T1 $\overline{g_i} = \frac{\partial \overline{r}}{\partial q^i}$, $(\overline{g_i}, \overline{g^j}) = \delta_i^j$, $g_{ij} = (\overline{g_i}, \overline{g_j})$, $g^{ij} = (\overline{g^i}, \overline{g^j})$

$a_i = (\overline{a}, \overline{g_i})$, $a^j = (\overline{a}, \overline{g^j})$

1) $g_{ik} g^{kj} = \sum_\alpha \frac{\partial x_\alpha}{\partial q^i} \cdot \frac{\partial x_\alpha}{\partial q^k} \cdot \sum_\beta \frac{\partial q^k}{\partial x_\beta} \cdot \frac{\partial q^j}{\partial x_\beta} = \sum_\alpha \sum_\beta \left(\frac{\partial x_\alpha}{\partial q^i} \cdot \frac{\partial x_\alpha}{\partial q^k} \cdot \frac{\partial q^k}{\partial x_\beta} \cdot \frac{\partial q^j}{\partial x_\beta} \right) = \sum_\alpha \sum_\beta \left(\frac{\partial x_\alpha}{\partial q^i} \cdot \delta_\beta^\alpha \cdot \frac{\partial q^j}{\partial x_\beta} \right) = \sum_\alpha \frac{\partial x_\alpha}{\partial q^i} \cdot \frac{\partial q^j}{\partial x_\alpha} = \delta_i^j$

2) $\overline{g^j} = g^{ji} \overline{g_i}$

$g^{ji} \overline{g_i} = (\overline{g^j}, \overline{g^i}) \overline{g_i} = \left(\sum_\alpha \frac{\partial q^j}{\partial x_\alpha} \cdot \frac{\partial q^i}{\partial x_\alpha} \right) \cdot \sum_\alpha \frac{\partial x_\alpha}{\partial q^i} \overline{i_\alpha} =$

$= \sum_\alpha \frac{\partial q^j}{\partial x_\alpha} \cdot \overline{i_\alpha} \sum_\alpha \frac{\partial q^i}{\partial x_\alpha} \cdot \frac{\partial x_\alpha}{\partial q^i} = \sum_\alpha \frac{\partial q^j}{\partial x_\alpha} \cdot \overline{i_\alpha} \cdot \delta_i^i = \sum_\alpha \frac{\partial q^j}{\partial x_\alpha} \overline{i_\alpha} = \overline{g^j}$

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T1

3) $\overline{a} = a^i \overline{g}_i = a_i \overline{g}^i$, $\exists\ \overline{a} = \sum_\alpha f_\alpha \overline{i}_\alpha$

$a^i \overline{g}_i = (\overline{a}, \overline{g}^i) \cdot \overline{g}_i = \left(f_1 \frac{\partial q^i}{\partial x_1} + f_2 \frac{\partial q^i}{\partial x_2} + f_3 \frac{\partial q^i}{\partial x^3}\right) \sum_\alpha \frac{\partial x_\alpha}{\partial q^i} \overline{i}_\alpha =$

$= \sum_\alpha f_\alpha \frac{\partial q^i}{\partial x^\alpha} \cdot \sum_\alpha \frac{\partial x_\alpha}{\partial q^i} \overline{i}_\alpha = \sum_\alpha f_\alpha \overline{i}_\alpha \sum_\alpha \frac{\partial x_\alpha}{\partial q^i} \cdot \frac{\partial q^i}{\partial x_\alpha} = \sum f_\alpha \delta^i_i \overline{i}_\alpha =$

$= \overline{a}$

$a_i \cdot \overline{g}^i = (\overline{a}, \overline{g}_i) \cdot \overline{g}^i = \sum_\alpha f_\alpha \frac{\partial x_\alpha}{\partial q^i} \cdot \sum_\alpha \frac{\partial q^i}{\partial x_\alpha} \overline{i}_\alpha = \sum_\alpha f_\alpha \overline{i}_\alpha \sum_\alpha \frac{\partial q^i}{\partial x_\alpha} \cdot \frac{\partial x_\alpha}{\partial q^i}$

$= \sum_\alpha f_\alpha \overline{i}_\alpha \delta^i_i = \overline{a}$

4) $a^i = (\overline{a}, \overline{g}^i) = \sum_\alpha f_\alpha \frac{\partial q^i}{\partial x_\alpha}$

$g^{ij} a_j = (\overline{g}^i, \overline{g}^j) \cdot (\overline{a}, \overline{g}_j) = \sum_\alpha \frac{\partial q^i}{\partial x_\alpha} \cdot \frac{\partial q^j}{\partial x_\alpha} \cdot \sum_\alpha f_\alpha \frac{\partial x_\alpha}{\partial q^j} =$

$= \sum_\alpha f_\alpha \frac{\partial q^i}{\partial x_\alpha} \cdot \sum_\alpha \frac{\partial q^j}{\partial x_\alpha} \cdot \frac{\partial x_\alpha}{\partial q^j} = \sum_\alpha f_\alpha \frac{\partial q^i}{\partial x_\alpha} \delta^j_j = \sum_\alpha f_\alpha \frac{\partial q^i}{\partial x_\alpha}$

T2 $x = \rho \sin\theta \cos\varphi$, $y = \rho \sin\theta \sin\varphi$, $z = \rho \cos\theta$

$$\overline{g}_\rho = \begin{pmatrix} \frac{\partial x}{\partial \rho} \\ \frac{\partial y}{\partial \rho} \\ \frac{\partial z}{\partial \rho} \end{pmatrix} = \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix}; \quad \overline{g}_\theta = \begin{pmatrix} \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \rho\cos\theta\cos\varphi \\ \rho\cos\theta\sin\varphi \\ -\rho\sin\theta \end{pmatrix}$$

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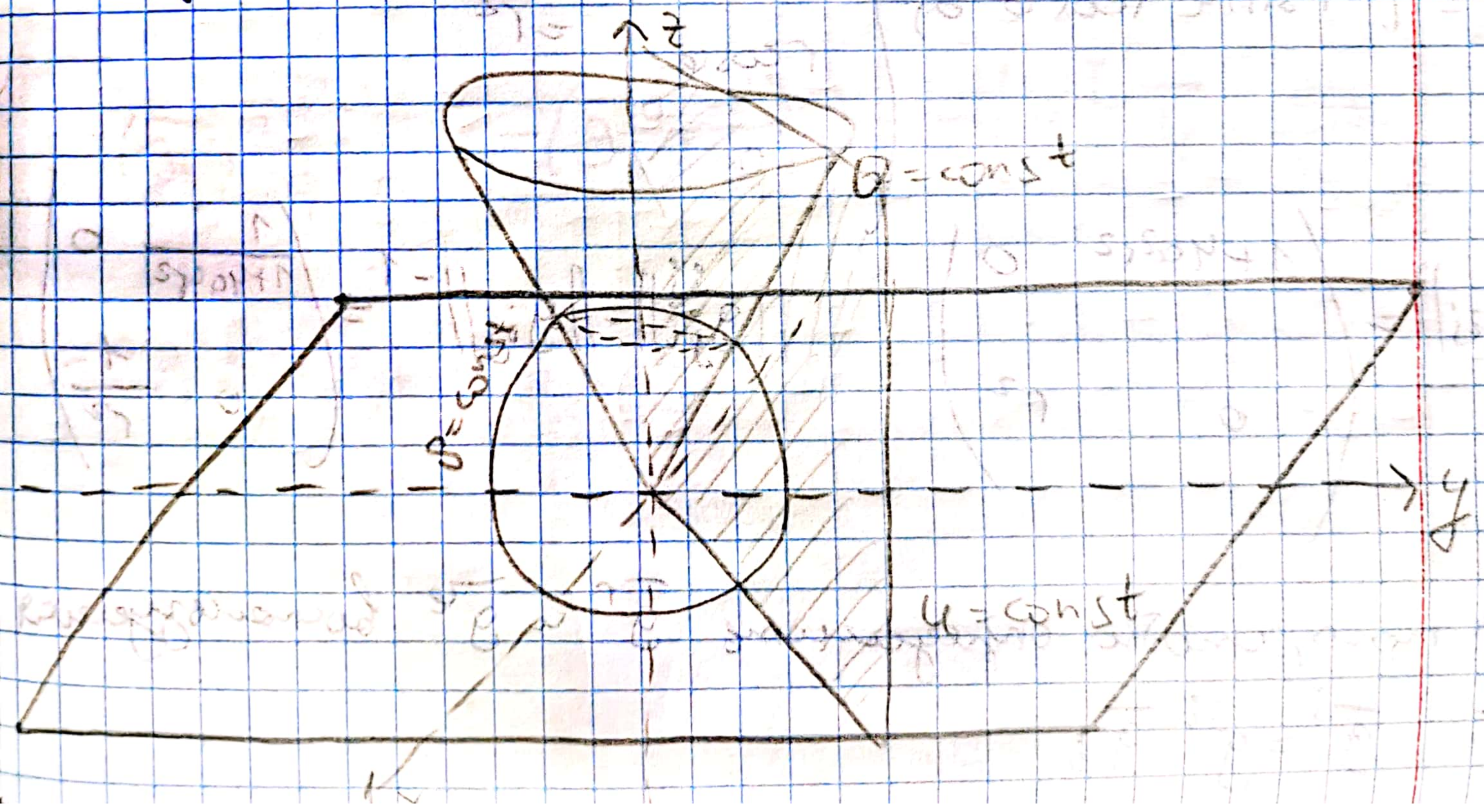
$$\overline{g_\varphi} = \begin{pmatrix} \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} -\rho \sin\theta \sin\varphi \\ \rho \sin\theta \cos\varphi \\ 0 \end{pmatrix}$$

нормировка $\frac{\overline{g_i}}{|\overline{g_i}|} = \overline{e_i}$

$|\overline{g_\rho}| = \rho \qquad |\overline{g_\theta}| = \rho \qquad |\overline{g_\varphi}| = \rho \sin\theta$

$$A = \begin{pmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\varphi\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$



T3 $x = r\cos\varphi$, $y = r\sin\varphi$, $z = a(x^2+y^2) = ar^2$

$$\overline{g_r} = \begin{pmatrix} \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial r} \\ \frac{\partial z}{\partial r} \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 2ar \end{pmatrix}, \quad \overline{g_\varphi} = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix}$$

$$g_{\varphi r} = (\cos\varphi \ \sin\varphi \ 2ar)\begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix} = 0 = g_{r\varphi}$$

$$g_{rr} = (\cos\varphi \ \sin\varphi \ 2ar)\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 2ar \end{pmatrix} = 1 + 4a^2r^2$$

$$g_{\varphi\varphi} = (-r\sin\varphi \ r\cos\varphi \ 0)\begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix} = r^2$$

$$\|g_{ij}\| = \begin{pmatrix} 1+4a^2r^2 & 0 \\ 0 & r^2 \end{pmatrix} \quad \|g^{ij}\| = \|g_{ij}\|^{-1} = \begin{pmatrix} \frac{1}{1+4a^2r^2} & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$$

Для того, чтобы определить $\overline{g}^r$ и $\overline{g}^\varphi$ воспользуемся
T1: $\overline{g}^j = g^{ji}\,\overline{g}_i$

$$\overline{g}^r = g^{rr}\overline{g}_r = \begin{pmatrix} \cos\varphi(1+4a^2r^2) \\ \sin\varphi(1+4a^2r^2) \\ 2ar(1+4a^2r^2) \end{pmatrix}$$

$$\overline{g}^\varphi = g^{\varphi\varphi}\overline{g}_\varphi = \begin{pmatrix} -r^3\sin\varphi \\ r^3\cos\varphi \\ 0 \end{pmatrix}$$

T4 $\Gamma_{ijk} = \left(\overline{g}_i, \dfrac{\partial \overline{g}_j}{\partial q^k}\right)$

1) Заметим, что $\dfrac{\partial \delta_j^i}{\partial q^k} = 0 = \dfrac{\partial(\overline{g}_i, \overline{g}^j)}{\partial q^k} =$

$= \left(\overline{g}_i, \dfrac{\partial \overline{g}^j}{\partial q^k}\right) + \left(\overline{g}^j, \dfrac{\partial \overline{g}_i}{\partial q^k}\right) \Rightarrow$

$\left(\overline{g}_i, \dfrac{\partial \overline{g}^j}{\partial q^k}\right) = -\left(\overline{g}^j, \dfrac{\partial \overline{g}_i}{\partial q^k}\right) = -\left(g^{jp}\overline{g}_p, \dfrac{\partial \overline{g}_i}{\partial q^k}\right) =$

$= -g^{jp}\Gamma_{pik}$