ECE4010J Probabilistic Methods in Engineering

Spring 2022 — Assigment 1

Date Due: 4:00 PM, Friday, the 25th of February 2022

This assignment has a total of **(25 Marks)**.

Exercise 1.1 Elementary Probability via Counting
A fair six-sided die[1] is rolled seven times. What is the probability that every possible result (1,2,3,4,5,6) appears at least once?
(2 Marks)

Exercise 1.2 Probability Function
A weighted six-sided die[1] satisfies $P[\{1,3,5\}] = P[\{1,2,3\}] = P[\{2,4,5\}]] =: \alpha$. Express the probabilities $P[\{2\}], \ldots, P[\{6\}]$ in terms of α and $P[\{1\}]$.
(3 Marks)

Exercise 1.3 A Weighted Die
Suppose[1] that a six-sided die is weighted so that $P[\{k\}] = 1/6 + \varepsilon_k$ with $|\varepsilon_k| < 1/12$, $k = 1, \ldots, 6$. (Perhaps due to minor production errors.)

This die is now rolled twice, the results are added and the remainder of the sum modulo 6 is retained, yielding a number between 0 and 5. Then, one is added to this result to give a final result between 1 and 6.

Show that this procedure halves the maximum deviation from fairness of the die result, i.e., $\max|\varepsilon_k|$ is halved.
(4 Marks)

Exercise 1.4 Two Children Paradox - Birthday Party!
At a meeting of boy scouts in 2010 (at the time, girls were not allowed to become boy scouts but could join the girl scouts instead), you meet a friendly lady, mother of one of the boys. She tells you that her son actually has a sibling, but neglects to mention whether he or she is younger or older. Being familiar with basic probability, you conclude that the sibling is most likely (with probability 2/3) a girl.

But then you recall that the meeting is actually a communal birthday party only for those boy scouts born in July, so that must be the month of birth of the boy. You assume that the probability for any given child of being born in July is 1/12, ignoring the variable lengths of the months as well as all sociological, biological or other distortions. What is the probability that the lady's other child is a girl?
(3 Marks)

Exercise 1.5 A Cooperative Monty Hall Problem
In this episode of the Monty Hall show there are again three closed doors in a room. Behind one of the doors is a safe containing a million RMB, behind another door there is a slip of paper with the combination for the safe and behind the last door there is a goat.

The game involves two players (Alice and Bob) who will either both win or both lose. To win, Alice must find the safe while Bob must find the safe combination. Each player may open two doors at random and if any door contains the desired object, the player will gain it. Alice enters the room first (she is alone) and opens two doors. if she is successful, she leaves, the doors are closed again and Bob enters the room. He may then open two doors of his choice. If he finds the combination, both win.

Alice and Bob may confer before opening any doors but may not communicate with each other in any way once the game starts.

i) If the Alice and Bob open doors at random, show that the probability of them winning is 4/9. **(1 Mark)**

ii) Find a strategy (which the Alice and Bob may decide on before starting the game) so that the probability of them winning is 2/3. **(3 Marks)**

iii) Why is 2/3 the greatest possible probability of both players winning? In other words, show that the above strategy is optimal. **(1 Mark)**

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[1] Adapted from U. Krengel, *Einführung in die Wahrscheinlichkeitstheorie und Statistik* [Introduction to Probability and Statistics], 5th Ed., Vieweg Verlag, 2000

In clinical testing for a disease, one is interested in whether a patient is healthy (h) or diseased (d). A test for the disease may be either positive (p), indicating that a patient is diseased, or negative (n), indicating that the patient is healthy. A test is never completely reliable and there are four possible outcomes:

- $n \mid h$ – A healthy person tests negative for the disease.
- $p \mid h$ – A healthy person tests positive for the disease (*false positive*).
- $n \mid d$ – A diseased person tests negative for the disease (*false negative*).
- $p \mid d$ – A diseased person tests positive for the disease.

The probability $P[n \mid h]$ is called the *specificity*, $P[p \mid d]$ the *sensitivity* of the test.

Exercise 1.6 Testing for COVID-19

During the COVID-19 pandemic of 2020, testing for presence of the disease quickly became critical. Since the genome of the virus was available early, initial efforts focussed on detecting the viral RNA in a patient's bloodstream using a "reverse transcriptase polymerase chain reaction" (RT-PCR) procedure, commonly referred to as a "PCR test". (Subsequently, other tests were developed.)

A guideline[2] published in May 2020 assumed a specificity of 70% and a sensitivity of 95% for PCR tests, based on early reports of the reliability of PCR tests.

i) Suppose that a random person from a population where the COVID-19 incidence is 1% is tested for the disease. Find $P[d \mid p]$, $P[d \mid n]$, $P[h \mid p]$, $P[h \mid n]$. **(2 Marks)**

ii) Interpret the above numbers and explain what this says about the reliability of the test in such a situation. What practical consequences would you recommend? **(2 Marks)**

iii) Suppose that a person with typical flu symptoms presents in a clinic and is tested for COVID-19. From experience, 30% of such patients are diseased. Find $P[d \mid p]$, $P[d \mid n]$, $P[h \mid p]$, $P[h \mid n]$. **(2 Marks)**

iv) Interpret the above numbers and explain what this says about the reliability of the test in such a situation. What practical consequences would you recommend? **(2 Marks)**

[2] Watson, J., Whiting, P. F., and Brush, J. E. (2020). *Interpreting a covid-19 test result.* BMJ (Clinical research ed.), 369, m1808. `https://doi.org/10.1136/bmj.m1808`