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Holographic entanglement entropy: an overview

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Abstract

In this paper, we review recent progress on the holographic understanding of the entanglement entropy in the anti-de Sitter space/conformal field theory (AdS/CFT) correspondence. In general, the AdS/CFT relates physical observables in strongly coupled quantum many-body systems to certain classical quantities in gravity plus matter theories. In the case of our holographic entanglement entropy, its gravity dual turns out to be purely geometric, i.e. the area of minimal area surfaces in AdS spaces. One interesting application is to study various phase transitions by regarding the entanglement entropy as order parameters. Indeed we will see that our holographic calculations nicely reproduce the confinement/deconfinement transition. Our results can also be applied to understanding the microscopic origins of black hole entropy.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In the recent developments of string theory, the idea of holography has obviously played crucial roles. Holography claims that the degrees of freedom in $(d + 2)$-dimensional quantum gravity are much more reduced than we naively think, and will be comparable to those of quantum many-body systems in $d + 1$ dimensions [1, 2]. This was essentially found by remembering that the entropy of a black hole is not proportional to its volume, but to the area of its event horizon Σ (the Bekenstein–Hawking formula [3]):

$$S_{\mathrm{BH}} = \frac{\mathrm{Area}(\Sigma)}{4G_N},\qquad(1.1)$$

where G_N is the Newton constant. Owing to the discovery of the anti-de Sitter space/conformal field theory (AdS/CFT) correspondence [4], we know explicit examples where the holography

is manifestly realized. The AdS/CFT argues that the quantum gravity on $(d + 2)$-dimensional anti-de Sitter spacetime (AdS_{d+2}) is equivalent to a certain conformal field theory in $d + 1$ dimensions (CFT_{d+1}) [4–7].

Even after quite active researches of the AdS/CFT over these 10 years, the fundamental mechanism of the AdS/CFT correspondence remains a mystery, in spite of much evidence in various examples. In particular, we cannot answer which region of AdS is responsible to particular information in the dual CFT. To make modest progress for this long-standing problem, we believe that it is important to understand and formulate the holography in terms of a universal observable, rather than quantities which depend on the details of theories such as specific operators or Wilson loops, etc. We only expect that a quantum gravity in some spacetime is dual to (i.e. equivalent to) a certain theory which is governed by the law of quantum mechanics. We would like to propose that an appropriate quantity which can be useful in this universal viewpoint is the entanglement entropy. Indeed, we can always define the entanglement entropy in any quantum mechanical system.

The entanglement entropy S_A in quantum field theories or quantum many-body systems is a non-local quantity as opposed to correlation functions. It is defined as the von Neumann entropy S_A of the reduced density matrix when we 'trace out' (or smear out) degrees of freedom inside a d-dimensional space-like submanifold B in a given $(d + 1)$-dimensional QFT, which is a complement of A. S_A measures how the subsystems A and B are correlated with each other. Intuitively, we can also say that this is the entropy for an observer in A who is not accessible to B as the information is lost by the smearing out in region B. This origin of entropy looks analogous to the black hole entropy. Indeed, this was the historical motivation for considering the entanglement entropy in quantum field theories [8–10]. Interestingly, the leading divergence of S_A is proportional to the area of the subsystem A when $d > 1$, called the area law [9, 10] (refer also to the review articles [11–20]).

Since S_A is defined as a von Neumann entropy, we expect that the entanglement entropy is directly related to the degrees of freedom. Indeed, in two-dimensional conformal field theory, the universal piece of entanglement entropy is proportional to the central charge in two-dimensional conformal field theories (2D CFTs) as shown in [21, 22], where a general prescription of computing the entropy in 2D CFTs is given. Also in the mass perturbed CFTs (massive QFTs), the same conclusion holds for a logarithmic term with respect to the correlation length [22–28]. Furthermore, our holographic result shows that a similar statement is also true in four or higher even-dimensional CFTs. As opposed to the thermal entropy, the entanglement entropy is non-vanishing at zero temperature. Therefore, we can employ it to probe the quantum properties of the ground state for a given quantum system. It is also a useful order parameter of a quantum phase transition at zero temperature as will be explained in section 4.

Now we come back to our original question where in AdS space given information in CFT is saved. Since the information included in a subsystem B is evaluated by the entanglement entropy S_A, we can formulate this question more concretely as follows: 'which part of AdS space is responsible for the calculation of S_A in the dual gravity side?' Two of the authors of this paper proposed a holographic formula of the entanglement entropy in [29, 30]:

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}}, \tag{1.2}$$

where γ_A is the d-dimensional minimal surface γ_A whose boundary is given by the $(d − 1)$-dimensional manifold $\partial \gamma_A = \partial A$ (see figure 3); the constant $G_N^{(d+2)}$ is the Newton constant of the general gravity in AdS_{d+2}. This formula can be applied equally well to asymptotically AdS static spacetimes. Originally, this formula (1.2) is speculated from the Bekenstein–

Hawking formula (1.1). Indeed, since the minimal surface tends to wrap the horizon in the presence of an event horizon, our formula (1.2) can be regarded as a generalization of the well-known formula (1.1). Also the area law of S_A [9, 10] can be automatically derived from our holographic description.

The purpose of this paper is to explain this holographic description and then to review its current status with recent progress and applications [31–84]. In AdS_3/CFT_2, we can confirm that formula (1.2) is precisely true by comparing the holographic result with the known 2D CFT results [29, 30]. In higher dimensional cases, however, the proposed formula (1.2) has not been derived rigorously from the bulk to boundary relation in AdS/CFT [5, 6] at present. Also the direct calculations of the entanglement entropy in the CFT side are very complicated in higher dimensions. Nevertheless, a heuristic derivation has been presented in [33] and much evidence [30, 34, 35, 38, 42, 63, 78] has been found. Our holographic formula has also been successfully applied to the explanation of the black hole and de Sitter entropy [31, 32, 34, 50] (see also [85, 86] for earlier pioneering ideas on the entanglement entropy in AdS/CFT with event horizon; see also [87]), and to an order parameter of a confinement/deconfinement phase transition [38, 49, 51, 64, 70, 71, 74, 79].

In condensed matter physics, the entanglement entropy is expected to be a key quantity to understand several aspects of quantum many-body physics. A central question in quantum many-body physics is how we can characterize different phases and phase transitions. While microscopic Hamiltonians in condensed matter systems (electronic systems, in particular) are quantum mechanical, a wide range of quantum phases turn out to have a classical analogue, and if so, they can be understood in terms of symmetry breaking of some kind, and in terms of classical order parameters. On the other hand, this paradigm, known from Landau and Ginzburg, does not always apply when phases of our interest are inherently quantum. Indeed, one of the main foci in modern condensed matter physics is to understand quantum phases of matter and phase transitions between them, which are beyond the Landau–Ginzburg paradigm. To name a few, relatively well-understood examples, the fractional quantum Hall effect, and quantum magnets on some geometrically frustrated lattices have attracted a lot of interest. Many-body wavefunctions of quantum ground states in these phases look featureless when one looks at correlation functions of local operators. They cannot be characterized by classical order parameters of some kind. Indeed, they should be distinguished by their pattern of entanglement rather than their pattern of symmetry breaking [89]. Thus, the entanglement entropy is potentially useful to characterize these exotic phases[4].

One can ask these questions from a slightly more practical, but ultimately fundamental, point of view; how can we simulate quantum states of matter efficiently by classical computers? The total dimension of the Hilbert space increases exponentially as we increase the system size, and hence brute force approaches (e.g. exact diagonalization) to quantum many-body systems are destined to fail. It turns out having a good understanding on how local regions of the whole quantum system are entangled to each other would help to find good algorithms for quantum many-body problems, such as the density matrix renormalization group (DMRG) [90]. To be more precise, the scaling of the entanglement entropy as a function of the size of a given subregion of the system of interest gives us a criterion for efficient approximability. In other words, the entanglement entropy tells us amount of information and degrees of freedom necessary to represent a quantum ground state efficiently.

Reversing the logic, one can distinguish different phases of quantum matter according to their computational complexity and hence from the scaling of the entanglement entropy. After

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[4] Recently, there has been some progress on holographic descriptions of various phase transitions analogous to the ones in condensed matter physics; e.g. refer to the review [88].

all, what makes simulation of quantum systems by classical computers difficult is nothing but entanglement. Indeed, this idea has been pushed extensively in recent years for several 1D quantum systems. It has been revealed that several quantum phases in 1D spin chains can be distinguished by different scaling of the entanglement entropy. See, for example, [23, 24, 91–93] and references in [22].

For higher dimensional condensed matter systems, there have been many recent attempts in this direction. In particular, the entanglement entropy was applied for the so-called topological phases in 2+1 dimensions [94, 95]. Typically, these phases have a finite gap and are accompanied by many exotic features such as fractionalization of quantum numbers, non-Abelian statistics of quasi-particles, topological degeneracy, etc. They can also be useful for fault-tolerant quantum computations. On the other hand, unconventional quantum liquid phases with gapless excitations, such as gapless spin liquid phases, seem to be, at least at present, more difficult to characterize in higher dimensions. Our results from the AdS/CFT correspondence can be useful to study these gapless (spin liquid) states (some of these phases have been suspected to be described by a relativistic gauge field theory of some sort [89]).

The organization of this paper is as follows. In section 2, we go through some basic properties of the entanglement entropy. In particular, we discuss how the entanglement entropy scales as a function of the size of the subsystem in quantum field theories and many-body systems when we take the UV cutoff to infinity. Section 3 presents our basic formula of the holographic entanglement entropy via AdS/CFT. Many results mentioned in section 2 are reproduced from the holographic point of view. In section 4, we apply the entanglement entropy as a non-local order parameter to confinement/deconfinement phase transitions. Section 5 reviews two connections between the entanglement entropy and the black hole entropy obtained from the holographic calculation of the entanglement entropy. In section 6, we will explain a covariant formulation of the holographic entanglement entropy. We conclude in section 7 with a summary and with possible future directions.

2. Basics of entanglement entropy

We start with a review of the definition and properties of the entanglement entropy.

2.1. Definition of entanglement entropy

Consider a quantum mechanical system with many degrees of freedom such as spin chains. More generally, we can consider arbitrary lattice models or quantum field theories (QFTs). We put such a system at zero temperature and then the total quantum system is described by the pure ground state $|\Psi\rangle$. We assume no degeneracy of the ground state. Then the density matrix is that of the pure state:

$$\rho_{\text{tot}} = |\Psi\rangle\langle\Psi|. \tag{2.1}$$

The von Neumann entropy of the total system is clearly zero: $S_{\text{tot}} = -\text{tr}\,\rho_{\text{tot}} \log \rho_{\text{tot}} = 0$.

Next we divide the total system into two subsystems A and B (see figure 1). In the spin chain example, we artificially cut off the chain at some point and divide the lattice points into two groups. Note that physically we do not do anything to the system and the cutting procedure is an imaginary process. Accordingly, the total Hilbert space can be written as a direct product of two spaces $\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B$ corresponding to those of subsystems A and B. The observer who is only accessible to the subsystem A will feel as if the total system is described by the reduced density matrix ρ_A:

$$\rho_A = \text{tr}_B\,\rho_{\text{tot}}, \tag{2.2}$$

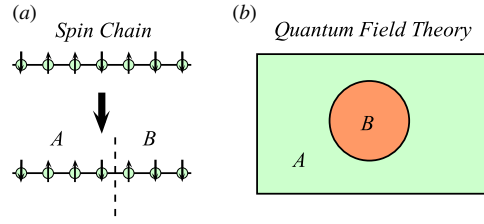where the trace is taken only over the Hilbert space $\mathcal{H}_B$.

Figure 1. Examples of bipartitioning for the entanglement entropy. A choice of the subsystems *A* and *B* is shown for each of the two examples: (*a*) a spin chain, (*b*) a quantum field theory.

Now we define the entanglement entropy of the subsystem *A* as the von Neumann entropy of the reduced density matrix ρ_A:

$$S_A = -\mathrm{tr}_A \, \rho_A \log \rho_A. \tag{2.3}$$

This quantity provides us with a convenient way to measure how closely entangled (or how 'quantum') a given wave function $|\Psi\rangle$ is.

In time-dependent backgrounds the density matrices ρ_{tot} and ρ_A are time dependent as dictated by the von Neumann equation. Thus, we need to specify the time $t = t_0$ when we measure the entropy. In this paper, we will always deal with static systems except in section 6.

It is also possible to define the entanglement entropy $S_A(\beta)$ at finite temperature $T = \beta^{-1}$. This can be done just by replacing (2.1) with the thermal one $\rho_{\mathrm{thermal}} = \mathrm{e}^{-\beta H}$, where H is the total Hamiltonian. When *A* is the total system, $S_A(\beta)$ is clearly the same as the thermal entropy. Also in general, if we take the high temperature limit $\beta \to 0$, then the difference $S_{A_1}(\beta) - S_{A_2}(\beta)$ approaches the difference of thermal entropy between A_1 and A_2. This subtraction is necessary to cancel the ultraviolet divergences as explained later.

2.2. Properties

There are several useful properties which the entanglement entropy enjoys generally. We summarize some of them as follows (the derivations and other properties of the entanglement entropy can be found in e.g. the textbook [96]).

- If the density matrix ρ_{tot} is pure such as in the zero temperature system, then we find the following relation assuming *B* is the complement of *A*:

$$S_A = S_B. \tag{2.4}$$

  This manifestly shows that the entanglement entropy is not an extensive quantity. This equality is violated at finite temperature.
- For any three subsystems *A*, *B* and *C* that do not intersect each other, the following inequalities hold:

$$S_{A+B+C} + S_B \leqslant S_{A+B} + S_{B+C}, \tag{2.5}$$

$$S_A + S_C \leqslant S_{A+B} + S_{B+C}. \tag{2.6}$$

These inequalities are called the strong subadditivity [97], which is the most powerful inequality obtained so far with respect to the entanglement entropy. In [98, 99] (see also [100]), the authors presented an entropic proof of the c-theorem by applying the strong subadditivity to 2*d* quantum field theories.

- By setting B empty in (2.5), we can find the subadditivity relation

$$S_{A+B} \leqslant S_A + S_B. \tag{2.7}$$

The subadditivity (2.7) allows us to define an interesting quantity called the mutual information $I(A, B)$ by

$$I(A, B) = S_A + S_B - S_{A+B} \geqslant 0. \tag{2.8}$$

2.3. Entanglement entropy in QFTs

Consider a QFT on a $(d + 1)$-dimensional manifold $\mathbb{R} \times N$, where $\mathbb{R}$ and N denote the time direction and the d-dimensional space-like manifold, respectively. We define the subsystem by a d-dimensional submanifold $A \subset N$ at a fixed time $t = t_0$. We call its complement the submanifold B. The boundary of A, which is denoted by ∂A, divides the manifold N into two submanifolds A and B. Then we can define the entanglement entropy S_A by the previous formula (2.3). Sometimes this kind of entropy is called the geometric entropy as it depends on the geometry of the submanifold A. Since the entanglement entropy is always divergent in a continuum theory, we introduce an ultraviolet cutoff a (or a lattice spacing). Then the coefficient in front of the divergence turns out to be proportional to the area of the boundary ∂A of the subsystem A (when $d > 1$) as first pointed out in [9, 10]:

$$S_A = \gamma \cdot \frac{\text{Area}(\partial A)}{a^{d-1}} + \text{subleading terms}, \tag{2.9}$$

where γ is a constant which depends on the system. This behavior can be intuitively understood since the entanglement between A and B occurs at the boundary ∂A most strongly. This result (2.9) was originally found from numerical computations [9, 10] and checked in many later arguments (see e.g. recent works [100–102]).

The simple area law (2.9), however, does not always describe the scaling of the entanglement entropy in generic situations. Indeed, the entanglement entropy of 2D CFT scales logarithmically with respect to the length l of A [21, 22]. If we assume the total system is infinitely long, it is given by the simple formula [21]

$$S_A = \frac{c}{3} \log \frac{l}{a}, \tag{2.10}$$

where c is the central charge of the CFT. As we will see in section 3, the scaling behavior (2.10) is consistent with the generic structure (2.15) expected from AdS/CFT.

Other situations such as a compactified circle at zero temperature or an infinite system at finite temperature can be treated by applying the conformal map technique and analytic formulas have been obtained in [22]. The results are given as follows:

$$S_A^{\text{c.c.}} = \frac{c}{3} \cdot \log \left(\frac{L}{\pi a} \sin \left(\frac{\pi l}{L} \right) \right), \tag{2.11}$$

$$S_A^{\text{f.t.}} = \frac{c}{3} \cdot \log \left(\frac{\beta}{\pi a} \sinh \left(\frac{\pi l}{\beta} \right) \right), \tag{2.12}$$

respectively, where L is the circumference of the circle.

The result for a finite size system at finite temperature has been obtained in [50] for a free Dirac fermion (i.e. $c = 1$) in two dimensions. In the high temperature expansion, the result

becomes (we set $L = 1$)

$$S_A(\beta, l) = \frac{1}{3} \log \left[\frac{\beta}{\pi a} \sinh \left(\frac{\pi l}{\beta} \right) \right] + \frac{1}{3} \sum_{m=1}^{\infty} \log \left[\frac{\left(1 - e^{2\pi \frac{l}{\beta}} e^{-2\pi \frac{m}{\beta}} \right) \left(1 - e^{-2\pi \frac{l}{\beta}} e^{-2\pi \frac{m}{\beta}} \right)}{\left(1 - e^{-2\pi \frac{m}{\beta}} \right)^2} \right]$$

$$+ 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \cdot \frac{\frac{\pi k l}{\beta} \coth \left(\frac{\pi k l}{\beta} \right) - 1}{\sinh \left(\pi \frac{k}{\beta} \right)}. \tag{2.13}$$

Using this expression, we can find the relation between the thermal entropy $S^{\text{thermal}}(\beta)$ and the entanglement entropy

$$S^{\text{thermal}}(\beta) = \lim_{\epsilon \to 0} (S(\beta, 1 - \epsilon) - S(\beta, \epsilon)). \tag{2.14}$$

For conformal field theories in higher dimensions ($d > 1$), our holographic method discussed in section 3.5 predicts the following general form of S_A for relativistic quantum field theories in the scaling limit $a \to 0$, assuming that ∂A is a smooth and compact manifold

$$S_A = p_1 (l/a)^{d-1} + p_3 (l/a)^{d-3} + \cdots + \begin{cases} p_{d-1} (l/a) + p_d, & d: \text{even} \\ p_{d-2} (l/a)^2 + \tilde{c} \log (l/a), & d: \text{odd} \end{cases} \tag{2.15}$$

where l is the typical length scale of ∂A. This result includes the known result for $d = 1$ (2.10) with $\tilde{c} = c/3$. Also, in the case of $(3 + 1)$-dimensional conformal field theories ($d = 3$), the scaling law (2.15) has been confirmed by direct field theoretical calculations based on Weyl anomaly [30], where again the coefficient of the logarithmic term $\tilde{c}$ is given in terms of central charges of $(3 + 1)$ CFTs. For $d =$ even, equation (2.15) has been the only known analytical result for (interacting) conformal field theories. Even though we assumed conformal field theories in the above, the same scaling formula (2.15) should be true for a quantum field theory with a UV fixed point, i.e. at a relativistic quantum critical point.

When the boundary ∂A is not a smooth manifold such as the one with cusp singularities, we will have other terms $(l/a)^{d-2}, (l/a)^{d-4}, \ldots$ which do not obey the scaling law in (2.15). For example, for a three-dimensional CFT ($d = 2$), if ∂A has a cusp with the angle Ω, then S_A includes a logarithmic term $\sim - f(\Omega) \log l/a$, for a certain function f [78, 112]. Refer also to section 3.5.3 for more details.

Also, if we consider a gapped system in three dimensions ($d = 2$) which is described (at low energies) by a topological field theory (called topologically ordered phase), the scaling of the entanglement entropy is the same as (2.15) with $d = 2$. The constant $p_d = p_2$ in a topologically ordered phase is, however, invariant under a smooth deformation of the boundary ∂A. In this case, $S_{\text{top}} = p_2$ is called the topological entanglement entropy [94, 95].

It has also been pointed out that the area law is corrected by a logarithmic factor as $S_A \propto (l/a)^{d-1} \log l/a +$ (subleading terms) for fermionic systems in the presence of a finite Fermi surface (where, again, l is the characteristic length scale of the $(d - 1)$-dimensional manifold ∂A) [103–106].

2.4. How to compute entanglement entropy in QFTs

It is helpful to know how to calculate the entanglement entropy generally in QFTs for later arguments. We will follow the method considered in [22] (see also the reviews [16, 17]). For this, we first evaluate $\text{tr}_A \rho_A^n$, differentiate it with respect to n and finally take the limit $n \to 1$ (remember that ρ_A is normalized such that $\text{tr}_A \rho_A = 1$)

$$S_A = -\frac{\partial}{\partial n} \text{tr}_A \rho_A^n |_{n=1} = -\frac{\partial}{\partial n} \log \text{tr}_A \rho_A^n |_{n=1}. \tag{2.16}$$
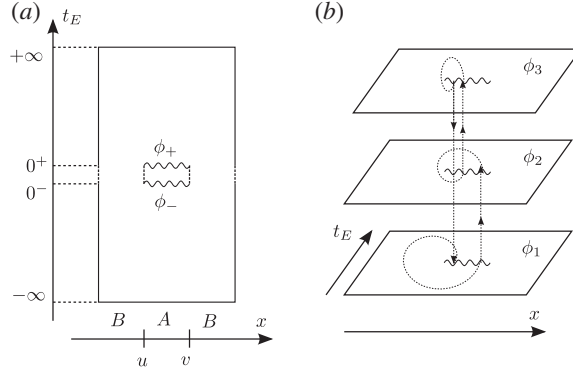
Figure 2. (*a*) The path integral representation of the reduced density matrix $[\rho_A]_{\phi_+\phi_-}$. (*b*) The *n*-sheeted Riemann surface $\mathcal{R}_n$. (Here we take $n = 3$ for simplicity.)

This is called the replica trick. Therefore, what we have to do is to evaluate $\text{tr}_A \rho_A^n$ in a given QFT.

This can be done in the path integral formalism as follows. First, assuming two-dimensional QFT just for simplicity, we take A to be the single interval $x \in [u, v]$ at $t_E = 0$ in the flat Euclidean coordinates $(t_E, x) \in \mathbb{R}^2$. The ground-state wave functional Ψ can be found by path integrating from $t_E = -\infty$ to $t_E = 0$ in the Euclidean formalism

$$\Psi(\phi_0(x)) = \int_{t_E=-\infty}^{\phi(t_E=0,x)=\phi_0(x)} D\phi\, e^{-S(\phi)}, \tag{2.17}$$

where $\phi(t_E, x)$ denotes the field which defines the 2D QFT. The values of the field at the boundary ϕ_0 depends on the spatial coordinate x. The total density matrix ρ is given by two copies of the wave functional $[\rho]_{\phi_0\phi_0'} = \Psi(\phi_0)\bar{\Psi}(\phi_0')$. The complex conjugate one $\bar{\Psi}$ can be obtained by path integrating from $t_E = \infty$ to $t_E = 0$. To obtain the reduced density matrix ρ_A, we need to integrate ϕ_0 on B with the condition $\phi_0(x) = \phi_0'(x)$ when $x \in B$

$$[\rho_A]_{\phi_+\phi_-} = (Z_1)^{-1} \int_{t_E=-\infty}^{t_E=\infty} D\phi\, e^{-S(\phi)} \prod_{x \in A} \delta(\phi(+0, x) - \phi_+(x)) \cdot \delta(\phi(-0, x) - \phi_-(x)), \tag{2.18}$$

where Z_1 is the vacuum partition function on $\mathbb{R}^2$ and we multiply its inverse in order to normalize ρ_A such that $\text{tr}_A \rho_A = 1$. This computation is sketched in figure 2(*a*).

To find $\text{tr}_A \rho_A^n$, we can prepare *n* copies of (2.18)

$$[\rho_A]_{\phi_{1+}\phi_{1-}} [\rho_A]_{\phi_{2+}\phi_{2-}} \cdots [\rho_A]_{\phi_{n+}\phi_{n-}}, \tag{2.19}$$

and take the trace successively. In the path integral formalism this is realized by gluing $\{\phi_{i\pm}(x)\}$ as $\phi_{i-}(x) = \phi_{(i+1)+}(x)$ ($i = 1, 2, \ldots, n$) and integrating $\phi_{i+}(x)$. In this way, $\text{tr}_A \rho_A^n$ is given in terms of the path integral on an *n*-sheeted Riemann surface $\mathcal{R}_n$ (see figure 2(*b*))

$$\text{tr}_A \rho_A^n = (Z_1)^{-n} \int_{(t_E,x)\in\mathcal{R}_n} D\phi\, e^{-S(\phi)} \equiv \frac{Z_n}{(Z_1)^n}. \tag{2.20}$$

Though we have assumed 2D QFTs so far, it can be straightforwardly generalized to higher dimensions. Then Z_n becomes a partition function on a singular space which is obtained by gluing *n* copies of the original space along ∂A. It has a negative deficit angle $2\pi(1 - n)$ along the surface ∂A. This becomes two end points of the cut in the two-dimensional example.

In 2D CFTs, it is possible to analytically calculate (2.20) to find formula (2.10) [21, 22, 46, 80, 107–109] as it essentially becomes products of two point functions of twisted vertex operators [22, 25]. However, in higher dimensions, analytical calculations of S_A become very complicated. Below we list some recent progress in this direction. The analytical results when A and B divide a flat spacetime along a flat plane have been found in [22, 38, 110]. When the subsystem A is a straight strip, numerical results are available in [30, 108]. Also, if we assume that A includes a cusp singularity, then the entanglement entropy of $(2 + 1)$-dimensional QFT has a logarithmic term. This is evaluated in [78, 112]. Lattice calculations in non-Abelian gauge theories have also been performed in [59, 64, 71, 79]. Generally speaking, there have been essentially no analytical calculations of S_A in interacting QFTs in dimensions greater than 2. Thus, our holographic approach which can be applied to strongly coupled theories will provide a powerful complementary method.

2.5. Entanglement entropy and black holes

It may be interesting to note that the area law (2.9) looks very similar to the Bekenstein–Hawking entropy of black holes which is proportional to the area of the event horizon (1.1). Intuitively, we can regard S_A as the entropy for an observer who is only accessible to the subsystem A and cannot receive any signals from B. In this sense, the subsystem B is analogous to the inside of a black hole horizon for an observer sitting in A, i.e. outside of the horizon. Indeed, this similarity was an original motivation of the entanglement entropy [8–10].

However, one may immediately note the discrepancy between them. The entanglement entropy is proportional to the number of matter fields, while the black hole entropy is not. Also the former includes ultraviolet divergences as opposed to the latter. The correct statement of this relation turns out to be that quantum corrections to the black hole entropy in the presence of matter fields are equal to the entanglement entropy [113–116]. In the setup of the induced gravity, where the Einstein–Hilbert action is assumed to be all generated from the quantum corrections to matter fields, we can fully identify the black hole entropy with the entanglement entropy [115]. This relation has been reconsidered in [117] recently by proposing the existence of a new gravitational cut-off.

In our holographic argument below, we will present an identification of the entanglement entropy in $(d+1)$-dimensional QFT with a certain geometrical quantity in $(d+2)$-dimensional gravity, which can be regarded as a generalization of the black hole entropy. This relation holds whenever holography dual of the QFT exists. In a particular case of the brane-world setup, our identification turns out to be reduced to the mentioned equivalence between the black hole entropy and entanglement entropy in the induced gravity [31, 34, 85] as we will review in section 5.1 and section 6.

3. Holographic entanglement entropy

Here, we would like to explain the holographic calculation of the entanglement entropy. In order to simplify the notations and reduce ambiguities, we consider the setup of the AdS/CFT correspondence, though it will be rather straightforward to extend our results to general holographic setups.

The AdS/CFT correspondence argues that (quantum) gravity in the $(d + 2)$-dimensional anti-de Sitter space AdS$_{d+2}$ is equivalent to a $(d + 1)$-dimensional conformal field theory

CFT$_{d+1}$ [4]. Below we mainly employ the Poincaré metric of AdS$_{d+2}$ with radius R:

$$ds^2 = R^2 \frac{dz^2 - dx_0^2 + \sum_{i=1}^{d-1} dx_i^2}{z^2}. \tag{3.1}$$

The dual CFT$_{d+1}$ is supposed to live on the boundary of AdS$_{d+2}$ which is $R^{1,d}$ at $z \to 0$ spanned by the coordinates (x^0, x^i). The extra coordinate z in AdS$_{d+2}$ is interpreted as the length scale of the dual CFT$_{d+1}$ in the RG sense. Since the metric diverges in the limit $z \to 0$, we put a cutoff by imposing $z \geqslant a$. Then the boundary is situated at $z = a$ and this cutoff a is identified with the ultraviolet cutoff in the dual CFT. Under this interpretation, a fundamental principle of AdS/CFT, known as the bulk to boundary relation [5, 6], is simply expressed by the equivalence of the partition functions in both theories:

$$Z_{\text{CFT}} = Z_{\text{AdS Gravity}}. \tag{3.2}$$

Since (non-normalizable) perturbations in the AdS background by exciting fields in the AdS side are dual to the shift of background in the CFT side, we can compute the correlation functions in the CFT by taking the derivatives with respect to the perturbations. In generic parameter regions, the gravity should be treated in string theory to take the quantum corrections into account. Nevertheless, in a particular interesting limit, typically the strong coupling limit of CFT, the quantum corrections become negligible and we can employ supergravity to describe AdS spaces. Moreover, most of our examples shown in this paper are simple enough that we can apply in general relativity. In this situation, the right-hand side of (3.2) is reduced to the exponential $e^{-S_{\text{EH}}}$ of the on-shell Einstein–Hilbert action.

So far we applied the AdS/CFT to the pure AdS spacetime (3.1). However, the AdS/CFT can be applied to any asymptotically AdS spacetimes including the AdS black holes.

3.1. Holographic formula

Now we are in a position to present how to calculate the entanglement entropy in CFT$_{d+1}$ from the gravity on AdS$_{d+2}$. This argument here can be straightforwardly generalized to any static backgrounds.

To define the entanglement entropy in the CFT$_{d+1}$, we divide the (boundary) time slice N into A and B as we explained before (see figure 3). In the Poincaré coordinate (3.1), we are setting $N = R^d$ and the CFT$_{d+1}$ is supposed to live on the boundary $z = a \to 0$ of AdS$_{d+2}$. To have its dual gravity picture, we need to extend this division $N = A \cup B$ to the time slice M of the bulk spacetime. In the setup (3.1), M is the $(d + 1)$-dimensional hyperbolic spacetime H_{d+1}. Thus, we extend ∂A to a surface γ_A in the entire M such that $\partial \gamma_A = \partial A$. Note that this is a surface in the time slice M, which is a Euclidean manifold. Of course, there are infinitely many different choices of γ_A. We claim that we have to choose the minimal area surface among them. This means that we require that the variation of the area functional vanishes; if there are multiple solutions, we choose the one whose area takes the minimum value. This procedure singles out a unique minimal surface and we call this γ_A again (see figure 3).

In this setup we propose that the entanglement entropy S_A in CFT$_{d+1}$ can be computed from the following formula [29, 30]:

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}}. \tag{3.3}$$

We stress again that the manifold γ_A is the d-dimensional minimal area surface in AdS$_{d+2}$ whose boundary is given by ∂A. Its area is denoted by $\text{Area}(\gamma_A)$. Also $G_N^{(d+2)}$ is the $(d+2)$-dimensional Newton constant of the AdS gravity. We can easily show that the leading divergence $\sim a^{-(d-1)}$
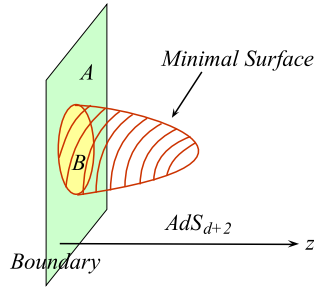
Figure 3. The holographic calculation of the entanglement entropy via the AdS/CFT.

in (3.3) is proportional to the area of the boundary ∂A and this immediately reproduces the area law property (2.9).

Though we assumed the pure AdS spaces for simplicity, we can equally apply formula (3.3) to any asymptotically AdS spaces which include the AdS black holes dual to finite temperature theories as is standard in AdS/CFT.

The appearance of formula (3.3) looks very similar to the area law of the Bekenstein–Hawking formula (1.1) of the black hole entropy. Indeed, we can regard our formula (3.3) as a generalization of (1.1) because in the presence of event horizon such as the AdS Schwarzschild black hole solutions, the minimal surface tends to wrap the horizon. Refer to section 3.4.2 for more details.

This formula (3.3) was originally motivated by the following intuitive interpretation [29, 30]. Since the entanglement entropy S_A is defined by smearing out the region B, the entropy is considered to be the one for an observer in A who is not accessible to B. The smearing process produces the fuzziness for the observer and that should be measured by S_A. In the higher dimensional perspective of the AdS space, the fuzziness appears by hiding a part of the bulk space AdS_{d+2} inside an imaginary horizon, which we call γ. It is clear that γ covers the smeared region B from the inside of the AdS space and thus we find $\partial\gamma = \partial B (=\partial A)$. To make this imaginary horizon more precise, we can employ the argument of the entropy bound [118]. This idea, roughly speaking, claims that the entropy contained in a certain space is bounded by the area of its surface (for details, see section 6). To choose the minimal surface as in (3.3) means that we are seeking the severest entropy bound [1, 2, 118] so that it has a chance to saturate the bound. Refer to section 6 for more details.

In the above we implicitly assume that the subsystem A is a connected manifold. When it is disconnected, we need to extend the holographic formula properly[5].

3.2. Heuristic derivation of holographic formula

In principle, we should be able to perform the holographic calculation of the entanglement entropy based on the first principle of the AdS/CFT correspondence known as the bulk to boundary relation (3.2). In the CFT side, the entanglement entropy can be found if we can compute the partition function on the $(d + 1)$-dimensional n-sheeted space (2.20) via formula (2.16). This space, called $\mathcal{R}_n$, is characterized by the presence of the deficit angle

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[5] This extension of the holographic formula to disconnected subsystems has not been settled down at present. A candidate of a formula in disconnected cases has been proposed in [54] based on the strong subadditivity. However, there have recently been several results [80, 119] from quantum field theory calculations which do not agree with this proposal.

$\delta = 2\pi(1 - n)$ on the surface ∂A. Therefore, we need to find a $(d + 2)$-dimensional back-reacted geometry $\mathcal{S}_n$ by solving the Einstein equation with the negative cosmological constant such that its metric approaches to that of $\mathcal{R}_n$ at the boundary $z \to 0$. This is a technically complicated mathematical problem if we try to solve it directly and it has not been completely solved at present.

To circumvent this situation, we make a following natural assumption [33]: the back-reacted geometry $\mathcal{S}_n$ is given by an n-sheeted AdS$_{d+2}$, which is defined by putting the deficit angle δ localized on a codimension two surface γ_A. This is clearly true in the three-dimensional pure gravity as the solution to the Einstein equation should be locally the same as AdS$_3$. However, this is not trivially obvious in higher dimensions. Under this assumption, the Ricci scalar behaves like a delta function:

$$R = 4\pi(1 - n)\delta(\gamma_A) + R^{(0)}, \tag{3.4}$$

where $\delta(\gamma_A)$ is the delta function localized on γ_A, $\delta(\gamma_A) = \infty$ for $x \in \gamma_A$ whereas $\delta(\gamma_A) = 0$ otherwise and $R^{(0)}$ is that of the pure AdS$_{d+2}$. Then we plug this in the supergravity action

$$S_{\text{AdS}} = -\frac{1}{16\pi G_N^{(d+2)}} \int_M \mathrm{d}x^{d+2} \sqrt{g}(R + \Lambda) + \cdots, \tag{3.5}$$

where we only make explicit the bulk Einstein–Hilbert action. This is because the other parts omitted in the above, such as kinetic terms of scalars, lead to extensive terms which are proportional to n and are canceled in the ratio (2.20). Now the bulk to boundary relation (3.2) equates the partition function of CFT with the one of AdS gravity. Thus, we can holographically calculate the entanglement entropy S_A as follows:

$$S_A = -\frac{\partial}{\partial n} \log \mathrm{Tr} \rho_A^n \big|_{n=1} = -\frac{\partial}{\partial n} \left[\frac{(1 - n)\text{Area}(\gamma_A)}{4G_N^{d+2}} \right]_{n=1} = \frac{\text{Area}(\gamma_A)}{4G_N^{d+2}}. \tag{3.6}$$

The action principle in the gravity theory requires that γ_A is the minimal area surface. In this way, we reproduced our holographic formula (3.3) [33]. Note that the presence of non-trivial minimal surfaces is a well-established property of asymptotically AdS spaces.

In this derivation of the holographic formula, the assumption about the back-reacted geometry $\mathcal{S}_n$ has been crucial. This assumption is clearly satisfied for the three-dimensional pure gravity as we noted in the above (see also [69] for detailed analysis). To explore this issue in higher dimensional AdS/CFT, hopefully demonstrating the proof of (3.3), is one of the most important future problems in the holographic entanglement entropy[6].

3.3. Holographic proof of strong subadditivity

One of the most important properties of the entanglement entropy is the strong subadditivity [97] given by the inequalities (2.5) and (2.6). This represents the concavity of the entropy and is somehow analogous to the second law of thermodynamics. Actually, it is possible to check that our holographic formula (3.3) satisfies this property in a rather simple argument as shown in [42] (see also [35] for the explicit numerical studies).

Let us start with three regions A, B and C on a time slice of a given CFT so that there are no overlaps between them. We extend this boundary setup toward the bulk AdS (see

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[6] Recently, a subtle disagreement about the logarithmic term of the entanglement entropy between the holographic result (3.3) and the CFT result has been pointed out based on the anomaly analysis in [61]. This occurs when the extrinsic curvature of ∂A is non-vanishing, where the geometric analysis gets quite complicated. However, it is possible that this problem arises from subtle differential geometric calculations in the presence of deficit angles as suggested in [63], where agreements between the gravity and CFT sides have been observed for the logarithmic term. This issue clearly deserves detailed future analysis.
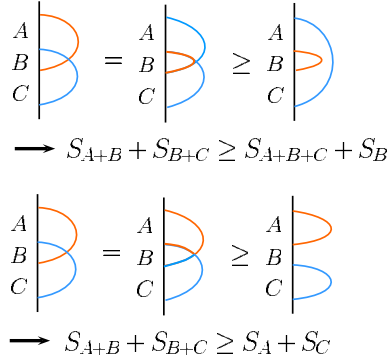
$$S_{A+B} + S_{B+C} \geq S_{A+B+C} + S_B$$

$$S_{A+B} + S_{B+C} \geq S_A + S_C$$

Figure 4. A holographic proof of the strong subadditivity of the entanglement entropy. To make the figures simple, we project the time slice of a $(d+2)$-dimensional AdS space onto a two-dimensional plane. This simplification does not change our result.

figure 4). Consider the entanglement entropies S_{A+B} and S_{B+C}. In the holographic description (3.3), they are given by the areas of the minimal area surfaces γ_{A+B} and γ_{B+C} which satisfy $\partial \gamma_{A+B} = \partial(A+B)$ and $\partial \gamma_{B+C} = \partial(B+C)$ as before. Then it is easy to see that we can divide these two minimal surfaces into four pieces and recombine into (i) two surfaces γ'_B and γ'_{A+B+C} or (ii) two surfaces γ'_A and γ'_C, corresponding to two different ways of the recombination. Here we again meant that γ'_X is a surface which satisfies $\partial \gamma'_X = \partial X$. Since in general the γ'_X are not the minimal area surface, we have $\mathrm{Area}(\gamma'_X) \geqslant \mathrm{Area}(\gamma_X)$. Therefore, as we can easily find from figure 4, this argument immediately leads to

$$\mathrm{Area}(\gamma_{A+B}) + \mathrm{Area}(\gamma_{B+C}) = \mathrm{Area}(\gamma'_B) + \mathrm{Area}(\gamma'_{A+B+C}) \geqslant \mathrm{Area}(\gamma_B) + \mathrm{Area}(\gamma_{A+B+C}),$$
$$\mathrm{Area}(\gamma_{A+B}) + \mathrm{Area}(\gamma_{B+C}) = \mathrm{Area}(\gamma'_A) + \mathrm{Area}(\gamma'_C) \geqslant \mathrm{Area}(\gamma_A) + \mathrm{Area}(\gamma_C). \tag{3.7}$$

In this way, we are able to check the strong subadditivities (2.5) and (2.6). Analogous inequalities have been discussed in [58] for the holographic Wilson loops.

3.4. Entanglement entropy from AdS$_3$/CFT$_2$

Consider AdS$_3$/CFT$_2$ as one of the simplest setups of AdS/CFT. Since the entanglement entropy in two-dimensional CFT can be analytically obtained as we mentioned, we can test our holographic formula explicitly. The central charge of CFT is related to radius of AdS$_3$ [120] via

$$c = \frac{3R}{2G_N^{(3)}}. \tag{3.8}$$

3.4.1. Entanglement entropy in CFT$_2$ at zero temperature.
We are interested in the entanglement entropy S_A in an infinitely long system when A is an interval of length l. To compute this via the holographic formula (3.3), we need to find geodesics between the two points $(x^1, z) = (-l/2, a)$ and $(x^1, z) = (l/2, a)$ in the Poincaré coordinate (3.1). It is actually given by the half circle

$$(x, z) = \frac{l}{2}(\cos s, \sin s) \quad (\epsilon \leqslant s \leqslant \pi - \epsilon), \tag{3.9}$$

where $\epsilon = \frac{2a}{l}$. The length of γ_A can be found as

$$\text{Length}(\gamma_A) = 2R \int_{\epsilon}^{\pi/2} \frac{\mathrm{d}s}{\sin s} = -2R \log(\epsilon/2) = 2R \log \frac{l}{a}. \tag{3.10}$$

Finally, the entropy can be obtained as follows:

$$S_A = \frac{\text{Length}(\gamma_A)}{4G_N^{(3)}} = \frac{c}{3} \log \frac{l}{a}. \tag{3.11}$$

This perfectly agrees with the result (2.10) in the CFT side [29, 30]. By starting from the global coordinate of AdS$_3$, we can also derive the result (2.11) similarly.

3.4.2. Entanglement entropy in CFT$_2$ at finite temperature. Next we consider how to explain the entanglement entropy (2.12) at finite temperature $T = \beta^{-1}$ from the viewpoint of the AdS/CFT correspondence. We assume that the spatial length of the total system L is infinite, i.e. $\beta/L \ll 1$. In such a high temperature region, the gravity dual of the conformal field theory is described by the Euclidean BTZ black hole [121]. Its metric looks like

$$\mathrm{d}s^2 = (r^2 - r_+^2) \, \mathrm{d}\tau^2 + \frac{R^2}{r^2 - r_+^2} \, \mathrm{d}r^2 + r^2 \, \mathrm{d}\varphi^2. \tag{3.12}$$

The Euclidean time is compactified as $\tau \sim \tau + \frac{2\pi R}{r_+}$ to obtain a smooth geometry. We also impose the periodicity $\varphi \sim \varphi + 2\pi$. By taking the boundary limit $r \to \infty$, we find the relation between the boundary CFT and the geometry (3.12):

$$\frac{\beta}{L} = \frac{R}{r_+} \ll 1. \tag{3.13}$$

The subsystem for which we consider the entanglement entropy is given by $0 \leqslant \varphi \leqslant 2\pi l/L$ at the boundary. Then by extending our formula (3.3) to asymptotically AdS spaces, the entropy can be computed from the length of the space-like geodesic starting from $\varphi = 0$ and ending at $\varphi = 2\pi l/L$ at the boundary $r = r_0 \to \infty$ at a fixed time. This geodesic distance can be found analytically as

$$\cosh\left(\frac{\text{Length}(\gamma_A)}{R}\right) = 1 + \frac{2r_0^2}{r_+^2} \sinh^2\left(\frac{\pi l}{\beta}\right). \tag{3.14}$$

The relation between the cutoff a in CFT and the one r_0 of AdS is given by $\frac{r_0}{r_+} = \frac{\beta}{a}$. Then it is easy to see that our area law (3.3) precisely reproduces the known CFT result (2.12).

It is also useful to understand these calculations geometrically. The geodesic line in the BTZ black hole takes the form shown in figure 5(*a*). When the size of A is small, it is almost the same as the one in the ordinary AdS$_3$. As the size becomes large, the turning point approaches the horizon and eventually, the geodesic line covers a part of the horizon. This is the reason why we find a thermal extensive behavior of the entropy when $l/\beta \gg 1$ in (2.12). The thermal entropy in a conformal field theory is dual to the black hole entropy in its gravity description via the AdS/CFT correspondence. In the presence of a horizon, it is clear that S_A is not equal to S_B (remember B is the complement of A) since the corresponding geodesic lines wrap different parts of the horizon (see figure 5(*b*)). This is a typical property of the entanglement entropy at finite temperature as we mentioned in section 2. We also expect that when A becomes very large before it coincides with the total system, γ_A becomes separated into the horizon circle and a small half circle localized on the boundary (see figure 5(*c*)). We can indeed confirm that this indeed happens in the dual CFT result (2.13) as shown in [50].
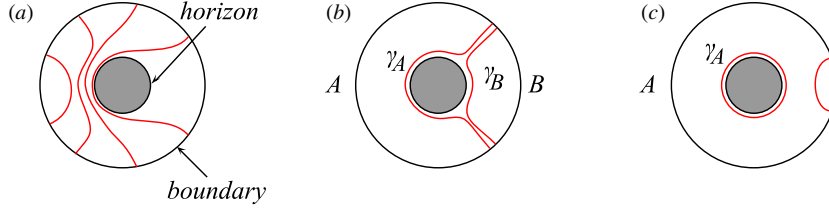
Figure 5. (*a*) Minimal surfaces γ_A in the BTZ black hole for various sizes of *A*. (*b*) γ_A and γ_B wrap the different parts of the horizon. (*c*) When ∂A gets larger, γ_A is separated into two parts: one is wrapped on the horizon and the other localized near the boundary.

3.4.3. Massive deformation. Massive quantum field theories can be obtained by perturbing two-dimensional conformal field theories by relevant perturbations. In the dual gravity side, this corresponds to an IR deformation of AdS$_3$ space. As in the well-known examples [122–125] of confining gauge theories, we expect the massive deformation caps off the IR region $z > z_{IR}$.

Consider a $(1 + 1)$-dimensional infinite system divided into two semi-infinite pieces and define the subsystem *A* to be one of them. The important quantity in the massive theory is the correlation length ξ, which is identified with $\xi \sim z_{IR}$ in AdS/CFT. Since we assumed that the subsystem *A* is infinite, we should take a geodesic (3.9) with a large value of $l(\gg \xi)$. Then the geodesic starts from the UV cutoff $z = a$ and ends at the IR cutoff $z = \xi$. Thus, we can estimate the length of this geodesic and finally the entanglement entropy as follows:

$$S_A = \frac{\text{Length}(\gamma_A)}{4G_N^{(3)}} = \frac{R}{4G_N^{(3)}} \int_{\epsilon = 2a/l}^{2\xi/l} \frac{\mathrm{d}s}{\sin s} = \frac{c}{6} \log \frac{\xi}{a}. \tag{3.15}$$

This agrees with the known result [22, 23] in the $(1 + 1)$-dimensional quantum field theory.

3.5. Holographic entanglement entropy in higher dimensions

Next we turn to the holographic computation of the entanglement entropy in higher dimensions. To obtain analytical results, we assume that the subsystem *A* is given either by (a) *d*-dimensional infinite strip (called A_S) with the width *l* in one direction and the width $L(\rightarrow \infty)$ in other $d - 1$ directions; (b) *d*-dimensional disk (called A_D) with radius *l*; or (c) *d*-dimensional wedge cone with a cusp with the angle Ω (called A_W). We can find their corresponding minimal surfaces in AdS, explicitly as depicted in figure 6.

3.5.1. Entanglement entropy for infinite strip A_S. The holographic entanglement entropy (3.3) for A_S is obtained from (3.3) as follows [30]:

$$S_{A_S} = \frac{1}{4G_N^{(d+2)}} \left[\frac{2R^d}{d - 1} \left(\frac{L}{a} \right)^{d-1} - \frac{2^d \pi^{d/2} R^d}{d - 1} \left(\frac{\Gamma\left(\frac{d+1}{2d}\right)}{\Gamma\left(\frac{1}{2d}\right)} \right)^d \left(\frac{L}{l} \right)^{d-1} \right]. \tag{3.16}$$

Note that the first divergent term is proportional to the area of ∂A, i.e. L^{d-1} as we expect from the known area law in the field theory computations (2.9). The second term is finite and thus is universal (i.e. does not depend on the cutoff). This is the quantity which we can directly compare with the field theory counterpart. The presence of these two terms agrees with the field theoretic results in [30, 108]. Note that our result (3.16) does not include the subleading divergent terms $\mathcal{O}(a^{-d+3})$.
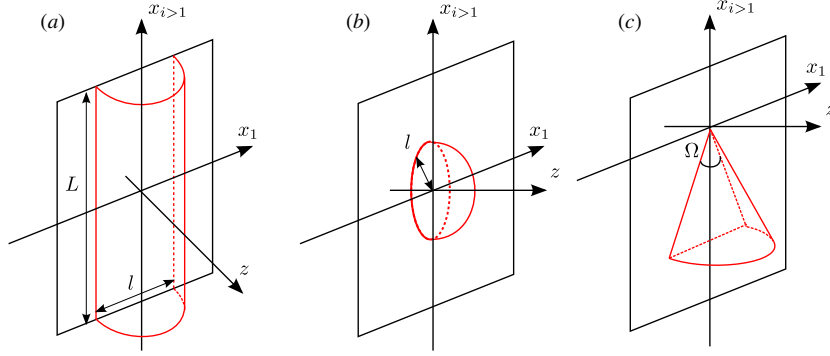
Figure 6. Minimal surfaces in AdS_{d+2}: (*a*) A_S (an infinite strip), (*b*) A_D (a disk) and (*c*) A_W (a wedge).

If we apply the above result (3.16) to AdS_5/CFT_4, we obtain the following prediction of the entanglement entropy for the $\mathcal{N} = 4$ $SU(N)$ super Yang–Mills theory[7]:

$$S_{A_S} = \frac{N^2 L^2}{2\pi a^2} - 2\sqrt{\pi} \left(\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{6}\right)} \right)^3 \frac{N^2 L^2}{l^2}. \tag{3.17}$$

Note that this is proportional to N^2 as expected since the number of fields in the $SU(N)$ gauge theory is proportional to N^2. Moreover, for general even-dimensional CFTs, we can show that the holographic entanglement entropy for any choice of A is always proportional to the central charge [30, 40].

As we mentioned, it is intriguing to compare the second finite term in (3.17) to that obtained from field theoretic calculations. The finite term in (3.17) is numerically expressed as

$$\left. S_{A_S}^{\text{Sugra}} \right|_{\text{finite}} \simeq -0.0510 \cdot \frac{N^2 L^2}{l^2}. \tag{3.18}$$

On the other hand, the free field theory results can be obtained by employing the method first considered in [108]. The $\mathcal{N} = 4$ super Yang–Mills consists of a gauge field A_μ, six real scalar fields $(\phi^1, \phi^2, \ldots, \phi^6)$ and four Majorana fermions $(\psi_\alpha^1, \psi_\alpha^2, \psi_\alpha^3, \psi_\alpha^4)$. The contribution from the gauge field is the same as those from two real scalar fields [110]. In this way, the total entropy in the free Yang–Mills theory is the same as those from eight real scalars and four Majorana fermions. In this way, we eventually obtain the numerical estimation [29, 30]

$$\left. S_{A_S}^{\text{FreeYM}} \right|_{\text{finite}} \simeq -(8 \times 0.0049 + 4 \times 0.0097) \cdot \frac{N^2 L^2}{l^2} = -0.078 \cdot \frac{N^2 L^2}{l^2}. \tag{3.19}$$

We observe that the free field result (3.19) is larger than the one (3.18) in the gravity dual by roughly 50%. The deviation itself is anticipated since our holographic computation should give the result in the strongly coupling limit and the entanglement entropy is not a protected quantity which does not depend on the coupling constant. This situation is very similar to the computation of thermal entropy [111], where we have a similar discrepancy (so-called $\frac{4}{3}$ problem). The fact that the discrepancy is of order 1, which was also found in our computation, can be considered as encouraging evidence for our argument.

---

[7] Here we employed the relation $G_5 = \frac{\pi}{2N^2} R^3$, where R is the radius of AdS_5.

We may also apply this holographic calculation to gauge theories in different dimensions. We can find holographic results of the entanglement entropy in [77] for 2D $\mathcal{N} = (4, 4)$ Yang–Mills, in [81] for 3D $\mathcal{N} = 4$ Yang–Mills and in [72] for 3D $\mathcal{N} = 6$ Chern–Simons theory. Refer to [57, 65] for the analysis in the presence of gauge fluxes.

3.5.2. Entanglement entropy for circular disk A_D.   The holographic entanglement entropy (3.3) for A_D is found as follows [30]

$$
\begin{aligned}
S_{A_D} &= \frac{2\pi^{d/2} R^d}{4 G_N^{(d+2)} \Gamma(d/2)} \int_{a/l}^1 dy \frac{(1 - y^2)^{(d-2)/2}}{y^d} \\
&= p_1 (l/a)^{d-1} + p_3 (l/a)^{d-3} + \cdots \\
&\quad + \begin{cases} p_{d-1} (l/a) + p_d + \mathcal{O}(a/l), & d: \text{even}, \\ p_{d-2}(l/a)^2 + q \log(l/a) + \mathcal{O}(1), & d: \text{odd}, \end{cases}
\end{aligned}
\tag{3.20}
$$

where the coefficients are defined by

$$
\begin{aligned}
&p_1/C = (d - 1)^{-1}, \qquad p_3/C = -(d - 2)/[2(d - 3)], \ldots \\
&p_d/C = (2\sqrt{\pi})^{-1} \Gamma(d/2) \Gamma((1 - d)/2) \quad (\text{if } d = \text{even}), \\
&q/C = (-)^{(d-1)/2} (d - 2)!!/(d - 1)!! \quad (\text{if } d = \text{odd}), \\
&\text{where} \qquad C \equiv \frac{\pi^{d/2} R^d}{2 G_N^{d+2} \Gamma(d/2)}.
\end{aligned}
\tag{3.21}
$$

We note that the result (3.21) includes a leading UV divergent term $\sim a^{-d+1}$ and its coefficient is proportional to the area of the boundary ∂A as expected from the area law [9, 10] in the field theories (2.9). We have also subleading divergent terms which reflect the form of the boundary ∂A.

In particular, we prefer a physical quantity that is independent of the cutoff (i.e. universal). The final term in (3.21) has such a property. When d is even, it is given by a constant p_d. This seems to be somewhat analogous to the topological entanglement entropy (or quantum dimension) in $(2+1)$D topological field theories [94, 95], though our theory is not topological. On the other hand, when d is odd, the coefficient q of the logarithmic term $\sim \log(l/a)$ is universal as in the 2D case (2.10). In higher dimensional CFTs, we can show that q is proportional to a certain linear combination of central charges.

This result is based on an explicit calculation when $A = A_D$. However, from [127], we find that the behavior (3.20) is also true for any compact submanifold A with different coefficients p_k and q depending on the shape of A.

3.5.3. Entanglement entropy and cusps.   In the third example $A = A_W$ with $d = 2$, we can obtain the following result [35]:

$$
S_A = \frac{R^2}{4 G_N^{(4)}} \left(\frac{2L}{a} - 2f(\Omega) \log \frac{L}{a} \right),
\tag{3.22}
$$

where the function $f(\Omega)$ is given by

$$
f(\Omega) = \int_0^\infty dz \left[1 - \sqrt{\frac{z^2 + g_0^2 + 1}{z^2 + 2g_0^2 + 1}} \right],
\tag{3.23}
$$

where g_0 is a function of the cusp angle Ω specified by

$$
\frac{\Omega}{2} = g_0 \sqrt{1 + g_0^2} \int_0^\infty \frac{dz}{(z^2 + g_0^2) \sqrt{(z^2 + g_0^2 + 1)(z^2 + 2g_0^2 + 1)}}.
\tag{3.24}
$$

The presence of the characteristic logarithmic term is due to the presence of the cusp singularity of the space ∂A. We can show that $f(\Omega)$ is a convex (or $-f(\Omega)$ is concave) function. This property $f''(\Omega) \geqslant 0$ is actually what the strong subadditivity requires (for details refer to [35]).

In free scalar and fermion theories, S_A with $A = A_W$ has been computed in 2+1 dimensions and the same scaling structure (3.22) has been found [78, 112], where the form of the function $f(\Omega)$ also turns out to agree semi-quantitatively with our strong coupling limit prediction (3.23).

4. Entanglement entropy as an order parameter

In recent discussions in condensed matter physics, the entanglement entropy is expected to play a role of an appropriate order parameter describing quantum phases and phase transitions. For example, it can be particularly useful for a system which realizes a topological order, such as fractional quantum Hall systems. At low energies, such systems can be described by a topological field theory, and the correlation functions are not useful order parameter as they are trivial. However, the entanglement entropy can capture important information of the topological ground state [94, 95]. Also it is interesting to note that the entanglement entropy has been employed to estimate efficiency of a numerical algorithm, such as DMRG [90], which makes use of the (reduced) density matrix as a criterion to discard unimportant information. This is because the entanglement entropy can measure the amount of lost information by the coarse-graining procedure or equally the renormalization flow [23, 128, 129].

The main purpose of this section is to apply the entanglement entropy to the confinement/deconfinement transition of gauge theories [38, 49, 51, 74]. It is much easier to employ our holographic calculation as we need to deal with strongly coupled gauge theories.

4.1. Confinement/deconfinement transition

One of the most interesting applications of the entanglement entropy is that it can be used as an order parameter for the confinement/deconfinement phase transition in the confining gauge theory. When we divide one of the spatial direction into a line segment with length l and its complement, the entanglement entropy between the two regions measures the effective degrees of freedom at the energy scale $\Lambda \sim 1/l$. Then, in the confining gauge theory, the behavior entanglement entropy should become trivial (i.e. S_A approaches to a constant) as l becomes large, i.e. the infrared limit $\Lambda \to 0$.

Such a transition can be captured by the holographic entanglement entropy if there are confining backgrounds dual to the confining gauge theories [38, 49]. We find that there are two candidates for the minimal surface with the same end points at the boundary in the confining background. One is the two disconnected straight lines extending from the end points of the line segment to inside the bulk, and the other is the curved line connecting these two points. The connected curve corresponds to the deconfinement phase in dual gauge theory because the entanglement entropy depends on the length l, while the disconnected lines independent of the length l correspond to the confinement phase. In general, there is a critical length l_c above which the disconnected lines are favored, and below which the connected curve is favored, as we will see below explicitly.

For example, we consider the AdS soliton solution [122]

$$ds^2 = R^2 \frac{dr^2}{r^2 f(r)} + \frac{r^2}{R^2}\left(-dt^2 + f(r)\, d\chi^2 + dx_1^2 + dx_2^2 \right), \tag{4.1}$$

where $f(r) = 1 - r_0^4/r^4$ and the χ direction is compactified with the radius $L = \pi R^2/r_0$ to avoid the conical singularity at $r = r_0$. This can be obtained from the double Wick rotation of the AdS Schwarzschild solution. The dual gauge theory is $\mathcal{N} = 4$ super Yang–Mills on $R^{1,2} \times S^1$, but the supersymmetry is broken due to the anti-periodic boundary condition for fermions along the χ direction. Then the scalar fields acquire non-zero masses from radiative corrections, and the theory becomes almost the same as the $(2+1)$-dimensional pure Yang–Mills, which shows the confinement behavior [122].

To define the entanglement entropy, let us divide the boundary region into two parts A and B: A is defined by $-l/2 \leqslant x_1 \leqslant l/2$, $0 \leqslant x_2 \leqslant V(\to \infty)$ and $0 \leqslant \chi \leqslant L$, and B is the complement of A. The minimal surface γ_A, whose boundary coincides with the end point ∂A, can be obtained by minimizing the area

$$\text{Area} = LV \int_{-l/2}^{l/2} \mathrm{d}x_1 \frac{r}{R} \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}x_1}\right)^2 + \frac{r^4 f(r)}{R^4}}. \tag{4.2}$$

Regarding x_1 as a time, then the energy conservation leads to

$$\frac{\mathrm{d}r}{\mathrm{d}x_1} = \frac{r^2}{R^2} \sqrt{f(r)\left(\frac{r^6 f(r)}{r_*^6 f(r_*)} - 1\right)}, \tag{4.3}$$

where r_* is the minimal value of r. When integrating this relation, we should also take the boundary condition into account

$$\frac{l}{2} = \int_{r_*}^{r_\infty} \mathrm{d}r \frac{R^2}{r^2 \sqrt{f(r)\left(\frac{r^6 f(r)}{r_*^6 f(r_*)} - 1\right)}}, \tag{4.4}$$

which relates r_* with l. Here we introduced the UV cutoff at $r = r_\infty$. After eliminating l in (4.2) and (4.4), we find the entanglement entropy as

$$S_A^{(\text{con})} = \frac{LV}{2R G_N^{(5)}} \int_{r_*}^{r_\infty} \frac{r^4 \sqrt{f(r)}}{\sqrt{r^6 f(r) - r_*^6 f(r_*)}}. \tag{4.5}$$

It is important that l is bounded from above due to relation (4.4):

$$l \leqslant l_{\max} \simeq 0.22L. \tag{4.6}$$

Then, when l becomes large, there is no minimal surface that connects the two boundaries of ∂A. Instead, the disconnected straight lines actually dominate before l becomes greater than $l_{\max}$. The entanglement entropy is easy to be found:

$$S_A^{(\text{discon})} = \frac{VL}{2G_N^{(5)}} \int_{r_0}^{r_\infty} \mathrm{d}r \frac{r}{R} = \frac{VL}{4G_N^{(5)} R}\left(r_\infty^2 - r_0^2\right). \tag{4.7}$$

We plot the difference of the entanglement entropy between the connected and disconnected surfaces $\Delta S_A \equiv S_A^{(\text{con})} - S_A^{(\text{discon})}$ as a function of length l of the subsystem A as shown in figure 7. Note that the physical solution shown in figure 7 (i.e. lower branch) is concave as a function of l, being consistent with the strong subadditivity of the von Neumann entropy (see section 2.2). When ΔS_A becomes positive at the critical length $l_c(<l_{\max})$, the disconnected surface dominates, i.e. becomes minimal. Then there happens a phase transition at $l = l_c$, which corresponds to the confinement/deconfinement phase transition in dual gauge theory [38, 49].

A similar analysis has been done in [49] in the more general backgrounds including the Klebanov–Strassler solution [123]. These results indicate that the entanglement entropy can be a good order parameter for a phase transition. A benefit of the holographic entanglement
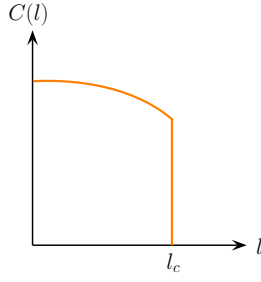
Figure 7. The entanglement entropy as a function of width l. There are three solutions which are locally minimal area surfaces: two connected surfaces and a disconnected one. We set $\Delta S_A = 0$ for the disconnected one. One of the connected ones has larger area than the other and is unphysical. When $0 < l < l_c$, the connected one is chosen, while when $l > l_c$, the disconnected one becomes dominant.

entropy is that in order to detect the confinement/deconfinement, we do not need finite temperature black brane solutions, which are often difficult to get analytically.

In summary, our holographic analysis predicts the following behavior of the finite part of the entanglement entropy in $(d+1)$-dimensional confining large N gauge theories at vanishing temperature (we subtracted the area law divergence $\sim a^{-(d-1)}$):

$$S_A(l)|_{\text{finite}} = -V F(l),$$
$$\text{where} \quad F(l) \simeq c_1 N^2 l^{-(d-1)} \quad (l \to 0), \tag{4.8}$$
$$F(l) = c_2 N^2 \quad (l > l_c),$$

and V is the volume of the non-compact $d - 2$ directions transverse to the separation. The numerical coefficients c_1 and c_2 depend on each theory.

Remarkably, the numerical computation of the entanglement entropy in the lattice gauge theory has been done in [59, 64, 71, 79], and the non-analytic behavior as shown figure 7 or (4.8) has been confirmed. These results would also support the validity of the holographic formula (3.3) of the entanglement entropy in the AdS/CFT correspondence.

To make the phase transition clear, we can define the following quantity called an 'entropic c-function':

$$C(l) \equiv \frac{l^d}{V} \frac{\mathrm{d}S_A(l)}{\mathrm{d}l}, \tag{4.9}$$

which does not depend on the UV cutoff. This is a natural generalization of the entropic c-function defined in two dimensions [98, 99]. We observe that $C(l)$ is a monotonically decreasing function of l, which is regarded as an entropy version of the c-theorem as $C(l)$ measures the degrees of freedom at the energy scale $\Lambda \sim 1/l$. Its explicit form is sketched in figure 8. The sharp dump of $C(l)$ is because the AdS bubble solution is completely cutoff in the IR region $r < r_0$ and represents the mass gap in dual gauge theory. For finite N gauge theories, the behavior may become milder.

Finally, it is also intriguing to mention a relation to closed string tachyon condensation. It is argued that in [130] the AdS soliton corresponds to the state after closed string tachyon condensation in the background of compactified AdS_5 with anti-periodic boundary conditions for fermions. In this interpretation, we observe that the entanglement entropy is decreased after the tachyon condensation like the ADM energy [131] of the background [38]. This might

Figure 8. The entropic c-function with respect to l. It jumps to zero at $l = l_c$, which is identified with the confinement/deconfinement transition.

suggest that the entanglement entropy is an important quantity which characterizes the closed string tachyon condensation.

4.2. Geometric entropy

In the previous sections, we always assumed the vanishing temperature and we found that the entanglement entropy can be an order parameter for a phase transition in confining gauge theory. It is also interesting to detect the confinement/deconfinement phase transition at finite temperature. Unfortunately, the entanglement entropy cannot probe the thermal phase transition because it is defined at the specific time and does not wind the thermal cycle (refer to [51] for detailed calculations). Instead, we can define the geometric entropy, which is regarded as the double Wick rotated version of the entanglement entropy [70]. The relation between the geometric entropy and the ordinary entanglement entropy is analogous to the one between the Polyakov loop and the Wilson loop.

To illustrate the definition of the geometric entropy, we consider the gauge theory on S^3 at finite temperature. We express the metric of S^3 as follows:

$$d\Omega_{(3)}^2 = d\theta^2 + \sin^2\theta(d\psi^2 + \sin^2\psi \, d\phi^2), \tag{4.10}$$

where $0 \leqslant \theta, \psi \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$. If we change the periodicity of ϕ to $0 \leqslant \phi \leqslant 2\pi/n$, there exists conical singularities at $\psi = 0$ and $\psi = \pi$ with the deficit angle $\delta = 2\pi(1 - 1/n)$. Then the gauge theory is defined on the orbifold S^3/Z_n. Considering the partition function on the orbifolded space, we can define the geometric entropy following the usual definition of the von Neumann entropy [70]:

$$S_G = -\frac{\partial}{\partial(1/n)} \log\left[\frac{Z_{YM}(S^3/Z_n)}{(Z_{YM}(S^3))^{1/n}}\right]\Bigg|_{n=1}. \tag{4.11}$$

If we have a dual geometry for the gauge theory, we can also perform the holographic calculation of the geometric entropy. When we require the boundary should be $S^1 \times S^3$, we have two solutions in the bulk space [122]: one is the thermal AdS space and the other is the Schwarzschild AdS black hole. Using the bulk to boundary relation in the supergravity approximation, we obtain the holographic formula for the geometric entropy similar to (3.6):

$$S_G = \frac{\text{Area}(\gamma)}{4G_N^5}, \tag{4.12}$$

where the surface γ is defined by $\sin\psi = 0$ in the bulk and it winds the thermal cycle τ.

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It is well known that there is a thermal phase transition between the thermal AdS space and the Schwarzschild AdS black hole, called the Hawking–Page transition [132], when we change the temperature or the period of the thermal cycle β. From the viewpoint of the AdS/CFT correspondence, there should be a corresponding phase transition in dual CFT on S^3 [122], and actually there exists a confinement/deconfinement phase transition in it [133, 134]. The geometric entropy can capture this phase transition in both gravity and gauge theory sides and then, this quantity can be a useful order parameter for a confinement/deconfinement transition at finite temperature. Indeed, the result in the gravity side qualitatively agrees with that in the free Yang–Mills theories as shown in [70]. See [74] for the application of the geometric entropy as an order parameter to the other backgrounds.

4.3. Topological entanglement entropy and boundary entropy

As we mentioned, in a gapped system whose low energy theory is described by a topological field theory, the entanglement entropy offers us important information about the ground state. This kind of systems is rather common in $(2 + 1)$-dimensional condensed matter systems. For example, the quantum Hall effect occurs in materials whose low-energy theory is described by an Abelian Chern–Simons gauge theory (see e.g. [89]). The appearances of non-Abelian Chern–Simons theories have also been discussed in a similar context [135].

In such $(2 + 1)$-dimensional systems with a mass gap, the entanglement entropy takes the form (assuming A is a disk)

$$S_A = \gamma \cdot \frac{l}{a} + S_{\text{top}}, \tag{4.13}$$

where the first term on the right-hand side is the area law divergence. The second term S_{top} is a finite quantity and is called the topological entanglement entropy. In [94, 95], this has been shown to be invariant under any smooth deformations of the subsystem A and S_{top} has been calculated explicitly. Calculations based on the surgery method in Chern–Simons gauge theory [136] have been performed in [137].

Therefore, it is intriguing to calculate S_{top} holographically. In the absence of the Chern–Simons term, it has been calculated in [68] for the pure Yang–Mills in $2 + 1$ dimensions and found that $S_{\text{top}} = 0$, which is consistent with the gauge theory side. To obtain non-trivial results, we need to include the Chern–Simons interaction. In [82], it has been clarified how the expected result of S_{top} can be holographically obtained by considering a D3–D7 system and by treating D7-branes as probes. A direct supergravity computation of S_{top} is still a future problem.

It is also intriguing to note that in these topologically ordered systems, there is a precise connection between physics in the bulk of the system and at the boundaries (this correspondence is sometimes called 'holography' in condensed matter physics) [36]. For example, in the Chern–Simons gauge theory, information in the bulk, such as the fractional charge and statistics of quasi particle excitations, can be mapped to chiral conformal field theory which is realized at the (1+1)-dimensional edge (boundary) of the $(2 + 1)$-dimensional system. In this correspondence, the topological entanglement entropy is equal to the boundary entropy in the conformal field theory [36]. In the language of AdS/CFT, we can indeed realize this bulk/edge duality as the AdS$_3$/CFT$_2$ correspondence [82].

The boundary entropy is originally defined as the ground-state degeneracy due to the presence of boundary in two-dimensional CFTs [138]. Actually, it also coincides with the finite part of the entanglement entropy which arises due to the presence of boundary [22, 28]. By using this relation, a holographic calculation of boundary entropy has successfully been done in [56] based on AdS$_3$/CFT$_2$.

5. BH entropy as entanglement entropy

An important original motivation for the entanglement entropy in quantum field theories has been the microscopic understanding of the black hole entropy. Even though the entropy of supersymmetric (BPS) black holes has been understood by explicitly counting the BPS states [139], the entropy of Schwarzschild black holes has not been well understood microscopically. It is natural that some sort of quantum entanglement between the inside and outside of the event horizon is relevant for the explanation of the entropy of the Schwarzschild black hole. Indeed, as we have explained in section 2.5, the entanglement entropy shares similar properties with the black hole entropy.

In the case of induced gravity, the entanglement is essentially equivalent to the black hole entropy (see e.g. [115]). Interestingly, we can confirm this holographically by considering an analogue of AdS/CFT in the brane-world setup (RS II [140]) [31, 32, 34, 85] as we review in section 5.1.

There is another way to relate the black hole entropy to the entanglement entropy. This is given by directly applying AdS/CFT to AdS black holes. The most well-known example is the AdS-Schwarzschild solution. It is clearly dual to a CFT at finite temperature. At the same time, we can start with a pure state (Hartle–Hawking state) in a pair of these CFTs (called CFT1 and CFT2)

$$|\Psi\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\beta E_n/2}|n\rangle_1 \otimes |n\rangle_2, \tag{5.1}$$

where E_n denote energy eigenvalues of the given CFT and we define $Z = \sum_n e^{-\beta E_n}$. Indeed, by tracing out one of the Hilbert spaces of the second CFT we get correctly the thermal density matrix:

$$\rho_1 = \text{Tr}_2|\Psi\rangle\langle\Psi| = \frac{1}{Z} \sum_n e^{-\beta E_n}|n\rangle_1\langle n|_1. \tag{5.2}$$

This happens exactly in the AdS-Schwarzschild black hole since its extended Penrose diagram has two boundaries, which are identified with CFT1 and CFT2 as found in [86]. In section 5.2, we will explain that a similar interpretation is also possible for a pure AdS_2 space and this enables us to understand the entropy of extremal black holes in flat spacetimes as the entanglement entropy of certain systems of conformal quantum mechanics assuming AdS_2/CFT_1 [50] (see also relevant discussions in [83]). For other recent progresses on the relation between the black hole entropy and entanglement entropy, refer to [37, 41, 45, 48, 55, 141, 142].

5.1. BH entropy as entanglement entropy via brane world

Let us recall that the AdS/CFT correspondence with a UV cutoff $z > a$ can be regarded as a brane-world setup (RS2 [140]). In this context, we usually generalize AdS/CFT so that the cutoff a to be of order R (AdS radius) and a Newton constant $G_N^{\text{brane}} \sim \frac{d-1}{R} G_N^{\text{bulk}}$ is induced on the brane. By assuming the extension of AdS/CFT to this system, we find that the $(d+1)$-dimensional quantum gravity on the brane is dual to the classical gravity on the $(d+2)$-dimensional AdS space with the cutoff. This description offers us an interesting way to treat a black hole including quantum corrections [143].

If one wants to work within the standard AdS/CFT conservatively, we can assume that the cutoff a is small $a \ll R$. Then the Newton constant on the brane becomes very small

$$\frac{1}{G_N^{\text{brane}}} \sim \frac{R^d}{G_N^{\text{bulk}}} \int_a^\infty \frac{\mathrm{d}z}{z^d} = \frac{R^d}{(d-1)a^{d-1}} \frac{1}{G_N^{\text{bulk}}} \gg \frac{R}{G_N^{\text{bulk}}}. \tag{5.3}$$

This weak gravity system is also enough for our purpose below.

In the paper [143], authors construct four-dimensional black hole solutions to the vacuum Einstein equation with the negative cosmological constant. The horizon Σ extends toward the $(2+1)$-dimensional brane and the induced metric on the brane looks like Schwarzschild metric[8]

$$\mathrm{d}s_{\text{brane}}^2 = -\left(1 - \frac{r_0}{r}\right) \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - r_0/r} + r^2 \mathrm{d}\phi^2. \tag{5.4}$$

In the middle of the bulk AdS$_4$, the size of the horizon shrinks to zero and thus its topology is a disk, which looks very similar to the setup (b) shown in figure 6. Recently, brane-world black hole for AdS$_5$ has been obtained in [144] by considering extremal brane-world black holes with the AdS$_2 \times S^2$ near-horizon geometry.

Now we would like to apply the holographic entanglement entropy to brane-world black holes. Let us choose the subsystem A is inside the horizon $r = r_0$ on the brane. Then the minimal surface γ which is the bulk extension of A is actually given by the horizon Σ of the bulk black hole solution [143]. Thus, we find that the holographic entanglement entropy (3.3) coincides with the Bekenstein–Hawking entropy of the AdS$_4$ black hole solution. The latter is considered to be equal to the quantum-corrected black hole entropy of the $(2+1)$-dimensional brane-world black hole via AdS/CFT. Therefore, we can conclude that the entropy for the entanglement between the inside and outside of the horizon is the same as the black hole entropy with quantum corrections [31, 85].

Note that the classical entropy $\frac{\text{Area}(\Sigma)}{4G_N^{\text{brane}}}$ in the brane gravity largely deviates from the quantum corrected one $\frac{\text{Area}(\gamma)}{4G_N^{\text{bulk}}}$ when $a \sim R$. We can see the above claim explicitly by computing the holographic entanglement entropy. Assuming that a is very small, the entanglement entropy S_A behaves like

$$S_A = \frac{\text{Area}(\gamma)}{4G_N^{\text{bulk}}} = \gamma \frac{\text{Area}(\Sigma)}{a^{d-1}} + O(a^{d-2}) = \frac{\text{Area}(\Sigma)}{4G_N^{\text{brane}}} + O(a^{d-2}), \tag{5.5}$$

where γ is a certain numerical factor. The subleading term $O(a^{d-2})$ can be interpreted as the quantum corrections to the classical Bekenstein–Hawking formula. In this weak gravity limit $a \to 0$, the leading term becomes dominant. It is amusing to note that in (5.5), the area law term of the entanglement entropy in quantum field theories essentially becomes equal to the black hole entropy. These arguments strongly suggest that some sort of induced gravity is realized in the brane-world setup.

A similar interpretation of two-dimensional black holes has been found in [34]. We can also apply the same brane-world argument to explain the entropy of de Sitter spacetime as discussed in [32, 85].

5.2. BH entropy as entanglement entropy via AdS$_2$/CFT$_1$

The pure AdS spacetime AdS$_{d+1}$ with $d \geqslant 2$ has no entropy as is also clear from its dual CFT$_d$ at zero temperature. To obtain non-zero entropy, we need to consider the AdS black hole as the dual geometry. On the other hand, we expect non-zero entropy for the pure AdS$_2$

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[8] On the brane, we expect no cosmological constant. In usual Einstein gravity with zero cosmological constant, there is no black hole solution. In our case, the result should be interpreted such that it already contains quantum corrections following the philosophy of AdS/CFT. We expect that large quantum corrections make such a black hole solution possible [143].

spacetime since it appears as the near-horizon limit of higher dimensional extremal black holes [145–148]. Thus, the microscopic interpretation of the Bekenstein–Hawking entropy of the extremal black holes would be related to the AdS_2/CFT_1 correspondence [150, 151]. Even though the AdS_2/CFT_1 has not been well understood compared to the higher dimensional AdS/CFT, below we assume that the gravity on *AdS₂* is dual to a certain conformal quantum mechanics (CFT₁). In other words, one may think that the following argument is indirect evidence for AdS_2/CFT_1. A formulation based on the entropy function has been done in [152]. Also the appearance of the AdS_2 spacetime plays an important role in the recent investigations of the attractor mechanism (see [153] and references therein), and more recently, in a new duality called the extremal black hole/CFT correspondence [154, 155].

The AdS_2 geometry has a special property such that it has two time-like boundaries in the global coordinate

$$ds^2 = \ell^2 \frac{-d\tau^2 + d\sigma^2}{\cos^2 \sigma}, \tag{5.6}$$

where ℓ is the radius of the AdS space and $-\frac{\pi}{2} \leqslant \sigma \leqslant \frac{\pi}{2}$. Then, according to the principle of AdS/CFT, we expect that there are two CFT₁s on the boundary $\sigma = \pm\frac{\pi}{2}$ of the AdS_2 space. Here we would like to show that the black hole entropy is exactly the same as the entanglement entropy between the two CFTs by using the AdS_2/CFT_1 correspondence. Actually we can show that the two CFTs are entangled applying the holographic formula of the entanglement entropy (1.2) as

$$S_{\text{ent}} = \frac{\text{Area}(\gamma_A)}{4G_N^{(2)}} = \frac{1}{G_N^{(2)}}. \tag{5.7}$$

This is because the minimal surface now becomes a point. Below we will give a clearer derivation of (5.7) based on AdS/CFT [50].

As we mentioned above, there are two independent CFTs on the boundaries of the AdS_2 space, namely CFT1 and CFT2. The Hilbert spaces of CFT1 and CFT2 are denoted by H_1 and H_2. The total Hilbert space looks like $H_{\text{tot}} = H_1 \otimes H_2$. We define the reduced density matrix from the total density matrix ρ_{tot}:

$$\rho_1 = \text{Tr}_{H_2} \rho_{\text{tot}}, \tag{5.8}$$

by tracing over the Hilbert space H_2. This is the density matrix for an observer who is blind to CFT2. It is natural to assume that ρ_{tot} is the one for a pure state.

The entanglement entropy for CFT1, when we assume that the opposite part CFT2 is invisible for the observer in CFT1, is defined by

$$S_{\text{ent}} = \text{Tr}[-\rho_1 \log \rho_1]. \tag{5.9}$$

We can obtain this by first computing $\text{Tr}(\rho_1)^n$, taking the derivative w.r.t. *n* and finally setting $n = 1$. In the path integral formalism of the quantum mechanics, ρ_1 and $\text{Tr}(\rho_1)^n$ are computed as shown in figure 9 (we perform the path integral along the thick lines and *a* and *b* are the boundary conditions).

By using the bulk-boundary relation of AdS/CFT [5, 6], we can compute the entanglement entropy holographically[9] as shown in the right panel of figure 10. The dual geometry is the *n*-sheeted Riemann surface [29, 30], assuming the Euclidean metric. The cut should end at a certain point in the bulk because there should not be any cut on the opposite boundary, which

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[9] Our derivation seems to be closely related to the conical defect argument of black hole entropy (see e.g. [115, 156]). However, note that in these arguments the authors consider the entanglement entropy for the total spacetime of non-extremal black holes, while in our argument we consider the entanglement entropy for the boundary of the extremal black hole geometry. See also [83] for a discussion on the relevance of the AdS_2 geometry.
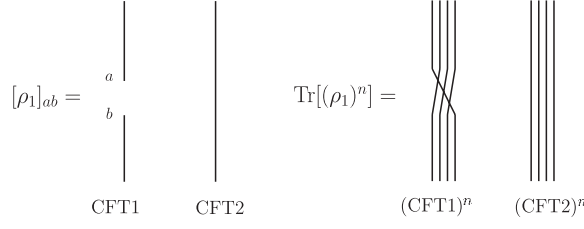
Figure 9. The calculation of reduced density matrix ρ_1.
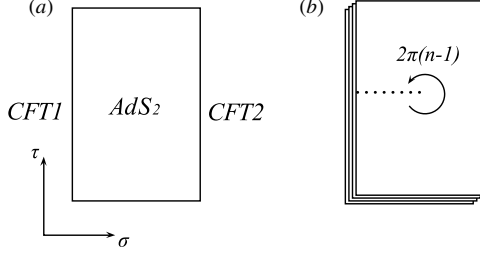
Figure 10. The geometry of AdS$_2$ with two boundaries (left) and the 2D spacetime (n-sheeted Riemann surface with a cut) which is dual to the computation $\text{Tr}(\rho_1)^n$ (right).

is first traced out. Note that the presence of two boundaries in AdS$_2$ plays a crucial role in this holographic computation. We would get the vanishing entropy if we were to start with the spacetime which has a single boundary such as the Poincaré metric of AdS$_2$.

Now we recall the Einstein–Hilbert action in the Euclidean space

$$I = -\frac{1}{16\pi G_N^{(2)}} \int \mathrm{d}x^2 \sqrt{g}(R + \Lambda). \tag{5.10}$$

The cosmological constant Λ is not important since the contribution to the Einstein–Hilbert action from the cosmological constant term is extensive and it will vanish in the end of the entropy computation. In the n-sheeted geometry we find $I = \frac{n-1}{4G_N^{(2)}}$ in the Euclidean formalism because the curvature behaves like a delta function $R = 4\pi(1 - n)\delta^2(x)$ (see e.g. [33, 156]). The entanglement entropy is obtained as follows:

$$S_{\text{ent}} = -\frac{\partial}{\partial n} \log \left(\mathrm{e}^{-I + nI^{(0)}} \right)\big|_{n=1} = \frac{1}{4G_N^{(2)}}, \tag{5.11}$$

where $I^{(0)}$ is the value of the Einstein–Hilbert action of a single sheet in the absence of the cut (or negative deficit angle).

Recently, it has been shown that extremal (rotating) black holes always have the $SO(2, 1)$ symmetry in the near-horizon limit [145–148]. For example, the near-horizon geometry of a four-dimensional extremal Kerr black hole is given by a warped product of AdS$_2$ and a two-dimensional manifold [149]. Then we can assume the near-horizon geometry of d-dimensional extremal black holes as

$$\mathrm{d}s^2 = f \, \mathrm{d}s^2_{\text{AdS}_2} + \mathrm{d}s^2_{M^{d-2}}, \tag{5.12}$$

where the first term is the AdS$_2$ space in the Poincaré coordinate $\mathrm{d}s^2_{\text{AdS}_2} = -\frac{r^2}{\ell^2} \mathrm{d}t^2 + \frac{\ell^2}{r^2} \mathrm{d}r^2$ with a warp factor f that depends on the coordinate of M^{d-2}; the second term is the metric

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of the compact manifold M^{d-2} of the horizon such as S^{d-2}. The horizon is at $r = 0$ in this coordinate and the Bekenstein–Hawking entropy is

$$S_{\text{BH}} = \frac{\text{Vol}(M^{d-2})}{4G_N^{(d)}}. \tag{5.13}$$

Finally, it is trivial to see that

$$S_{\text{ent}} = S_{\text{BH}}, \tag{5.14}$$

because the Newton constant in two dimensions is defined as $\frac{1}{G_N^{(2)}} \equiv \frac{\text{Vol}(M^{d-2})}{G_N^{(d)}}$. This means that the entanglement between CFT1 and CFT2 is precisely the source of the extremal black hole entropy.

Moreover, we can take curvature corrections into account. We assume that the near-horizon geometry is of the form $\text{AdS}_2 \times M^{d-2}$ even in the presence of the higher derivative corrections. Even though we start with the Lagrangian $\mathcal{L}$ that includes the curvature tensor $R_{\mu\nu\rho\sigma}$ and their covariant derivatives, we can neglect the covariant derivative of curvature tensors because the near-horizon geometry has the constant curvature. In this case, the black hole entropy with the curvature corrections is given by Wald's formula [157–159]

$$S_{\text{BH}} = -2\pi \int_{\mathcal{H}} \sqrt{h} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma}, \tag{5.15}$$

where $\epsilon_{\mu\nu} = \xi_\mu\eta_\nu - \xi_\nu\eta_\mu$ by using the Killing vector ξ_μ of the Killing horizon and its normal η_ν, normalized such that $\xi \cdot \eta = 1$; $\mathcal{H}$ represents the horizon and h is the metric on it. Reducing the d-dimensional metric to AdS_2 space, the action might change into that with higher derivative corrections and non-gravitational fields such as gauge fields. Fortunately, the non-gravitational fields do not contribute to Wald's formula and we can neglect these terms and concentrate on the higher derivative action even in two dimensions.

Now we would like to compare the Wald entropy with the entanglement entropy computed holographically via $\text{AdS}_2/\text{CFT}_1$. We consider the n-sheeted AdS_2, where the Riemann tensor behaves as follows [156]:

$$R_{abcd} = R_{abcd}^{(0)} + 2\pi(1 - n) \cdot (g_{ac}g_{bd} - g_{ad}g_{bc}) \cdot \delta_H. \tag{5.16}$$

Here δ_H is the delta function localized at the (codimension 2) horizon (H is actually a point in AdS_2 and is related to the original horizon $\mathcal{H}$ as $\mathcal{H} = H \times M^{d-2}$). $R_{abcd}^{(0)}$ represents the constant curvature contribution from the cosmological constant. a, b run the coordinate in the AdS_2. Note also that if we employ the relation $g_{ab} = \xi_a\eta_b + \xi_b\eta_a$, we obtain $\epsilon_{ab}\epsilon_{cd} = -(g_{ac}g_{bd} - g_{ad}g_{bc})$. Now we consider the perturbative expansions of the Lagrangian with respect to the (delta functional) deviation of R_{abcd} from $R_{abcd}^{(0)}$. Then the quadratic and higher order terms do not contribute since $\lim_{n \to 1} \frac{d}{dn}(1 - n)^p = 0$ for $p \geqslant 2$. Therefore, we can find

$$I_n = -\log Z_n = 2\pi(1 - n) \int_H \sqrt{h} \frac{\partial \mathcal{L}}{\partial R_{abcd}} \epsilon_{ab}\epsilon_{cd}. \tag{5.17}$$

Thus, this agrees with Wald's formula in two dimensions:

$$S_{\text{ent}} = -\frac{\partial}{\partial n} \log Z_n|_{n=1} = -2\pi \int_H \sqrt{h} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma} = S_{\text{BH}}. \tag{5.18}$$

After introducing the 2D Newton constant as before, the Wald entropy in two dimensions is exactly the same as that in d dimensions, and we complete the proof of the equivalence of the black hole entropy and the entanglement entropy even in the presence of the higher derivative correction.

6. Covariant holographic entanglement entropy

So far we have only discussed static spacetimes. It is straightforward to extend our holographic formula to static spacetimes which are not asymptotically AdS as long as we have its holographic dual theory. However, it may be more interesting to consider holography in a time-dependent spacetime as eventually we would like to understand cosmological backgrounds such as the de Sitter space from a holographic viewpoint.

6.1. Covariant entropy bound

In the previous argument of section 3, we assumed a time slice on which we can define minimal surfaces since its signature is Euclidean. However, in the time-dependent case there is no longer a natural choice of the time slices as we have infinitely many different ways of defining the time slices. Thus, we need to consider the entire Lorentzian spacetime. However, there is a problem since in Lorentzian geometry there is no minimal area surface as the area vanishing if the surface extends in the light-like direction. In order to resolve this issue, let us remember an analogous problem: the covariant entropy bound so called the Bousso bound [118].

In general, if we get heavy objects together in a small region and continue to bring another one into the region, this system eventually experiences the gravitational collapse. Therefore, we have an upper bound of the mass and entropy which can be included inside of the surface Σ. The bound for the entropy in flat spacetime is called the Bekenstein bound and it is given by

$$S_\Sigma \leqslant \frac{\text{Area}(\Sigma)}{4G_N},\tag{6.1}$$

where Σ is a codimension two closed surface in the spacetime. It is also more interesting to generalize this bound to any time-dependent backgrounds such as the cosmological ones. This requires to find a covariant description. It is obvious that the Bekenstein bound (6.1) is not covariant since the definition of the entropy included inside Σ is not covariant but depends on the choice of the time slice. The covariant entropy bound was eventually formulated by Bousso [118] and it is given by

$$S_{L(\Sigma)} \leqslant \frac{\text{Area}(\Sigma)}{4G_N}.\tag{6.2}$$

The light-like manifold $L(\Sigma)$ is called the light sheet of Σ. This is defined by the manifold which is generated by the null geodesics starting from the surface Σ. We require that the expansion θ of the null geodesic is non-positive $\theta \leqslant 0$. In the flat spacetime, this is just a half of light cone and the same is true for the AdS spacetime as it is conformally flat. Then the quantity $S_{L(\Sigma)}$ means the entropy which passes through the light sheet $L(\Sigma)$, which is covariantly well defined. One more interesting thing on the Bousso bound is that we can apply the bound even if the surface Σ has boundaries, which is quite useful in the holographic setup as we employ below.

6.2. Covariant holographic entanglement entropy

Now we would like to return to our original question of the covariant holographic entanglement entropy. Our final claim [43] is given by

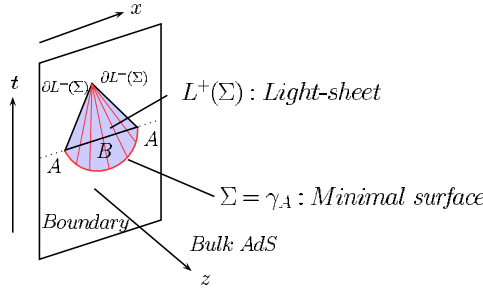$$S_A(t) = \frac{\text{Area}(\gamma_A(t))}{4G_N^{d+2}},\tag{6.3}$$

Figure 11. The setup of the Bousso bound applied to the AdS$_3$/CFT$_2$ in the Poincaré coordinate $ds^2 = \frac{R^2}{z^2}(-dt^2 + dz^2 + dx^2)$. In this simplified case, the future Cauchy horizon coincides with the future light sheet $\partial L^+(\Sigma)$. In this figure, we only write the future light sheet and not the past one (i.e. $L^-(\Sigma)$) just for simplicity.

where $\gamma_A(t)$ is the extremal surface in the entire Lorentzian spacetime $\mathcal{M}$ with the boundary condition $\partial \gamma_A(t) = \partial A(t)$. The time t is the time on the time slice in the boundary $\partial \mathcal{M} = R^{1,d}$ and there is no unique way to extend it to the bulk spacetime $\mathcal{M}$. This formula, for example, when applied to rotating BTZ black holes correctly reproduces the entanglement entropy expected from CFT$_2$ [43].

This covariant formula (6.3) has originally been motivated from the Bousso bound (6.2) in [43]. To see this let us again remember the fact that the AdS/CFT correspondence with a UV cutoff $z > a$ can be regarded as a brane-world setup (RS2 [140]). Assuming that the cutoff is close to the UV $a \ll R$, the gravity on the $(d+1)$-dimensional brane theory is very weak as in (5.3). In this setup, we would like to ask what is the Bousso bound on the brane gravity theory (see figure 11 in the simplest case of AdS$_3$/CFT$_2$). We expect that the brane theory with quantum corrections taken into account is dual to the bulk gravity theory which is classical, based on the standard idea of the AdS/CFT correspondence. Therefore, we argue that the quantum-corrected Bousso bound on the brane can be found as the classical Bousso bound on the brane.

First we start with the setup of Bousso bound at the boundary $\partial \mathcal{M}$. We pick up a (closed) surface $\partial \Sigma$ which separates a time slice into the subsystems A and B such that $\partial A = \partial \Sigma$. See figure 11. Now we define their light sheets. We consider both ones directed to the future and past and call them $\partial L^+(\Sigma)$ and $\partial L^-(\Sigma)$, respectively. The reason why we put the symbol ∂ is that we are interested in their bulk extensions $L^\pm(\Sigma)$. Again there are infinitely many different ways of extending the boundary light sheets toward the bulk. We define the surface Σ by the intersection $L^+(\Sigma) \cap L^-(\Sigma)$. For each choice of such a Σ, we get the Bousso bound (6.2).

Here the condition of non-positive expansions of the null geodesics on the light sheets i.e. $\theta^\pm \leqslant 0$ comes into play. If it were not for this condition, we could choose arbitrary Σ and take them to be light-like. However, the condition is rather strong enough that the area of allowed Σ takes a non-trivial minimum and therefore we can define an analogue of the minimal surface in this Lorentzian spacetime. The minimum of the area corresponds to the most strict Bousso bound for a given boundary surface $\partial \Sigma$ or equally the choice of the subsystem A.

This minimum of the area occurs when the expansions on the two light sheets are both vanishing $\theta^\pm = 0$. This condition is actually equal to the statement that the surface Σ is an extremal surface again called γ_A, which is defined by the saddle point of the area functional in the Lorentzian spacetime [43].

The final assumption is that the quantum Bousso bound on the brane will be saturated by the entanglement entropy. This is because the entanglement entropy represents a thermal entropy plus quantum corrections and it is defined by assuming that the subsystem *B* is completely smeared, which will be expected to lead to the maximal entropy allowed in the region. If we assume this, then we immediately reach the holographic entanglement entropy formula (6.3).

In this way, we can extract the entropy in the dual time-dependent background by looking at our holographic entanglement entropy. One may think that the opposite may be true: the extraction of metric from the information of the entanglement entropy in CFT. For recent discussions in this direction refer to [44, 73].

6.3. Applications to black holes

In black hole backgrounds, taking the limit where *A* is the total space at the boundary, S_A becomes equal to the thermal entropy. In generic time-dependent backgrounds, the thermal entropy is also time dependent and is clearly defined uniquely as the von Neumann entropy of the thermal density matrix. On the other hand, in the dual gravity side, it corresponds to the black hole entropy which is obtained from the area of black hole horizon. However, we have two candidates of horizon: event horizon and apparent horizon. These two horizons are different in time-dependent spacetimes. Our formula (6.3) of the holographic entanglement entropy selects the latter, i.e. the apparent horizon [43]. This is because the extremal surface condition requires the vanishing of null expansion and this precisely match the definition of apparent horizon. Note also that the definition of event horizon is global. Also our formula offers us to uniquely determine the time slice which is required to define the apparent horizon. Indeed, the recent work [84] shows that the area of apparent horizons behaves in a sensible way as the thermal entropy, while that of even horizons does not.

Finally, let us discuss an example where we can apply the above covariant formula. We consider the AdS Vaidya solution (see e.g. [160])

$$ds^2 = -(r^2 - m(v))\,dv^2 + 2\,dv\,dr + r^2\,d\phi^2. \tag{6.4}$$

This is the solution to the Einstein equation with the negative cosmological constant in the presence of null matter whose EM tensor looks like $T_{vv} = \frac{1}{2r}\frac{dm(v)}{dv}$. The null energy condition requires $T_{vv} \geqslant 0$ and thus we find that $m(v)$ is a monotonically increasing function of the (light-cone) time v.

This background is asymptotically AdS$_3$ and if we assume that $m(v)$ is a constant, then it is equivalent to the static BTZ black hole [121] with the mass *m*. Thus, our background (6.4) describes an idealized collapse of a radiating star in the presence of negative cosmological constant. The dual theory is expected to be a CFT in a time-dependent background. The time dependence comes from the time-dependent temperature. We can now apply the covariant entanglement entropy formula (6.3) and in the end we find

$$S_A(v) = \frac{c}{3}\left[\log\frac{l}{a} + \frac{m(v)l^2}{6} + \cdots\right], \tag{6.5}$$

as the expansion of small $m(v)$. The null energy condition guarantees that this is a monotonically increasing function of time. This shows that the entanglement entropy in this background is a monotonically increasing function of time as is so in the second law of the thermal entropy. We believe this behavior of the entanglement entropy in black hole formation processes is rather general. However, we would like to stress that we are not claiming that the entanglement entropy is always increasing. For example, if we start with the system

maximally entangled, the entanglement entropy will decrease after a small perturbation due to the decoherence phenomenon.

7. Conclusions

In this paper, we reviewed the recent progress of holographic understanding of the entanglement entropy, starting from the main holographic formula (3.3). We mainly employed the setup of AdS/CFT, though we can straightforwardly extend our results to more general spacetimes with their holographic duals. Even though formula (3.3) has not been rigorously proven, there has been much evidence so far. Including those which we did not discuss in this paper, we can list important pieces of evidence as follows.

- The area law (2.9) known in QFT can be easily reproduced holographically. The warp factor in the AdS space leads to the UV divergence of the dual CFT as explained in section 3.1.
- We can holographically derive the strong subadditivity (2.7) in a very simple way as we reviewed in section 3.3.
- We find perfect agreements between the AdS and CFT sides in the AdS_3/CFT_2 setup as in section 3.4.
- In higher dimensions, it is not easy to calculate the entanglement entropy in QFTs analytically. Still we can show the semi-qualitative agreements between the CFT and AdS calculations as reviewed in section 3.5. Moreover, for the logarithmic terms of the entropy we can show the precise agreement as its coefficient under the condition that the extrinsic curvature of ∂A is vanishing [30].
- In the example of $\mathcal{N} = 4$ Yang–Mills theory compactified on a Scherk–Schwarz circle dual to the AdS soliton, we can holographically detect a confinement/deconfinement transition as reviewed in section 4. This qualitatively agrees both with the free Yang–Mills result and with recent lattice simulations. Moreover, in an almost supersymmetric limit, the holographic result of the entanglement entropy quantitatively agrees with the free Yang–Mills result [38].
- In the presence of a horizon, the minimal surface γ_A tends to wrap the (apparent) horizon. Then the wrapped part gives an extensive contribution to the holographic entanglement entropy. This shows that our holographic formula generalizes the Bekenstein–Hawking entropy formula, as explained in sections 3.1 and 5.1.
- We can regard black hole entropy as the entanglement entropy by applying the idea of the holographic entanglement entropy to either brane-world black holes or AdS_2/CFT_1 as reviewed in section 5.
- The holographic formula is non-trivially consistent with the covariant entropy bound (Bousso bound) as explained in section 6.

There are many future directions. One of the most important future problems is to derive the holographic entanglement entropy from the first principle of AdS/CFT (i.e. bulk to boundary relation). On the other hand, to confirm the AdS/CFT correspondence from the viewpoint of the entanglement entropy, we need to develop methods of calculations of the entanglement entropy in quantum field theories. Since the holographic entanglement entropy is expected to be a universal observable in holography, this quantity may be useful when we try to extend the holography to other spacetimes such as de Sitter spaces. The precise relevance of the entanglement entropy to the understanding of black hole entropy is also an interesting future problem. The fact that our covariant formula is closely related to the Bousso bound may be a clue to this problem.

Acknowledgments

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