12.13 $\exists$ u - дифференцируемое поле, $f(t)$ - дифференцируемая функция, $t \in \mathbb{R}$. Доказать: $\operatorname{grad} f(u) = f'(u) \operatorname{grad} u$

Введем ДСК. $u = u(x, y, z)$

$f(u) = f(u(x, y, z))$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y}, \quad \frac{\partial f}{\partial z} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial z}$$

$$\operatorname{grad} f(u) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \frac{\partial f}{\partial u} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} = f'(u) \operatorname{grad} u$$

12.15 $\bar{a}, \bar{b}$ = const векторы, $\bar{r} = \bar{i}x + \bar{j}y + \bar{k}z$

Найти $\operatorname{grad} u$, если

1) $u = r = \sqrt{x^2+y^2+z^2}$

$$\operatorname{grad} u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{2x}{2\sqrt{x^2+y^2+z^2}} \\ \frac{2y}{2\sqrt{x^2+y^2+z^2}} \\ \frac{2z}{2\sqrt{x^2+y^2+z^2}} \end{pmatrix} = \begin{pmatrix} \frac{x}{r} \\ \frac{y}{r} \\ \frac{z}{r} \end{pmatrix}$$

3) $u = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}$

$$\operatorname{grad} u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{-2x}{2(x^2+y^2+z^2)^{3/2}} \\ \frac{-2y}{2(x^2+y^2+z^2)^{3/2}} \\ \frac{-2z}{2(x^2+y^2+z^2)^{3/2}} \end{pmatrix} = \begin{pmatrix} -\frac{x}{r^3} \\ -\frac{y}{r^3} \\ -\frac{z}{r^3} \end{pmatrix}$$

5) $u = (\bar{a}, \bar{r}) = a_1 x + a_2 y + a_3 z$

$$\operatorname{grad} u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \bar{a}$$

8) $u = |[\bar{a} \times \bar{r}]| = ([\bar{a} \times \bar{r}], [\bar{a} \times \bar{r}])$

$$\mathrm{grad}\,u = \nabla\left([\bar a \times \overset{\downarrow}{\bar r}], [\bar a \times \bar r]\right) + \nabla\left([\bar a \times \bar r], [\bar a \times \overset{\downarrow}{\bar r}]\right) =$$

$$= 2\nabla\left(\underbrace{[\bar a \times \bar r]}_{= c}, [\bar a \times \overset{\downarrow}{\bar r}]\right) = 2\nabla\left(\bar c, [\bar a \times \overset{\downarrow}{\vec r}]\right) \underset{a = const}{=\!=}$$

$$= 2\nabla\left(\bar c, [\bar a \times \overset{\downarrow}{\bar r}]\right) = 2\nabla\left(\overset{\downarrow}{\bar r}, \bar c, \bar a\right) = 2\nabla\left(\overset{\downarrow}{\bar r}, [\bar c \times \bar a]\right) =$$

$$\overset{(5)}{=} 2\left[[\bar a \times \bar r] \times \bar a\right] = -2[\bar a \times [\bar a \times \bar r]] = -2\bar a(\bar a, \bar r) +$$

$$+ 2\bar r(\bar a, \bar a)$$

12.40 2) найти $\mathrm{div}(u\,\mathrm{grad}\,v) = (\nabla, u\nabla v) =$

$$= (\nabla, \overset{\downarrow}{u}\nabla v) + (\nabla, u\overset{\downarrow}{\nabla v}) = \nabla u \nabla v + u\,\mathrm{div}\,\nabla v =$$

$$= (\mathrm{grad}\,u, \mathrm{grad}\,v) + u\,\mathrm{div}\,\mathrm{grad}\,v$$

12.42 2) решить уравнение ($\bar r = x\bar i + y\bar j + z\bar k$, $r = |\bar r|$)

$$\mathrm{div}\,\mathrm{grad}\,u(r) = 0$$

$$\left.\begin{aligned} &\mathrm{grad}\,u(r) \overset{12.13}{=\!=} u'\,\mathrm{grad}\,r \\ &\mathrm{grad}\,r \overset{12.15(1)}{=\!=} \begin{pmatrix} \frac{x}{r} \\ \frac{y}{r} \\ \frac{z}{r} \end{pmatrix} \end{aligned}\right\} \Rightarrow \mathrm{grad}\,u(r) = \frac{u'}{r}\bar r$$

$$\operatorname{div}\left(\frac{u'}{r}\bar r\right)=\left(\nabla,\frac{u'}{r}\bar r\right)=\left(\nabla,\overset{\downarrow}{\frac{u'}{r}}\bar r\right)+\left(\nabla,\frac{u'}{r}\overset{\downarrow}{\bar r}\right)=$$

$$=\left(\nabla\frac{u'}{r},\bar r\right)+\frac{u'}{r}\left(\nabla,\overset{\downarrow}{\bar r}\right)=\left(\operatorname{grad}\frac{u'}{r},\bar r\right)+\frac{u'}{r}\operatorname{div}\bar r$$

$$\operatorname{div}\bar r=\frac{\partial}{\partial x}x+\frac{\partial}{\partial y}y+\frac{\partial}{\partial z}z=3$$

$$\operatorname{grad}\frac{u'}{r}\overset{12.13}{=\!=}\left(\frac{u'}{r}\right)'_r\operatorname{grad}r=\frac{u''r-u'}{r^2}\cdot\frac{\bar r}{r}=$$

$$=\frac{u''r-u'}{r^3}\bar r$$

$$\operatorname{div}\operatorname{grad}u(r)=\operatorname{div}\left(\frac{u'}{r}\bar r\right)=\left(\frac{u''r-u'}{r^3}\bar r,\bar r\right)+\frac{u'}{r}\cdot 3=$$

$$=\frac{u''r+2u'}{r}=0\Rightarrow u''r+2u'=0$$

$$\exists\, V=u' \qquad V'r+2V=0 \qquad V'=-\frac{2V}{r}\Rightarrow V=\frac{\tilde C}{r^2}$$

$$\Rightarrow u=\frac{C_1}{r}+C_2\ ,\quad C_1,C_2\in\mathbb{R}$$

12.38 $\bar r=\begin{pmatrix}x\\y\\z\end{pmatrix}$ найти $\operatorname{div}\bar a$

1) $\bar a=\bar r$

$$\operatorname{div}\bar r=\frac{\partial}{\partial x}x+\frac{\partial}{\partial y}y+\frac{\partial}{\partial z}z=3$$

3) $\bar a=\frac{\bar r}{r}=\frac{1}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix}x\\y\\z\end{pmatrix}$

$$\operatorname{div}\bar{a} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) +$$

$$+ \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \left(\frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{2x^2}{2(x^2+y^2+z^2)^{3/2}}\right) +$$

$$+ \left(\frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{2y^2}{2(x^2+y^2+z^2)^{3/2}}\right) + \left(\frac{1}{\sqrt{x^2+y^2+z^2}} -\right.$$

$$\left. - \frac{2z^2}{2(x^2+y^2+z^2)^{3/2}}\right) = \frac{3}{r} - \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}} = \frac{3}{r} - \frac{1}{r} =$$

$$= \frac{2}{r}$$

12.49 проверить равенства, используя оператор ∇ и правила действия с ним. α, β - числа $u, \bar{a}, \bar{b}$ - дифференцируемые скалярное и векторные поля

3) $\operatorname{rot}(u\bar{a}) = u \operatorname{rot} \bar{a} + [\operatorname{grad} u \times \bar{a}]$

$$\operatorname{rot}(u\bar{a}) = [\nabla \times u\bar{a}] = [\nabla \times \overset{\downarrow}{u}\bar{a}] + [\nabla \times u\overset{\downarrow}{\bar{a}}] =$$

$$= u[\nabla \times \bar{a}] + [\nabla \overset{\downarrow}{u} \times \bar{a}] = u \operatorname{rot} \bar{a} + [\operatorname{grad} u \times \bar{a}]$$

б) $\operatorname{div}[\bar{a} \times \bar{b}] = (\bar{b}, \operatorname{rot} \bar{a}) - (\bar{a}, \operatorname{rot} \bar{b})$

$$\operatorname{div}[\bar{a} \times \bar{b}] = (\nabla, [\bar{a} \times \bar{b}]) = (\nabla, [\overset{\downarrow}{\bar{a}} \times \bar{b}]) + (\nabla, [\bar{a} \times \overset{\downarrow}{\bar{b}}]) =$$

$$= (\nabla, \overset{\downarrow}{\bar{a}}, \bar{b}) - (\nabla, \overset{\downarrow}{\bar{b}}, \bar{a}) = (\bar{b}, \nabla, \bar{a}) - (\bar{a}, \nabla, \overset{\downarrow}{\bar{b}}) =$$

$$= (\bar{b}, [\nabla \times \bar{a}]) - (\bar{a}, [\nabla \times \bar{b}]) = (\bar{b}, \operatorname{rot} \bar{a}) - (\bar{a}, \operatorname{rot} \bar{b})$$

T30 Для гладкого векторного поля $\bar{V}$ на $\mathbb{R}^3$ доказать

а) если $\operatorname{rot} \bar{V} = 0$ на всём $\mathbb{R}^3$, то $\bar{V} = \operatorname{grad} f$ для некоторой функции f

б) если $\operatorname{div} \bar{V} = 0$ на всём $\mathbb{R}^3$, то $\bar{V} = \operatorname{rot} \bar{a}$ для некоторого векторного поля $\vec{v}$

a) $\overline{V} = \begin{pmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{pmatrix}$ $\quad rot\,\overline{V} = \begin{pmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\forall$ гладкая замкнутая кривая L — (по теореме Жордана) край некоторой ограниченной области S

По теореме Стокса

$$\int_L \overline{V} = \int_S d\overline{v} \Rightarrow \int_L P\,dx + Q\,dy + R\,dz =$$

$$= \int_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial z}\right) dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy =$$

$= 0 \Leftrightarrow$ поле $\overline{V}$ потенциально, т.е. $\exists f$:

$\overline{V} = grad f$

б) $\operatorname{div} \overline{V} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$

Рассмотрим форму $\omega = P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy$

$d\omega = \operatorname{div} \overline{V}\, dx \wedge dy \wedge dz = 0 \Rightarrow \omega$ - замкнута $\Rightarrow$

(т.к. $\mathbb{R}^3$ выпукло, т.е. $\mathbb{R}^3$ стягиваемое) по лемме Пуанкаре, ω точна $\Rightarrow \exists \alpha : d\alpha = \omega$

$\exists \alpha = A\, dx + B\, dy + C\, dz$

$$d\alpha = \frac{\partial A}{\partial y} dy \wedge dx + \frac{\partial A}{\partial z} dz \wedge dx + \frac{\partial B}{\partial x} dx \wedge dy + \frac{\partial B}{\partial z} dz \wedge dy + \frac{\partial C}{\partial x} dx \wedge dz + \frac{\partial C}{\partial y} dy \wedge dz =$$

$$= \left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) dy \wedge dx + \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dz \wedge dx = \omega$$

$$\begin{pmatrix} P \\ Q \\ R \end{pmatrix} = \begin{pmatrix} \frac{\partial C}{\partial y} - \frac{\partial B}{\partial z} \\ \frac{\partial A}{\partial z} - \frac{\partial B}{\partial x} \\ \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \end{pmatrix} = \operatorname{rot} \alpha$$

12.50 $\bar r = x\bar i + y\bar j + z\bar k$, $u(r)$ - диф-мое поле

найти $\mathrm{rot}\,(u(r)\bar r) = \begin{vmatrix} \bar i & \bar j & \bar k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u(r)x & u(r)y & u(r)z \end{vmatrix} =$

$$= \bar i\left(\frac{\partial u}{\partial x}y - \frac{\partial u}{\partial y}x\right) - \bar j\left(\frac{\partial u}{\partial x}z - \frac{\partial u}{\partial z}x\right) +$$

$$+\bar k\left(\frac{\partial u}{\partial y}z - \frac{\partial u}{\partial z}y\right) = \frac{\partial u}{\partial r}\begin{pmatrix} \frac{\partial r}{\partial x}y - \frac{\partial r}{\partial y}x \\ \frac{\partial r}{\partial x}z - \frac{\partial r}{\partial z}x \\ \frac{\partial r}{\partial y}z - \frac{\partial r}{\partial z}y \end{pmatrix} =$$

$$= \frac{\partial u}{\partial r}\begin{pmatrix} \frac{xy}{\sqrt{x^2+y^2+z^2}} - \frac{yx}{\sqrt{x^2+y^2+z^2}} \\ \frac{xz}{\sqrt{x^2+y^2+z^2}} - \frac{zx}{\sqrt{x^2+y^2+z^2}} \\ \frac{yz}{\sqrt{x^2+y^2+z^2}} - \frac{zy}{\sqrt{x^2+y^2+z^2}} \end{pmatrix} = \bar 0$$