Technical University of Denmark

Assignment 2: ARMA Processes and Seasonal Processes

02417 Time Series Analysis

Jonas Wiendl, s243543
Tais Abrahamsson, s193900

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Contents

1 Stability

In this chapter an AR(2) process is investigated wrt. its stationarity, invertibility as well as its autocorrelation. The process is given in Equation 1.

$$X_t + \phi_1 X_{t-1} + \phi_2 X_{t-2} = \varepsilon_t \tag{1}$$

1.1 Is the process stationary?

The characteristic equation is given and solved in Equation 2

$$1 - 0.7z^{-1} - 0.2z^{-2} = 0 \rightarrow z = \{0.918, -0.2179\} \tag{2}$$

Both roots of the characteristic equation are thus inside the unit circle ($|z| > 1$), meaning the process is stationary.[1, p.120]

1.2 Is the process invertible?

The process consist only of auto-regressive terms. Thus it is necessarily invertible.[1, p.120]

1.3 ACF of the process

Using the Yule Walker equation [1, p.122], we can first find the autocorrelation for $k = \{1,2\}$.

$$\rho(1) = -(\phi_1 + \rho(1) \cdot \phi_2) \tag{3}$$

$$\rho(2) = -(\rho(1) \cdot \phi_1 + \phi_2) \tag{4}$$

Using equations 3 and 4 and simple algebra, we can isolate the expressions for $\rho(1)$ and $\rho(2)$.

$$\rho(1) = -\frac{\phi_1}{1+\phi_2} \tag{5}$$

$$\rho(2) = \frac{\phi_1^2}{1+\phi_2} - \phi_2 \tag{6}$$

Any $\rho(k \geq 3)$ can be found recursively by the formula in Equation 7.

$$\rho(k) = -\phi_1(k-1) - \phi_2(k-2) \tag{7}$$

1.4 ACF Plot

Below a plot of the ACF for $k = \{1,...,30\}$ is shown.

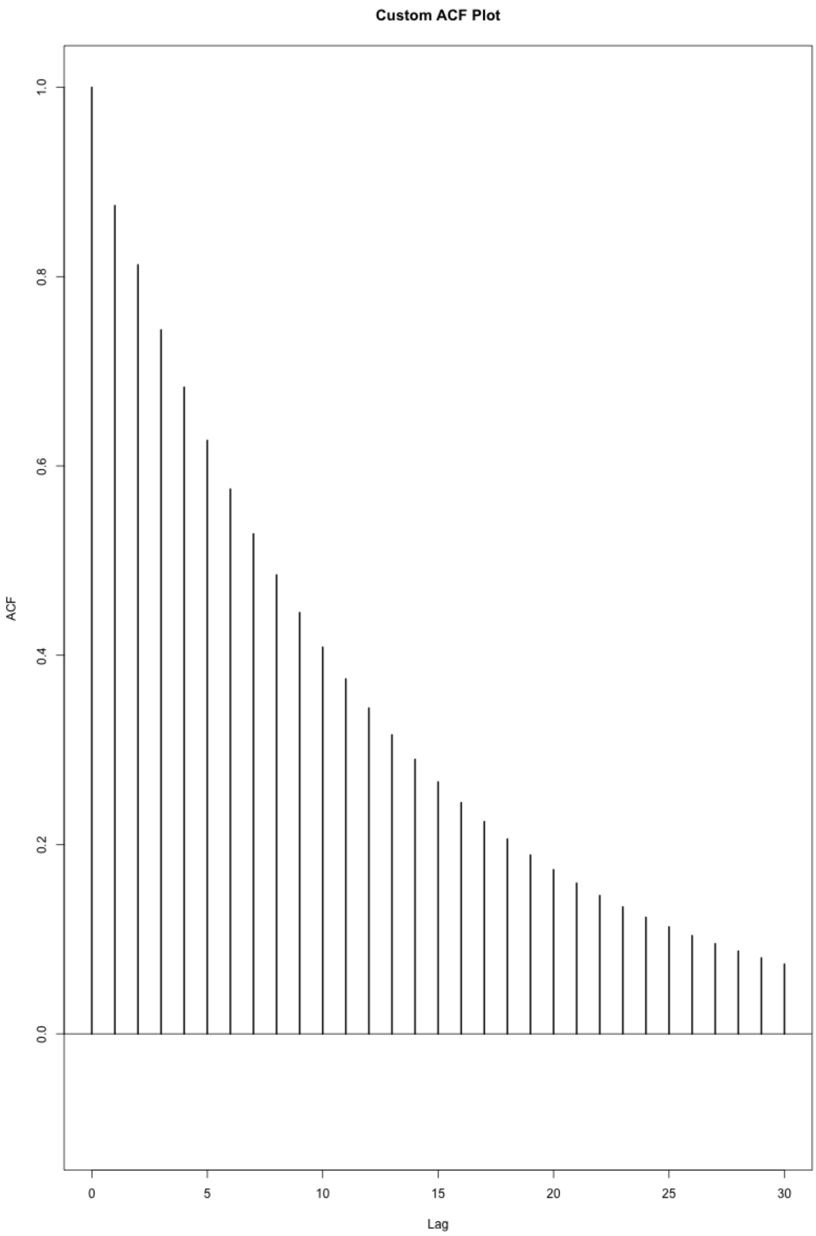

Figure 1: ACF of process with $\phi_1 = -0.7$ *and* $\phi_2 = -0.2$

2 Simulating seasonal processes

In this chapter, various ARIMA models and their corresponding Autocorrolation Function (ACF) and Partial Autocorrolation Function PACF are analysed.
The general ARIMA model is defined as

$$\Phi(B^s)\phi(B)\nabla^d\nabla_s^D Y_t = \Theta(B^s)\theta(B)\varepsilon_t \tag{8}$$

where:

- Y_t is the observed time series at time t.
- B is the backshift operator, where $BY_t = Y_{t-1}$.
- $\nabla = 1 - B$ is the differencing operator used to remove trends.
- $\nabla_s = 1 - B^s$ is the seasonal differencing operator.
- ε_t is white noise.
- $\phi(B)$ and $\theta(B)$ are polynomials for the non-seasonal autoregressive (AR) and moving average (MA) components.
- $\Phi(B^s)$ and $\Theta(B^s)$ are polynomials for the seasonal autoregressive (SAR) and seasonal moving average (SMA) components.

The ARIMA model is written as:

$$\text{ARIMA}(p,d,q) \times (P,D,Q)_s \tag{9}$$

where:

- p = Number of non-seasonal autoregressive (AR) terms.
- d = Number of non-seasonal differencing steps to make the series stationary.
- q = Number of non-seasonal moving average (MA) terms.
- P = Number of seasonal autoregressive (SAR) terms.
- D = Number of seasonal differencing steps.
- Q = Number of seasonal moving average (SMA) terms.
- s = Length of the seasonality period (e.g., $s = 12$ for monthly data with yearly seasonality).

The behavior of the autocorrelation function (ACF) and partial autocorrelation function (PACF) can help identify the structure of a time series. The following table summarizes their expected patterns for different types of models:

Model	ACF Behavior	PACF Behavior
AR(p)	Decays exponentially or oscillates	Cuts off after lag p
MA(q)	Cuts off after lag q	Decays exponentially or oscillates
ARMA(p,q)	Decays exponentially	Decays exponentially

Table 1: ACF and PACF behavior for different ARIMA model types, from Lecture 6

2.1 ARIMA(1,0,0) $\times$ (0,0,0)$_{12}$ with $\phi_1 = 0.6$

The first model to be simulated is a simple non-seasonal AR(1) model. The time series along the ACF and PACF is shown in Figure 2. The ACF shows an exponential decay and the PACF cuts off after the first lag, oscillating insignificantly afterwards.

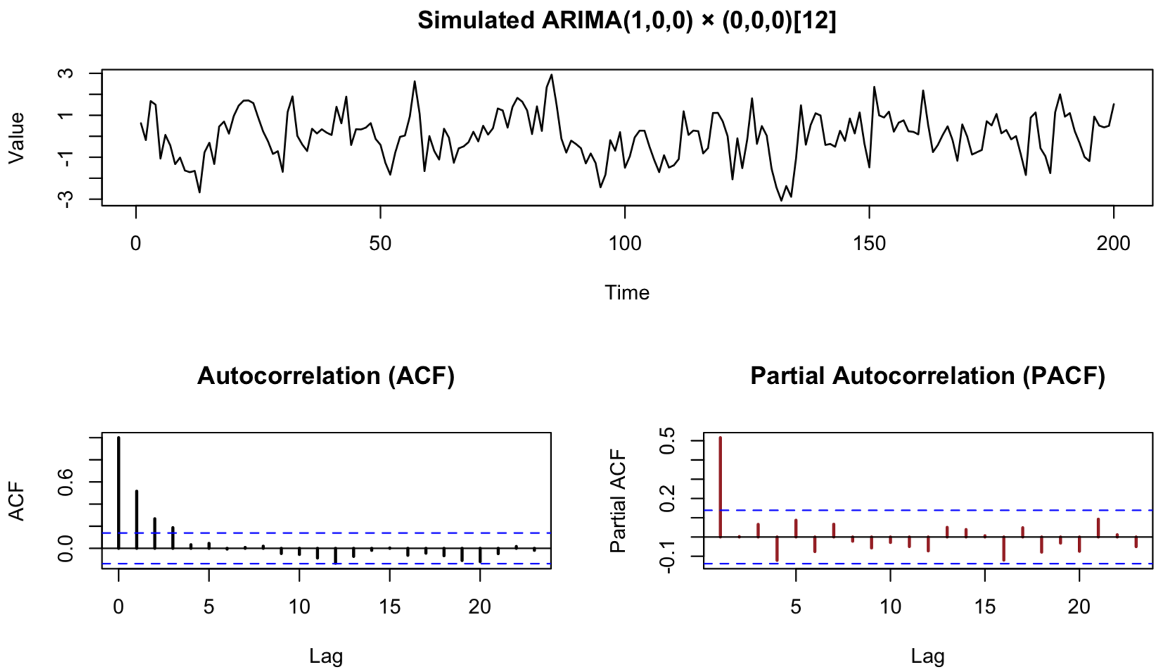

Figure 2: A (1,0,0) $\times$ (0,0,0)$_{12}$ model with the parameter $\phi_1 = 0.6$

2.2 ARIMA(0,0,0) $\times$ (1,0,0)$_{12}$ with $\Phi_1 = -0.9$

The second model is a seasonal AR(1) model, with a negative polynomial for the SAR term and a seasonal period of 12. In comparison to the first model, there is no exponential decay of the ACF, but rather it is cut off after the first leg, implying MA(q) behavior. As for the PACF, the values oscillate within the confidence bound. Also, we cannot see the expected seasonal pattern in the seasonal legs, suggesting that the seasonal AR effect is wek and the model fit might not be ideal.

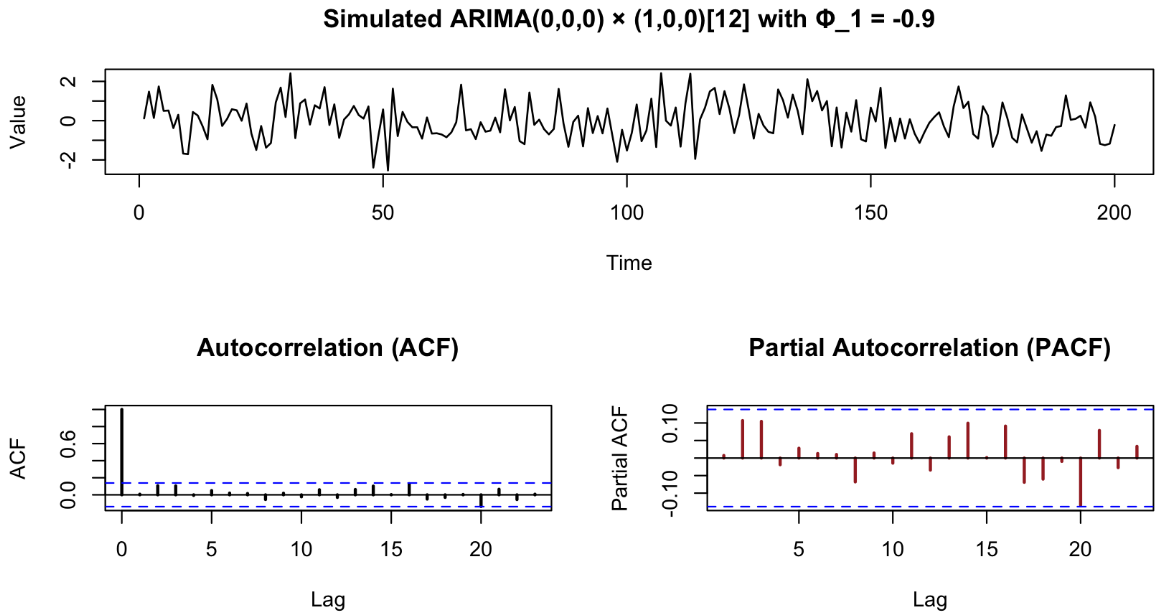

Figure 3: A (0,0,0) $\times$ (1,0,0)$_{12}$ model with the parameter $\Phi_1 = -0.9$.

2.3 ARIMA(1,0,0) $\times$ (0,0,1)$_{12}$ with $\phi_1 = 0.9$ and $\Theta_1 = -0.7$

The third model now includes a non-seasonal AR(1) and a seasonal MA(1) at lag 12. The ACF is decaying slowly due to the strong positive AR coefficient, indicating a strong autocorrelation at short lags. After leg 12 the rate of decay increases. Contrary, the PACF is cut off after the first leg gain. The time series suggests a seasonal pattern.

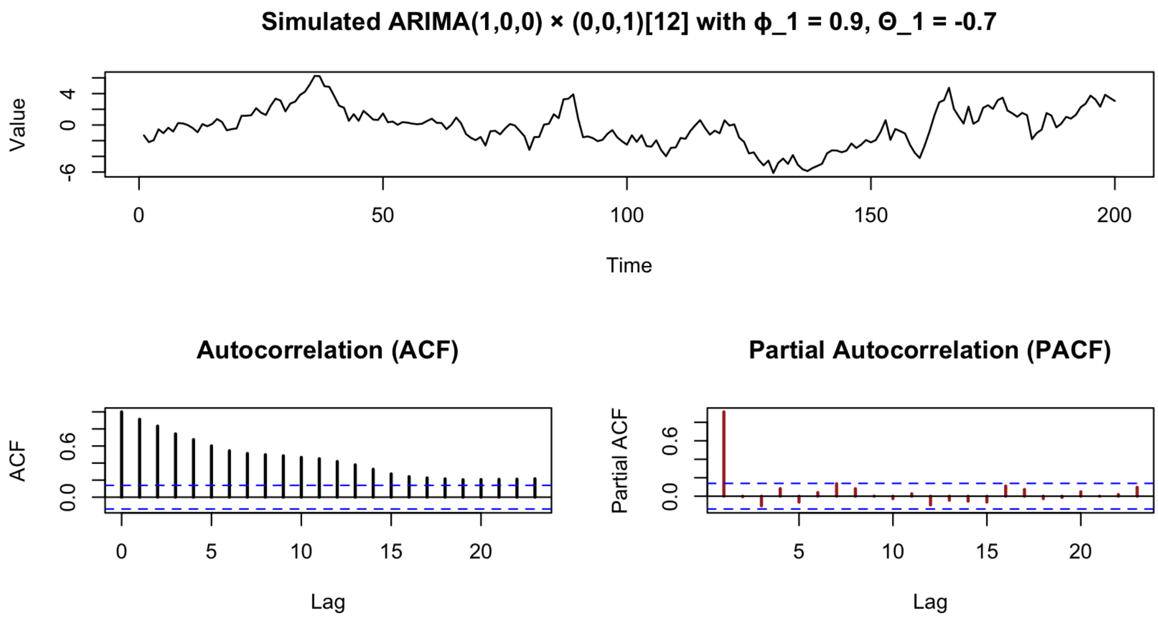

Figure 4: A (1,0,0) $\times$ (0,0,1)$_{12}$ model with the parameters $\phi_1 = 0.9$ and $\Theta_1 = -0.7$.

2.4 ARIMA(1,0,0) $\times$ (1,0,0)$_{12}$ with $\phi_1 = -0.6$ and $\Phi_1 = -0.8$

The next model has both, a seasonal and non-seasonal AR(1) part with negative polynomials for both AR and SAR terms. This leads to oscillating behavior instead of an exponential decay, with spikes due to the seasonal lags in the ACF. As for the PACF, it is cut off after the first lag demonstrating the AR(q) behavior.

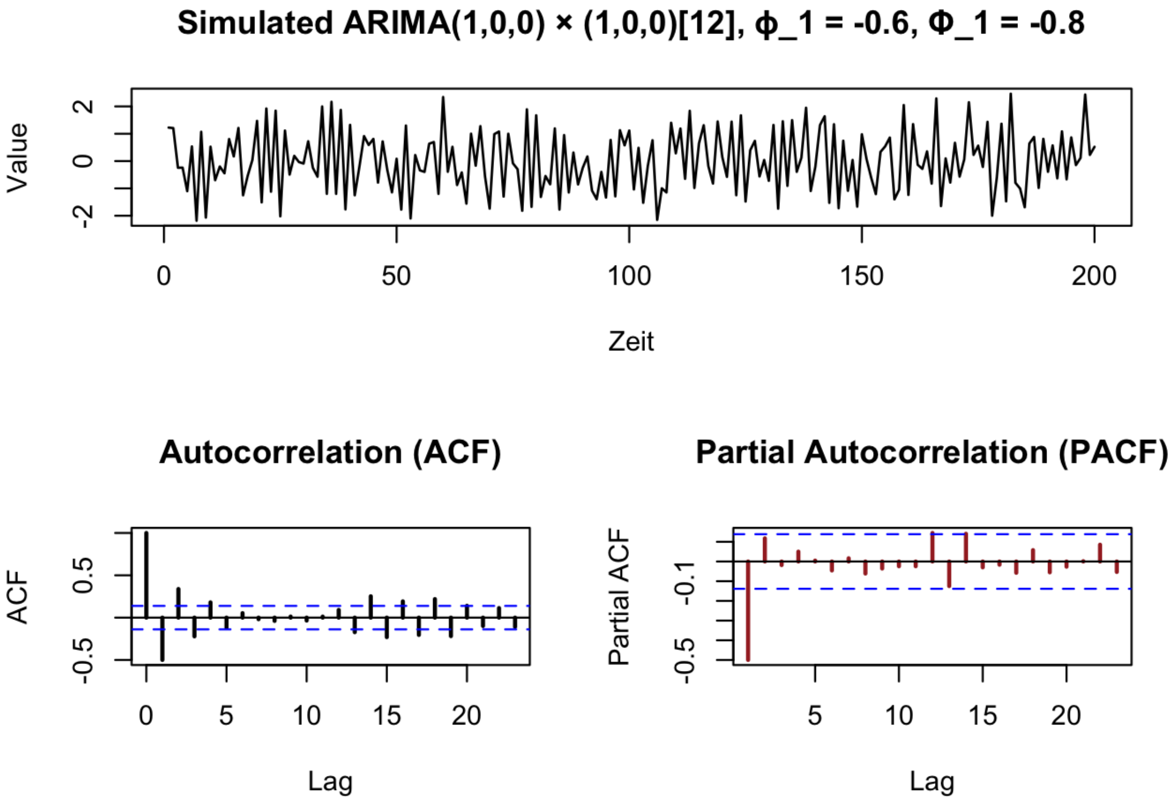

Figure 5: A (1,0,0) $\times$ (1,0,0)$_{12}$ model with the parameters $\phi_1 = -0.6$ and $\Phi_1 = -0.8$.

2.5 ARIMA(0,0,1) $\times$ (0,0,1)$_{12}$ with $\theta_1 = 0.4$ and $\Theta_1 = -0.8$

This model is set together with both, a seasonal and non-seasonal MA(1) part. This is demonstrated in a cut off ACF agter lag 1, whereas the PACF oscillates. Also, the polynominal of the seasonal part is strongly negative, leading to sign-changing seasonal patterns visible in the PACF.

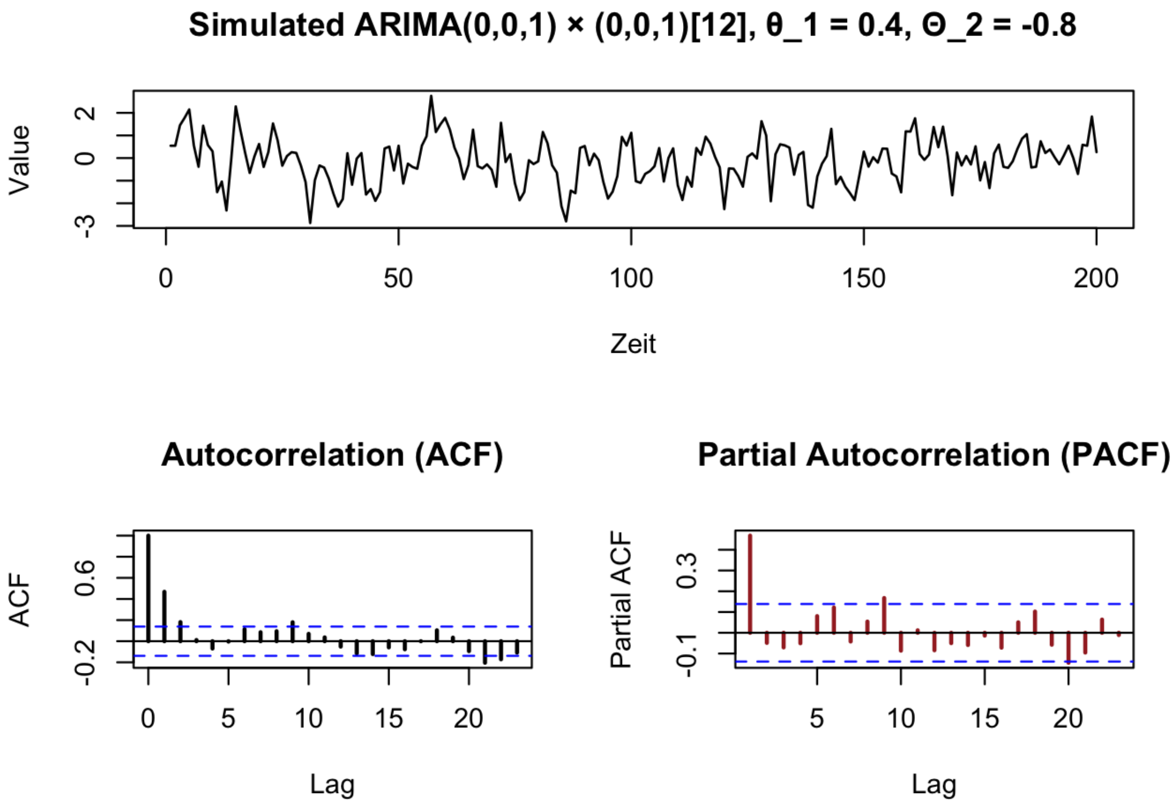

Figure 6: A (0,0,1) $\times$ (0,0,1)$_{12}$ model with the parameters $\theta_1 = 0.4$ and $\Theta_1 = -0.8$.

2.6 ARIMA(0,0,1) $\times$ (1,0,0)$_{12}$ with $\theta_1 = -0.4$ and $\Phi_1 = 0.7$

The next model is combined by a non-seasonal MA(1) and a seasonal AR(1) part with a negative polynomial for the non-seasonal MA(1) part. The ACF is cut off after the first lag, whereas the PACF is oscillating.

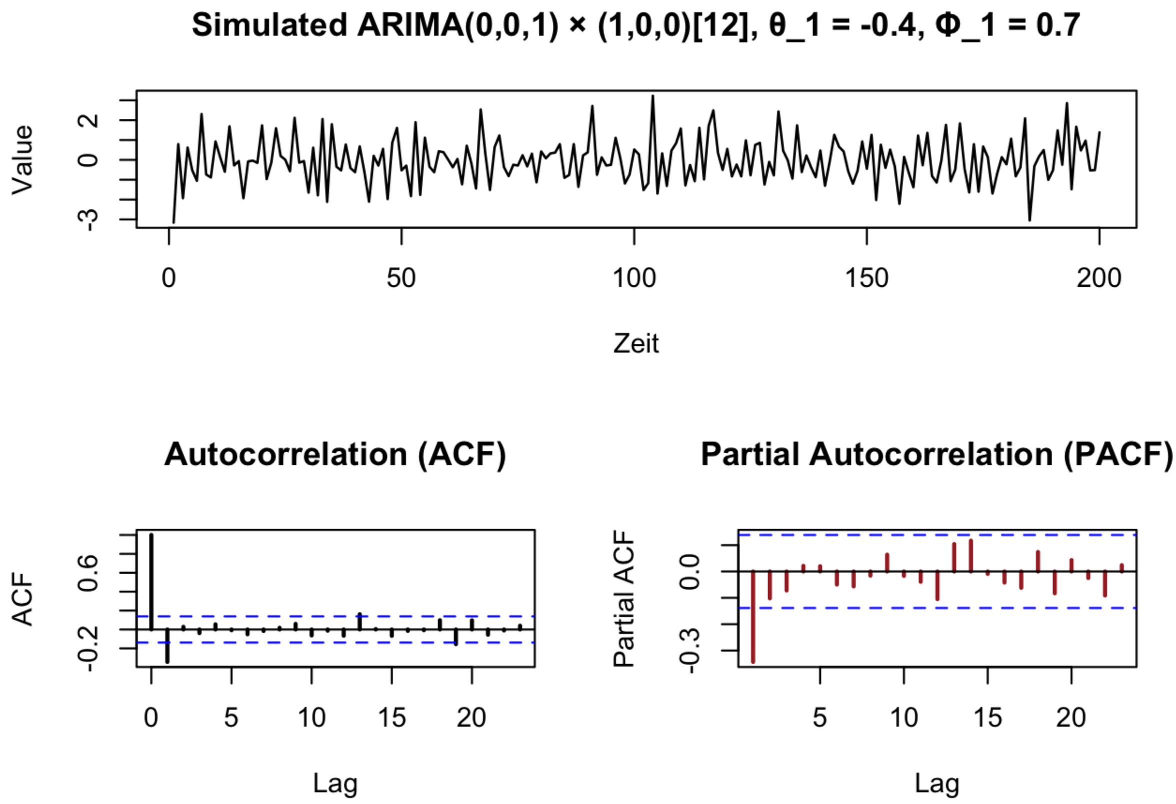

Figure 7: A (0,0,1) $\times$ (1,0,0)$_{12}$ model with the parameters $\theta_1 = -0.4$ *and* $\Phi_1 = 0.7$.

2.7 Summary of Observations and Conclusions on Seasonal Processes

Across the simulated models, the non-seasonal components strongly affected the character of the ACF and PACF. Depending on whether an AR or MA part was present, the autocorrelations either decayed gradually, were cut off after a few lags, or oscillated. AR(1) models typically showed slowly decaying ACFs and PACFs with a cut-off after the first lag, while MA(1) models showed the opposite — a cut-off in the ACF and gradually decaying PACF. Negative coefficients tended to introduce oscillations.

As for the seasonal component, the direction and strength of the seasonal effect was mainly determined by the seasonal AR or MA coefficient, particularly the polynomial Φ_1. Seasonal AR(1) components produced seasonal behavior, resulting in slowly decaying autocorrelations at seasonal lags. In contrast, seasonal MA(1) components created seasonal spikes in the ACF, which were typically followed by a cut-off.

3 Identifying ARMA processes

In this chapter 3 processes are analysed through visual inspection and guesses are made as to the nature of the processes in accordance with ARMA characteristics[1, p.155].

3.1 Process 1

Process 1 looks like white noise. There seems to be no strong features in the ACF nor PACF.

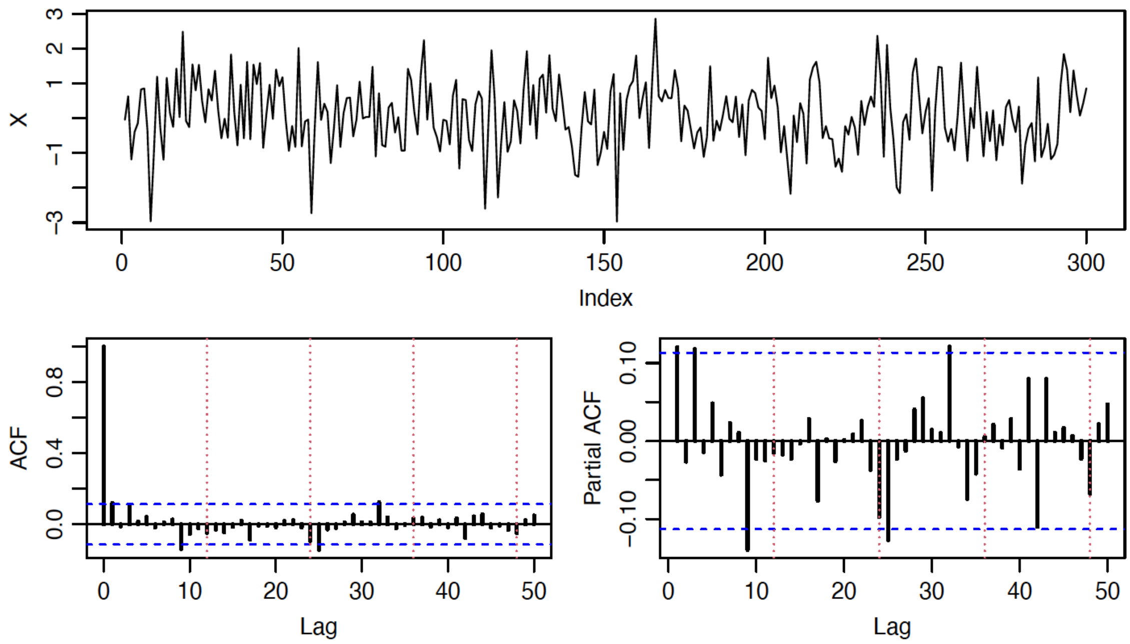

Figure 8: Process 1

3.2 Process 2

Process 2 looks like an AR(2) process. We see a $\rho(k) \approx 0$ for $k > 2$ in the PACF-plot and the damped ACF characteristic of AR-processes.

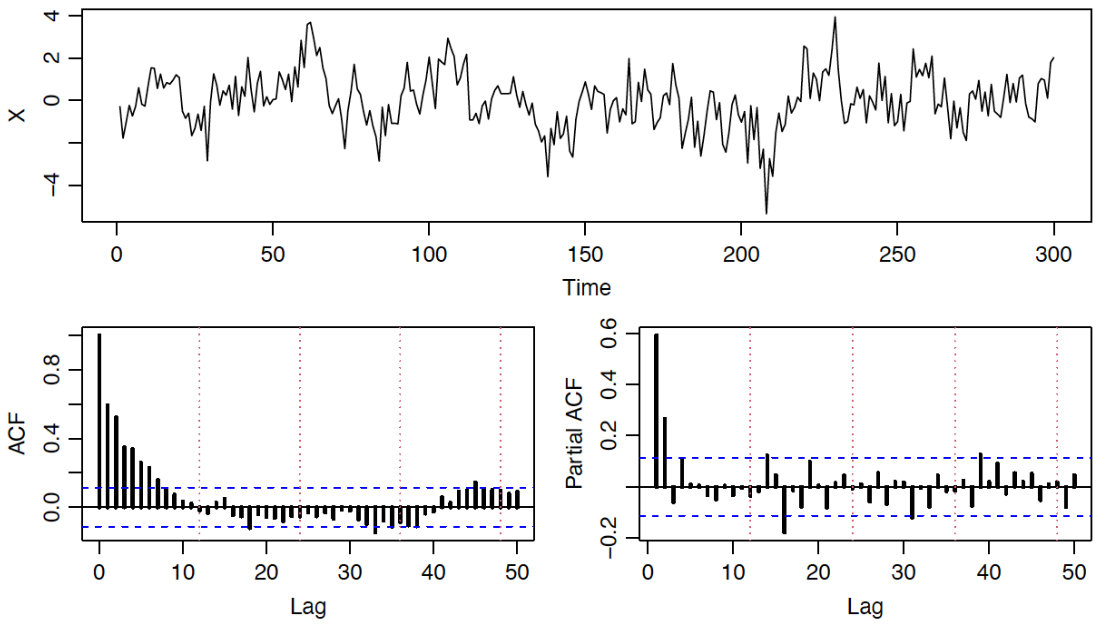

Figure 9: Process 2

3.3 Process 3

Process 3 looks to be a combination (ARMA), since both ACF and PACF look to be decaying exponentially. The sinusoid nature of the PACF seems to suggest a $q = 1$ due to the decaying sinusoidal look. p is hard to determine only through visual inspection due to the interference of AR and MA.

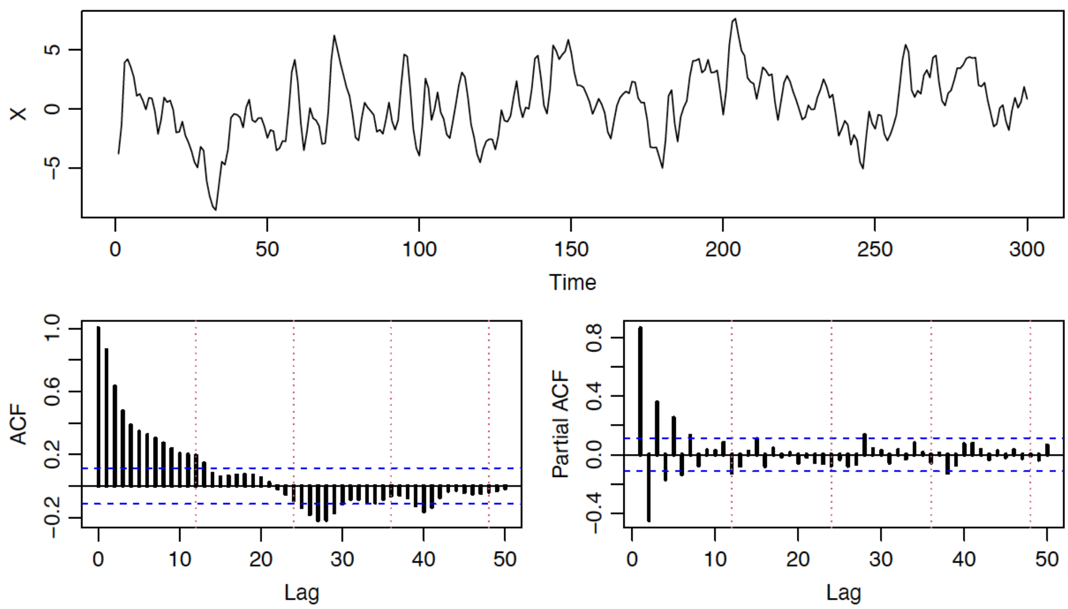

Figure 10: Process 3

4 References

[1] Madsen, H. (2007). *Time series analysis*. Chapman Hall/CRC.
[2] Statistics Denmark. (2025). Statistikbanken [Accessed: February 21, 2025]. https://www.statistikbanken.dk/statbank5a/default.asp?w=1512