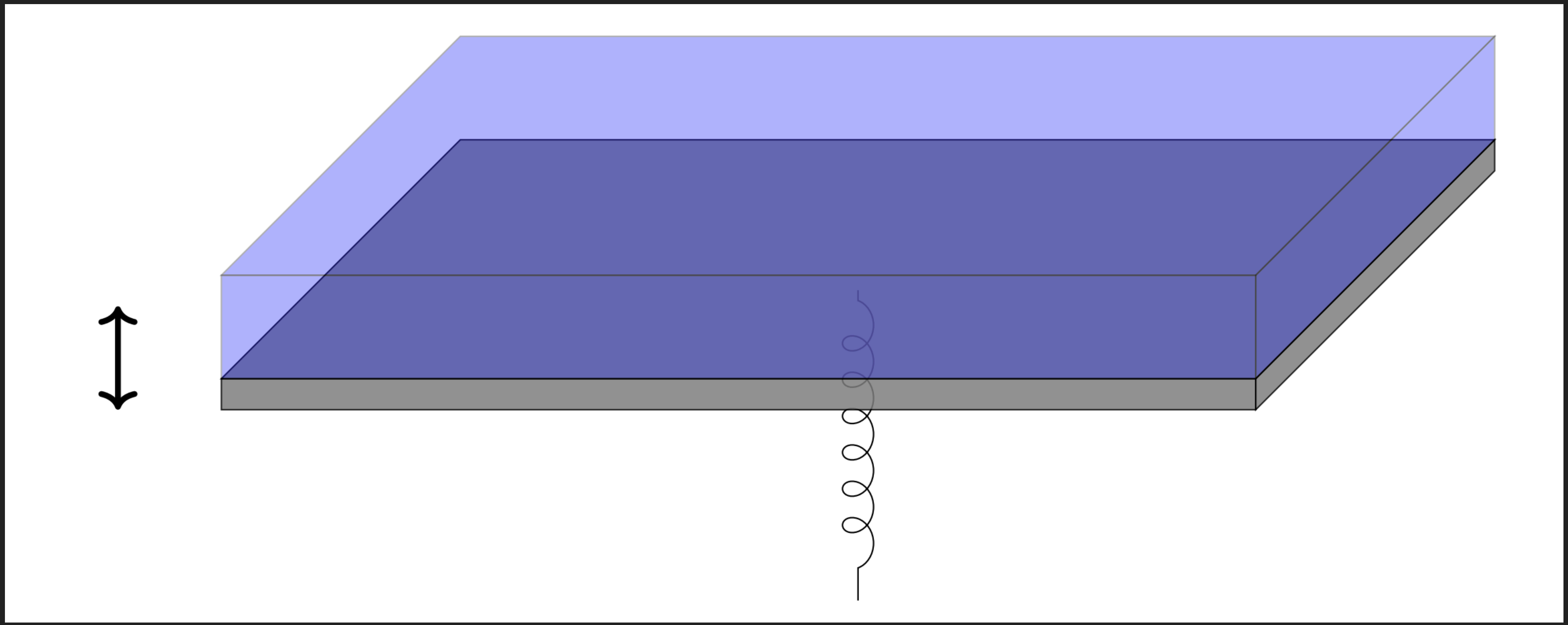
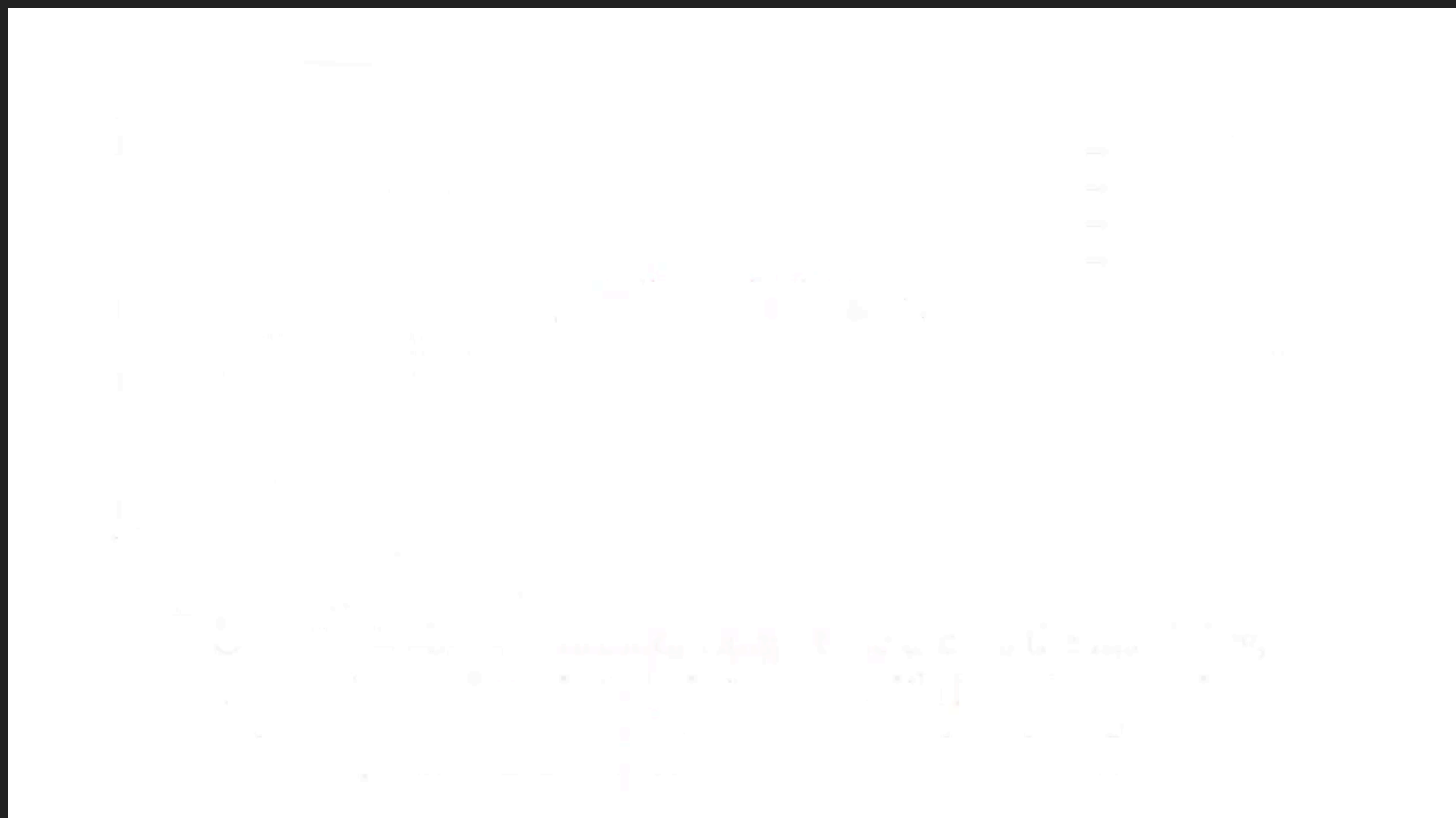
TAISUKE YASUDA

ASYMPTOTIC STABILITY OF THE FARADAY WAVE PROBLEM

FARADAY WAVES

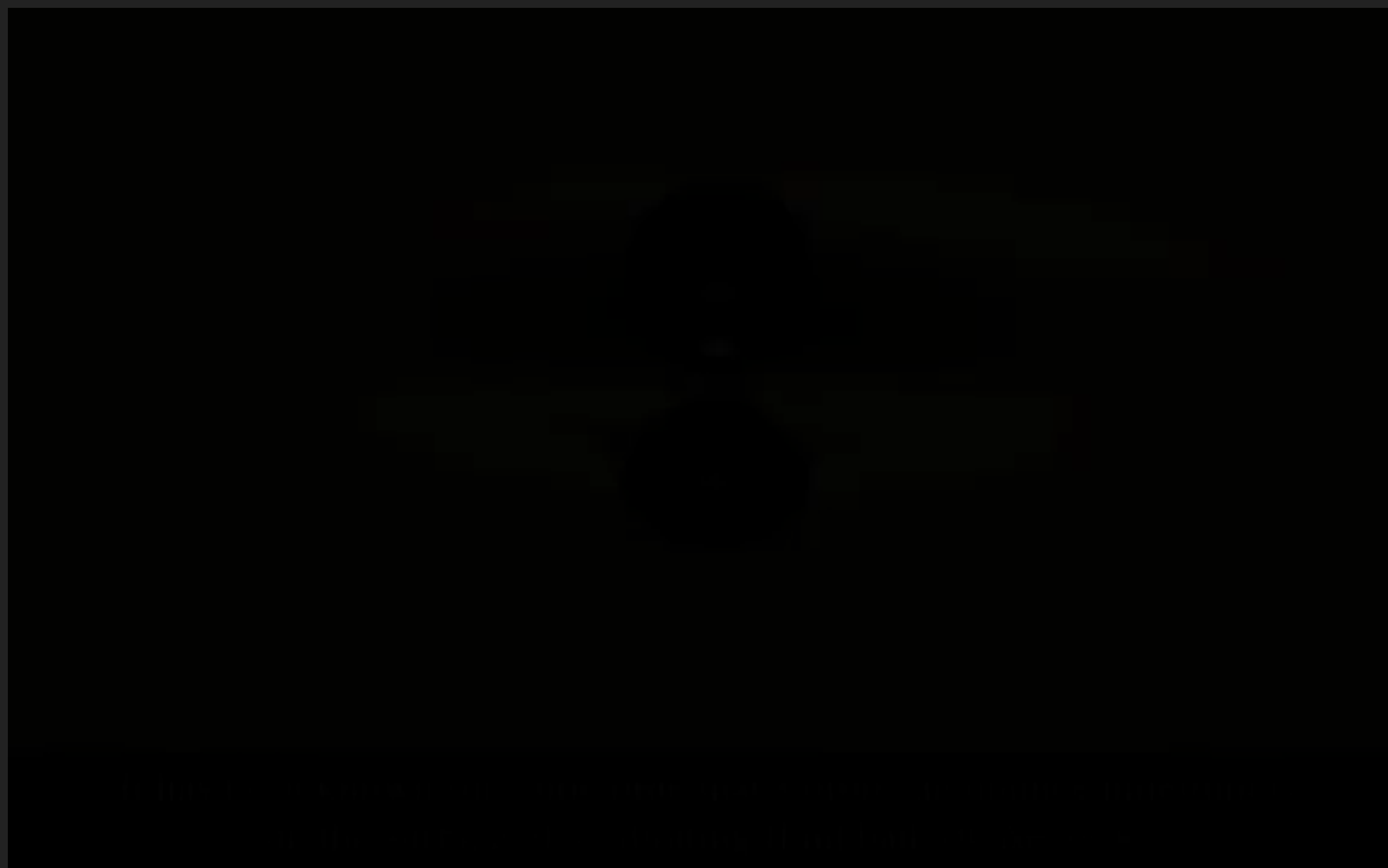


WHAT'S NOT IN THIS TALK



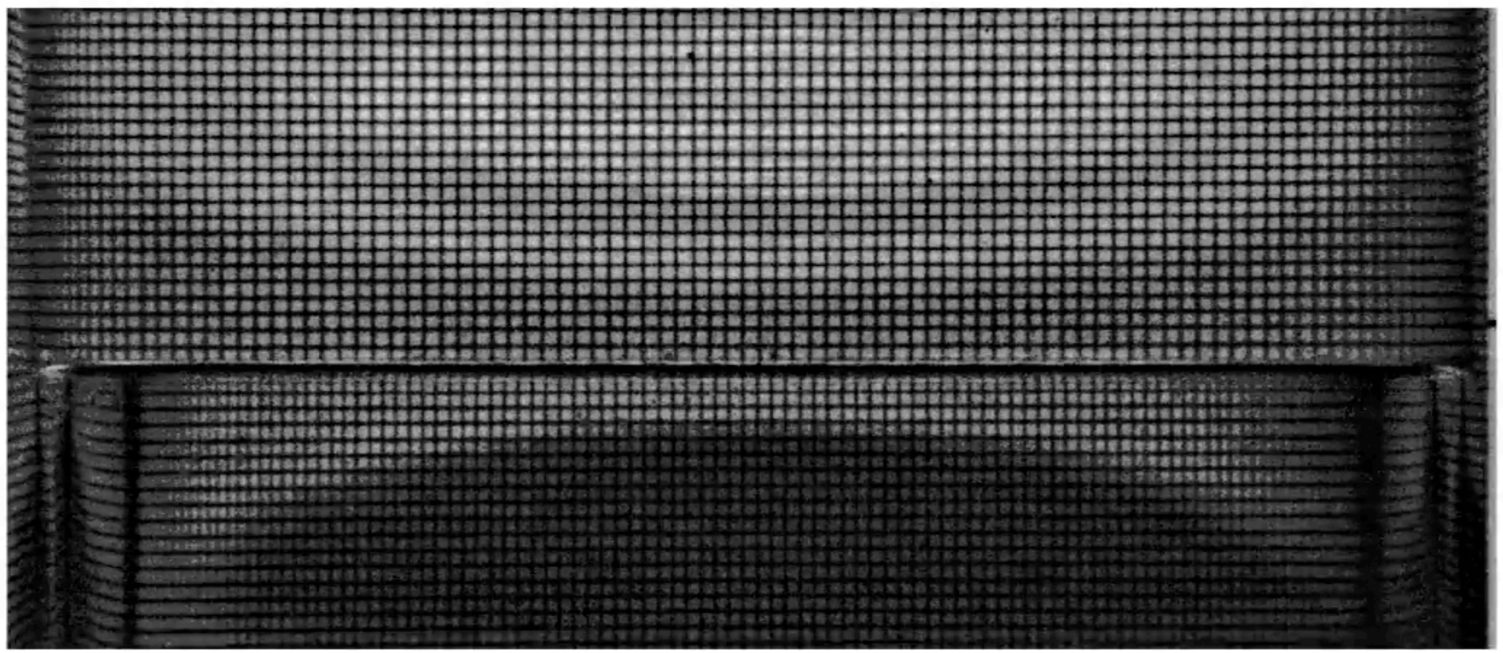
Dynamics of the Faraday instability
in a small cylinder (William Batson)

WHAT'S NOT IN THIS TALK



The pilot-wave dynamics of walking droplets
(Daniel M. Harris & John W. M. Bush)

WHAT'S IN THIS TALK

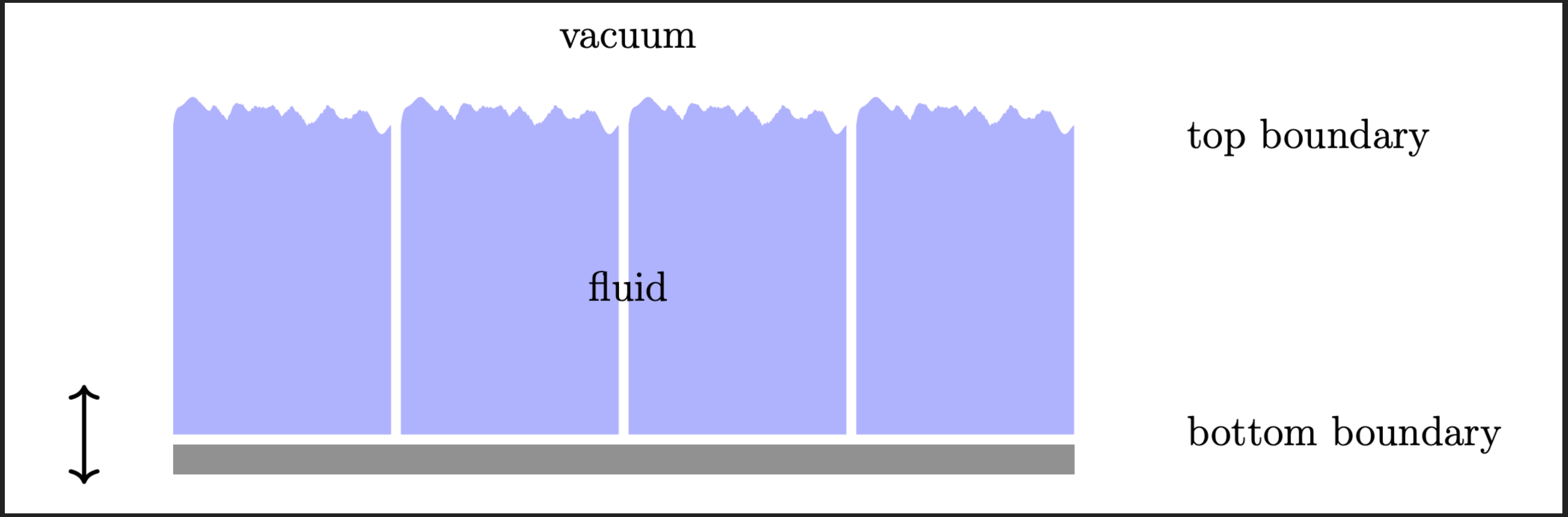


~~Oscillation above a threshold amplitude excites the instability,~~

Oscillation below a threshold amplitude is stable

Dynamics of the Faraday instability
in a small cylinder (William Batson)

EQUATIONS OF MOTION


vacuum
top boundary
fluid
bottom boundary

ASSUMPTIONS

- Oscillation profile $f : \mathbb{T} \to [-1,1]$ with amplitude A and frequency ω (i.e. $Af(\omega t)$)
- Horizontally periodic domain $\Sigma = \mathbb{T} \times \mathbb{T}$

ASSUMPTIONS

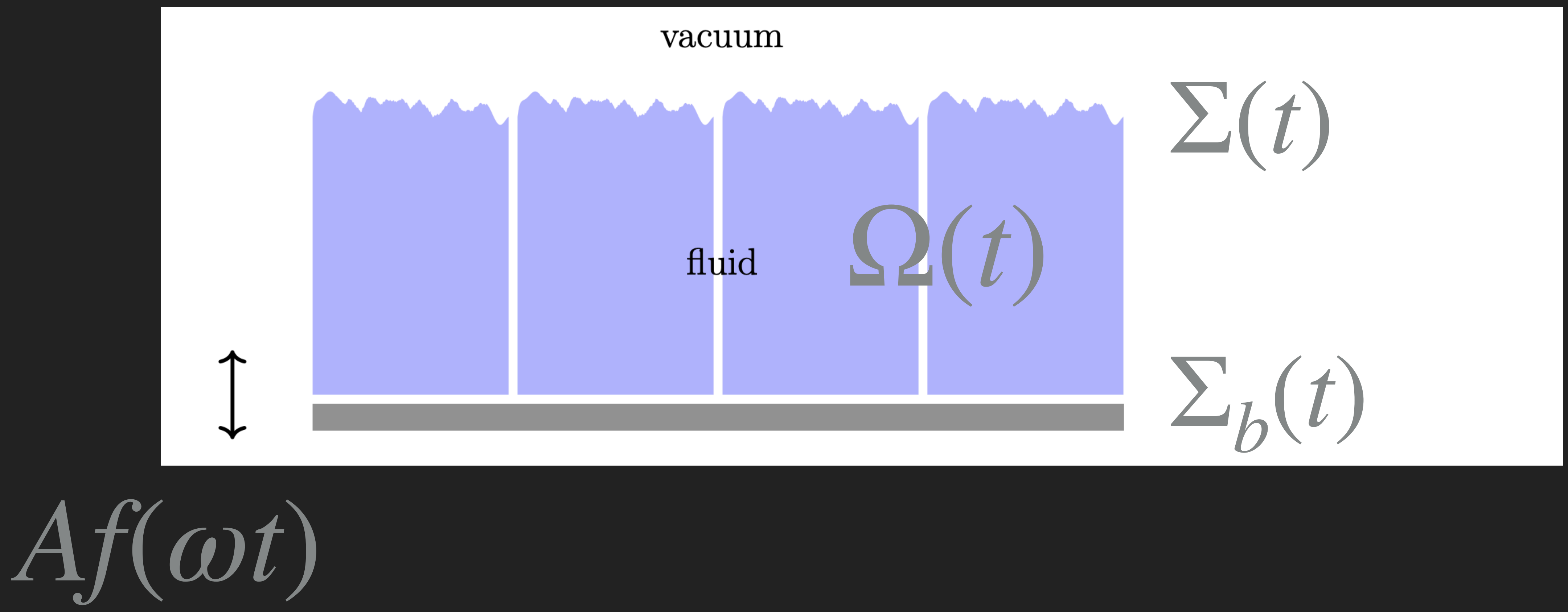
- Graph of top free boundary $\eta : \Sigma \times [0,\infty) \to \mathbb{R}$
- Top free boundary

$$\Sigma(t) = \{(x', x_3) \in \Sigma \times \mathbb{R} : x_3 = \eta(x', t)\}$$

- Oscillating lower boundary

$$\Sigma_b(t) = \{(x', x_3) \in \Sigma \times \mathbb{R} : x_3 = Af(\omega t) - b\}$$

- Domain $\Omega(t) = \{(x', x_3) : Af(\omega t) - b < x_3 < \eta(x', t)\}$



vacuum
$\Sigma(t)$
fluid
$\Omega(t)$
$\Sigma_b(t)$
$Af(\omega t)$

ASSUMPTIONS

- Gravitational force $-ge_3$
- Constant external pressure P_{ext}
- Surface tension $-\sigma\mathfrak{H}(\eta)$
- Viscosity μ

MAIN CHARACTERS

- Fluid velocity field $u : \Omega(t) \times (0,\infty) \to \mathbb{R}^3$
- Pressure $p : \Omega(t) \times (0,\infty) \to \mathbb{R}$

INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p - \mu \Delta u = F & \text{in } \Omega(t) \\ \operatorname{div} u = 0 & \text{in } \Omega(t) \end{cases}$$

BOUNDARY CONDITIONS

$$\begin{cases} \partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3 & \text{on } \Sigma(t) \\ (pI - \mu \mathbb{D} u)\nu = (P_{\text{ext}} - \sigma \mathfrak{H}(\eta))\nu & \text{on } \Sigma(t) \\ u = A\omega f'(\omega t) e_3 & \text{on } \Sigma_b(t) \end{cases}$$

THE FULL PDE

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p - \mu \Delta u = - g e_3 & \text{in } \Omega(t) \\ \operatorname{div} u = 0 & \text{in } \Omega(t) \\ \partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3 & \text{on } \Sigma(t) \\ (pI - \mu \mathbb{D} u)\nu = (P_{\text{ext}} - \sigma \mathfrak{H}(\eta))\nu & \text{on } \Sigma(t) \\ u = A\omega f'(\omega t) e_3 & \text{on } \Sigma_b(t) \end{cases}$$

ABSORBING THE GRAVITY

- Set $p_{\text{new}} = p_{\text{old}} + gx_3 - P_{\text{ext}}$

$$
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla p - \mu \Delta u = 0 & \text{in } \Omega(t) \\
\operatorname{div} u = 0 & \text{in } \Omega(t) \\
\partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3 & \text{on } \Sigma(t) \\
(pI - \mu \mathbb{D} u)\nu = (-\sigma \mathfrak{H}(\eta) + g\eta)\nu & \text{on } \Sigma(t) \\
u = A\omega f'(\omega t) e_3 & \text{on } \Sigma_b(t)
\end{cases}
$$

CHANGE COORDINATES TO THE FLUID FRAME

- Set

$$u_{\text{old}}(x,t) = u_{\text{new}}(x', x_3 - Af(\omega t), t) + A\omega f'(\omega t)$$
$$p_{\text{old}}(x,t) = p_{\text{new}}(x', x_3 - Af(\omega t), t)$$
$$\eta_{\text{old}}(x',t) = \eta_{\text{new}}(x', t) + Af(\omega t)$$

- Domain is now

$$\Omega(t) = \{x = (x', x_3) \in \Sigma \times \mathbb{R} : -b < x_3 < \eta(x',t)\}$$
$$\Sigma(t) = \{x = (x', x_3) \in \Sigma \times \mathbb{R} : x_3 = \eta(x',t)\}$$
$$\Sigma_b = \{x = (x', x_3) \in \Sigma \times \mathbb{R} : x_3 = -b\}$$

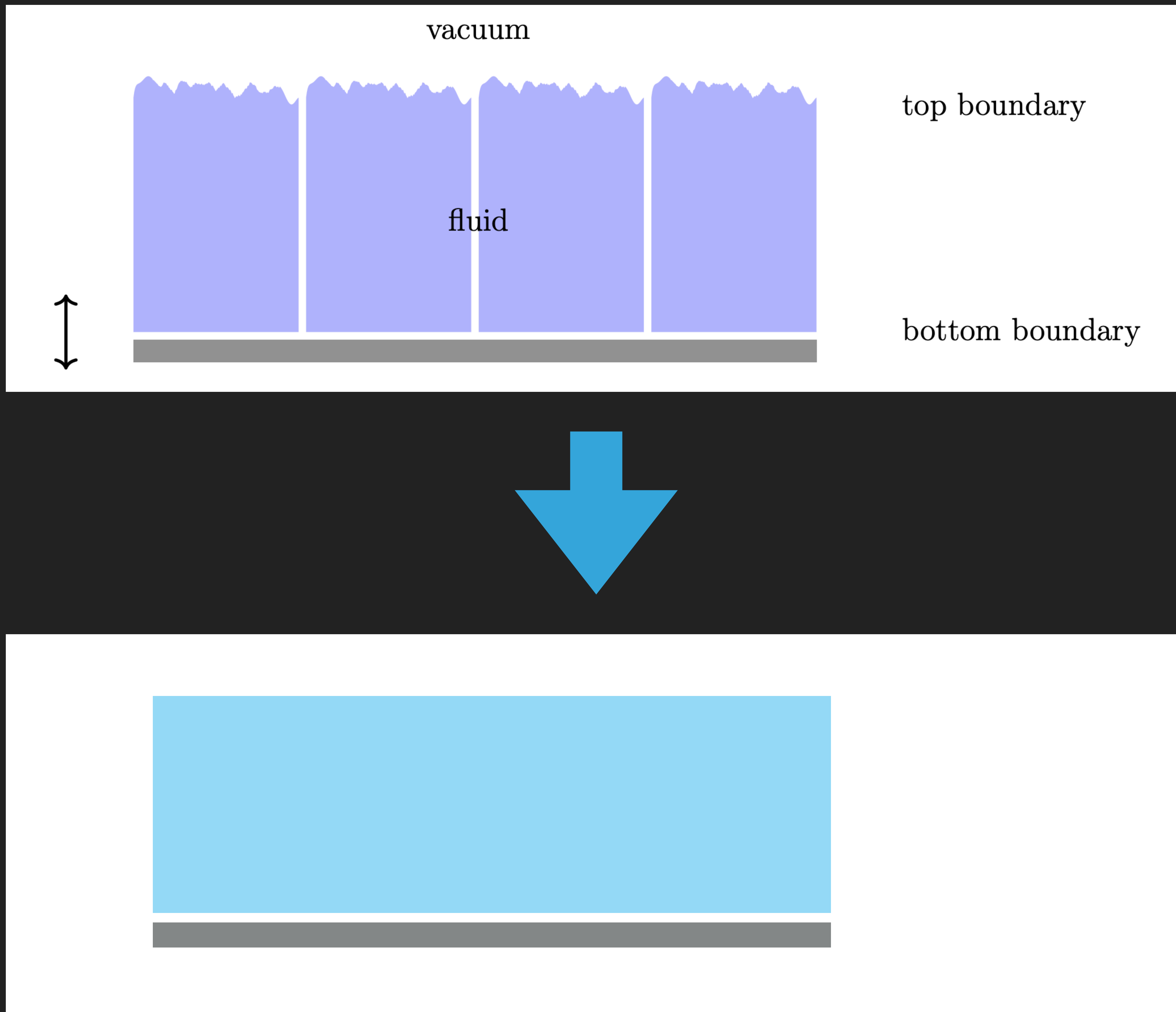
CHANGE COORDINATES TO THE FLUID FRAME

$$
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla p - \mu \Delta u + A\omega^2 f''(\omega t) e_3 = 0 & \text{in } \Omega(t) \\
\operatorname{div} u = 0 & \text{in } \Omega(t) \\
\partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3 & \text{on } \Sigma(t) \\
(pI - \mu \mathbb{D} u)\nu = (-\sigma \mathfrak{H}(\eta) + g(\eta + Af(\omega t)))\nu & \text{on } \Sigma(t) \\
u = 0 & \text{on } \Sigma_b(t)
\end{cases}
$$

ABSORBING THE OSCILLATION ACCELERATION

- Set $p_{\text{new}} = p_{\text{old}} + A\omega^2 f''(\omega t)x_3 - gAf(\omega t)$

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p - \mu \Delta u = 0 & \text{in } \Omega(t) \\ \operatorname{div} u = 0 & \text{in } \Omega(t) \\ \partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3 & \text{on } \Sigma(t) \\ (pI - \mu \mathbb{D} u)\nu = (-\sigma \mathfrak{H}(\eta) + (g + A\omega^2 f''(\omega t))\eta)\nu & \text{on } \Sigma(t) \\ u = 0 & \text{on } \Sigma_b(t) \end{cases}$$


vacuum
top boundary
fluid
bottom boundary

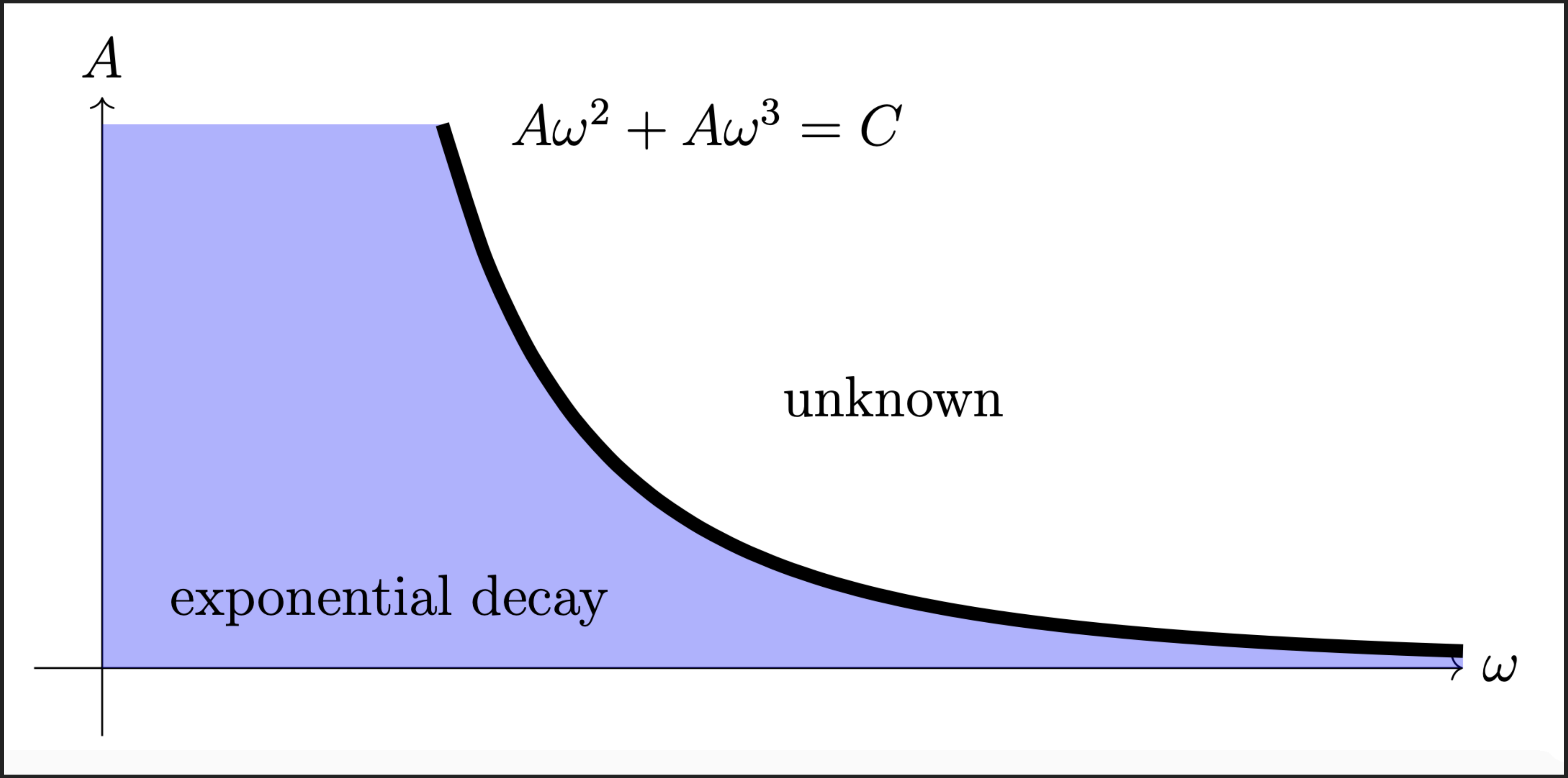
FLATTENING

$$\begin{cases} \partial_t u - \partial_t \hat{\eta} \tilde{b} K \partial_3 u + u \cdot \nabla_{\mathscr{A}} u + \operatorname{div}_{\mathscr{A}} S_{\mathscr{A}}(u,p) = 0 & \text{in } \Omega \\ \operatorname{div}_{\mathscr{A}} u = 0 & \text{in } \Omega \\ \partial_t \eta = u \cdot \mathscr{N} & \text{on } \Sigma \\ S_{\mathscr{A}}(u,p)\mathscr{N} = \left(-\sigma \mathfrak{H}(\eta) + \left(g + A\omega^2 f''(\omega t) \right) \eta \right) \mathscr{N} & \text{on } \Sigma \\ u = 0 & \text{on } \Sigma_b \end{cases}$$

MAIN THEOREM

There is a parameter regime in which the PDE for the Faraday wave problem is asymptotically stable.

MAIN THEOREM



Proof (sketch).

PROOF OVERVIEW

Global Existence of
Decaying Solutions

=

Local Existence + A Priori Estimates

PROOF OVERVIEW

Global Existence of
Decaying Solutions

=

Local Existence + A Priori Estimates

BASIC ENERGY-DISSIPATION ESTIMATES

LINEARIZED EQUATIONS

$$
\begin{cases}
\partial_t u + \nabla p - \mu \Delta u = 0 & \textbf{in} \quad \Omega \\
\operatorname{div} u = 0 & \textbf{in} \quad \Omega \\
S\nu = -\left[\sigma \Delta \eta - \left(g + A\omega^2 f''(\omega t)\right) \eta\right] \nu & \textbf{on} \quad \Sigma \\
\partial_t \eta = u_3 & \textbf{on} \quad \Sigma \\
u = 0 & \textbf{on} \quad \Sigma_b
\end{cases}
$$

$$
\begin{cases}
\partial_t u + \nabla p - \mu \Delta u = 0 & \textbf{in} \quad \Omega \\
\operatorname{div} u = 0 & \textbf{in} \quad \Omega \\
S\nu = -\left[\sigma \Delta \eta - \left(g + A\omega^2 f''(\omega t)\right)\eta\right]\nu & \textbf{on} \quad \Sigma \\
\partial_t \eta = u_3 & \textbf{on} \quad \Sigma \\
u = 0 & \textbf{on} \quad \Sigma_b
\end{cases}
$$

$$
\begin{cases}
\partial_t u + \nabla p - \mu \Delta u = 0 & \textbf{in} \quad \Omega \\
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S\nu = -\left[\sigma \Delta \eta - \left(g + A\omega^2 f''(\omega t)\right) \eta\right] \nu & \textbf{on} \quad \Sigma \\
\partial_t \eta = u_3 & \textbf{on} \quad \Sigma \\
u = 0 & \textbf{on} \quad \Sigma_b
\end{cases}
$$

$$\partial_t u + \nabla p - \mu \Delta u = 0$$

$$\int_\Omega u \cdot (\partial_t u + \nabla p - \mu \Delta u) = 0$$

$$
\begin{cases}
\partial_t u + \nabla p - \mu \Delta u = 0 & \textbf{in} \quad \Omega, \\
\operatorname{div} u = 0 & \textbf{in} \quad \Omega, \\
S\nu = -\left[\sigma \Delta \eta - \left(g + A\omega^2 f''(\omega t) \right) \eta \right] \nu & \textbf{on} \quad \Sigma, \\
\partial_t \eta = u_3 & \textbf{on} \quad \Sigma, \\
u = 0 & \textbf{on} \quad \Sigma_b
\end{cases}
$$

$$\int_{\Omega} u \cdot (\partial_t u + \nabla p - \mu \Delta u) = 0$$

$$\Longrightarrow$$

$$\partial_t \left(\int_{\Omega} \frac{|u|^2}{2} + \int_{\Sigma} \frac{\sigma |\nabla \eta|^2}{2} + \frac{\big(g + A\omega^2 f''(\omega t)\big) |\eta|^2}{2} \right) + \mu \int_{\Omega} \frac{|\mathbb{D} u|^2}{2} = \int_{\Sigma} \frac{A\omega^3 f'''(\omega t) |\eta|^2}{2}$$

$$\int_\Omega u \cdot (\partial_t u + \nabla p - \mu \Delta u) = 0$$

$$\Longrightarrow$$

$$\partial_t \left(\int_\Omega \frac{|u|^2}{2} + \int_\Sigma \frac{\sigma |\nabla \eta|^2}{2} + \frac{\big(g + A\omega^2 f''(\omega t)\big)|\eta|^2}{2} \right) + \mu \int_\Omega \frac{|\mathbb{D}u|^2}{2} = \int_\Sigma \frac{A\omega^3 f'''(\omega t)|\eta|^2}{2}$$

$$\partial_t \mathscr{E} + \mathscr{D} = \mathscr{F}$$

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If

$$\lambda \mathscr{E} \leq \mathscr{D}$$

and

$$|\mathscr{F}| \leq \frac{\lambda}{2} \mathscr{E}$$

then

$$\partial_t \mathscr{E} + \frac{\lambda}{2} \mathscr{E} \leq 0$$

$$\partial_t \mathscr{E} + \frac{\lambda}{2} \mathscr{E} \leq 0$$

$$\partial_t \mathscr{E} + \frac{\lambda}{2}\mathscr{E} \leq 0$$

By Gronwall's inequality,

$$\mathscr{E}(t) \leq \mathscr{E}(0) \cdot \exp\left(-\frac{\lambda}{2}t\right)$$

But this isn't quite true yet ☹️

MORE ENERGY-DISSIPATION ESTIMATES

$$
\begin{cases}
\partial_t u + \nabla p - \mu \Delta u = 0 & \textbf{in } \ \Omega \\
\operatorname{div} u = 0 & \textbf{in } \ \Omega \\
S\nu = -\left[\sigma \Delta \eta - \left(g + A\omega^2 f''(\omega t)\right) \eta\right] \nu & \textbf{on } \ \Sigma \\
\partial_t \eta = u_3 & \textbf{on } \ \Sigma \\
u = 0 & \textbf{on } \ \Sigma_b
\end{cases}
$$

$$
\partial_t \begin{cases} \partial_t u + \nabla p - \mu \Delta u = 0 & \textbf{in } \Omega \\ \operatorname{div} u = 0 & \textbf{in } \Omega \\ S\nu = -\left[\sigma \Delta \eta - \left(g + A\omega^2 f''(\omega t)\right) \eta\right] \nu & \textbf{on } \Sigma \\ \partial_t \eta = u_3 & \textbf{on } \Sigma \\ u = 0 & \textbf{on } \Sigma_b \end{cases}
$$

$$
\begin{cases}
\partial_t \partial_t u + \nabla \partial_t p - \mu \Delta \partial_t u = 0 & \textbf{in } \Omega \\
\operatorname{div} \partial_t u = 0 & \textbf{in } \Omega \\
\partial_t S \nu = -\partial_t \left[\sigma \Delta \eta - \left(g + A\omega^2 f''(\omega t) \right) \eta \right] \nu & \textbf{on } \Sigma \\
\partial_t \partial_t \eta = \partial_t u_3 & \textbf{on } \Sigma \\
\partial_t u = 0 & \textbf{on } \Sigma_b
\end{cases}
$$

$$\partial_t\left(\int_\Omega \frac{|\partial_t u|^2}{2} + \int_\Sigma \frac{\sigma|\nabla\partial_t\eta|^2}{2} + \frac{\big(g + A\omega^2 f''(\omega t)\big)|\partial_t\eta|^2}{2} + A\omega^3 f'''(\omega t)\eta\partial_t\eta\right) + \mu\int_\Omega \frac{|\mathbb{D}\partial_t u|^2}{2}$$

$$= \int_\Sigma \frac{3A\omega^3 f'''(\omega t)|\partial_t\eta|^2}{2} + A\omega^4 f''''(\omega t)\eta\partial_t\eta$$

$$\partial^{\alpha} \begin{cases} \partial_t u + \nabla p - \mu \Delta u = 0 & \textbf{in} \ \ \Omega \\ \operatorname{div} u = 0 & \textbf{in} \ \ \Omega \\ S\nu = -\left[\sigma \Delta \eta - \left(g + A\omega^2 f''(\omega t)\right)\eta\right]\nu & \textbf{on} \ \ \Sigma \\ \partial_t \eta = u_3 & \textbf{on} \ \ \Sigma \\ u = 0 & \textbf{on} \ \ \Sigma_b \end{cases}$$

$$\alpha_3 = 0$$

$$\partial^\alpha \begin{cases} \partial_t u + \nabla p - \mu \Delta u = 0 & \textbf{in} \quad \Omega \\ \operatorname{div} u = 0 & \textbf{in} \quad \Omega \\ S\nu = -\left[\sigma \Delta \eta - \left(g + A\omega^2 f''(\omega t)\right) \eta\right] \nu & \textbf{on} \quad \Sigma \\ \partial_t \eta = u_3 & \textbf{on} \quad \Sigma \\ u = 0 & \textbf{on} \quad \Sigma_b \end{cases}$$

$$\partial_t \left(\int_\Omega \frac{|\partial^\alpha u|^2}{2} + \int_\Sigma \frac{\sigma |\nabla \partial^\alpha \eta|^2}{2} + \frac{\big(g + A\omega^2 f''(\omega t)\big)|\partial^\alpha \eta|^2}{2} \right) + \mu \int_\Omega \frac{|\mathbb{D}\partial^\alpha u|^2}{2}$$

$$= \int_\Sigma \frac{A\omega^3 f'''(\omega t)|\partial^\alpha \eta|^2}{2}$$

$$\partial_t\left(\int_\Omega \frac{|u|^2}{2}+\int_\Sigma \frac{\sigma|\nabla\eta|^2}{2}+\frac{\left(g+A\omega^2 f''(\omega t)\right)|\eta|^2}{2}\right)+\mu\int_\Omega \frac{|\mathbb{D}u|^2}{2}$$

$$=\int_\Sigma \frac{A\omega^3 f'''(\omega t)|\eta|^2}{2}$$

$$\partial_t\left(\int_\Omega \frac{|\partial_t u|^2}{2}+\int_\Sigma \frac{\sigma|\nabla\partial_t\eta|^2}{2}+\frac{\left(g+A\omega^2 f''(\omega t)\right)|\partial_t\eta|^2}{2}+A\omega^3 f'''(\omega t)\eta\partial_t\eta\right)+\mu\int_\Omega \frac{|\mathbb{D}\partial_t u|^2}{2}$$

$$=\int_\Sigma \frac{3A\omega^3 f'''(\omega t)|\partial_t\eta|^2}{2}+A\omega^4 f''''(\omega t)\eta\partial_t\eta$$

$$\partial_t\left(\int_\Omega \frac{|\partial^\alpha u|^2}{2}+\int_\Sigma \frac{\sigma|\nabla\partial^\alpha\eta|^2}{2}+\frac{\left(g+A\omega^2 f''(\omega t)\right)|\partial^\alpha\eta|^2}{2}\right)+\mu\int_\Omega \frac{|\mathbb{D}\partial^\alpha u|^2}{2}$$

$$=\int_\Sigma \frac{A\omega^3 f'''(\omega t)|\partial^\alpha\eta|^2}{2}$$

$$\mathscr{E} := \int_\Omega \frac{|\partial_t u|^2}{2} + \sum_{\substack{|\alpha| \le 2 \\ \alpha_3 = 0}} \frac{|\partial^\alpha u|^2}{2}$$

$$+ \int_\Sigma \frac{\sigma |\nabla \partial_t \eta|^2}{2} + \frac{\left(g + A\omega^2 f''(\omega t)\right) |\partial_t \eta|^2}{2} + A\omega^3 f'''(\omega t)\eta \partial_t \eta$$

$$+ \sum_{\substack{|\alpha| \le 2 \\ \alpha_3 = 0}} \frac{\sigma |\nabla \partial^\alpha \eta|^2}{2} + \frac{\left(g + A\omega^2 f''(\omega t)\right) |\partial^\alpha \eta|^2}{2}$$

$$\mathscr{D} := \mu \int_\Omega \frac{|\mathbb{D} \partial_t u|^2}{2} + \sum_{\substack{|\alpha| \le 2 \\ \alpha_3 = 0}} \frac{|\mathbb{D} \partial^\alpha u|^2}{2}$$

$$\mathscr{F} := \int_\Sigma \frac{3A\omega^3 f'''(\omega t) |\partial_t \eta|^2}{2} + A\omega^4 f''''(\omega t)\eta \partial_t \eta + \sum_{\substack{|\alpha| \le 2 \\ \alpha_3 = 0}} \frac{A\omega^3 f'''(\omega t) |\partial^\alpha \eta|^2}{2}$$

ESTIMATES

$$\partial_t \mathscr{E} + \mathscr{D} = \mathscr{F}$$

If

$$\lambda \mathscr{E} \leq \mathscr{D}$$

and

$$|\mathscr{F}| \leq \frac{\lambda}{2} \mathscr{E}$$

then

$$\partial_t \mathscr{E} + \frac{\lambda}{2} \mathscr{E} \leq 0$$

$$\boxed{\begin{aligned}\mathscr{E} := &\int_\Omega \frac{|\partial_t u|^2}{2} + \sum_{\substack{|\alpha|\leq 2\\ \alpha_3=0}} \frac{|\partial^\alpha u|^2}{2}\\ &+ \int_\Sigma \frac{\sigma|\nabla\partial_t\eta|^2}{2} + \frac{\left(g + A\omega^2 f''(\omega t)\right)|\partial_t\eta|^2}{2} + A\omega^3 f'''(\omega t)\eta\partial_t\eta\\ &+ \sum_{\substack{|\alpha|\leq 2\\ \alpha_3=0}} \frac{\sigma|\nabla\partial^\alpha\eta|^2}{2} + \frac{\left(g + A\omega^2 f''(\omega t)\right)|\partial^\alpha\eta|^2}{2}\end{aligned}}$$

$$\mathscr{D} := \mu\int_\Omega \frac{|\mathbb{D}\partial_t u|^2}{2} + \sum_{\substack{|\alpha|\leq 2\\ \alpha_3=0}} \frac{|\mathbb{D}\partial^\alpha u|^2}{2}$$

$$\mathscr{F} := \int_\Sigma \frac{3A\omega^3 f'''(\omega t)|\partial_t\eta|^2}{2} + A\omega^4 f''''(\omega t)\eta\partial_t\eta + \sum_{\substack{|\alpha|\leq 2\\ \alpha_3=0}} \frac{A\omega^3 f'''(\omega t)|\partial^\alpha\eta|^2}{2}$$

$$
\begin{aligned}
\mathscr{E} := \int_\Omega \frac{|\partial_t u|^2}{2} + \sum_{\substack{|\alpha| \le 2 \\ \alpha_3 = 0}} \frac{|\partial^\alpha u|^2}{2} \\
+ \int_\Sigma \frac{\sigma |\nabla \partial_t \eta|^2}{2} + \frac{\left(g + A\omega^2 f''(\omega t)\right) |\partial_t \eta|^2}{2} + A\omega^3 f'''(\omega t)\eta \partial_t \eta \\
+ \sum_{\substack{|\alpha| \le 2 \\ \alpha_3 = 0}} \frac{\sigma |\nabla \partial^\alpha \eta|^2}{2} + \frac{\left(g + A\omega^2 f''(\omega t)\right) |\partial^\alpha \eta|^2}{2}
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{E} &:= \int_{\Omega} \frac{|\partial_t u|^2}{2} + \sum_{\substack{|\alpha| \leq 2 \\ \alpha_3 = 0}} \frac{|\partial^\alpha u|^2}{2} \\
&\quad + \int_{\Sigma} \frac{\sigma |\nabla \partial_t \eta|^2}{2} + \frac{\left(g + A\omega^2 f''(\omega t)\right) |\partial_t \eta|^2}{2} + A\omega^3 f'''(\omega t)\eta \partial_t \eta \\
&\quad + \sum_{\substack{|\alpha| \leq 2 \\ \alpha_3 = 0}} \frac{\sigma |\nabla \partial^\alpha \eta|^2}{2} + \frac{\left(g + A\omega^2 f''(\omega t)\right) |\partial^\alpha \eta|^2}{2} \\
&\gtrsim \|u\|^2_{L^2(\Omega)} + \|\partial_t u\|^2_{L^2(\Omega)} + \|\eta\|^2_{H^3(\Sigma)} + \|\partial_t \eta\|^2_{H^1(\Sigma)}
\end{aligned}
$$

Elliptic estimates for the Stokes operator with stress conditions (Beale '81)

If

$$\begin{cases} -\Delta u + \nabla p = f & \textbf{in } \Omega \\ \operatorname{div} u = h & \textbf{in } \Omega \\ (pI - \mathbb{D}u)e_3 = \psi & \textbf{on } \Sigma \\ u = 0 & \textbf{on } \Sigma_b \end{cases}$$

then

$$\|u\|^2_{H^{s+2}(\Omega)} + \|p\|^2_{H^{s+1}(\Omega)} \lesssim \|f\|^2_{H^s(\Omega)} + \|h\|^2_{H^{s+1}(\Omega)} + \|\psi\|^2_{H^{s+1/2}(\Sigma)}$$

$$\begin{cases} -\Delta u + \nabla p = f & \textbf{in } \Omega \\ \operatorname{div} u = h & \textbf{in } \Omega \\ (pI - \mathbb{D}u)e_3 = \psi & \textbf{on } \Sigma \\ u = 0 & \textbf{on } \Sigma_b \end{cases} \implies \|u\|^2_{H^{s+2}(\Omega)} + \|p\|^2_{H^{s+1}(\Omega)} \lesssim \|f\|^2_{H^{s}(\Omega)} + \|h\|^2_{H^{s+1}(\Omega)} + \|\psi\|^2_{H^{s+1/2}(\Sigma)}$$

$$\begin{cases} \partial_t u + \nabla p - \mu \Delta u = 0 & \textbf{in } \ \Omega \\ \operatorname{div} u = 0 & \textbf{in } \ \Omega \\ (pI - \mathbb{D}u)e_3 = -\left[\sigma \Delta \eta - \left(g + A\omega^2 f''(\omega t)\right)\eta\right] e_3 & \textbf{on } \ \Sigma \\ \partial_t \eta = u_3 & \textbf{on } \ \Sigma \\ u = 0 & \textbf{on } \ \Sigma_b \end{cases}$$

$$\begin{cases} -\Delta u + \nabla p = f & \textbf{in } \Omega \\ \operatorname{div} u = h & \textbf{in } \Omega \\ (pI - \mathbb{D}u)e_3 = \psi & \textbf{on } \Sigma \\ u = 0 & \textbf{on } \Sigma_b \end{cases} \implies \|u\|^2_{H^{s+2}(\Omega)} + \|p\|^2_{H^{s+1}(\Omega)} \lesssim \|f\|^2_{H^s(\Omega)} + \|h\|^2_{H^{s+1}(\Omega)} + \|\psi\|^2_{H^{s+1/2}(\Sigma)}$$

$$\begin{cases} \partial_t u + \nabla p - \mu \Delta u = 0 & \textbf{in } \ \Omega \\ \operatorname{div} u = 0 & \textbf{in } \ \Omega \\ (pI - \mathbb{D}u)e_3 = -\left[\sigma \Delta \eta - \left(g + A\omega^2 f''(\omega t)\right) \eta\right] e_3 & \textbf{on } \ \Sigma \\ \partial_t \eta = u_3 & \textbf{on } \ \Sigma \\ u = 0 & \textbf{on } \ \Sigma_b \end{cases}$$

$$\begin{aligned} \|u\|^2_{H^2(\Omega)} + \|p\|^2_{H^1(\Omega)} &\lesssim \|\partial_t u\|^2_{L^2(\Omega)} + \|\Delta \eta\|^2_{H^{1/2}(\Sigma)} + \|\eta\|^2_{H^{1/2}(\Sigma)} \\ &\lesssim \mathscr{E} \end{aligned}$$

$$
\begin{aligned}
\mathscr{E} := \int_\Omega \frac{|\partial_t u|^2}{2} + \sum_{\substack{|\alpha| \le 2 \\ \alpha_3 = 0}} \frac{|\partial^\alpha u|^2}{2} \\
+ \int_\Sigma \frac{\sigma |\nabla \partial_t \eta|^2}{2} + \frac{\left(g + A\omega^2 f''(\omega t)\right) |\partial_t \eta|^2}{2} + A\omega^3 f'''(\omega t) \eta \partial_t \eta \\
+ \sum_{\substack{|\alpha| \le 2 \\ \alpha_3 = 0}} \frac{\sigma |\nabla \partial^\alpha \eta|^2}{2} + \frac{\left(g + A\omega^2 f''(\omega t)\right) |\partial^\alpha \eta|^2}{2} \\
\gtrsim \|u\|^2_{L^2(\Omega)} + \|\partial_t u\|^2_{L^2(\Omega)} + \|\eta\|^2_{H^3(\Sigma)} + \|\partial_t \eta\|^2_{H^1(\Sigma)} \\
\gtrsim \|u\|^2_{H^2(\Omega)} + \|\partial_t u\|^2_{L^2(\Omega)} + \|\eta\|^2_{H^3(\Sigma)} + \|\partial_t \eta\|^2_{H^1(\Sigma)} + \|p\|^2_{H^1(\Omega)}
\end{aligned}
$$

$$\mathscr{D} := \mu \int_{\Omega} \frac{|\mathbb{D}\partial_t u|^2}{2} + \sum_{\substack{|\alpha| \le 2 \\ \alpha_3 = 0}} \frac{|\mathbb{D}\partial^\alpha u|^2}{2}$$

similar tricks...

$$\gtrsim \|u\|^2_{H^3(\Omega)} + \|\partial_t u\|^2_{H^1(\Omega)} + \|\eta\|^2_{H^{7/2}(\Sigma)} + \|\partial_t \eta\|^2_{H^{5/2}(\Sigma)} + \|p\|^2_{H^2(\Omega)}$$

$$
\begin{aligned}
\mathscr{F} &:= \int_{\Sigma} \frac{3A\omega^3 f'''(\omega t)\,|\,\partial_t \eta\,|^2}{2} + A\omega^4 f''''(\omega t)\eta\partial_t\eta + \sum_{\substack{|\,\alpha\,|\,\le 2 \\ \alpha_3 = 0}} \frac{A\omega^3 f'''(\omega t)\,|\,\partial^\alpha \eta\,|^2}{2} \\
&\lesssim \left\| A\omega^3 f''' + A\omega^4 f'''' \right\|_{L^\infty(\mathbb{T})} \left(\|\partial_t \eta\|^2_{L^2(\Sigma)} + \|\eta\|^2_{H^2(\Sigma)} \right)
\end{aligned}
$$

$$\partial_t \mathscr{E} + \mathscr{D} = \mathscr{F}$$

$$\mathscr{E} \asymp \|u\|_{H^2(\Omega)}^2 + \|\partial_t u\|_{L^2(\Omega)}^2 + \|\eta\|_{H^3(\Sigma)}^2 + \|\partial_t \eta\|_{H^1(\Sigma)}^2 + \|p\|_{H^1(\Omega)}^2$$

$$\mathscr{D} \asymp \|u\|_{H^3(\Omega)}^2 + \|\partial_t u\|_{H^1(\Omega)}^2 + \|\eta\|_{H^{7/2}(\Sigma)}^2 + \|\partial_t \eta\|_{H^{5/2}(\Sigma)}^2 + \|p\|_{H^2(\Omega)}^2$$

$$\mathscr{F} \lesssim \left\| A\omega^3 f''' + A\omega^4 f'''' \right\|_{L^\infty(\mathbb{T})} \left(\|\partial_t \eta\|_{L^2(\Sigma)}^2 + \|\eta\|_{H^2(\Sigma)}^2 \right)$$

$$\partial_t \mathscr{E} + \mathscr{D} = \mathscr{F}$$

$$\mathscr{E} \asymp \|u\|^2_{H^2(\Omega)} + \|\partial_t u\|^2_{L^2(\Omega)} + \|\eta\|^2_{H^3(\Sigma)} + \|\partial_t \eta\|^2_{H^1(\Sigma)} + \|p\|^2_{H^1(\Omega)}$$

$$\mathscr{D} \asymp \|u\|^2_{H^3(\Omega)} + \|\partial_t u\|^2_{H^1(\Omega)} + \|\eta\|^2_{H^{7/2}(\Sigma)} + \|\partial_t \eta\|^2_{H^{5/2}(\Sigma)} + \|p\|^2_{H^2(\Omega)}$$

$$\mathscr{F} \lesssim \left\| A\omega^3 f''' + A\omega^4 f'''' \right\|_{L^\infty(\mathbb{T})} \left(\|\partial_t \eta\|^2_{L^2(\Sigma)} + \|\eta\|^2_{H^2(\Sigma)} \right)$$

so

$$\begin{aligned} \lambda \mathscr{E} &\le \mathscr{D} \\ |\mathscr{F}| &\le \frac{\lambda}{2}\mathscr{E} \end{aligned} \quad \Longrightarrow \quad \partial_t \mathscr{E} + \frac{\lambda}{2}\mathscr{E} \le 0$$

$$\Longrightarrow \quad \mathscr{E}(t) \le \mathscr{E}(0) \cdot \exp\left(-\frac{\lambda}{2}t\right)$$

😊

LINEARIZED MAIN THEOREM

$$\begin{cases} \partial_t u + \nabla p - \mu \Delta u = 0 & \textbf{in } \Omega \\ \operatorname{div} u = 0 & \textbf{in } \Omega \\ S\nu = -\left[\sigma \Delta \eta - \left(g + A\omega^2 f''(\omega t)\right)\eta\right]\nu & \textbf{on } \Sigma \\ \partial_t \eta = u_3 & \textbf{on } \Sigma \\ u = 0 & \textbf{on } \Sigma_b \end{cases}$$

$$\Longrightarrow$$

$$\mathscr{E}(t) \leq \mathscr{E}(0) \cdot \exp\left(-\frac{\lambda}{2} t\right)$$

NONLINEAR ANALYSIS

NONLINEAR MAIN THEOREM

$$\begin{cases} \partial_t u - \partial_t \hat{\eta} \tilde{b} K \partial_3 u + u \cdot \nabla_{\mathscr{A}} u + \operatorname{div}_{\mathscr{A}} S_{\mathscr{A}}(u,p) = 0 & \text{in } \Omega \\ \operatorname{div}_{\mathscr{A}} u = 0 & \text{in } \Omega \\ \partial_t \eta = u \cdot \mathscr{N} & \text{on } \Sigma \\ S_{\mathscr{A}}(u,p)\mathscr{N} = \left(-\sigma \mathfrak{H}(\eta) + \left(g + A\omega^2 f''(\omega t) \right) \eta \right) \mathscr{N} & \text{on } \Sigma \\ u = 0 & \text{on } \Sigma_b \end{cases}$$

$$\Longrightarrow$$

$$\mathscr{E}(t) \leq \mathscr{E}(0) \cdot \exp\left(-\frac{\lambda}{2} t \right)$$

$$
\begin{cases}
\partial_t u - \partial_t \hat{\eta} \tilde{b} K \partial_3 u + u \cdot \nabla_{\mathscr{A}} u + \operatorname{div}_{\mathscr{A}} S_{\mathscr{A}}(u,p) = 0 & \text{in } \Omega \\
\operatorname{div}_{\mathscr{A}} u = 0 & \text{in } \Omega \\
\partial_t \eta = u \cdot \mathscr{N} & \text{on } \Sigma \\
S_{\mathscr{A}}(u,p)\mathscr{N} = \left(-\sigma \mathfrak{H}(\eta) + \left(g + A\omega^2 f''(\omega t) \right) \eta \right) \mathscr{N} & \text{on } \Sigma \\
u = 0 & \text{on } \Sigma_b
\end{cases}
$$

$$
\begin{cases}
\partial_t u - \partial_t \hat{\eta} \tilde{b} K \partial_3 u + u \cdot \nabla_{\mathscr{A}} u + \operatorname{div}_{\mathscr{A}} S_{\mathscr{A}}(u,p) = 0 & \text{in } \Omega \\
\operatorname{div}_{\mathscr{A}} u = 0 & \text{in } \Omega \\
\partial_t \eta = u \cdot \mathscr{N} & \text{on } \Sigma \\
S_{\mathscr{A}}(u,p)\mathscr{N} = \left(-\sigma \mathfrak{H}(\eta) + \left(g + A\omega^2 f''(\omega t)\right)\eta\right)\mathscr{N} & \text{on } \Sigma \\
u = 0 & \text{on } \Sigma_b
\end{cases}
$$

$$=$$

$$
\begin{cases}
\partial_t u + \nabla p - \mu \Delta u = 0 & \textbf{in } \ \Omega \\
\operatorname{div} u = 0 & \textbf{in } \ \Omega \\
S\nu = -\left[\sigma \Delta \eta - \left(g + A\omega^2 f''(\omega t)\right)\eta\right]\nu & \textbf{on } \ \Sigma \\
\partial_t \eta = u_3 & \textbf{on } \ \Sigma \\
u = 0 & \textbf{on } \ \Sigma_b
\end{cases}
\quad + \quad \text{Nonlinear terms}
$$

$$
\begin{cases}
\partial_t u - \partial_t \hat{\eta} \tilde{b} K \partial_3 u + u \cdot \nabla_{\mathscr{A}} u + \operatorname{div}_{\mathscr{A}} S_{\mathscr{A}}(u,p) = 0 & \text{in } \Omega \\
\operatorname{div}_{\mathscr{A}} u = 0 & \text{in } \Omega \\
\partial_t \eta = u \cdot \mathscr{N} & \text{on } \Sigma \\
S_{\mathscr{A}}(u,p)\mathscr{N} = \left(-\sigma \mathfrak{H}(\eta) + \left(g + A\omega^2 f''(\omega t) \right) \eta \right) \mathscr{N} & \text{on } \Sigma \\
u = 0 & \text{on } \Sigma_b
\end{cases}
$$

$$=$$

$$
\begin{cases}
\partial_t u + \nabla p - \mu \Delta u = 0 & \textbf{in } \Omega \\
\operatorname{div} u = 0 & \textbf{in } \Omega \\
S\nu = -\left[\sigma \Delta \eta - \left(g + A\omega^2 f''(\omega t) \right) \eta \right] \nu & \textbf{on } \Sigma \\
\partial_t \eta = u_3 & \textbf{on } \Sigma \\
u = 0 & \textbf{on } \Sigma_b
\end{cases}
\quad + \quad \boxed{\text{Nonlinear terms}}
$$

NONLINEAR ESTIMATES

- Nonlinear terms are at least quadratic
- E.g. $u \cdot \nabla u$

$$\|u \cdot \nabla u\|^2 \lesssim \|u\|^2 \cdot \|\nabla u\|^2 \lesssim \mathscr{E} \cdot \mathscr{E} \ll \mathscr{E}$$

- If $\mathscr{E}$ is small enough, these terms can be absorbed

$$\begin{cases} \partial_t u - \partial_t \hat{\eta} \tilde{b} K \partial_3 u + u \cdot \nabla_{\mathscr{A}} u + \operatorname{div}_{\mathscr{A}} S_{\mathscr{A}}(u,p) = 0 & \text{in } \Omega \\ \operatorname{div}_{\mathscr{A}} u = 0 & \text{in } \Omega \\ \partial_t \eta = u \cdot \mathscr{N} & \text{on } \Sigma \\ S_{\mathscr{A}}(u,p)\mathscr{N} = \left(-\sigma \mathfrak{H}(\eta) + \left(g + A\omega^2 f''(\omega t) \right) \eta \right) \mathscr{N} & \text{on } \Sigma \\ u = 0 & \text{on } \Sigma_b \end{cases}$$

$$=$$

$$\begin{cases} \partial_t u + \nabla p - \mu \Delta u = 0 & \textbf{in } \Omega \\ \operatorname{div} u = 0 & \textbf{in } \Omega \\ S\nu = -\left[\sigma \Delta \eta - \left(g + A\omega^2 f''(\omega t) \right) \eta \right] \nu & \textbf{on } \Sigma \\ \partial_t \eta = u_3 & \textbf{on } \Sigma \\ u = 0 & \textbf{on } \Sigma_b \end{cases} \quad + \quad \text{Nonlinear terms}$$

$$
\begin{cases}
\partial_t u - \partial_t \hat{\eta} \tilde{b} K \partial_3 u + u \cdot \nabla_{\mathscr{A}} u + \operatorname{div}_{\mathscr{A}} S_{\mathscr{A}}(u,p) = 0 & \text{in } \Omega \\
\operatorname{div}_{\mathscr{A}} u = 0 & \text{in } \Omega \\
\partial_t \eta = u \cdot \mathcal{N} & \text{on } \Sigma \\
S_{\mathscr{A}}(u,p)\mathcal{N} = \left(-\sigma \mathfrak{H}(\eta) + \left(g + A\omega^2 f''(\omega t)\right) \eta\right) \mathcal{N} & \text{on } \Sigma \\
u = 0 & \text{on } \Sigma_b
\end{cases}
$$

$$
=
$$

$$
\begin{cases}
\partial_t u + \nabla p - \mu \Delta u = 0 & \text{in } \Omega \\
\operatorname{div} u = 0 & \text{in } \Omega \\
S\nu = -\left[\sigma \Delta \eta - \left(g + A\omega^2 f''(\omega t)\right) \eta\right] \nu & \text{on } \Sigma \\
\partial_t \eta = u_3 & \text{on } \Sigma \\
u = 0 & \text{on } \Sigma_b
\end{cases}
\quad + \quad \text{Nonlinear terms}
$$

In conclusion, …

ACTUAL MAIN THEOREM

Theorem 3.3.1. *Suppose that initial data* (u_0, η_0) *satisfy* $\mathcal{E}_1^\sigma(0) < \infty$ *as well as the compatibility conditions of theorem 3.2.1. There exist constants* $\gamma_0 \in (0,1)$, $\kappa_0 \in (0,1)$, *and* $0 < c < g$ *such that if* $\mathcal{E}_1^\sigma(0) \leq \kappa_0$, $A\omega^2 + A\omega^3 \leq \gamma_0$, *and* $g - A\omega^2 > c$, *then there exists a unique solution* (u, p, η) *solving eq.* (2.3.10) *on the temporal interval* $(0, \infty)$, *achieves the initial data, and there exists constants* $\lambda > 0$ *and* $C > 0$, *depending on* A, ω *and* σ *such that the solution obeys the energy estimate*

$$\sup_{0 \leq t \leq \infty} e^{\lambda t} \mathcal{E}_1^\sigma(t) + \int_0^\infty \mathcal{D}_1^\sigma(t)\, dt \leq C \mathcal{E}_1^\sigma(0). \tag{3.3.1}$$

Questions?