12.13 $\exists\, u$ - дифференцируемое поле, $f(t)$ - дифференцируемая функция, $t \in \mathbb{R}$. Доказать: $\operatorname{grad} f(u) = f'(u) \operatorname{grad} u$

Введем ДСК. $u = u(x,y,z)$

$f(u) = f(u(x,y,z))$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y}, \quad \frac{\partial f}{\partial z} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial z}$$

$$\operatorname{grad} f(u) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \frac{\partial f}{\partial u} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} = f'(u) \operatorname{grad} u$$

12.15 $\bar{a}, \bar{b}$ = const векторы, $\bar{r} = \bar{i}x + \bar{j}y + \bar{k}z$

Найти $\operatorname{grad} u$, если

1) $u = r = \sqrt{x^2+y^2+z^2}$

$$\operatorname{grad} u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{2x}{2\sqrt{x^2+y^2+z^2}} \\ \frac{2y}{2\sqrt{x^2+y^2+z^2}} \\ \frac{2z}{2\sqrt{x^2+y^2+z^2}} \end{pmatrix} = \begin{pmatrix} \frac{x}{r} \\ \frac{y}{r} \\ \frac{z}{r} \end{pmatrix}$$

3) $u = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}$

$$\operatorname{grad} u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{-2x}{2(x^2+y^2+z^2)^{3/2}} \\ \frac{-2y}{2(x^2+y^2+z^2)^{3/2}} \\ \frac{-2z}{2(x^2+y^2+z^2)^{3/2}} \end{pmatrix} = \begin{pmatrix} -\frac{x}{r^3} \\ -\frac{y}{r^3} \\ -\frac{z}{r^3} \end{pmatrix}$$

5) $u = (\bar{a}, \bar{r}) = a_1 x + a_2 y + a_3 z$

$$\operatorname{grad} u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \bar{a}$$

8) $u = |[\bar{a} \times \bar{r}]| = ([\bar{a} \times \bar{r}], [\bar{a} \times \bar{r}])$

$$\operatorname{grad} u = \nabla\left([\bar a \times \overset{\downarrow}{\bar r}], [\bar a \times \bar r]\right) + \nabla\left([\bar a \times \bar r], [\bar a \overset{\downarrow}{\times} \bar r]\right) =$$

$$= 2\nabla\left(\underbrace{[\bar a \times \bar r]}_{\bar c}, [\bar a \overset{\downarrow}{\times} \bar r]\right) = 2\nabla\left(\bar c, [\bar a \times \overset{\downarrow}{\bar r}]\right) \underset{a = const}{=\!=}$$

$$= 2\nabla\left(\bar c, [\bar a \times \overset{\downarrow}{\bar r}]\right) = 2\nabla\left(\overset{\downarrow}{\bar r}, \bar c, \bar a\right) = 2\nabla\left(\overset{\downarrow}{\bar r}, [\bar c \times \bar a]\right) =$$

$$\overset{(5)}{=} 2\left[[\bar a \times \bar r] \times \bar a\right] = -2\left[\bar a \times [\bar a \times \bar r]\right] = -2\bar a(\bar a, \bar r) +$$

$$+ 2\bar r(\bar a, \bar a)$$

12.40 2) найти $\operatorname{div}(u \operatorname{grad} v) = (\nabla, u\nabla v) =$

$$= (\nabla, \overset{\downarrow}{u}\nabla v) + (\nabla, u\overset{\downarrow}{\nabla v}) = \nabla u \nabla v + u \operatorname{div} \nabla v =$$

$$= (\operatorname{grad} u, \operatorname{grad} v) + u \operatorname{div} \operatorname{grad} v$$

12.42 2) решить уравнение ($\bar r = x\bar i + y\bar j + z\bar k$, $r = |\bar r|$)

$$\operatorname{div} \operatorname{grad} u(r) = 0$$

$$\left.\begin{array}{l} \operatorname{grad} u(r) \overset{12.13}{=\!=} u' \operatorname{grad} r \\ \operatorname{grad} r \overset{12.15(1)}{=\!=} \begin{pmatrix} \frac{x}{r} \\ \frac{y}{r} \\ \frac{z}{r} \end{pmatrix} \end{array}\right\} \Rightarrow \operatorname{grad} u(r) = \frac{u'}{r}\bar r$$

$$\operatorname{div}\left(\frac{u'}{r}\bar{r}\right) = \left(\nabla, \frac{u'}{r}\bar{r}\right) = \left(\nabla, \frac{\overset{\downarrow}{u'}}{r}\bar{r}\right) + \left(\nabla, \frac{u'}{r}\overset{\downarrow}{\bar{r}}\right) =$$

$$= \left(\nabla \frac{u'}{r}, \bar{r}\right) + \frac{u'}{r}\left(\nabla, \overset{\downarrow}{\bar{r}}\right) = \left(\operatorname{grad}\frac{u'}{r}, \bar{r}\right) + \frac{u'}{r}\operatorname{div}\bar{r}$$

$$\operatorname{div}\bar{r} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3$$

$$\operatorname{grad}\frac{u'}{r} \overset{12.13}{=} \left(\frac{u'}{r}\right)'_r \operatorname{grad} r = \frac{u''r - u'}{r^2}\cdot\frac{\bar{r}}{r} =$$

$$= \frac{u''r - u'}{r^3}\bar{r}$$

$$\operatorname{div}\operatorname{grad} u(r) = \operatorname{div}\left(\frac{u'}{r}\bar{r}\right) = \left(\frac{u''r - u'}{r^3}\bar{r}, \bar{r}\right) + \frac{u'}{r}\cdot 3 =$$

$$= \frac{u''r + 2u'}{r} = 0 \Rightarrow u''r + 2u' = 0$$

$$\exists V = u' \qquad V'r + 2V = 0 \qquad V' = -\frac{2V}{r} \Rightarrow V = \frac{\tilde{C}}{r^2}$$

$$\Rightarrow u = \frac{C_1}{r} + C_2 \quad , \quad C_1, C_2 \in \mathbb{R}$$

12.38 $\bar{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ найти $\operatorname{div}\bar{a}$

1) $\bar{a} = \bar{r}$

$$\operatorname{div}\bar{r} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3$$

3) $\bar{a} = \frac{\bar{r}}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

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$$\operatorname{div}\bar{a}=\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)+\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)+$$

$$+\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)=\left(\frac{1}{\sqrt{x^2+y^2+z^2}}-\frac{2x^2}{2(x^2+y^2+z^2)^{3/2}}\right)+$$

$$+\left(\frac{1}{\sqrt{x^2+y^2+z^2}}-\frac{2y^2}{2(x^2+y^2+z^2)^{3/2}}\right)+\left(\frac{1}{\sqrt{x^2+y^2+z^2}}-\right.$$

$$\left.-\frac{2z^2}{2(x^2+y^2+z^2)^{3/2}}\right)=\frac{3}{r}-\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}}=\frac{3}{r}-\frac{1}{r}=$$

$$=\frac{2}{r}$$

12.49 проверить равенства, используя оператор ∇ и правила действия с ним. α, β - числа $u, \bar{a}, \bar{b}$ - дифференцируемые скалярное и векторные поля

3) $\operatorname{rot}(u\bar{a})=u\operatorname{rot}\bar{a}+[\operatorname{grad}u\times\bar{a}]$

$$\operatorname{rot}(u\bar{a})=[\nabla\times u\bar{a}]=[\nabla\times\overset{\downarrow}{u}\bar{a}]+[\nabla\times u\overset{\downarrow}{\bar{a}}]=$$

$= u[\nabla \times \bar{a}] + [\nabla \overset{\downarrow}{u} \times \bar{a}] = u \operatorname{rot} \bar{a} + [\operatorname{grad} u \times \bar{a}]$

6) $\operatorname{div}[\bar{a} \times \bar{b}] = (\bar{b}, \operatorname{rot} \bar{a}) - (\bar{a}, \operatorname{rot} \bar{b})$

$\operatorname{div}[\bar{a} \times \bar{b}] = (\nabla, [\bar{a} \times \bar{b}]) = (\nabla, [\overset{\downarrow}{\bar{a}} \times \bar{b}]) + (\nabla, [\bar{a} \times \overset{\downarrow}{\bar{b}}]) =$

$= (\nabla, \overset{\downarrow}{\bar{a}}, \bar{b}) - (\nabla, \overset{\downarrow}{\bar{b}}, \bar{a}) = (\bar{b}, \nabla, \bar{a}) - (\bar{a}, \nabla, \overset{\downarrow}{\bar{b}}) =$

$= (\bar{b}, [\nabla \times \bar{a}]) - (\bar{a}, [\nabla \times \bar{b}]) = (\bar{b}, \operatorname{rot} \bar{a}) - (\bar{a}, \operatorname{rot} \bar{b})$

12.50 5) $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, $r = |\bar{r}|$, $\bar{a}$ и $\bar{b}$ = const векторы, $u(\bar{r})$ - дифференцируемое поле

$$\operatorname{rot}(u(\bar{r})\bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u(\bar{r})x & u(\bar{r})y & u(\bar{r})z \end{vmatrix} = \bar{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)u(\bar{r}) -$$

$$- \bar{j}\, u(\bar{r})\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \bar{k}\, u(\bar{r})\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$$

T30 Для гладкого векторного поля $\bar{V}$ на $\mathbb{R}^3$ доказать

а) если $\operatorname{rot} \bar{V} = 0$ на всём $\mathbb{R}^3$, то $\bar{V} = \operatorname{grad} f$ для некоторой функции f

б) если $\operatorname{div} \bar{V} = 0$ на всём $\mathbb{R}^3$, то $\bar{V} = \operatorname{rot} \bar{a}$ для некоторого векторного поля $\bar{v}$

a) $\bar{v} = \begin{pmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{pmatrix}$ $\quad \operatorname{rot} \bar{v} = \begin{pmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\forall$ гладкая замкнутая кривая L – (по теореме Жордана) край некоторой ограниченной области S

По теореме Стокса

$$\int_L \bar{v} = \int_S d\bar{v} \Rightarrow \int_L P\,dx + Q\,dy + R\,dz =$$

$$= \int_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial z}\right) dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy =$$

$= 0 \Leftrightarrow$ поле $\bar{v}$ потенциально, т.е. $\exists f$:

$\bar{v} = \operatorname{grad} f$

б) $\operatorname{div} \overline{V} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$

Рассмотрим форму $\omega = P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy$

$d\omega = \operatorname{div} \overline{V}\, dx \wedge dy \wedge dz = 0 \Rightarrow \omega$ - замкнута $\Rightarrow$

(т.к. $\mathbb{R}^3$ выпукло, т.е. $\mathbb{R}^3$ стягиваемое) по лемме Пуанкаре, ω точна $\Rightarrow \exists\, \alpha : d\alpha = \omega$

$\exists\, \alpha = A\, dx + B\, dy + C\, dz$

$d\alpha = \frac{\partial A}{\partial y} dy \wedge dx + \frac{\partial A}{\partial z} dz \wedge dx + \frac{\partial B}{\partial x} dx \wedge dy +$

$+ \frac{\partial B}{\partial z} dz \wedge dy + \frac{\partial C}{\partial x} dx \wedge dz + \frac{\partial C}{\partial y} dy \wedge dz =$

$= \left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) dy \wedge dx + \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) dy \wedge dz +$

$+ \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dz \wedge dx = \omega$

$$\begin{pmatrix} P \\ Q \\ R \end{pmatrix} = \begin{pmatrix} \frac{\partial C}{\partial y} - \frac{\partial B}{\partial z} \\ \frac{\partial A}{\partial z} - \frac{\partial B}{\partial x} \\ \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \end{pmatrix} = \operatorname{rot} \alpha$$