asymptotic behaviour of $h_\lambda(\rho)$ is given by

$$h_\lambda(\rho) \underset{\rho\to 0}{\sim} \tfrac{1}{2}\sqrt{\pi}\,\frac{1}{\sin h\pi\lambda}\,\frac{1}{\rho}\,\left[\mathrm{e}^{\frac{1}{2}\pi\lambda}\,\frac{(\frac{1}{2}m\rho)^{-i\lambda}}{\Gamma(-i\lambda+1)} - \mathrm{e}^{-\frac{1}{2}\pi\lambda}\,\frac{(\frac{1}{2}m\rho)^{i\lambda}}{\Gamma(i\lambda+1)}\right], \tag{2.22}$$

$$h_\lambda(\rho) \underset{\rho\to\infty}{\sim} \frac{\mathrm{e}^{-\frac{1}{4}i\pi}}{\sqrt{2m}}\,\rho^{-\frac{3}{2}}\,\mathrm{e}^{-im\rho}\ . \tag{2.23}$$

Summarizing we have found that the functions

$$\phi_{\lambda lm}(\rho,\xi,\theta,\varphi) = h_\lambda(\rho) f_{\lambda l}(\xi)\, Y^m_l(\theta,\varphi) \tag{2.24}$$

together with their complex conjugate form a complete set of solutions of the Klein-Gordon equation for $0 \leqslant \lambda \leqslant \infty$, $l = 0, 1, 2, \ldots$, and $m = -l, -l+1, \ldots, l$. They satisfy the orthogonality relation

$$i\int \mathrm{d}\sigma\, \phi^*_{\lambda' l' m'}(x)\overleftrightarrow{\frac{\partial}{\partial\rho}}\phi_{\lambda lm}(x) = \delta_{ll'}\,\delta_{mm'}\,\delta(\lambda'-\lambda)\ , \tag{2.25}$$

as one readily verifies using eqs. (2.7), (2.8), (2.17), and (2.21). Here x stands for the variables ρ, ξ, θ, φ and

$$\mathrm{d}\sigma = \rho^3\sqrt{\xi^2-1}\ \mathrm{d}\xi\ \mathrm{d}\cos\theta\ \mathrm{d}\varphi \tag{2.26}$$

is the invariant volume element on the hyperboloid $x^2 = \rho^2 > 0$. Now any free scalar field $\phi(x)$ of mass m can be expanded in terms of our complete set of solutions $\phi_{\lambda lm}$ and $\phi^*_{\lambda lm}$:

$$\phi(x) = \sum_{l,m}\int_0^\infty \mathrm{d}\lambda\ \{a_{\lambda lm}\phi_{\lambda lm}(x) + b^*_{\lambda lm}\phi^*_{\lambda lm}(x)\}\ . \tag{2.27}$$

The inversion formulae for $a_{\lambda lm}$ and $b_{\lambda lm}$ are

$$a_{\lambda lm} = i\int \mathrm{d}\sigma\,\phi^*_{\lambda lm}(x)\overleftrightarrow{\frac{\partial}{\partial\rho}}\phi(x)\ ,\ b^*_{\lambda lm} = i\int \mathrm{d}\sigma\,\phi(x)\overleftrightarrow{\frac{\partial}{\partial\rho}}\phi_{\lambda lm}(x)\ , \tag{2.28}$$

where $\mathrm{d}\sigma$ is the invariant volume element (2.26). They follow immediately from the orthogonality relation (2.25). The integrals (2.28) are independent of ρ since $\phi(x)$ and $\phi_{\lambda lm}(x)$ satisfy the Klein-Gordon equation. Because of our choice of basis functions (2.24) the right hand side of (2.27) is a decomposition of $\phi(x)$ in positive and negative frequency components. To prove this we show that $b_{\lambda lm} = 0$ if $\phi(x) = e^{-ikx}$ from (2.28) we have that

$$b^*_{\lambda lm} = i \int d\sigma \left\{ e^{-i\rho [k^0 \xi - |\boldsymbol{k}| \sqrt{\xi^2 - 1} \cos \theta_{kr}]} \overleftrightarrow{\frac{\partial}{\partial \rho}} \phi_{\lambda lm}(\rho, \xi, \theta, \varphi) \right\} = 0 , \qquad (2.29)$$

where θ_{kr} is the angle between $\boldsymbol{k}$ and $\boldsymbol{r}$. Now the integral is an analytic function of ρ in the lower half plane, and furthermore independent of ρ on the positive real axis. It is hence a constant in the entire lower half plane and in fact vanishes as can be seen by taking the limit $\mathrm{Im}\, \rho \to -\infty$.

We now consider the conserved current

$$j^\mu(x) = i\phi^*(x) \overleftrightarrow{\partial}^\mu \phi(x) . \qquad (2.30)$$

For an arbitrary space-like surface σ, we can define the charge

$$Q_\sigma = \int_\sigma d\sigma^\mu j_\mu(x) ,$$

where

$$d\sigma^\mu = n^\mu \, d\sigma , \qquad (2.31)$$

n^μ being the unit normal vector to σ. In fact the charge is independent of the special surface chosen, since $Q_{\sigma_1} - Q_{\sigma_2}$ can be written as a 4-dimensional integral over the divergence of the current provided the boundary terms vanish. For the special case where σ is the hyperboloid $x^2 = \rho^2$,

$$n^\mu = \frac{x^\mu}{\rho} , \qquad (2.32)$$

and $d\sigma$ is given by eq. (2.26). Hence making use of the identity $x^\mu \partial_\mu = \rho\, \partial/\partial\rho$ we obtain

$$Q = i \int d\sigma\, \phi^*(x) \overleftrightarrow{\frac{\partial}{\partial \rho}} \phi(x) . \qquad (2.33)$$

We have dropped the subscript on Q since (2.33) is identical with the usual charge. The integral (2.33) may also be written in terms of the expansion coefficients $a_{\lambda lm}$ and $b_{\lambda lm}$. Using eqs. (2.27) and (2.25) we find

$$Q = \sum_{l,m} \int_0^\infty d\lambda\, [a^*_{\lambda lm} a_{\lambda lm} - b^*_{\lambda lm} b_{\lambda lm}] . \qquad (2.34)$$

3. The Dirac field *

In this section we extend the formalism to the case of the free Dirac field. To this effect we rewrite the Dirac equation in terms of the coordinates (2.1), making use of the identity

$$\gamma^{\mu} \frac{\partial}{\partial x^{\mu}} = \gamma x \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{2\rho^2} \sigma^{\mu\nu} M_{\mu\nu} \right), \tag{3.1}$$

where

$$\sigma^{\mu\nu} = \tfrac{1}{2} i \, [\gamma^{\mu}, \gamma^{\nu}]$$

and $M_{\mu\nu}$ are the generators of the homogeneous Lorentz group in the spinless case (cf. eq. (2.6)). The Dirac equation then becomes:

$$\left(i\rho \frac{\partial}{\partial \rho} - \tfrac{1}{2} i \, \sigma^{\mu\nu} M_{\mu\nu} - m \gamma x \right) \psi(x) = 0 \, . \tag{3.2}$$

Introducing the operator

$$K = \tfrac{1}{2} (L + i \gamma_5 N) \, , \tag{3.3}$$

where L is the orbital angular momentum operator, and N the generator of boosts in the spin-zero case (cf. eq. (2.7)), we can rewrite (3.2) as follows:

$$\left(\rho \frac{\partial}{\partial \rho} - 2 \, \Sigma \cdot K + im \, \gamma x \right) \psi(x) = 0 \, , \tag{3.4}$$

where

$$\Sigma = \begin{pmatrix} \sigma & \\ & \sigma \end{pmatrix} .$$

Here σ are the Pauli matrices. The operator K satisfies angular momentum commutation relations

$$[K_i, K_j] = i \epsilon_{ijk} K_k \, . \tag{3.5}$$

The generators of the homogeneous Lorentz group in the spin $\frac{1}{2}$ cases are

$$\mathcal{M}_{\mu\nu} = M_{\mu\nu} + \tfrac{1}{2} \sigma_{\mu\nu} \, , \tag{3.6}$$

and the irreducible representations of the group are characterized by the eigenvalues of two Casimir operators

$$\mathcal{F} = -\tfrac{1}{2} \mathcal{M}_{\mu\nu} \mathcal{M}^{\mu\nu} = \mathcal{N}^2 - J^2 \, , \tag{3.7}$$

$$\mathcal{G} = \tfrac{1}{8} \epsilon^{\mu\nu\lambda\rho} \mathcal{M}_{\mu\nu} \mathcal{M}_{\lambda\rho} = J \cdot \mathcal{N} \, , \tag{3.8}$$

* Throughout this paper we use the conventions of Bjorken-Drell.

where $J = (\mathcal{M}_{23}, \mathcal{M}_{31}, \mathcal{M}_{12})$ and $\mathcal{N} = (\mathcal{M}_{01}, \mathcal{M}_{02}, \mathcal{M}_{03})$, or written out explicitly

$$J = L + \tfrac{1}{2}\Sigma\,;\; \mathcal{N} = N - i\gamma_5 \tfrac{1}{2}\Sigma\,. \tag{3.9}$$

Alternatively (3.7) and (3.8) may be written in the form

$$\mathcal{F} = F - \tfrac{1}{2}\Sigma^2 - 2\,\Sigma\cdot K\,,\quad \mathcal{G} = -i\gamma_5(\Sigma\cdot K + \tfrac{1}{4}\Sigma^2)\,, \tag{3.10}$$

where F is defined by eq. (2.4). We now construct a complete set of basis functions, characterized by the eigenvalues of the commuting operators $\mathcal{F}$, $\mathcal{G}$, J^2, J_3, γ_5, which transform under a unitary irreducible representation of the homogeneous Lorentz group with space reflections.

From our considerations in sect. 1 (cf. eqs. (2.13), (2.11) and (2.10)) we see immediately that

$$\zeta^{\eta}_{\lambda jlM}(\xi,\theta,\varphi) = i^l \frac{(i\lambda + l + \frac{1}{2})!}{(i\lambda - \frac{1}{2})!}\,\frac{1}{(\xi^2 - 1)^{\frac{1}{4}}}\,\mathcal{P}^{-(l+\frac{1}{2})}_{i\lambda}(\xi)\,\zeta^{\eta}_{jlM}(\theta,\varphi) \tag{3.11}$$

is an eigenfunction of the operator F,

$$F\zeta^{\eta}_{\lambda jlM} = [1 + (\lambda - \tfrac{1}{2}i)^2]\,\zeta^{\eta}_{\lambda jlM}\,, \tag{3.12}$$

and also of J^2, L^2, J_3 and γ_5, provided ζ^{η}_{jlM} is an eigenfunction of these operators. The superscript $\eta = \pm$ denotes the eigenvalue of γ_5:

$$\gamma_5\,\zeta^{\eta}_{jlM} = \eta\,\zeta^{\eta}_{jlM}\,. \tag{3.13}$$

λ is so far an arbitrary complex number. Notice that l can take on only the values $j \pm \frac{1}{2}$. The corresponding functions ζ^{η}_{jlM} have the explicit form

$$\zeta^{\pm}_{jlM} = \begin{pmatrix} \varphi_{jlM} \\ \pm\varphi_{jlM} \end{pmatrix}, \tag{3.14}$$

where φ_{jlM} are the usual two component spin-angular functions. They satisfy the relation

$$\varphi_{j,j+\frac{1}{2},M}(\theta,\varphi) = -\frac{\sigma\cdot r}{r}\,\varphi_{j,j-\frac{1}{2},M}(\theta,\varphi)\,. \tag{3.15}$$

One can now easily construct simultaneous eigenfunctions of $\mathcal{F}$, $\mathcal{G}$, J^2, J_3 and γ_5. From the commutation relations (3.5) it follows that

$$(\Sigma\cdot K)(\Sigma\cdot K) = K^2 - \Sigma\cdot K\,. \tag{3.16}$$

Making use of (3.12) and of the fact that K commutes with $F = -4K^2$, we find that for a given $\zeta^{\eta}_{\lambda jlM}$ the spinors

$$(\tfrac{1}{4} - i\tfrac{1}{2}\lambda + \Sigma\cdot K)\,\zeta^{\eta}_{\lambda jlM}\,, \tag{3.17}$$

and $(\frac{3}{4} + i\frac{1}{2}\lambda + \Sigma\cdot K)\,\zeta^{\eta}_{\lambda jlM}$ are eigenfunctions of the operator $\Sigma\cdot K$ with eigenvalues $-\frac{3}{4} - i\frac{1}{2}\lambda$ and $-\frac{1}{4} + i\frac{1}{2}\lambda$, respectively. Because of the identity $\mathcal{P}^{\mu}_{\nu} = \mathcal{P}^{\mu}_{-\nu-1}$, it is only necessary to consider (3.17) since λ is so far an arbitrary complex parameter. Using eq. (3.10) and (3.11) we find that (3.17) is a simultaneous eigenfunction of the operators $\mathcal{F}$, $\mathcal{G}$, J^2, J_3 and γ_5 with eigenvalues which do not depend on l. Since the set of commuting operators is complete, the spinors (3.17) with $l = j + \frac{1}{2}$ and $l = j - \frac{1}{2}$ must be proportional. This may be confirmed by an explicit calculation. Hence it is sufficient to consider the functions

$$U^{\eta}_{\lambda jM}(\xi, \theta, \varphi) = N_0(\tfrac{1}{4} - i\tfrac{1}{2}\lambda + \Sigma \cdot K)\, \zeta^{\eta}_{\lambda, j, j-\frac{1}{2}, M}(\xi, \theta, \varphi)\,. \tag{3.18}$$

They satisfy the eigenvalue equations

$$\begin{aligned}
\mathcal{F}\, U^{\eta}_{\lambda jM} &= (\lambda^2 + \tfrac{3}{4})\, U^{\eta}_{\lambda jM}\,,\\
\mathcal{G}\, U^{\eta}_{\lambda jM} &= -\tfrac{1}{2}\lambda\eta\, U^{\eta}_{\lambda jM}\,,\\
\gamma_5\, U^{\eta}_{\lambda jM} &= \eta\, U^{\eta}_{\lambda jM}\,,\\
J^2\, U^{\eta}_{\lambda jM} &= j(j+1)\, U^{\eta}_{\lambda jM}\,,\\
J_3\, U^{\eta}_{\lambda jM} &= M\, U^{\eta}_{\lambda jM}\,.
\end{aligned} \tag{3.19}$$

Since we want the spinors $U^{\eta}_{\lambda jM}$ to form the basis of a unitary representation of the homogeneous Lorentz group, the eigenvalues of $\mathcal{F}$ and $\mathcal{G}$ must be real. Hence λ is restricted to the real axis.

To obtain an explicit expression for (3.18) we must know how $\gamma_5\, \Sigma \cdot N$ acts on $\zeta^{\eta}_{\lambda, j, j-\frac{1}{2}, M}$. One finds after some algebra

$$\gamma_5\, \Sigma \cdot N\, \zeta^{\eta}_{\lambda, j, j-\frac{1}{2}, M} = \eta(j - i\lambda)\zeta^{\eta}_{\lambda, j, j+\frac{1}{2}, M}\,. \tag{3.20}$$

Performing the remaining trivial operations we obtain

$$U^{\eta}_{\lambda jM}(\xi, \theta, \varphi) = \tfrac{1}{2}N_0(j - i\lambda)\, \{\zeta^{\eta}_{\lambda, j, j-\frac{1}{2}, M}(\xi, \theta, \varphi) + i\eta\, \zeta_{\lambda, j, j+\frac{1}{2}, M}(\xi, \theta, \varphi)\}\,. \tag{3.21}$$

Choosing an appropriate normalization factor (3.21) has the explicit form

$$U^{\eta}_{\lambda jM}(\xi, \theta, \varphi) = \tfrac{1}{2}\, i^{j-\frac{1}{2}} \left|\frac{(i\lambda + j)!}{(i\lambda - \frac{1}{2})!}\right| \begin{pmatrix} \varphi^{\eta}_{\lambda jM}(\xi, \theta, \varphi) \\ \eta\varphi^{\eta}_{\lambda jM}(\xi, \theta, \varphi) \end{pmatrix}, \tag{3.22}$$

where

$$\varphi^{\eta}_{\lambda jM}(\xi, \theta, \varphi) = \frac{1}{(\xi^2 - 1)^{\frac{1}{4}}} \left\{ \mathcal{P}^{-j}_{i\lambda}(\xi)\, \varphi_{j, j-\frac{1}{2}, M}(\theta, \varphi) - \eta(i\lambda + j + 1)\, \mathcal{P}^{-j-1}_{i\lambda}(\xi)\varphi_{j, j+\frac{1}{2}, M}(\theta, \varphi) \right\} \tag{3.23}$$

Concerning the action of the generators (3.9) on $U^{\eta}_{\lambda jM}(\xi, \theta, \varphi)$ one finds after some algebra, using eqs. (2.16) and the explicit expressions (3.23), the following transformation equations:

$$J_3 U^\eta_{\lambda jM} = M U^\eta_{\lambda jM}\,,$$

$$(J_1 \pm iJ_2) U^\eta_{\lambda jM} = [(j \mp M)(j \pm M + 1)]^{\frac{1}{2}} U^\eta_{\lambda jM\pm 1}\,,$$

$$\mathcal{N}_3 U^\eta_{\lambda jM} = -\frac{1}{2(j+1)} [((j+1)^2 + \lambda^2)(j-M+1)(j+M+1)]^{\frac{1}{2}} U^\eta_{\lambda,j+1,M}$$

$$-\frac{1}{2j} [(j^2+\lambda^2)(j-M)(j+M)]^{\frac{1}{2}} U^\eta_{\lambda,j-1,M} - \frac{\lambda\eta}{2j(j+1)} M U^\eta_{\lambda jM}\,,$$

$$(\mathcal{N}_1 \pm i\mathcal{N}_2) U^\eta_{\lambda jM} = \pm \frac{1}{2(j+1)} [((j+1)^2+\lambda^2)(j\pm M+2)(j\pm M+1)]^{\frac{1}{2}} U^\eta_{\lambda,j+1,M\pm 1}$$

$$\mp \frac{1}{2j} [(j^2+\lambda^2)(j\mp M-1)(j\mp M)]^{\frac{1}{2}} U^\eta_{\lambda,j-1,M\pm 1}$$

$$-\frac{\eta\lambda}{2j(j+1)} [(j\pm M+1)(j\mp M)]^{\frac{1}{2}} U^\eta_{\lambda,j,M\pm 1} \tag{3.24}$$

From (3.24) we see that the generators are hermitian operators in the space of functions $U^\eta_{\lambda jM}$, and that the transformations are diagonal in η. For some applications it may be more convenient to work with basis functions which are eigenfunctions of the parity operator rather than of γ_5. These are readily obtained by taking the linear combinations

$$\mathcal{W}^\pm_{\lambda jM} = \frac{1}{\sqrt{2}} (U^+_{\lambda jM} \pm U^-_{\lambda jM})\,. \tag{3.25}$$

As seen from their explicit structure

$$\mathcal{W}^+_{\lambda jM}(\xi,\theta,\varphi) = \frac{i^{j-\frac{1}{2}}}{\sqrt{2}} \left|\frac{(i\lambda+j)!}{(i\lambda-\frac{1}{2})!}\right| \frac{1}{(\xi^2-1)^{\frac{1}{4}}} \begin{pmatrix} \mathcal{P}^{-j}_{i\lambda}(\xi)\,\varphi_{j,j-\frac{1}{2},M}(\theta,\varphi) \\ -(i\lambda+j+1)\,\mathcal{P}^{-j-1}_{i\lambda}(\xi)\varphi_{j,j+\frac{1}{2},M}(\theta,\varphi) \end{pmatrix},$$

$$\mathcal{W}^-_{\lambda jM}(\xi,\theta,\varphi) = \gamma_5 \mathcal{W}^+_{\lambda jM}(\xi,\theta,\varphi)\,. \tag{3.26}$$

$\mathcal{W}^+_{\lambda jM}(\xi,\theta,\varphi)$ and $\mathcal{W}^-_{\lambda jM}(\xi,\theta,\varphi)$ have parity $(-1)^{j-\frac{1}{2}}$ and $(-1)^{j+\frac{1}{2}}$ respectively. As shown in the appendix they satisfy the following orthogonality and completeness relations:

$$\int \mathrm{d}\Omega \int_1^\infty \mathrm{d}\xi \sqrt{\xi^2-1}\, \overline{\mathcal{W}}^{\eta'}_{\lambda' j' M'}(\xi,\theta,\varphi) \frac{\gamma\cdot x}{\sqrt{x^2}} \mathcal{W}^\eta_{\lambda jM}(\xi,\theta,\varphi) = \delta_{jj'}\delta_{MM'}\delta_{\eta\eta'}\delta(\lambda'-\lambda)\,, \tag{3.27}$$

$$\sum_{j,M,\eta} \int_{-\infty}^{+\infty} d\lambda\, \mathcal{W}^{\eta}_{\lambda jM}(\xi,\theta,\varphi) \otimes \overline{\left(\frac{\gamma\cdot x'}{\sqrt{x'^2}} \mathcal{W}^{\eta}_{\lambda jM}(\xi',\theta',\varphi')\right)}$$

$$= \frac{1}{\sqrt{\xi^2-1}}\, \delta(\xi'-\xi)\, \delta(\cos\theta'-\cos\theta)\delta(\varphi'-\varphi) \cdot \mathbf{1}\,, \tag{3.28}$$

where $\mathbf{1}$ is the 4×4 unit matrix. The same relations apply with $\mathcal{W}^{\eta}_{\lambda jM}$ replaced by the basis functions $U^{\eta}_{\lambda jM}$. Starting from the explicit expression (3.26), and making use of (3.15) and of the relations between contiguous Legendre functions one readily proves the useful relation

$$\frac{\gamma\cdot x}{\sqrt{x^2}} \mathcal{W}^{\eta}_{\lambda jM}(\xi,\theta,\varphi) = \eta \mathcal{W}^{\eta}_{-\lambda,j,M}(\xi,\theta,\varphi)\,. \tag{3.29}$$

The transformation properties of the new basis functions (3.26) are immediately obtained from eqs. (3.25) and (3.24). The Lorentz transformations now mix the states of opposite parity.

3.1. Solution of the Dirac equation

Having obtained a complete set of basis functions with respect to ξ, θ, φ, we now look for solutions of the Dirac equation with definite parity. To this effect we make the ansatz

$$\psi^{\eta}_{\lambda jM}(\rho,\xi,\theta,\varphi) = \left[f_\lambda(\rho) + g_\lambda(\rho)\frac{\gamma\cdot x}{\sqrt{x^2}}\right] \mathcal{W}^{\eta}_{\lambda jM}(\xi,\theta,\varphi)\,, \tag{3.30}$$

where $\rho = \sqrt{x^2}$. Because of eq. (3.29) we can restrict ourselves to $\lambda \geqslant 0$. Substituting (3.30) into (3.4) and making use of the equations

$$\Sigma\cdot K\gamma\cdot x = -\gamma\cdot x(\Sigma\cdot K + \tfrac{3}{2})\,, \tag{3.31}$$

$$\Sigma\cdot K \mathcal{W}^{\eta}_{\lambda jM} = -\tfrac{1}{2}(\tfrac{3}{2} + i\lambda)\mathcal{W}^{\eta}_{\lambda jM}\,, \tag{3.32}$$

we arrive at the following system of coupled differential equations for $f_\lambda(\rho)$ and $g_\lambda(\rho)$:

$$\left(\rho\frac{\partial}{\partial\rho} + \tfrac{3}{2} + i\lambda\right) f_\lambda(\rho) + im\rho g_\lambda(\rho) = 0\,,$$

$$\left(\rho\frac{\partial}{\partial\rho} + \tfrac{3}{2} - i\lambda\right) g_\lambda(\rho) + im\rho f_\lambda(\rho) = 0\,. \tag{3.33}$$

A set of suitably normalized solutions of (3.33) is given by

$$f_\lambda(\rho) = \sqrt{m}\, h_{\lambda - \frac{i}{2}}(\rho)\,,$$

$$g_\lambda(\rho) = \sqrt{m}\, h_{\lambda + \frac{i}{2}}(\rho)\,, \tag{3.34}$$

where $h_\beta(\rho)$ is given by (2.19). Two independent solutions to the Dirac equation with fixed parity are

$$\psi^\eta_{\lambda jM}(\rho, \xi, \theta, \varphi) = \sqrt{m} \left\{ h_{\lambda - \frac{i}{2}}(\rho) + h_{\lambda + \frac{i}{2}}(\rho) \frac{\gamma \cdot x}{\sqrt{x^2}} \right\} \mathcal{W}^\eta_{\lambda jM}(\xi, \theta, \varphi)\,. \tag{3.35}$$

and its charge conjugate

$$(\psi^\eta_{\lambda jM})^c = i\gamma_2\, \psi^{\eta *}_{\lambda jM}\,. \tag{3.36}$$

As shown in the appendix they satisfy the following orthogonality relation for λ, $\lambda' \geqslant 0$:

$$\int d\sigma\, \bar{\psi}^{\eta'}_{\lambda' j' M'}(x) \frac{\gamma \cdot x}{\sqrt{x^2}}\, \psi^\eta_{\lambda jM}(x) = \delta_{jj'}\, \delta_{MM'}\, \delta_{\eta\eta'}\, \delta(\lambda' - \lambda)\,, \tag{3.37}$$

where x stands for the variables ρ, ξ, θ, φ and $d\sigma$ is the invariant volume element (2.26). The most general solution to the Dirac equation can be expanded in terms of (3.35) and (3.36) as follows

$$\psi(x) = \sum_{j, M, \eta} \int_0^\infty d\lambda\, \{a^\eta_{\lambda jM}\, \psi^\eta_{\lambda jM}(x) + b^{\eta *}_{\lambda jM}\, (\psi^\eta_{\lambda jM}(x))^c\}\,. \tag{3.38}$$

As in the Klein-Gordon case, the first and second integrals contain only positive and negative frequencies respectively. The inversion formulae for $a^\eta_{\lambda jM}$ and $b^\eta_{\lambda jM}$ are:

$$a^\eta_{\lambda jM} = \int d\sigma\, \psi^\eta_{\lambda jM}(x) \frac{\gamma \cdot x}{\sqrt{x^2}}\, \psi(x)\,, \quad b^\eta_{\lambda jM} = \int d\sigma\, \bar{\psi}^\eta_{\lambda jM}(x) \frac{\gamma \cdot x}{\sqrt{x^2}}\, \psi^c(x)\,, \tag{3.39}$$

where $\psi^c(x) = i\gamma_2 \psi^*(x)$. For the same reasons discussed in the case of the scalar field, the integrals (3.39) are independent of $\rho = \sqrt{x^2}$ as long as $\psi(x)$ is a free field.

4. Quantization

For the quantization of the Klein-Gordon and Dirac fields we start from the respective Lagrangians written in the coordinates ρ, ξ, θ, φ:

$$L_{\mathrm{KG}} = \int \rho^3 \,\mathrm{d}\xi \sqrt{\xi^2-1}\,\mathrm{d}\Omega \left\{ \frac{\partial\phi^+}{\partial\rho}\frac{\partial\phi}{\partial\rho} - \frac{\xi^2-1}{\rho^2}\frac{\partial\phi^+}{\partial\xi}\frac{\partial\phi}{\partial\xi} - \frac{1}{\rho^2(\xi^2-1)}\frac{\partial\phi^+}{\partial\theta}\frac{\partial\phi}{\partial\theta} \right.$$

$$\left. - \frac{1}{\rho^2(\xi^2-1)\sin^2\theta}\frac{\partial\phi^+}{\partial\varphi}\frac{\partial\phi}{\partial\varphi} - m^2\phi^+\phi \right\}$$

$$= \int \mathrm{d}\sigma \left\{ \frac{\partial\phi^+}{\partial\rho}\frac{\partial\phi}{\partial\rho} - \frac{\phi^+ F\phi}{\rho^2} - m^2\phi^+\phi \right\}, \tag{4.1}$$

$$L_{\mathrm{Dirac}} = \int \mathrm{d}\sigma \left\{ i\bar{\psi}\,\frac{\gamma\cdot x}{\rho}\left(\frac{\partial}{\partial\rho} - \frac{1}{2\rho}\sigma^{\mu\nu}M_{\mu\nu}\right)\psi - m\bar{\psi}\psi \right\}, \tag{4.2}$$

where F is the operator defined by eq. (2.9). The derivative transverse to the hyperboloid is $n_\mu\partial^\mu = \partial/\partial\rho$ where n_μ is given by eq. (2.32); hence the canonically conjugate momenta to the fields ϕ and ψ are

$$\pi_\phi = \rho^3\sqrt{\xi^2-1}\,\frac{\partial\phi^+}{\partial\rho}\,, \quad \pi_\psi = i\rho^3\sqrt{\xi^2-1}\,\bar{\psi}\,\frac{\gamma\cdot x}{\rho}\,. \tag{4.3}$$

Consequently the canonical commutators on the surface x^2 = const. are given by

$$\left[\frac{\partial\phi}{\partial\rho}(x), \phi^+(x')\right]_{\rho=\rho'} = \left[\frac{\partial\phi^+}{\partial\rho}(x), \phi(x')\right]_{\rho=\rho'}$$

$$= \frac{1}{i\rho^3}\frac{1}{\sqrt{\xi^2-1}}\,\delta(\xi'-\xi)\delta(\cos\theta'-\cos\theta)\delta(\varphi'-\varphi)\,, \tag{4.4}$$

$$\{\psi(x), \bar{\psi}(x')\}_{\rho=\rho'} = \gamma\cdot x\,\frac{\delta(\xi'-\xi)}{\rho^4\sqrt{\xi^2-1}}\,\delta(\cos\theta'-\cos\theta)\delta(\varphi'-\varphi)\,. \tag{4.5}$$

All other canonical field commutators vanish. The commutation relations (4.4) and (4.5) are compatible with the equations of motion. From (2.28) and (3.39) one then readily obtains the commutation relations for the a's and b's:

$$[a_{\lambda lm}, a^+_{\lambda' l' m'}] = [b_{\lambda lm}, b^+_{\lambda' l' m'}] = \delta_{ll'}\delta_{mm'}\delta(\lambda-\lambda')\,, \tag{4.6}$$

$$[a^\eta_{\lambda jM}, a^{\eta',+}_{\lambda' j' M'}] = [b^\eta_{\lambda jM}, b^{\eta',+}_{\lambda' j' M'}] = \delta_{\eta\eta'}\delta_{jj'}\delta_{MM'}\delta(\lambda-\lambda')\,. \tag{4.7}$$

In terms of these operators the charge operator Q takes the usual diagonal form: for example in the Dirac case we have

$$Q = \sum_{j,M,\eta} \int_0^\infty \mathrm{d}\lambda \; \{a^{\eta\,+}_{\lambda jM}\, a^{\eta}_{\lambda jM} \quad b^{\eta\,+}_{\lambda jM}\, b^{\eta}_{\lambda jM}\} \tag{4.8}$$

A similar expression holds in the Klein-Gordon case. The "Hamiltonians" obtained from the Lagrangians (4.1) and (4.2) are

$$\hat{H}_{\text{K.G.}}(\rho) = \int \mathrm{d}\sigma \left\{ \frac{\pi^+\pi}{\rho^6(\xi^2-1)} + \frac{\phi^+ F\phi}{\rho^2} + m^2\phi^+\phi \right\}, \tag{4.9}$$

$$\hat{H}_{\text{Dirac}}(\rho) = \int \mathrm{d}\sigma \left\{ i\bar{\psi}\, \frac{\gamma\cdot x}{2\rho^2}\, \sigma^{\mu\nu} M_{\mu\nu} + m\bar{\psi}\psi \right\}. \tag{4.10}$$

Using these expressions which are invariant under the homogeneous Lorentz group, the Hamiltonian equations lead to the correct field equations. However, because of the explicit ρ dependence the operators (4.7) and (4.8) are not conserved, not even in the zero-mass limit. Another operator which leads from one hyperboloid to another is the dilatation operator.

$$D_{\text{K.G.}} = \int \mathrm{d}\sigma \left[\rho\, \frac{\partial\phi^+}{\partial\rho}\frac{\partial\phi}{\partial\rho} + \frac{\partial\phi^+}{\partial\rho}\,\phi + \phi^+\,\frac{\partial\phi}{\partial\rho} + \frac{\phi^+ F\phi}{\rho} + m^2\rho\phi^+\phi \right], \tag{4.11}$$

$$D_{\text{Dirac}} = \int \mathrm{d}\sigma \left[i\bar{\psi}\, \frac{\gamma\cdot x}{2\rho}\, \sigma^{\mu\nu} M_{\mu\nu}\, \psi + i\tfrac{3}{2}\bar{\psi}\, \frac{\gamma\cdot x}{\rho}\, \psi + m\rho\bar{\psi}\psi \right]. \tag{4.12}$$

Their commutation relations with the fields are

$$i\,[D_{\text{K.G.}}, \phi] = \left(\rho\,\frac{\partial}{\partial\rho} + 1\right)\phi, \tag{4.13}$$

$$i\,[D_{\text{K.G.}}, \pi] = \left(\rho\,\frac{\partial}{\partial\rho} - 1\right)\pi, \tag{4.14}$$

$$i\,[D_{\text{Dirac}}, \psi] = \left(\rho\,\frac{\partial}{\partial\rho} + \tfrac{3}{2}\right)\psi. \tag{4.15}$$

The derivative of the dilatation operators with respect to ρ is

$$\frac{\mathrm{d}D_{\text{K.G.}}}{\mathrm{d}\rho} = 2m^2 \int \mathrm{d}\sigma\, \phi^+\phi, \tag{4.16}$$

$$\frac{\mathrm{d}D_{\text{Dirac}}}{\mathrm{d}\rho} = m \int \mathrm{d}\sigma\, \bar{\psi}\psi. \tag{4.17}$$

Consequently, these operators are conserved for $m = 0$, as expected: the zero mass limit, however, cannot be simply obtained by taking $m \to 0$ in the expansion formula for the fields (cf. eqs. (2.27) and (3.38)) since the functions $H^{(2)}_{\beta}(m\rho)$ occuring in the definition of $\phi_{\lambda lm}$ and $\psi^{\eta}_{\lambda jM}$ do not approach a definite limit. For the mass zero case we therefore go back to the original equations (2.12) and (3.33). For the Klein-Gordon equation, the positive frequency solution of eq. (2.12) with $m = 0$, normalized according to (2.21) is given by

$$h_\lambda(\rho) = \frac{1}{\sqrt{2\lambda}} \frac{e^{-i\lambda \ln \rho}}{\rho} . \tag{4.18}$$

In the Dirac case we see from (3.33) that for $m = 0$ the differential equations decouple, and that the positive frequency part is now given by (3.35) with

$$f_\lambda(\rho) = \frac{1}{\rho^{\frac{3}{2}}} e^{-i\lambda \ln \rho} , \quad g_\lambda(\rho) = 0 . \tag{4.19}$$

In terms of the corresponding annihilation and creation operators the dilatation operators are given by

$$D_{\mathrm{KG}} = \sum_{l,m} \int_0^\infty \lambda \, d\lambda \, \{a^\dagger_{\lambda lm} a_{\lambda lm} + b^\dagger_{\lambda lm} b_{\lambda lm}\} , \tag{4.20}$$

$$D_{\mathrm{Dirac}} = \sum_{jM\eta} \int_0^\infty \lambda \, d\lambda \, \{a^{\eta\dagger}_{\lambda jM} a^\eta_{\lambda jM} + b^{\eta\dagger}_{\lambda jM} b^\eta_{\lambda jM}\} . \tag{4.21}$$

From these equations we recognize the significance of the continuous parameter λ as the eigenvalue of the dilatation operators.

Finally, we wish to make a remark about the wave function description of state vectors. The wave function giving the two particle distribution (on each hyperboloid) contained in a state $|\alpha\rangle$ is defined by

$$A_\alpha(\rho; \xi_1, \theta_1, \varphi_1; \xi_2, \theta_2, \varphi_2) = \langle 0| \psi(\rho, \xi_1, \theta_1, \varphi_1) \, \psi(\rho, \xi_2, \theta_2, \varphi_2)|\alpha\rangle . \tag{4.22}$$

In this example we use spin $\frac{1}{2}$ fields. For large values of ρ and localized states this wave function is essentially an equal time correlation function. Since the Heisenberg state $|\alpha\rangle$ is independent of the space and time variables the field equation leads to a Lorentz invariant wave equation for A_α. In the free field case A_α is a solution to

$$\left(i\rho \frac{\partial}{\partial \rho} - \tfrac{1}{2} i \, \sigma^{\mu\nu}_{(1)} M^{(1)}_{\mu\nu} - \tfrac{1}{2} i \, \sigma^{\mu\nu}_{(2)} M^{(2)}_{\mu\nu} - m \, \gamma_{(1)} \cdot x_{(1)} - m \, \gamma_{(2)} \cdot x_{(2)}\right) A_\alpha = 0 . \tag{4.23}$$

For interacting fields one obtains a coupled set of equations for different multi-particle wave functions. We hope to come back to the question of interaction in a future publication.

Appendix A. Orthogonality relation

To prove the orthogonality relation (3.27) we first rewrite the integral making use of relation (3.29), and perform the angular integrations using the orthogonality properties of $\varphi_{j,j+\frac{1}{2},M}$ (cf. eqs. (3.26)):

$$\int \mathrm{d}\Omega\, \varphi^*_{j'l'M'}(\theta,\varphi)\, \varphi_{jlM}(\theta,\varphi) = \delta_{jj'}\delta_{ll'}\delta_{MM'} \,. \tag{A.1}$$

It follows that the integral (3.27) is diagonal in j, M, and η. Hence we only have to prove that:

$$\frac{1}{2}\left|\frac{(-i\lambda+j)!\,(i\lambda'+j)!}{(-i\lambda-\frac{1}{2})!\,(i\lambda'-\frac{1}{2})!}\right| \int_1^\infty \mathrm{d}\xi \left[\mathcal{P}^{\,j}_{-i\lambda'}(\xi)\,\mathcal{P}^{-j}_{i\lambda}(\xi)\right.$$

$$\left. -(i\lambda'+j+1)(-i\lambda+j+1)\,\mathcal{P}^{-j}_{-i\lambda'}{}^{1}(\xi)\,\mathcal{P}^{\,j}_{-i\lambda}{}^{1}(\xi)\right] = \delta(\lambda'-\lambda)\,. \tag{A.2}$$

The integral may be evaluated with the help of the formula [11]

$$\int_1^\xi \mathrm{d}\xi'\, \mathcal{P}^{\mu}_{\nu}(\xi')\,\mathcal{P}^{\mu}_{\sigma}(\xi') = \frac{1}{(\nu-\sigma)(\nu+\sigma+1)}\left[(\nu-\sigma)\,\xi\,\mathcal{P}^{\mu}_{\nu}(\xi)\,\mathcal{P}^{\mu}_{\sigma}(\xi)\right.$$

$$\left. +(\sigma+\mu)\,\mathcal{P}^{\mu}_{\nu}(\xi)\,\mathcal{P}^{\mu}_{\sigma}(\xi) - (\nu+\mu)\,\mathcal{P}^{\mu}_{-\nu}(\xi)\,\mathcal{P}^{\mu}_{\sigma}(\xi)\right] . \tag{A.3}$$

One then finds that in the limit $\xi\to\infty$ the linear divergent term in (A.3) drops out in the difference (A.2) and one is left with the following expression for the integral (A.2)

$$\lim_{\xi\to\infty} \frac{1}{2\pi i(\lambda-\lambda')\,\{1-i(\lambda+\lambda')\}}\left[2^{-i(\lambda+\lambda')}\,\frac{(-i\lambda'-\frac{1}{2})!\,(i\lambda-\frac{1}{2})!}{(-i\lambda'+j)!\,\overline{(i\lambda+j)!}}\,(1-2i\lambda)\,\xi^{i(\lambda-\lambda')}\right.$$

$$\left. -(\lambda\leftrightarrow\lambda')\right] . \tag{A.4}$$

For $\xi\to\infty$ only $\lambda=\lambda'$ contributes (in the distribution sense) and (A.4) approaches $|(i\lambda-\frac{1}{2})!/(i\lambda+j)!|^2\,\delta(\lambda'-\lambda)$. Hence the right hand side of (A.2) follows. Using the result (3.27) one then also readily verifies the orthogonality relation (3.37), where $\psi^\eta_{\lambda jM}$ is defined by eqs. (3.30) and (3.34) by making use of the identity

$$|h_{\lambda-\frac{1}{2}i}|^2 + |h_{\lambda+\frac{1}{2}i}|^2 = \frac{1}{m\rho^3}\,, \tag{A.5}$$

which follows from well known relations between Hankel functions (cf. eqs. (2.19) for the definition of h_β).

Appendix B. Completeness relation

To prove the completeness relation (3.28) we notice that the left and right hand sides of this equation are invariant under Lorentz transformations. This is obvious for the right hand side. For the left hand side it follows from the fact that the spinors (3.22), and hence (3.25) form the basis of a unitary representation. Hence it is sufficient to prove (3.28) in a special Lorentz frame. To this effect we first rewrite (3.28) with the help of relation (3.29), and introduce the variable u defined by eq.(2.1);

$$\sum_{j,M,\eta} \eta \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathcal{W}^{\eta}_{\lambda jM}(\cosh u, \theta, \varphi) \otimes (\sinh u')^2 \overline{\mathcal{W}}^{\eta}_{-\lambda jM}(\cosh u', \theta', \varphi')$$

$$= 1\ \delta(u'-u)\delta(\cos\theta' - \cos\theta)\delta(\varphi'-\varphi)\,, \tag{A.6}$$

with $u, u' \geqslant 0$. We now evaluate the left hand side of (A.6) in the special Lorentz frame $\xi = 1$, i.e. $u = 0$. Making use of the fact that

$$\frac{1}{(\sinh u)^{\frac{1}{2}}} \mathcal{P}^{\mu}_{\nu}(\cosh u) \underset{u\to 0}{\longrightarrow} \frac{2^{\mu}}{\Gamma(1-\mu)}\, u^{-\frac{1}{2}\mu}\,,$$

we see that only the first term with $j = \frac{1}{2}$ contributes to the sum (A.6). The corresponding basis functions $\mathcal{W}^{\eta}_{\lambda jM}$ are given by

$$\mathcal{W}^{+}_{\lambda,\frac{1}{2},M}(\cosh u = 1, \theta, \varphi) = \frac{1}{2\pi}\,|i\lambda + \tfrac{1}{2}| \begin{pmatrix} \chi_M \\ 0 \end{pmatrix}, \tag{A.7}$$

$$\mathcal{W}^{+}_{\lambda,\frac{1}{2},M}(\cosh u', \theta', \varphi') = \frac{|i\lambda + \frac{1}{2}|}{2\sqrt{2\pi}}\, \frac{1}{(\sinh u')^{\frac{1}{2}}} \begin{pmatrix} \mathcal{P}^{-\frac{1}{2}}_{i\lambda}(\cosh u')\chi_M \\ (-i\lambda + \frac{3}{2})\, \mathcal{P}^{\frac{3}{2}}_{i\lambda}(\cosh u') \frac{\sigma\cdot r}{r}\chi_M \end{pmatrix},$$

where χ_M are the usual Pauli spinors. The corresponding expressions for $\mathcal{W}_{\lambda jM}$ are obtained from eq. (3.26). From the explicit expression (A.7) one readily verifies that the diagonal terms in the spinor indices of (A.6) are all identical and given by

$$\frac{(\sinh u')^{\frac{3}{2}}}{2(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \mathrm{d}\lambda\, (\lambda^2 + \tfrac{1}{4})\, \mathcal{P}^{-\frac{1}{2}}_{i\lambda}(\cosh u')\,. \tag{A.8}$$

Substituting the explicit expression for the Legendre function

$$\mathcal{P}^{-\frac{1}{2}}_{i\lambda}(\cosh u') = i\sqrt{\frac{2}{\pi}}\, \frac{\sin[u'(\lambda - \frac{1}{2}i)]}{(i\lambda + \frac{1}{2})(\sinh u')^{\frac{1}{2}}}\,,$$

(A.8) becomes

$$\frac{\sinh u'}{4\pi^2} \int_{-\infty}^{\infty} \mathrm{d}\lambda\, (\lambda + \tfrac{1}{2}i) \sin [u'(\lambda - \tfrac{1}{2}i)]$$

$$= \lim_{x\to\infty} \frac{\sinh u'}{2\pi^2 u'} \left\{ \frac{\sin (xu')}{u'} - x \cos (xu') \right\} = \frac{1}{4\pi}\, \delta(u') . \qquad \text{(A.9)}$$

Notice that in the last equation we have set

$$\lim_{x\to\infty} \frac{1}{u'} \sin (x u') = \tfrac{1}{2}\pi\, \delta(u') ,$$

rather than $\pi\, \delta(u')$, since all integrations over the variable u' are restricted to the range $u' \geqslant 0$.

Concerning the off-diagonal elements of (A.6), some of them are trivially zero, while the remaining ones are all proportional to

$$(\sinh u')^{\frac{3}{2}} \int_{-\infty}^{\infty} \mathrm{d}\lambda\, (\lambda^2 + \tfrac{1}{4})(i\lambda + \tfrac{3}{2})\, \mathcal{P}_{i\lambda}^{-\frac{3}{2}} (\cosh u') ,$$

which may be shown to vanish by making use of the relation

$$\mathcal{P}_{i\lambda}^{-\frac{3}{2}}(\xi) = -\frac{1}{1+2i\lambda} \frac{1}{(\xi^2-1)^{\frac{1}{2}}} \{ \mathcal{P}_{i\lambda-1}^{-\frac{1}{2}}(\xi) - \mathcal{P}_{i\lambda+1}^{-\frac{1}{2}}(\xi) \} ,$$

and proceeding as before. Hence in the special Lorentz frame chosen, the left hand side of eq. (A.6) becomes

$$\frac{1}{4\pi}\, \delta(u') \cdot \mathbf{1} .$$

This result we claim agrees with the right hand side, for in the limit $u \to 0$ the δ function $\delta(u' - u)$ forces also u' to be zero. This corresponds to the singular point $\boldsymbol{x} = \boldsymbol{x}' = 0$, where θ and φ become undefined, and furthermore irrelevant. As is well known, in this case one can replace $\delta(\Omega'-\Omega)$ by $1/4\pi$. This completes the proof.

References

[1] H. Kendall, Proc. of the Int. Symposium on electron and photon interactions at high energies, ed. N.B. Mistry (Cornell Univ., Ithaca, New York 1972).
[2] R.P. Feynmann, Third Topical Conf. in high energy collisions of hadrons, Stony Brook, N.Y. Sept. 1969.
[3] S.D. Drell, D.J. Levy and T.M. Yan, Phys. Rev. Letters 22 (1969) 744; Phys. Rev. 187 (1969) 2159; D1 (1970) 1617;
T.M. Yan and S.D. Drell, Phys. Rev. D1 (1970) 2402.

[4] J.D. Bjorken and E.A. Paschos, Phys. Rev. 185 (1969) 1975;
J. Kuti and V.F. Weisskopf, Phys. Rev. D4 (1971) 3418;
M. Böhm, H. Joos and M. Krammer DESY 72/62 (1972).
[5] M. Gell-Mann, Acta Physica Austriaca, Suppl. IX (Schladming Lectures 1972) (1972) p. 733.
[6] B. Stech, Nucl. Phys. B64 (1973) 194.
[7] F. Rohrlich, Acta Phys. Austriaca, Suppl. VIII (1971) p. 277;
J.R. Klauder, H. Leutwyler and L. Streit, Nuovo Cimento 66A (1970) 536;
J.B. Kogut and D.E. Soper, Phys. Rev. D1 (1970) 2901;
J.D. Bjorken, J.B. Kogut and D.E. Soper, Phys. Rev. D3 (1971) 1382.
[8] P.A.M. Dirac, Rev. Mod. Phys. 21 (1949) 392.
[9] S. Fubini, A.J. Hanson and R. Jackiw, Phys. Rev. D7 (1973) 1732.
[10] C.M. Sommerfield, Quantization on space-time hyperboloids, Yale preprint (July, 1973).
[11] W. Magnus, F. Oberhettinger and R.P. Soni, Formulas and theorems for the special functions of mathematical physics, (Springer, Berlin, 1966).
[12] H. Joos, Fortschr. der Phys. 10 (1962) 65.