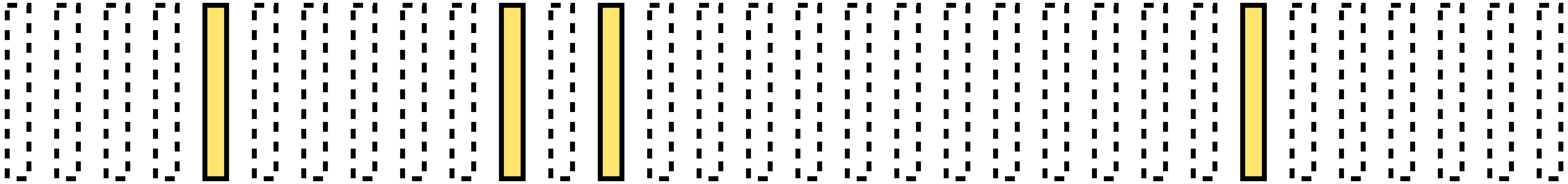

Thesis Proposal:

Advances in Algorithms for Matrix Approximation

via **Sampling** and **Sketching**

Taisuke (Tai) Yasuda

Carnegie Mellon University
Computer Science Department

April 10, 2023

Thesis Committee:

- David P. Woodruff (Carnegie Mellon University, Chair)
- Anupam Gupta (Carnegie Mellon University)
- Richard Peng (Carnegie Mellon University)
- Cameron Musco (University of Massachusetts Amherst)

Matrix Approximation

“Turning big data into tiny data”

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- Modern large-scale machine learning problems deal with huge matrices!
  - Billions of training examples and labels
  - Thousands of features
- Goal: replace a large dataset with a smaller dataset to **improve efficiency of data analytic tasks**

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Randomized Numerical Linear Algebra

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  - Key techniques: **sampling**, **sketching**, and **optimization**

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4. **Applications**: can techniques for matrix approximation be applied to solve problems in adjacent areas of computer science?

Subspace Embeddings and Linear Regression

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  - Can we design efficient approximation algorithms for linear regression?

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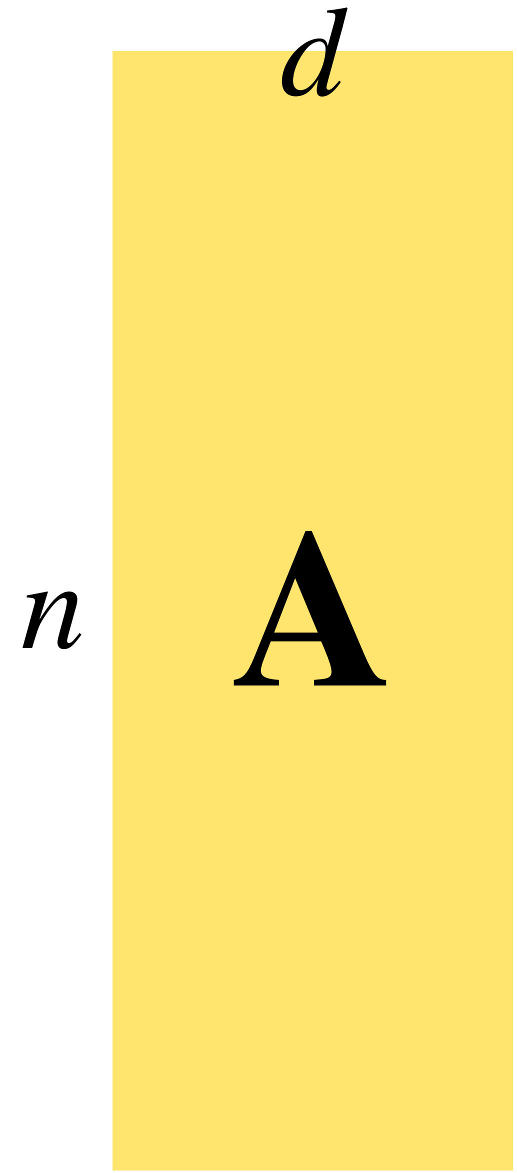
Approximating the Linear Regression Loss Function



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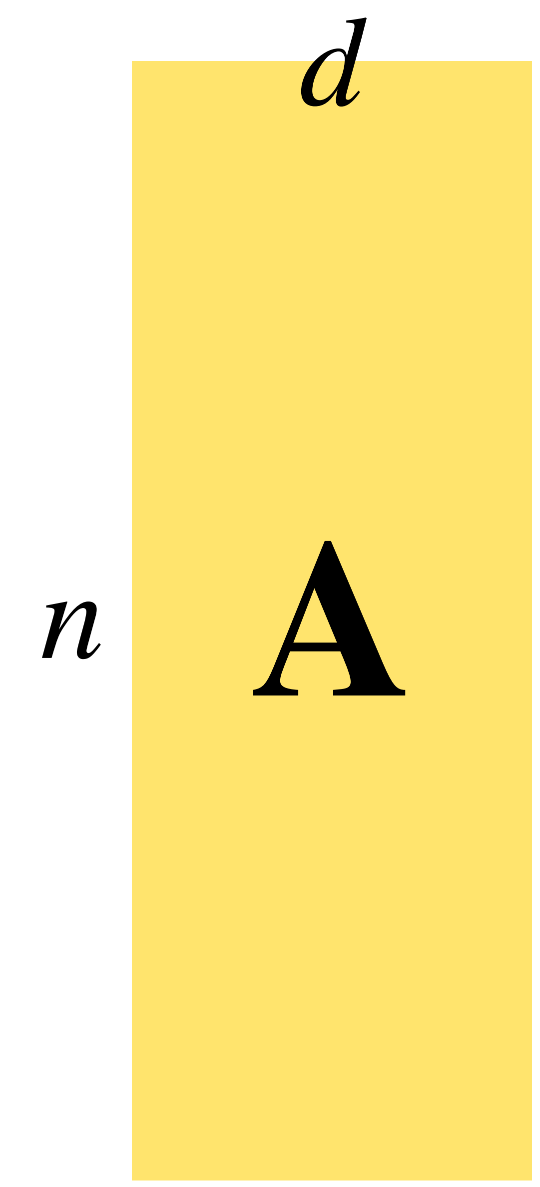

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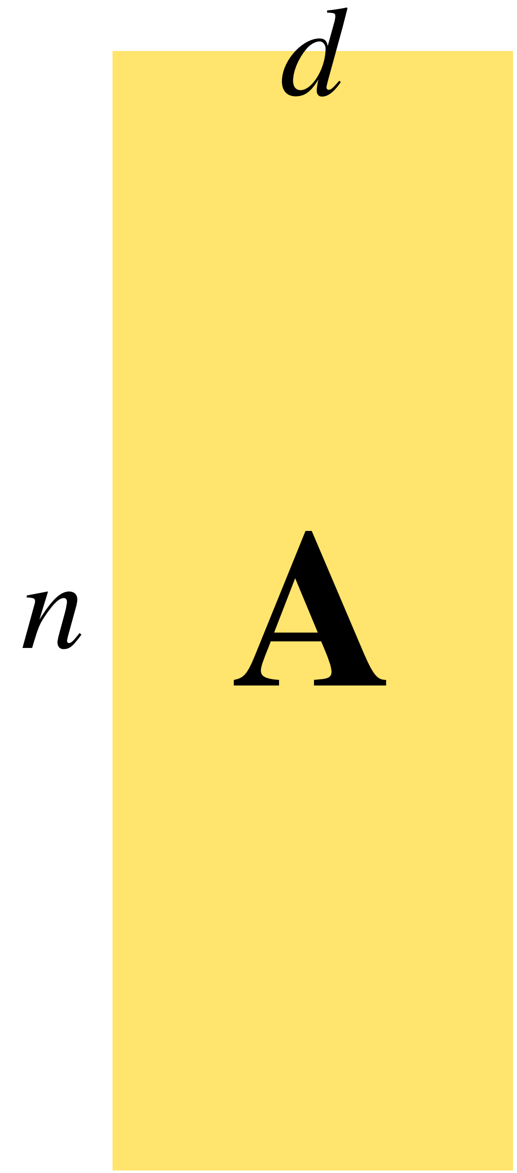

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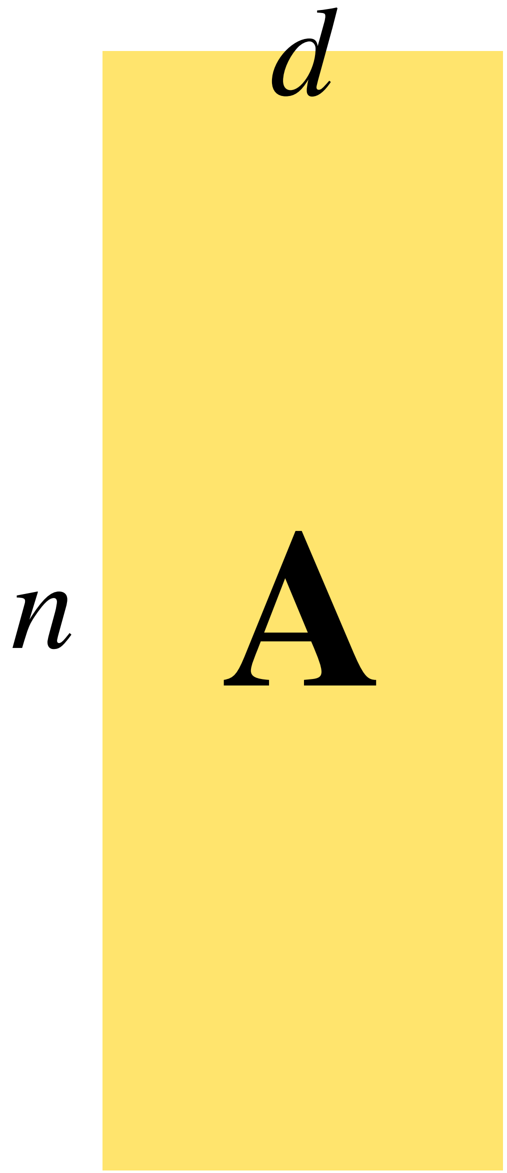

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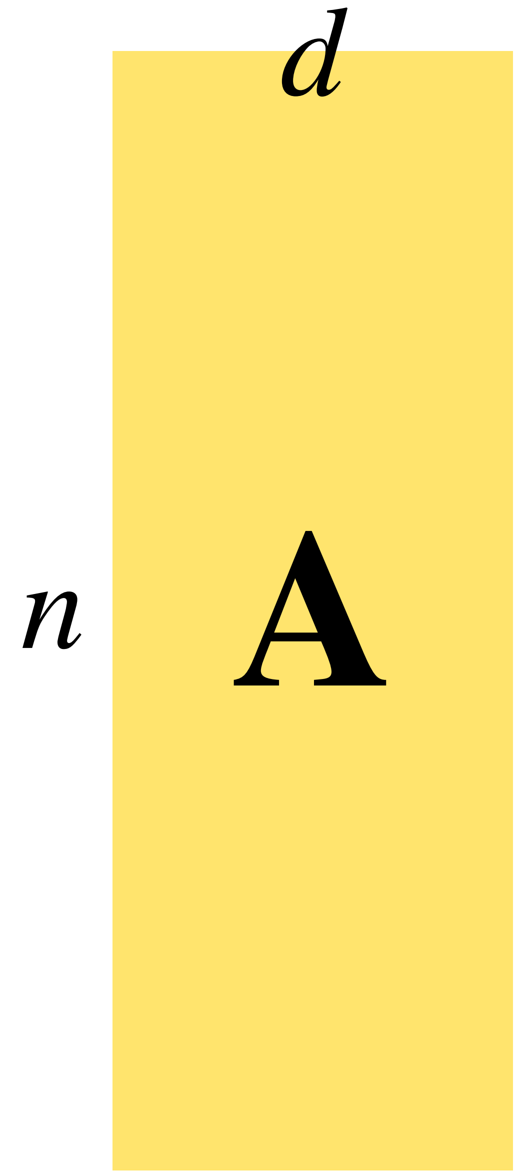

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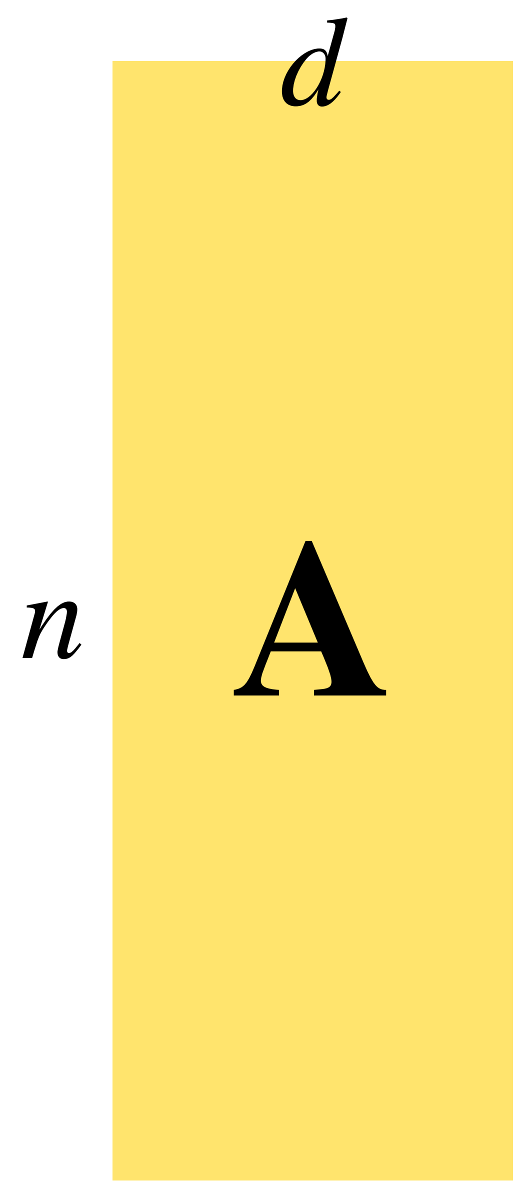

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“Sketch"

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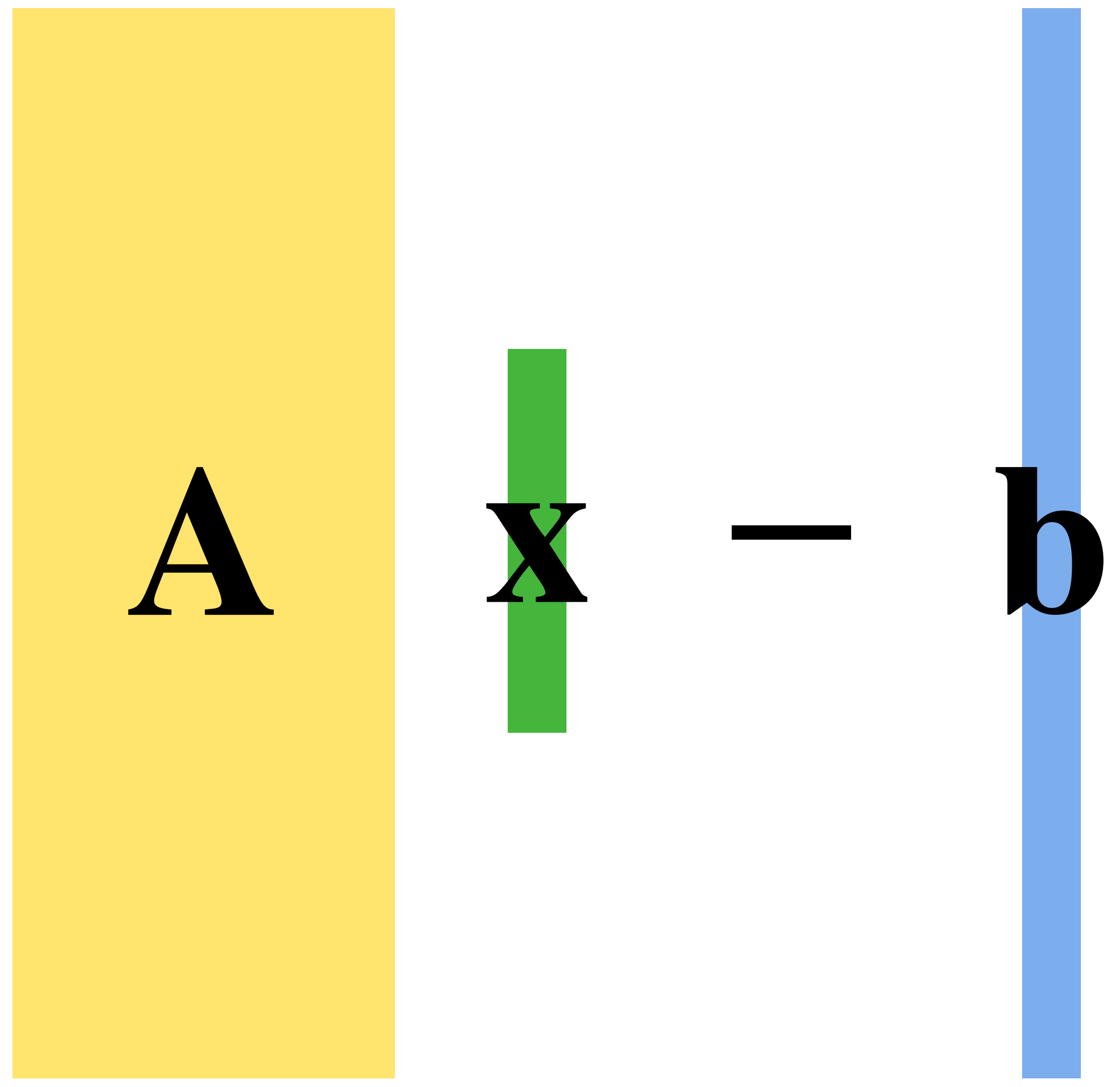

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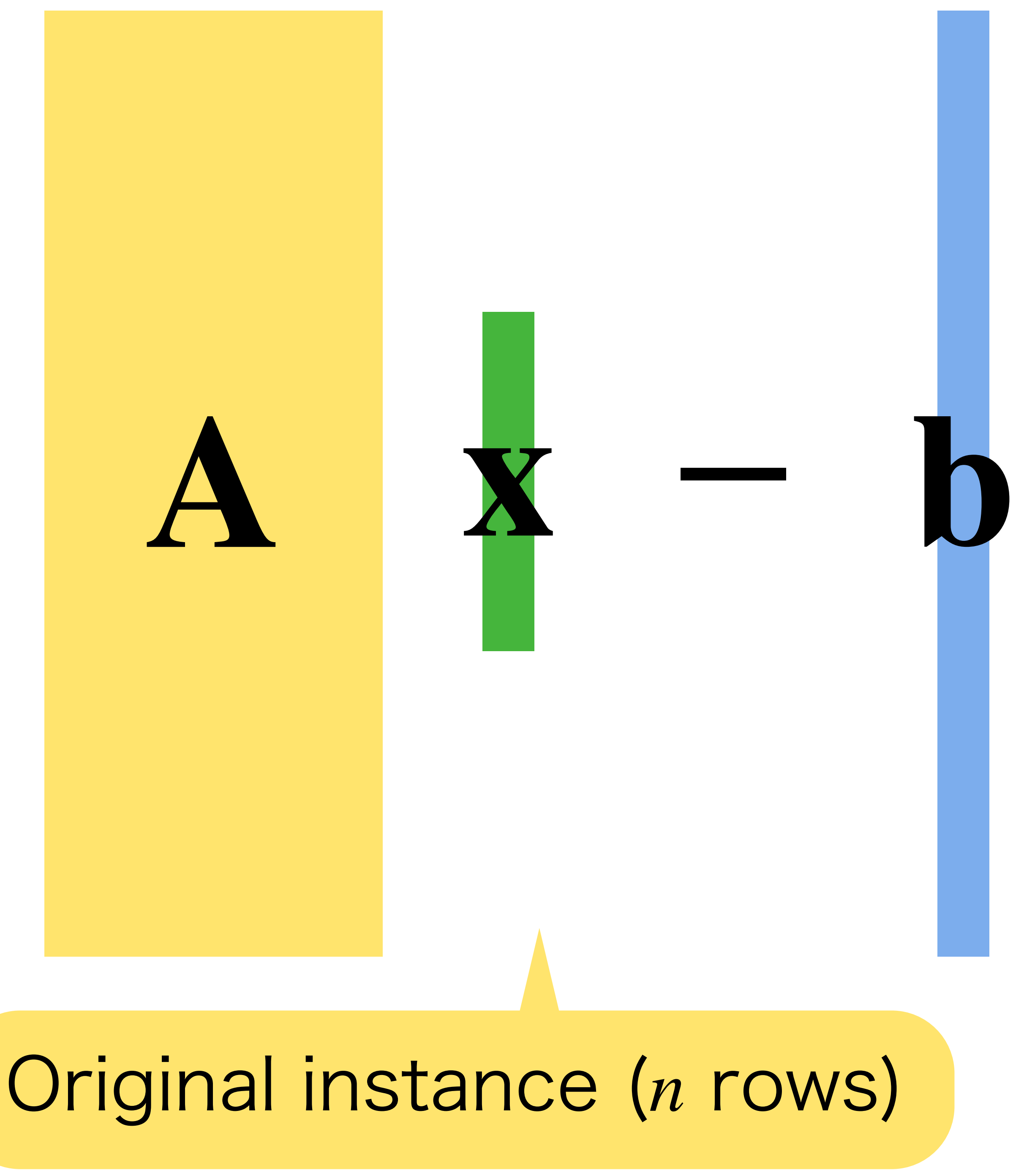

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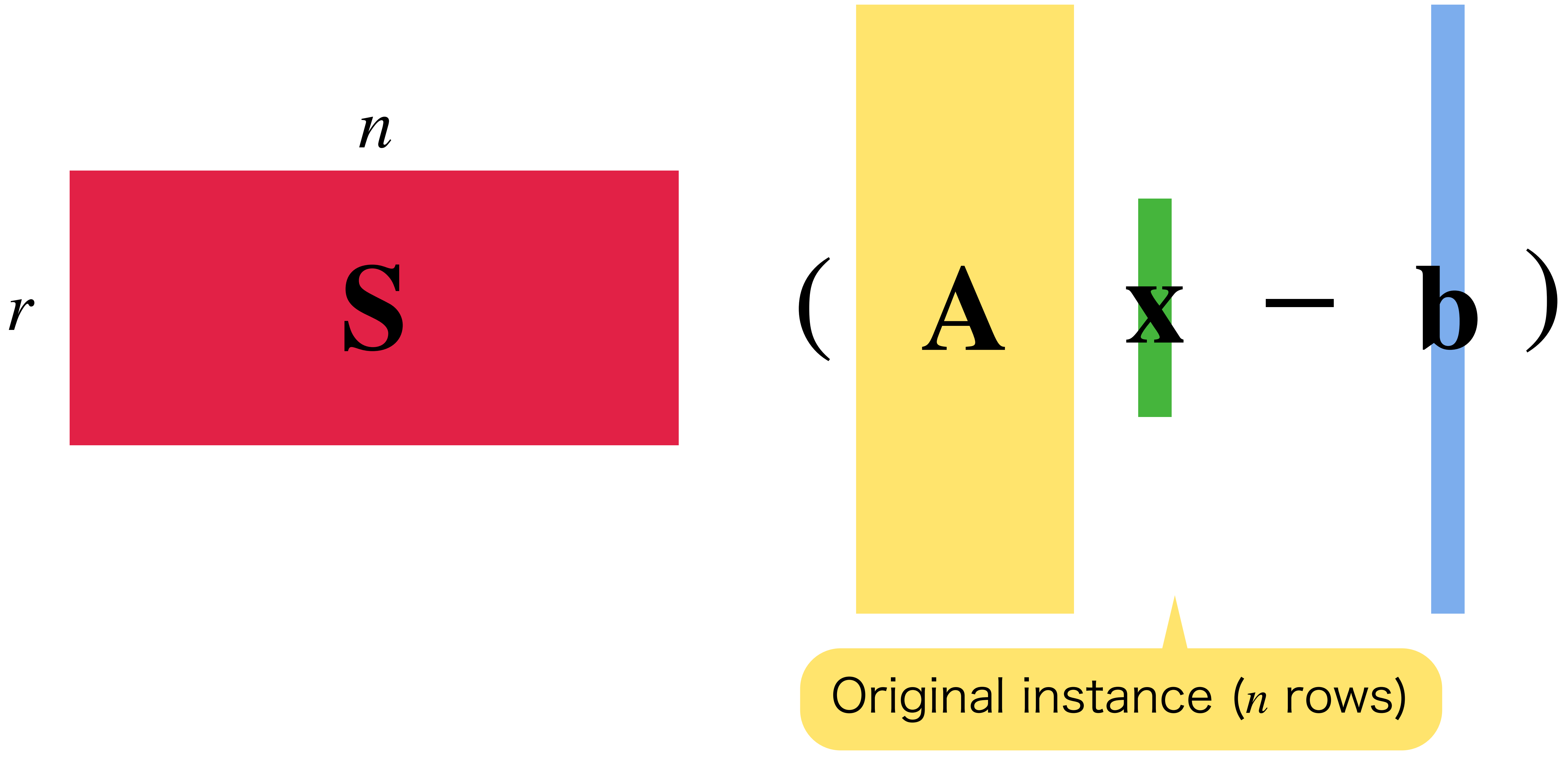

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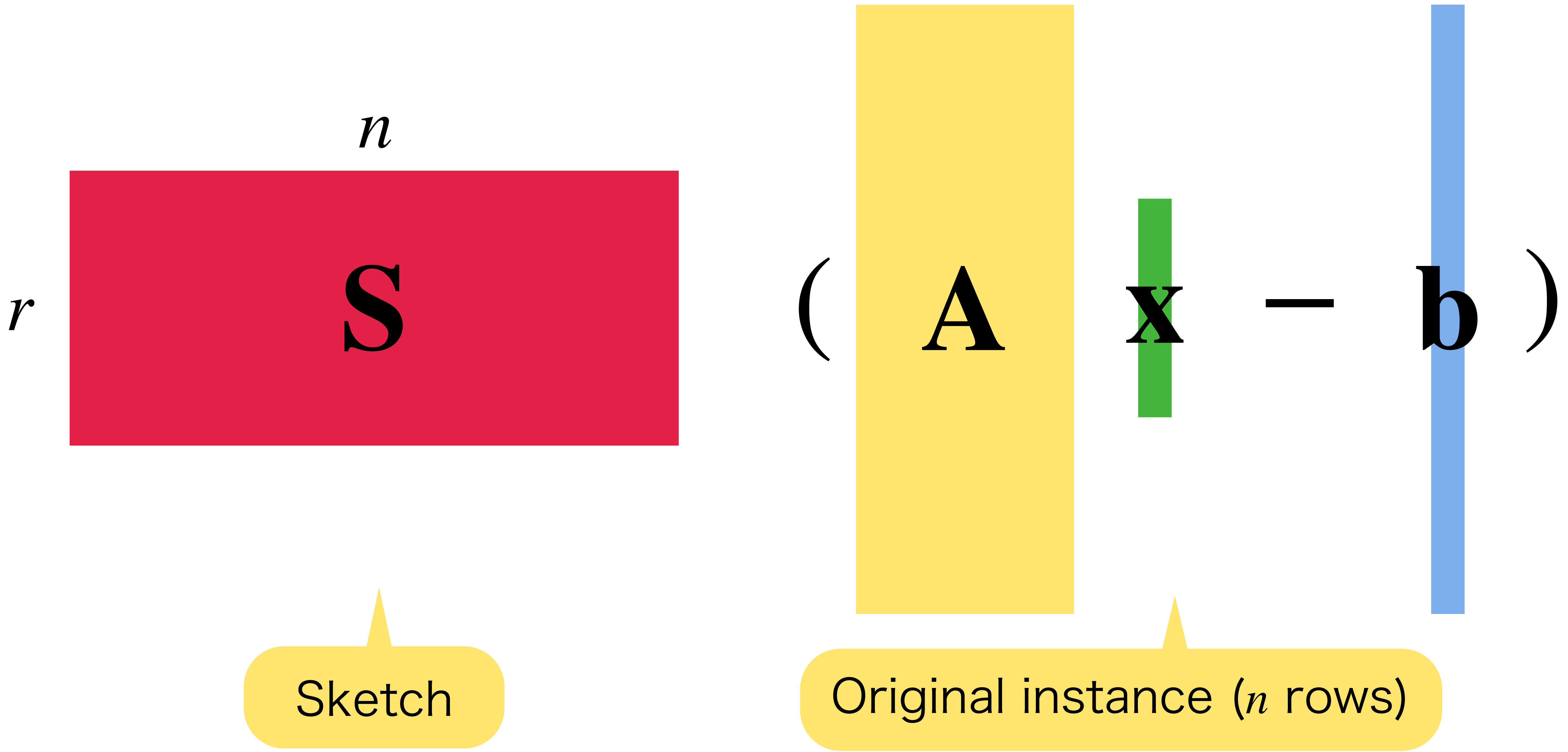

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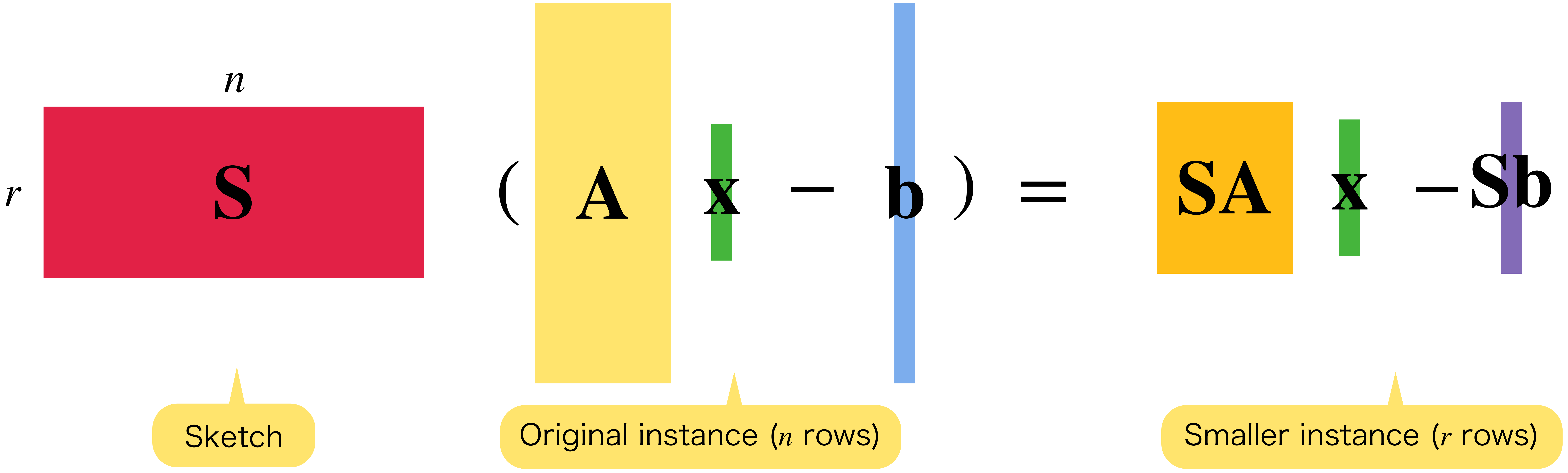

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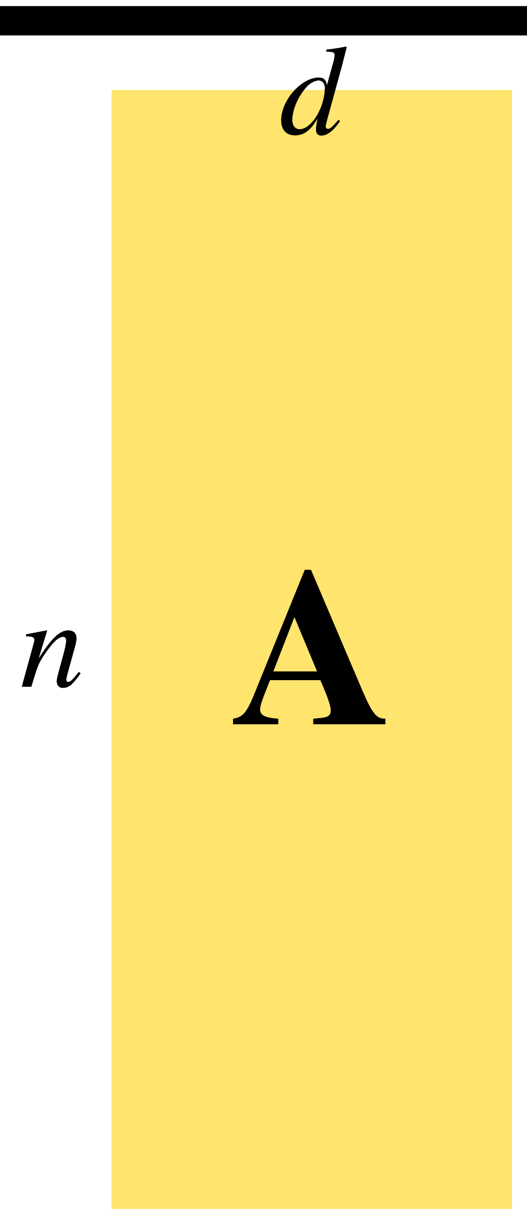
Cost of $\mathbf{SA}$ and $\mathbf{Sb}$ approximates cost of $\mathbf{A}$ and $\mathbf{b}$

- Solve linear regression on $r \times d$ matrix $\mathbf{SA}$ instead of $n \times d$ matrix $\mathbf{A}$!

Matrix Approximation

Oblivious Subspace Embeddings

Theorem (Sarlos 2006). Let $\kappa = (1+\varepsilon)$. Let $r = \tilde{O}(\varepsilon^{-2}d)$. If $\mathbf{S}$ is an $r \times n$ Gaussian matrix, then $\mathbf{S}$ is a subspace embedding for any $\mathbf{A}$ with distortion κ, with probability 99%.



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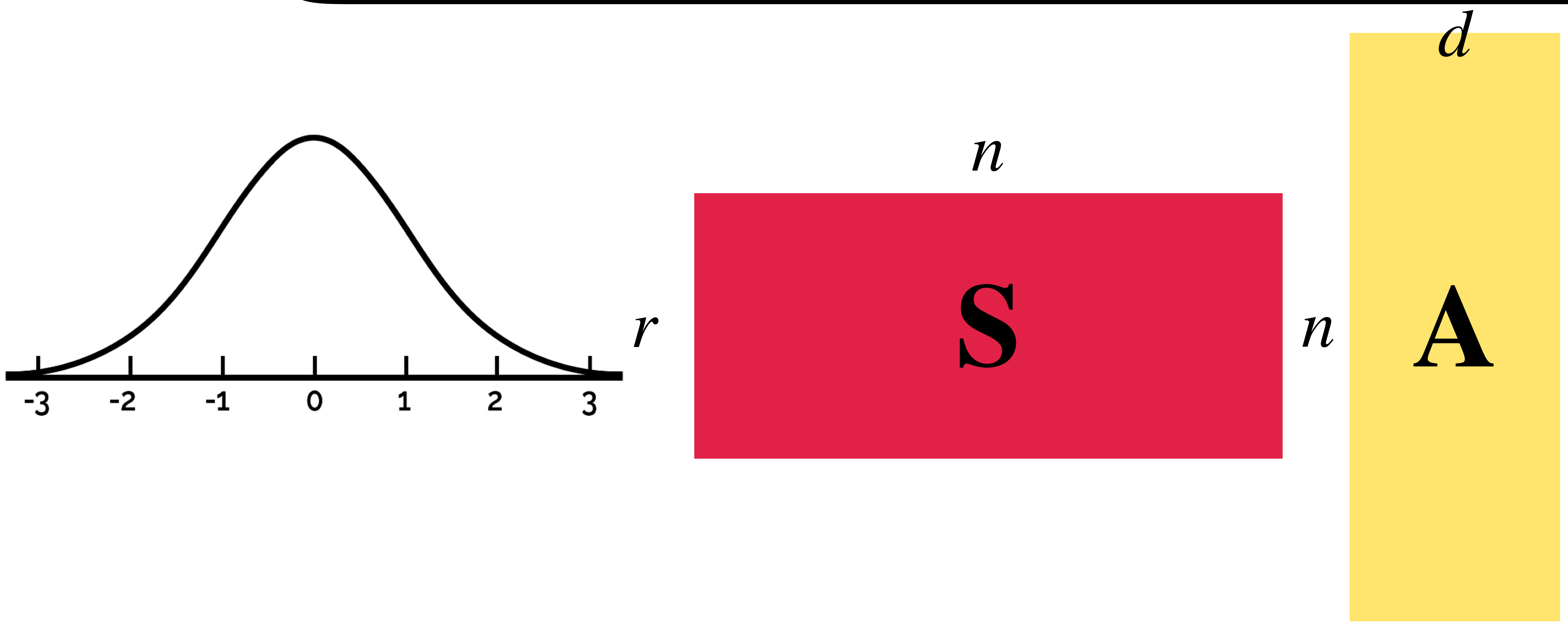

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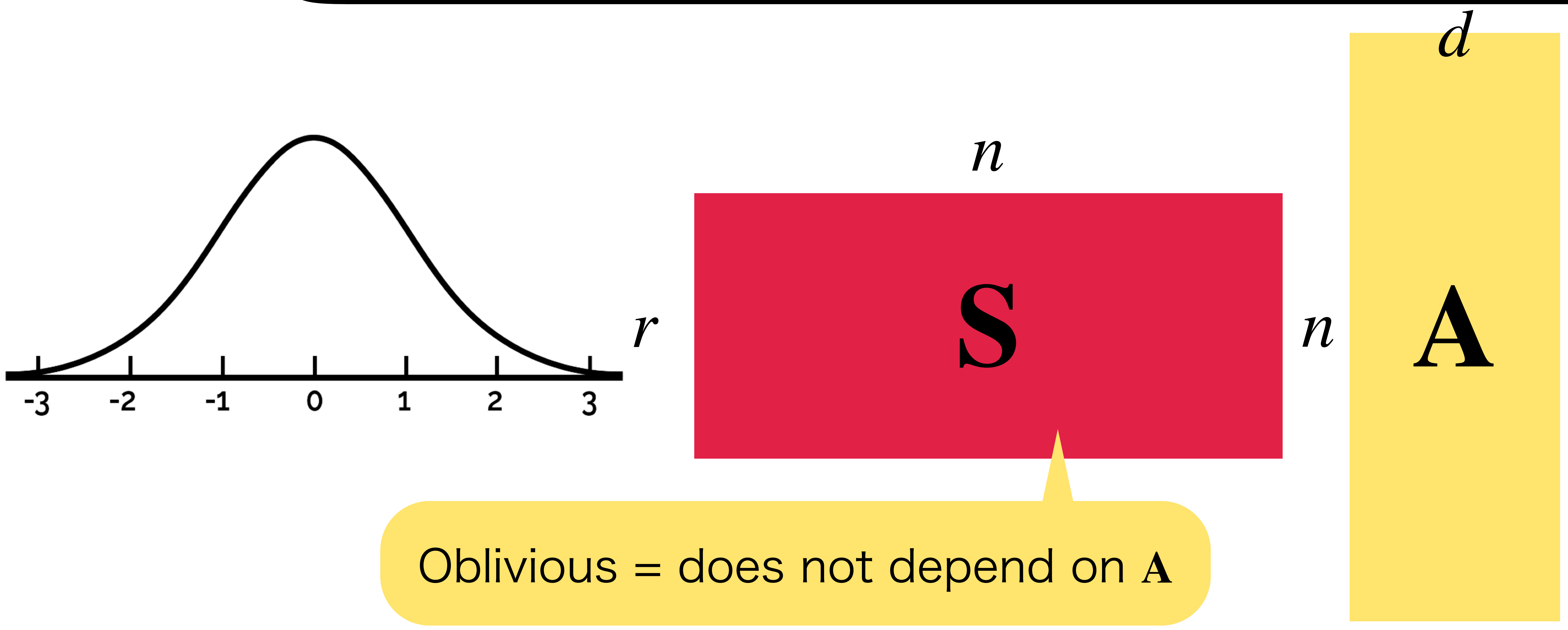

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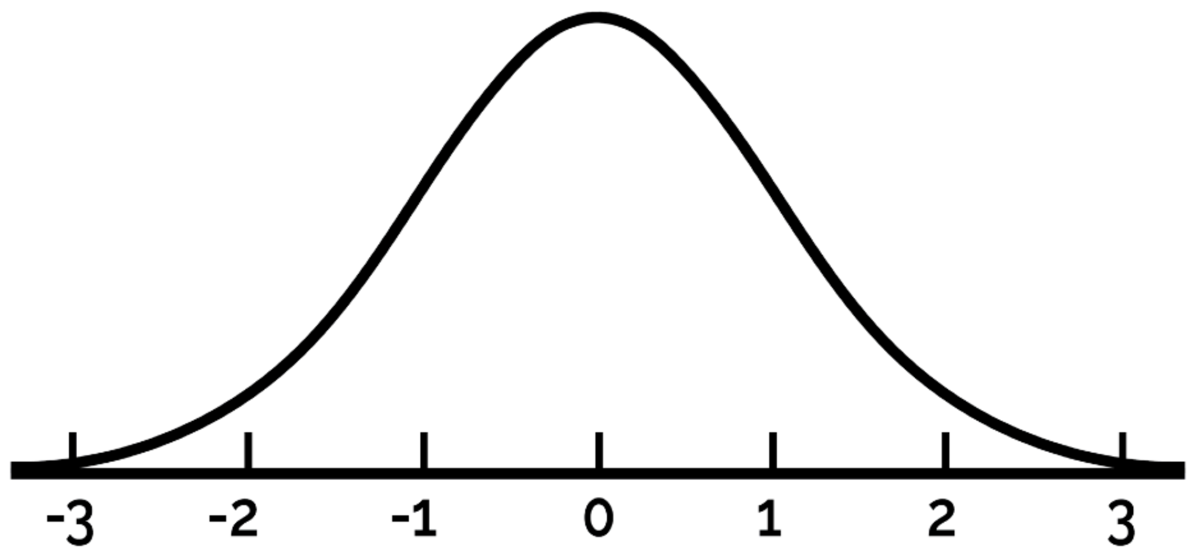

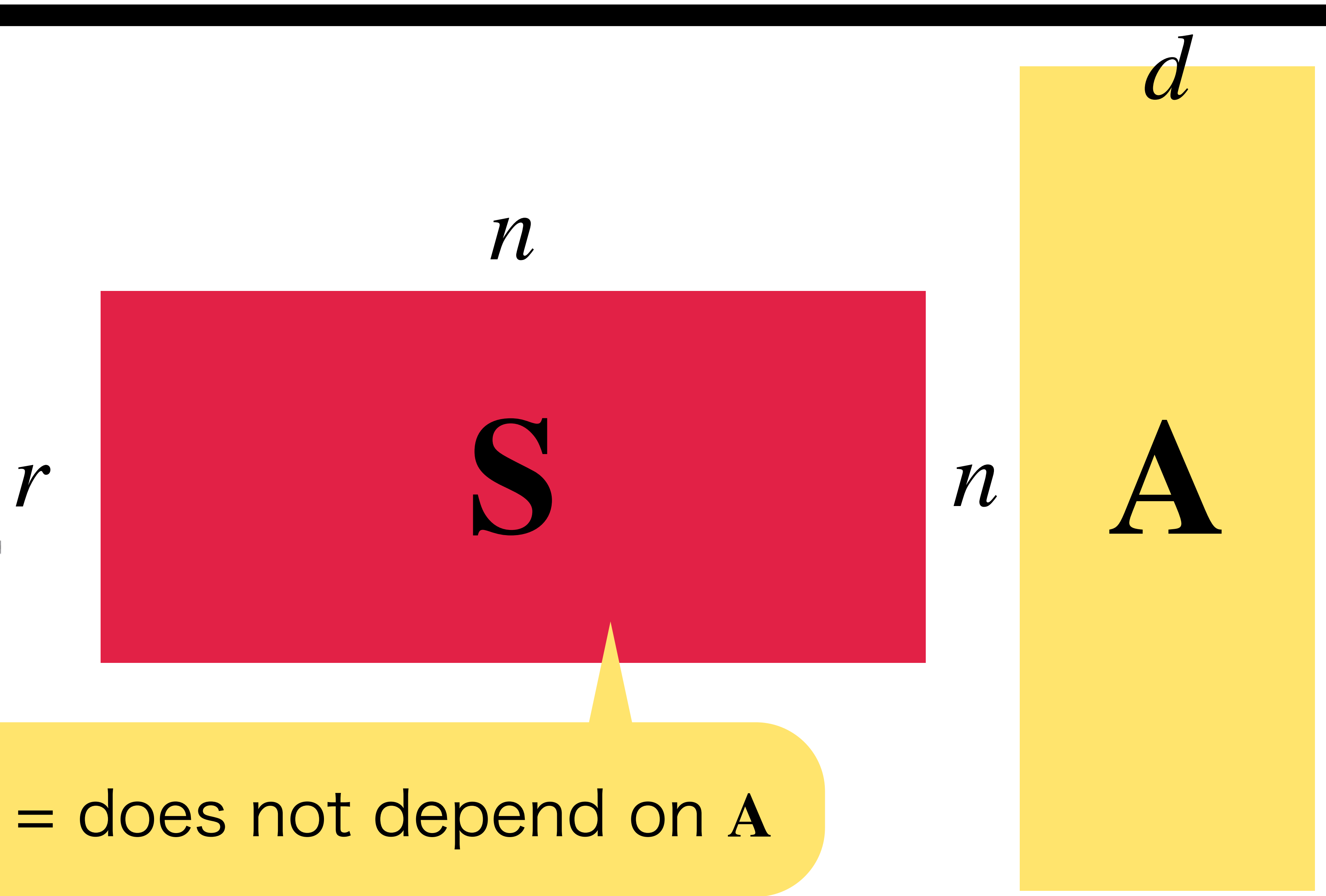

Oblivious = does not depend on $\mathbf{A}$

So ℓ_2 regression is resolved.
What's next?

Matrix Approximation

ℓ_p Linear Regression

Matrix Approximation

ℓ_p Linear Regression

ℓ_2 linear regression

$$\min_{\mathbf{x}\in\mathbb{R}^d} \|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2$$

- Minimize the **sum of squares** of errors

Matrix Approximation

ℓ_p Linear Regression

ℓ_1 linear regression

$$\min_{\mathbf{x}\in\mathbb{R}^d} \| \mathbf{A}\mathbf{x} - \mathbf{b} \|_1$$

- Minimize the **average** error
- **Robust** loss function

ℓ_2 linear regression

$$\min_{\mathbf{x}\in\mathbb{R}^d} \| \mathbf{A}\mathbf{x} - \mathbf{b} \|_2^2$$

- Minimize the **sum of squares** of errors

Matrix Approximation

ℓ_p Linear Regression

ℓ_1 linear regression

$$\min_{\mathbf{x}\in\mathbb{R}^d} \|\mathbf{Ax}-\mathbf{b}\|_1$$

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ℓ_∞ linear regression

$$\min_{\mathbf{x}\in\mathbb{R}^d} \|\mathbf{Ax}-\mathbf{b}\|_\infty$$

- Minimize the **worst-case** error
- **Sensitive** loss function

Matrix Approximation

ℓ_p Linear Regression

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$$\min_{\mathbf{x}\in\mathbb{R}^d} \|\mathbf{A}\mathbf{x}-\mathbf{b}\|_p^p$$

Matrix Approximation

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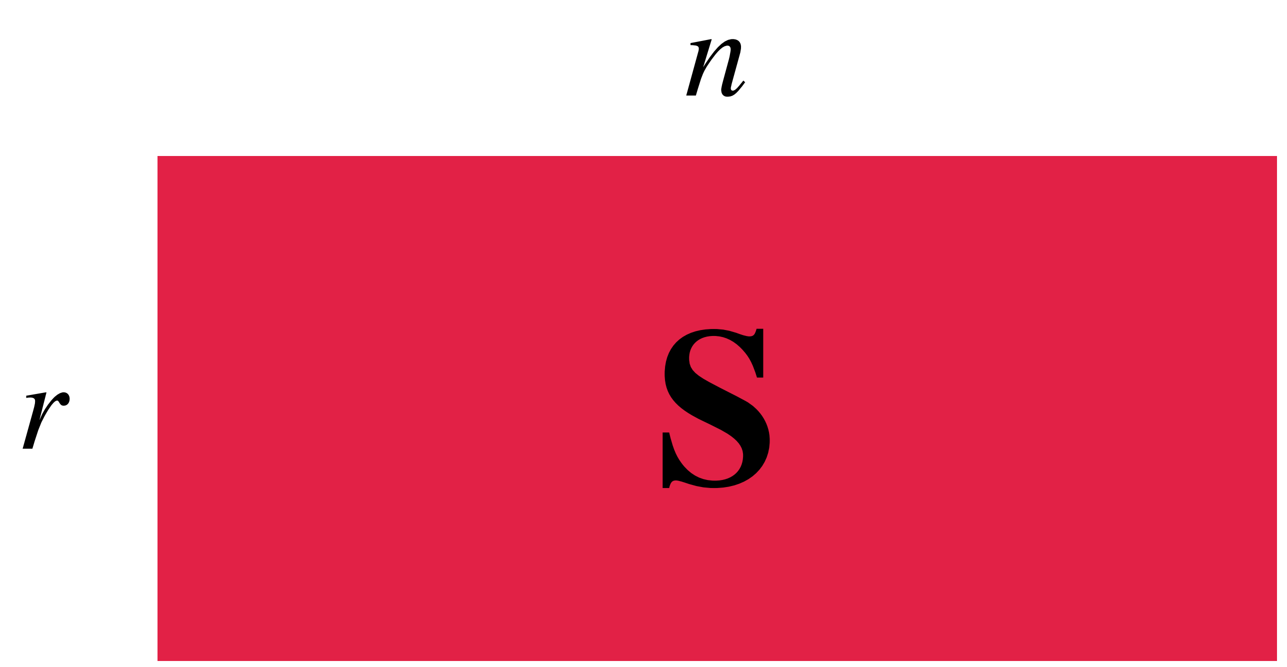
Question. What trade-offs are possible for oblivious subspace embeddings under the ℓ_p loss?

Matrix Approximation

Oblivious ℓ_p Subspace Embeddings

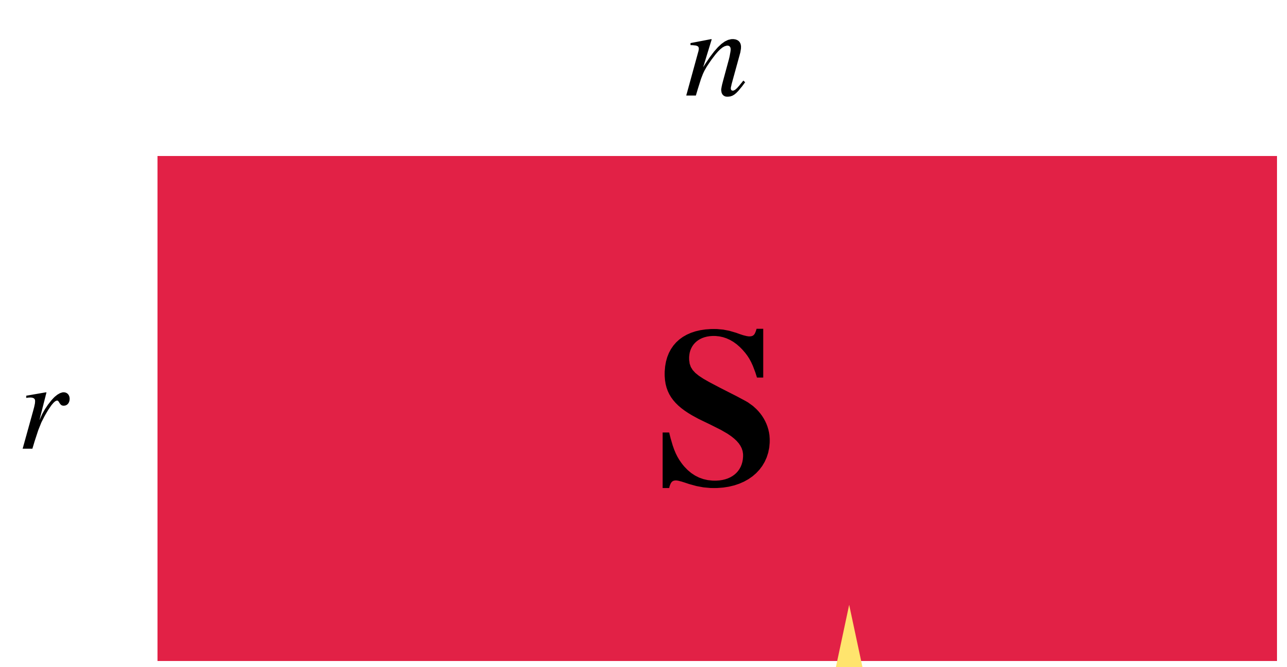
Matrix Approximation

Oblivious ℓ_p Subspace Embeddings



Matrix Approximation

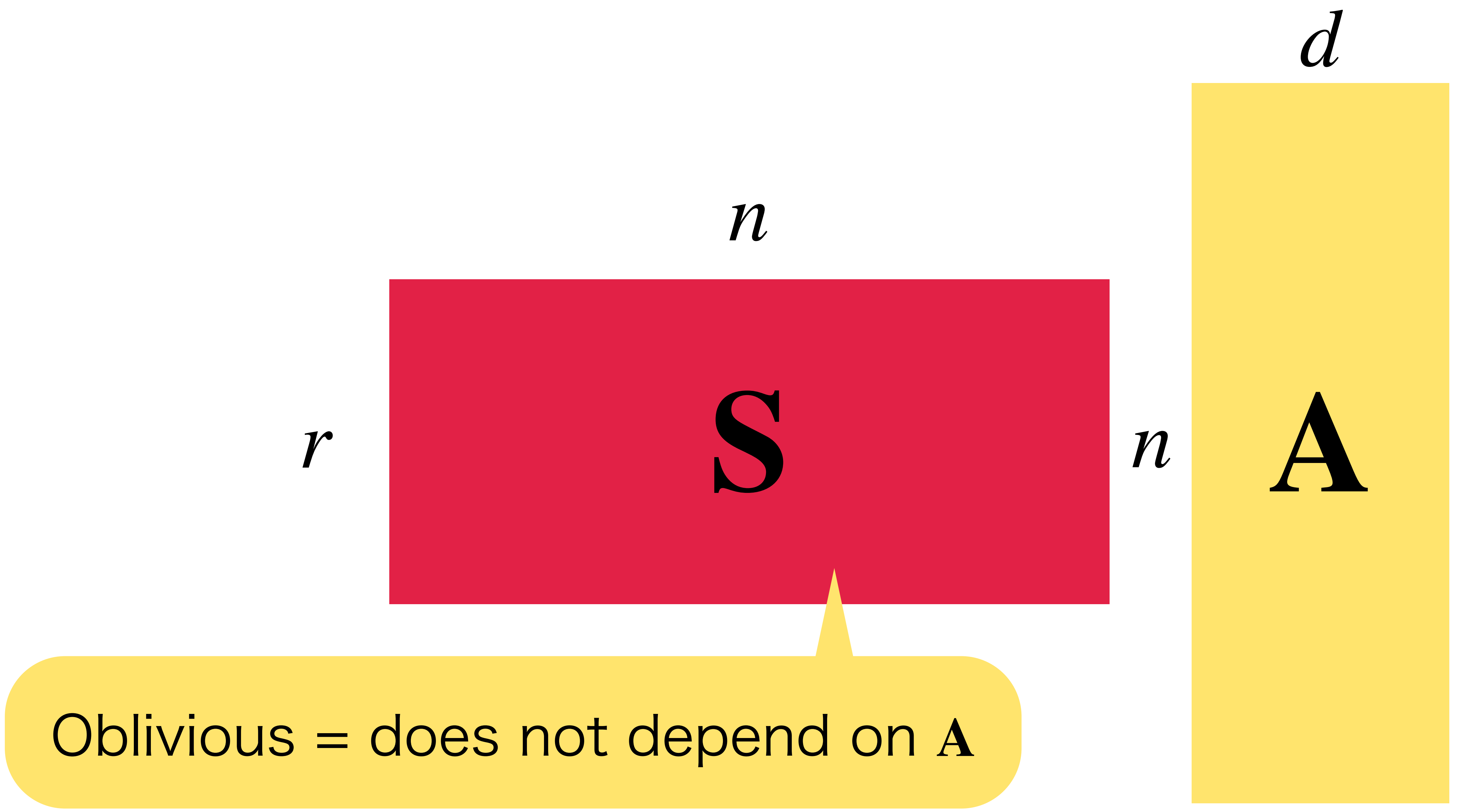
Oblivious ℓ_p Subspace Embeddings



Oblivious = does not depend on $\mathbf{A}$

Matrix Approximation

Oblivious ℓ_p Subspace Embeddings



Matrix Approximation

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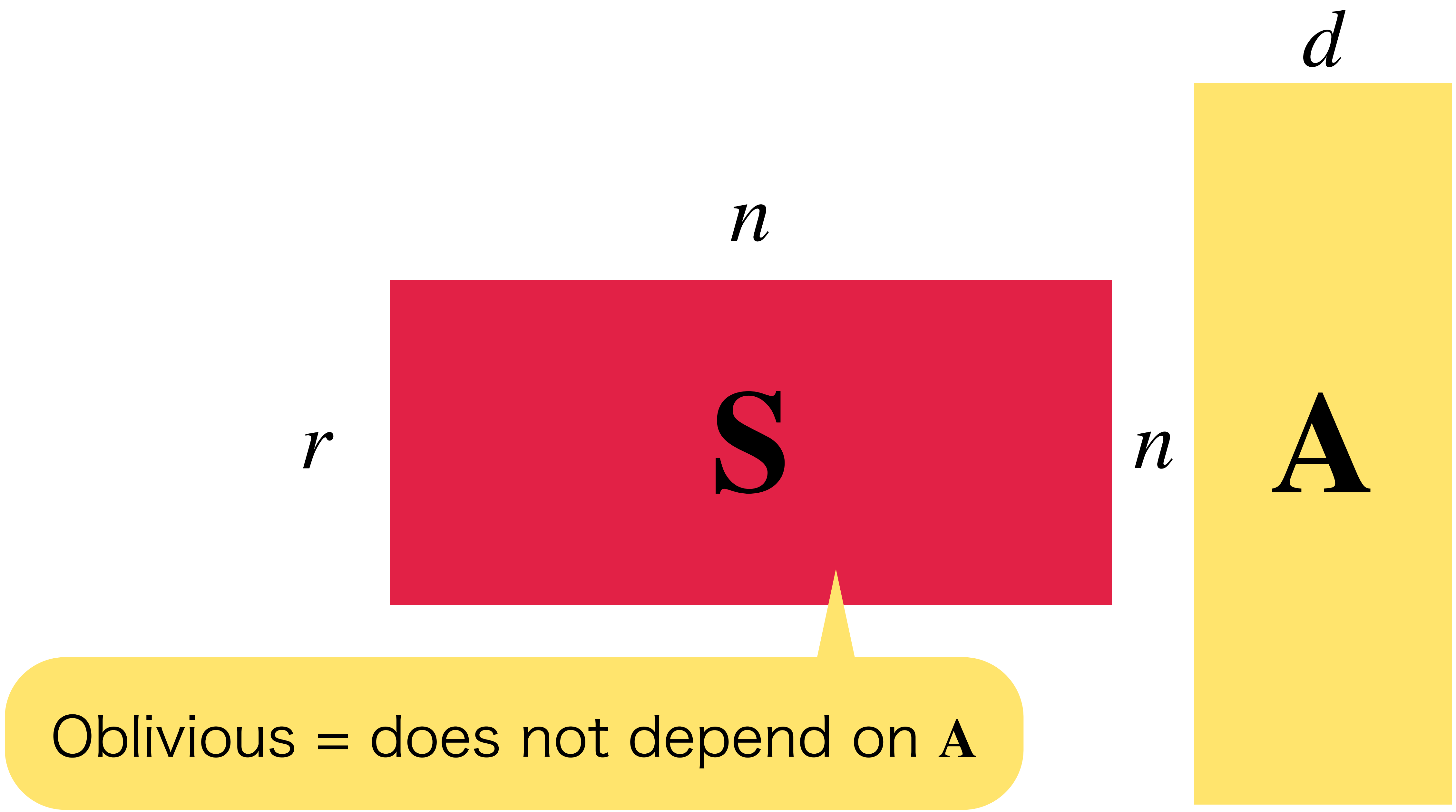


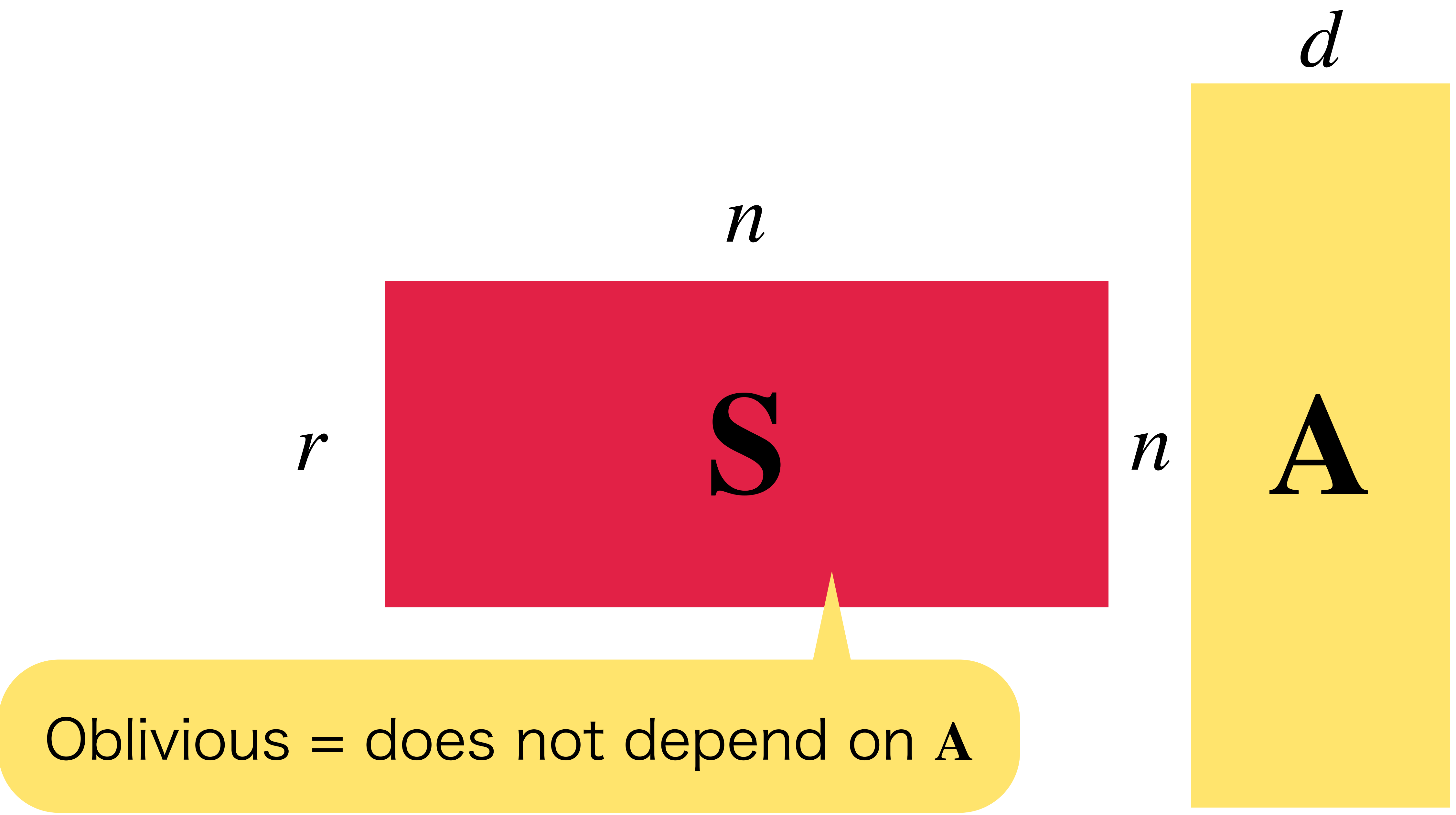
$$\|\mathbf{Ax}\|_p \leq \|\mathbf{SAx}\|_p \leq \kappa\|\mathbf{Ax}\|_p$$

for every $\mathbf{x} \in \mathbb{R}^d$

Matrix Approximation

Oblivious ℓ_p Subspace Embeddings

Two regimes for $1 \leq p < 2$



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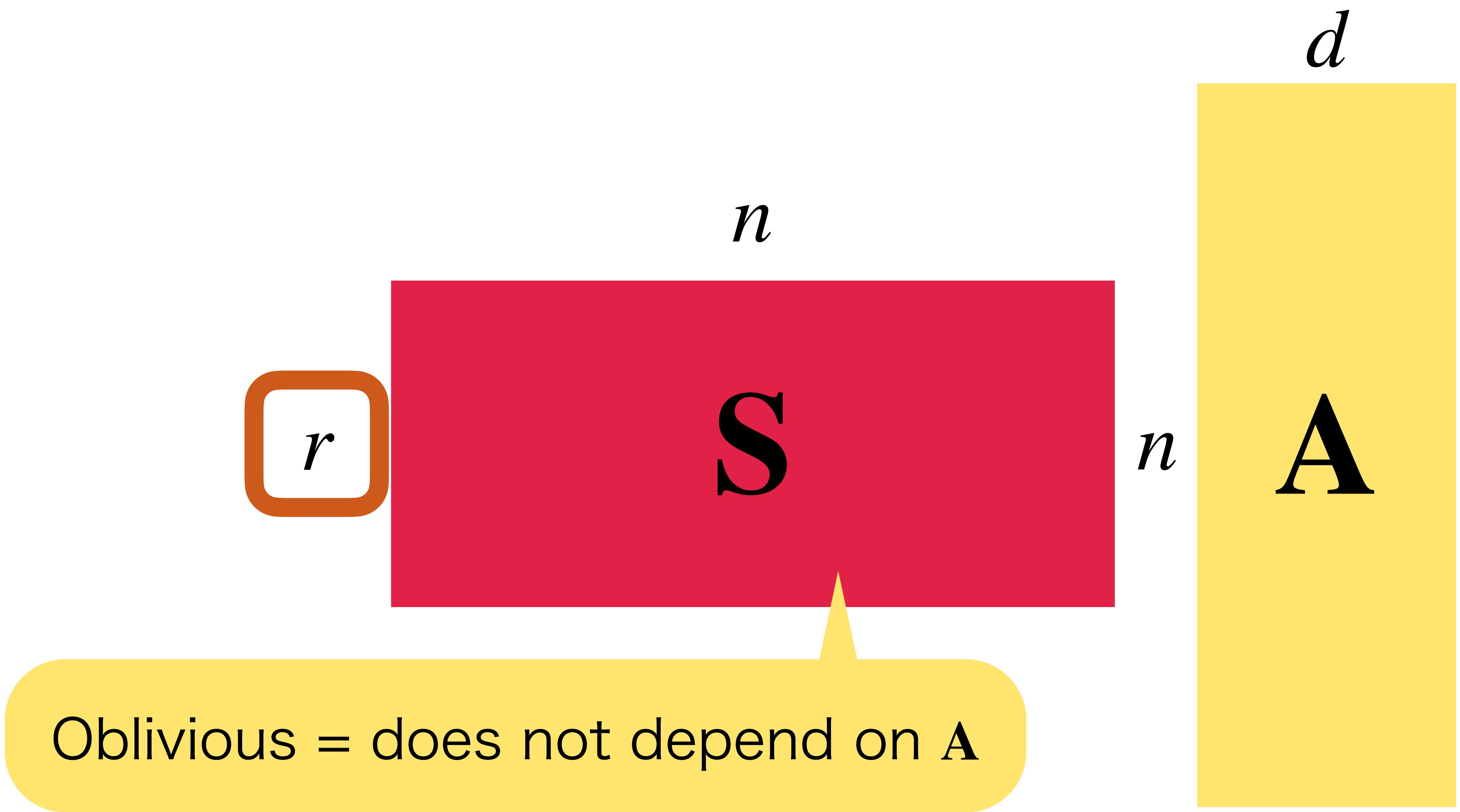

$$\|\mathbf{A}\mathbf{x}\|_p \leq \|\mathbf{S}\mathbf{A}\mathbf{x}\|_p \leq \kappa\|\mathbf{A}\mathbf{x}\|_p$$

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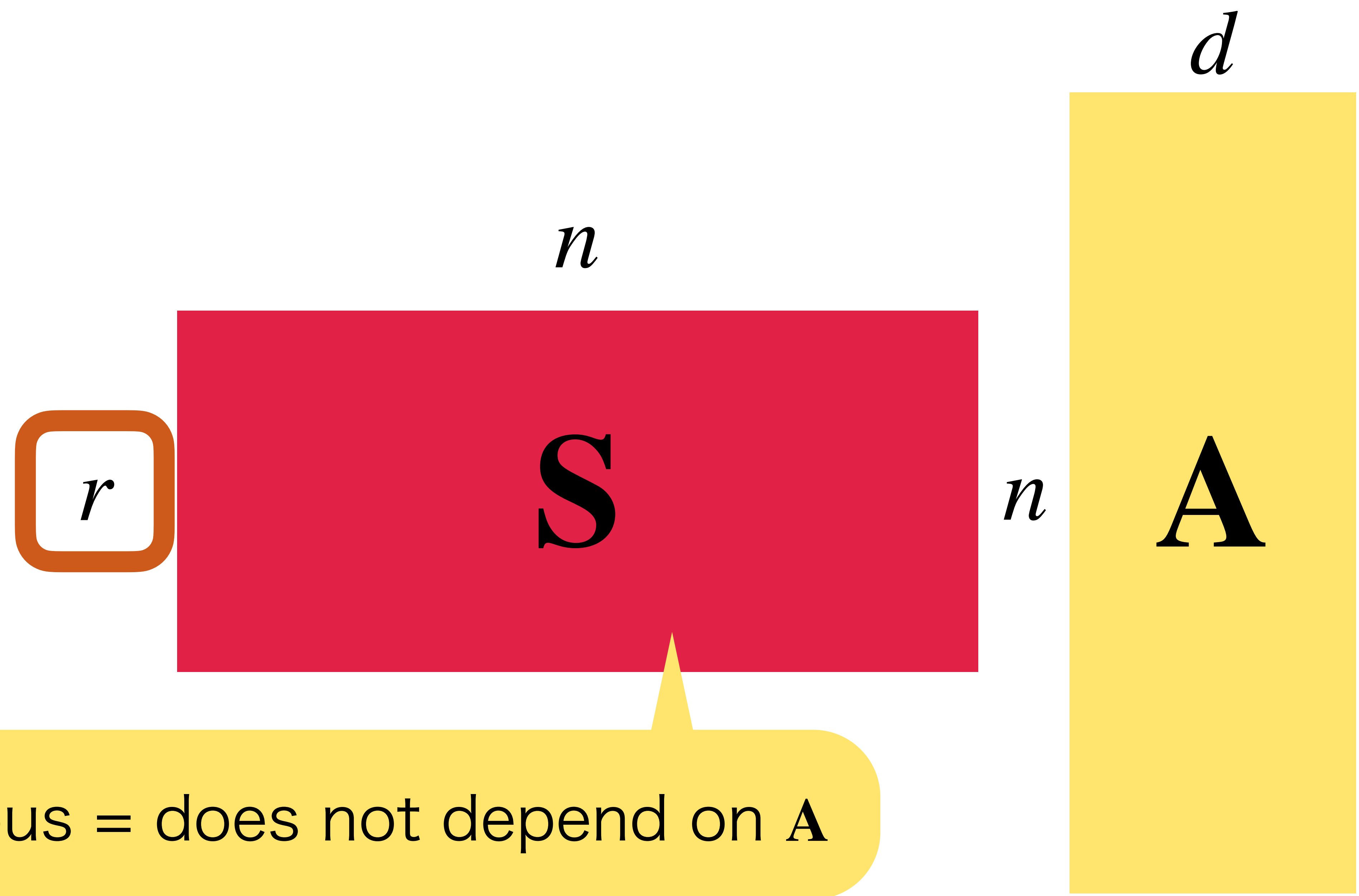

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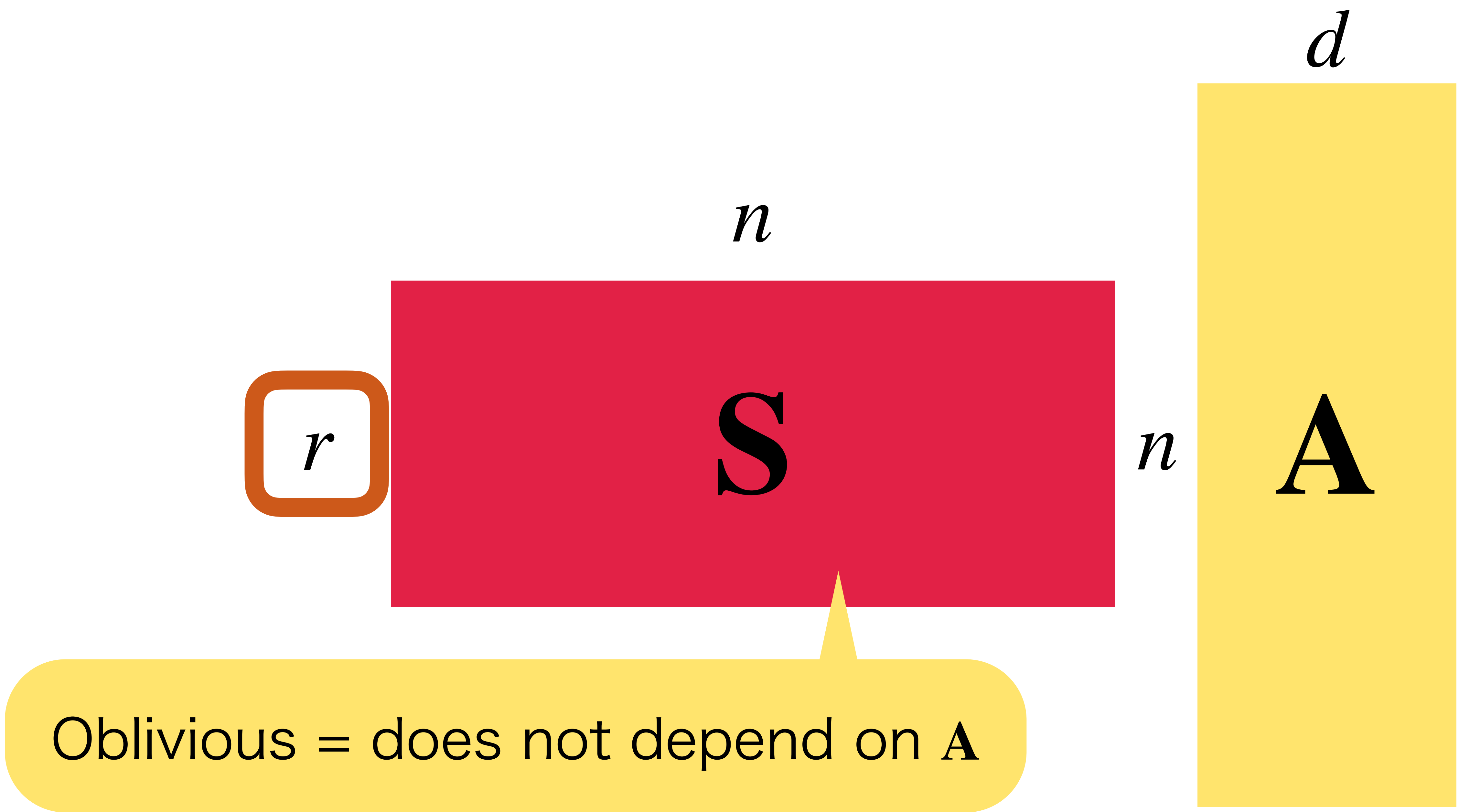

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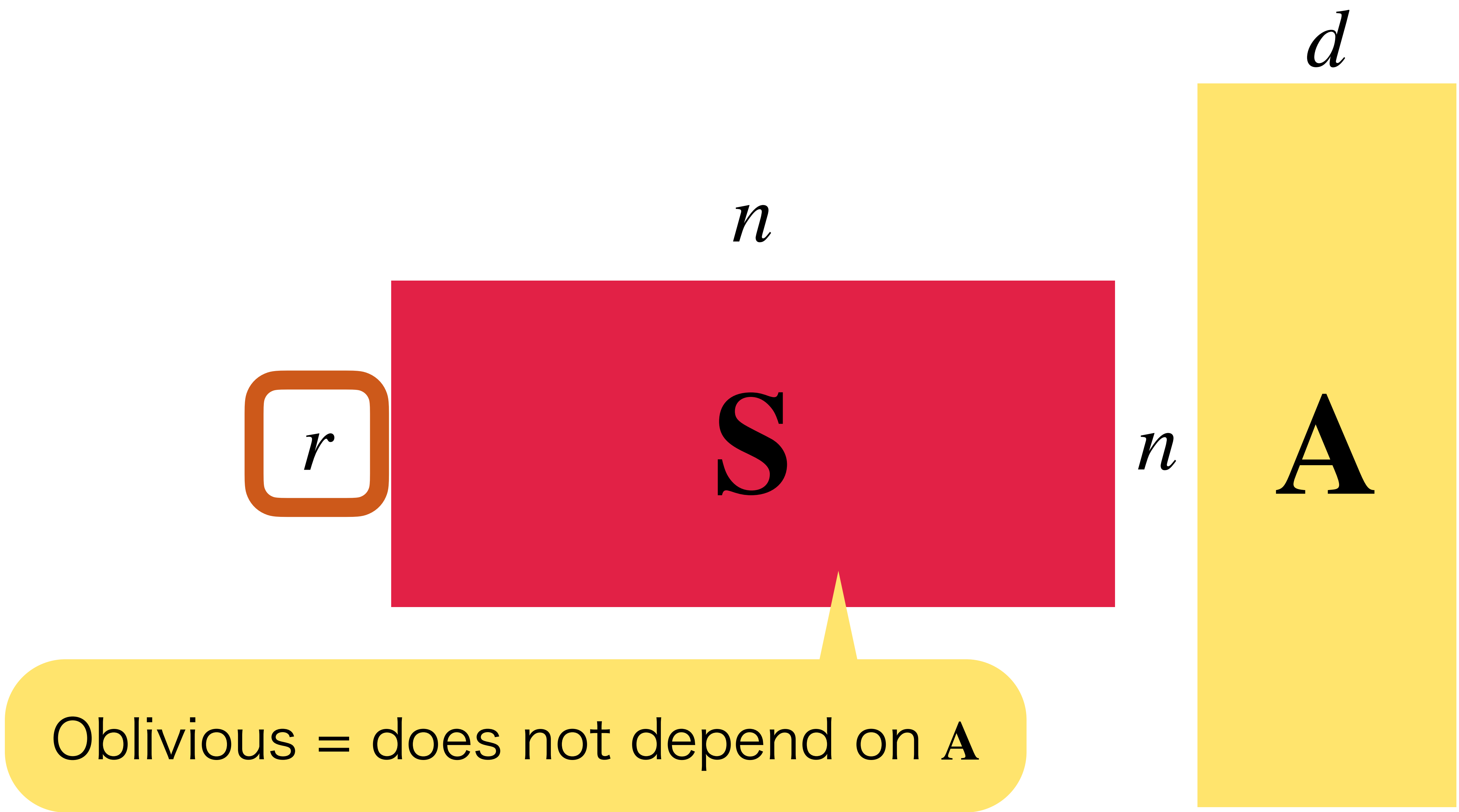

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Matrix Approximation

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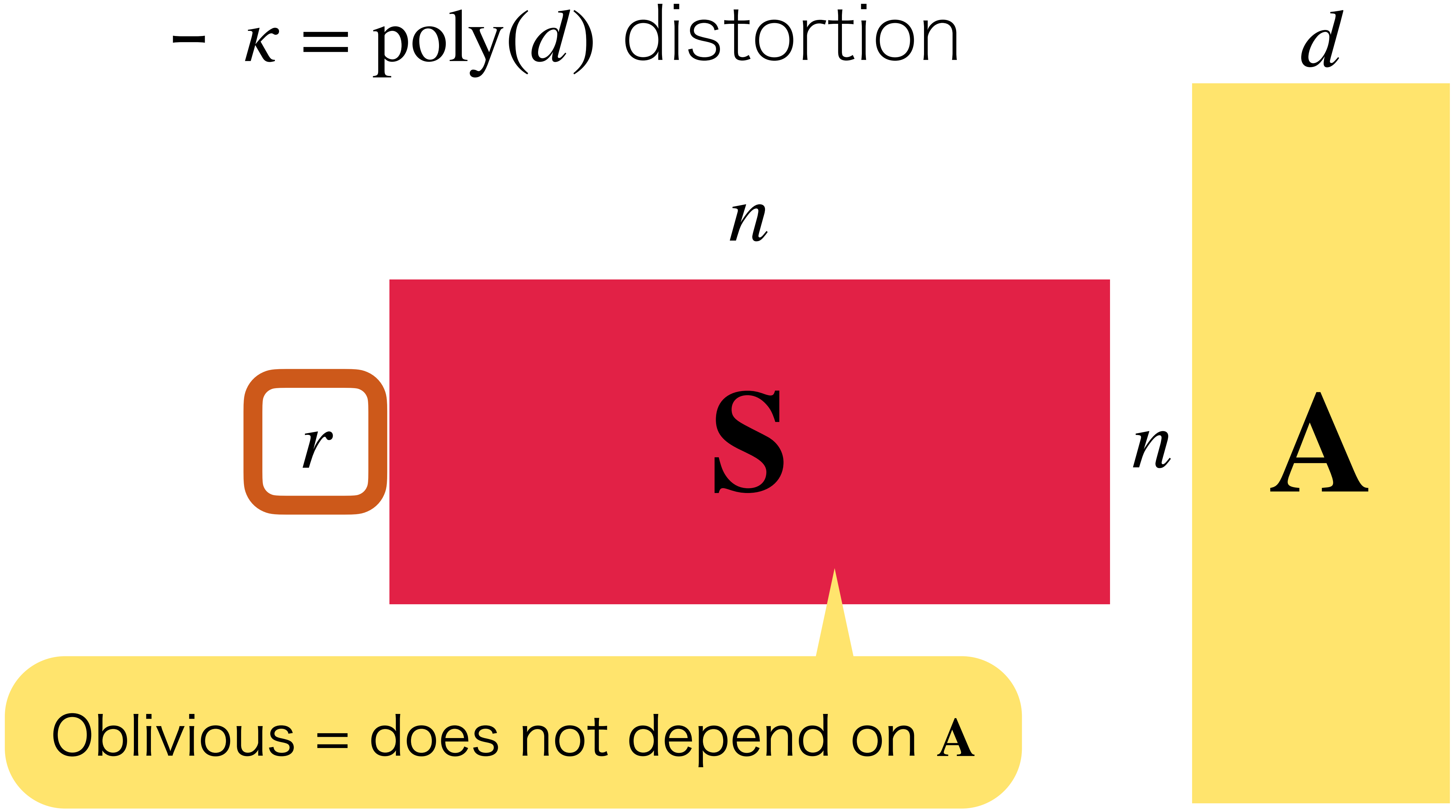

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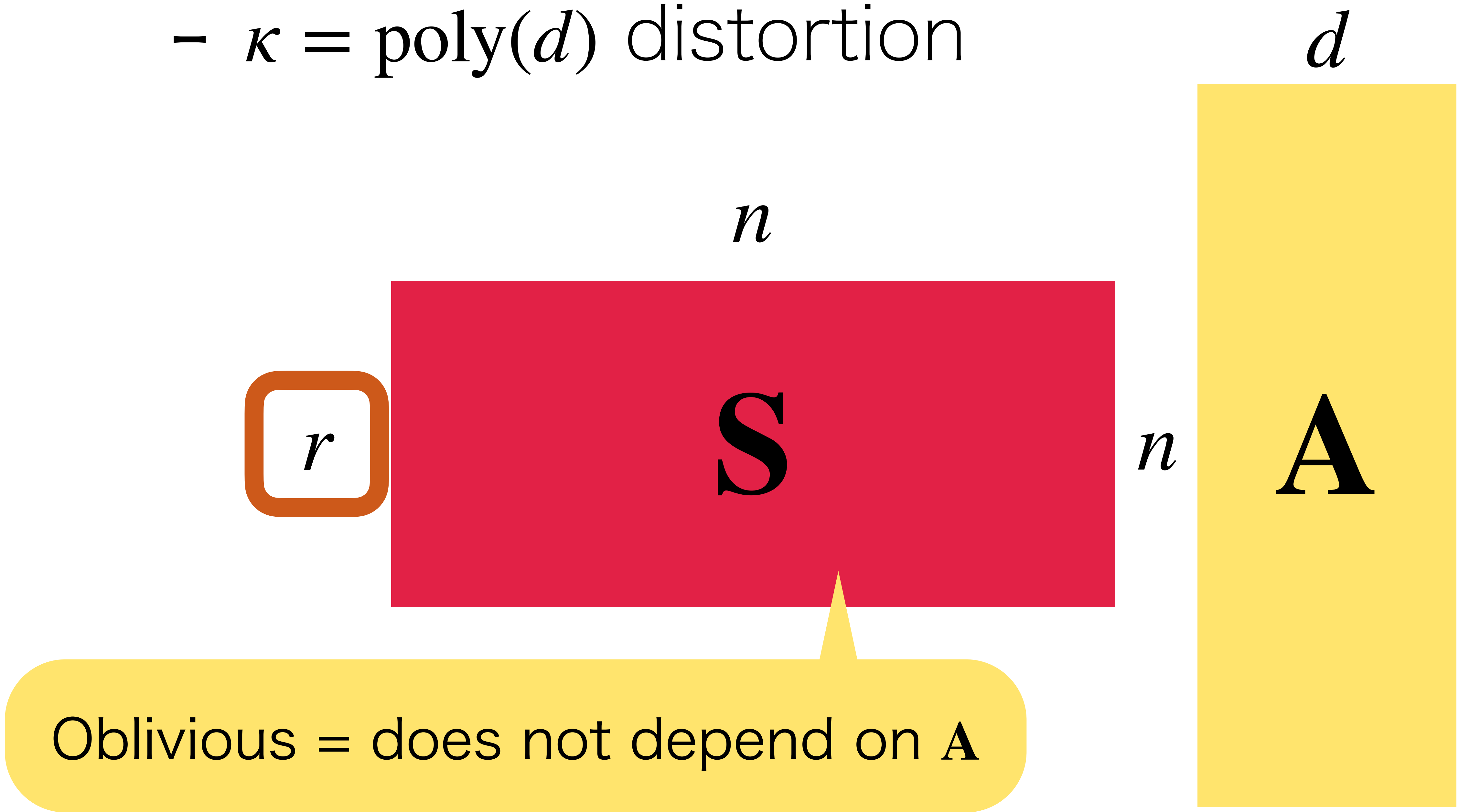

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Matrix Approximation

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Matrix Approximation

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Matrix Approximation

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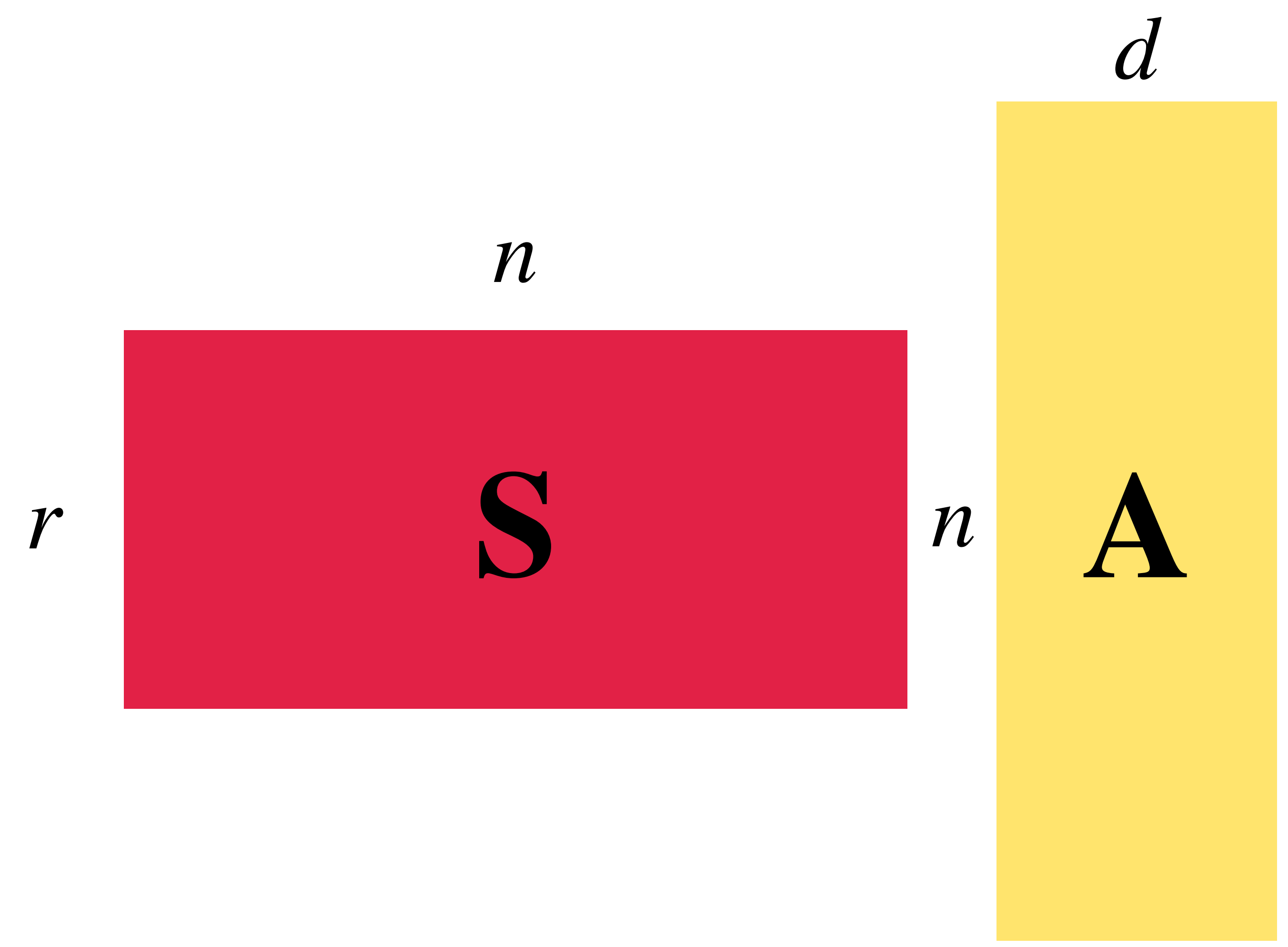
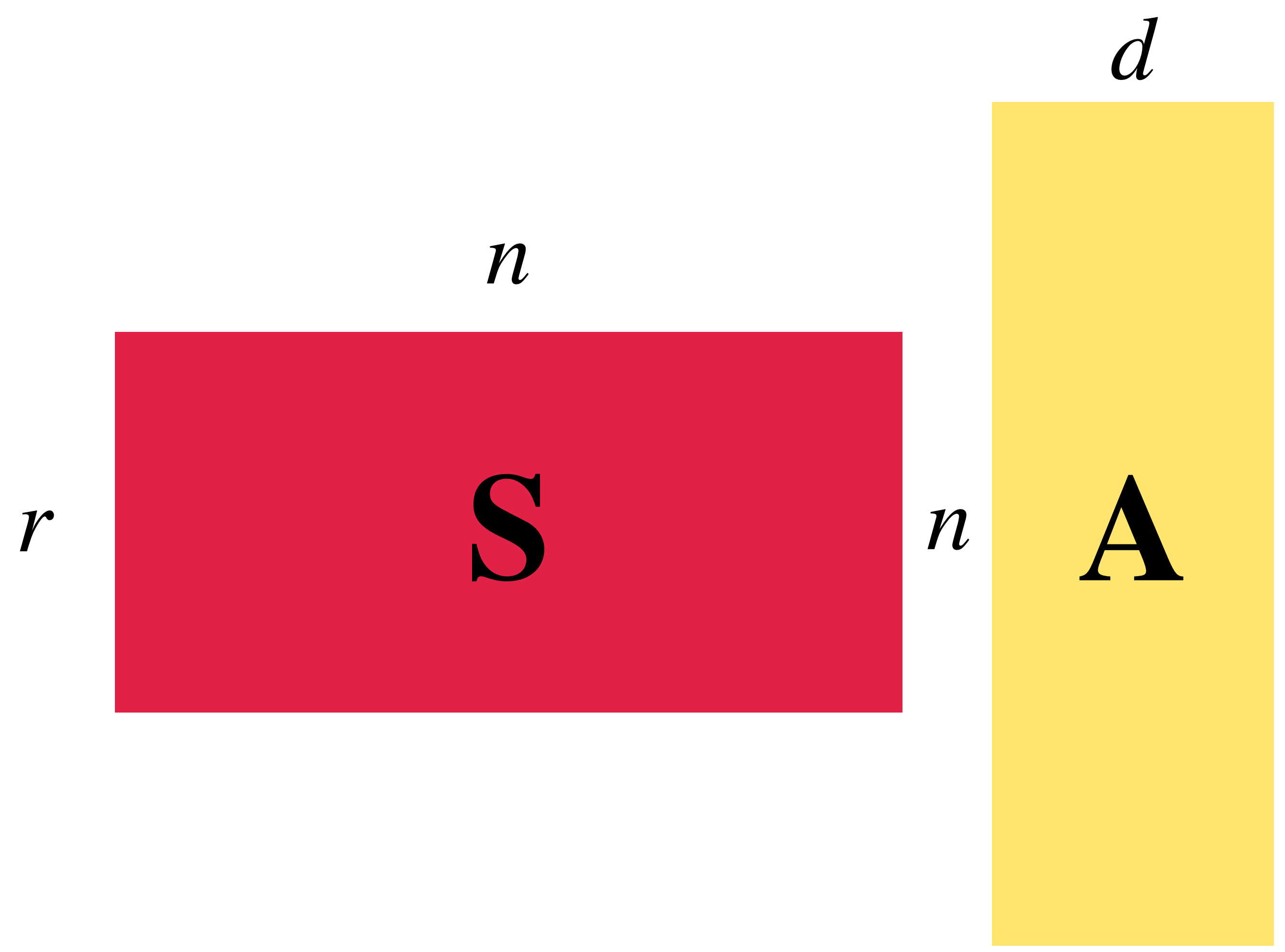


Matrix Approximation

Oblivious ℓ_p Subspace Embeddings

Fact. Oblivious ℓ_p subspace embeddings reduce to constructing **well-conditioned bases** for subspaces



Matrix Approximation

Oblivious ℓ_p Subspace Embeddings

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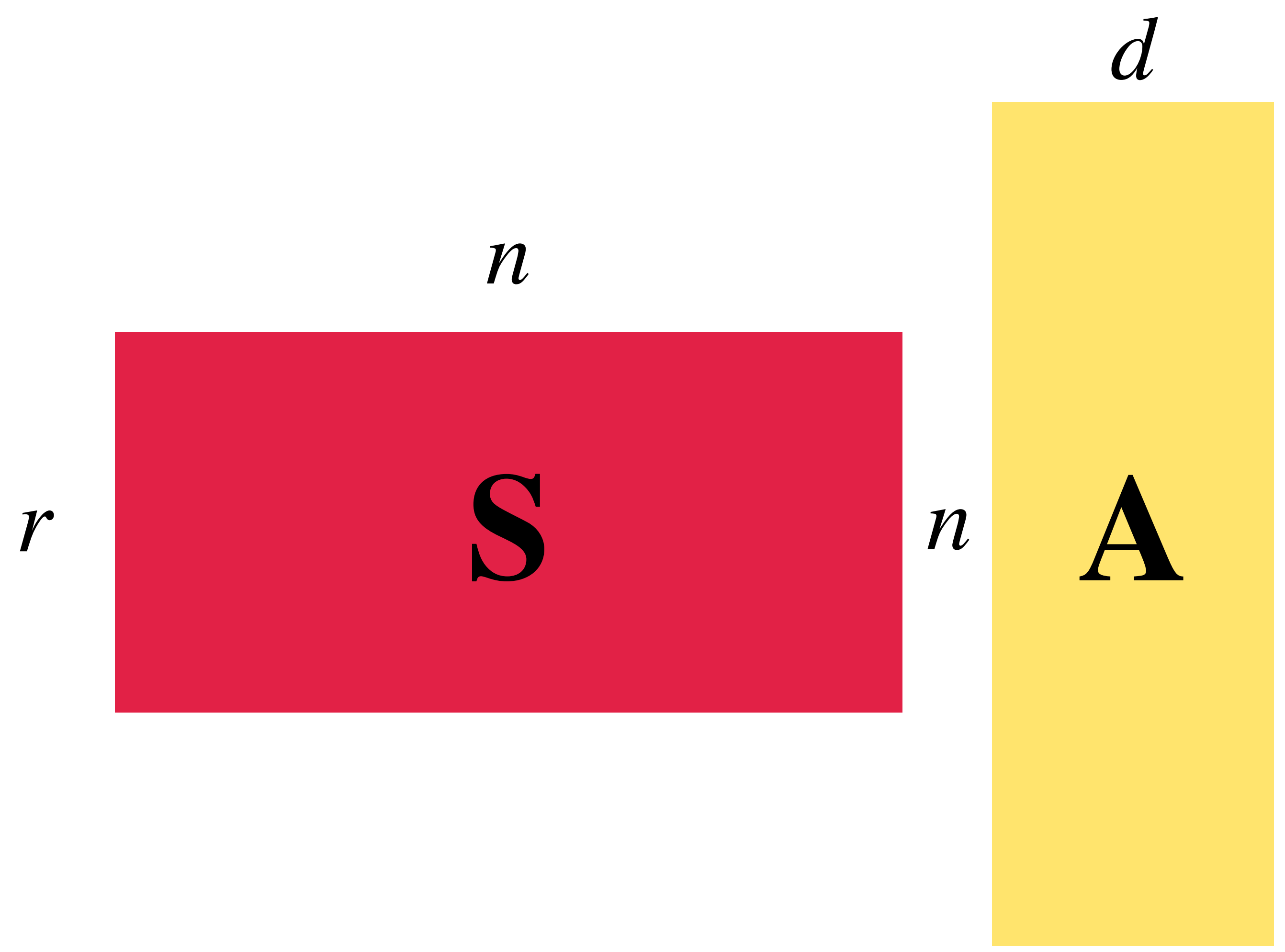
≈ orthonormal bases for ℓ_p norms



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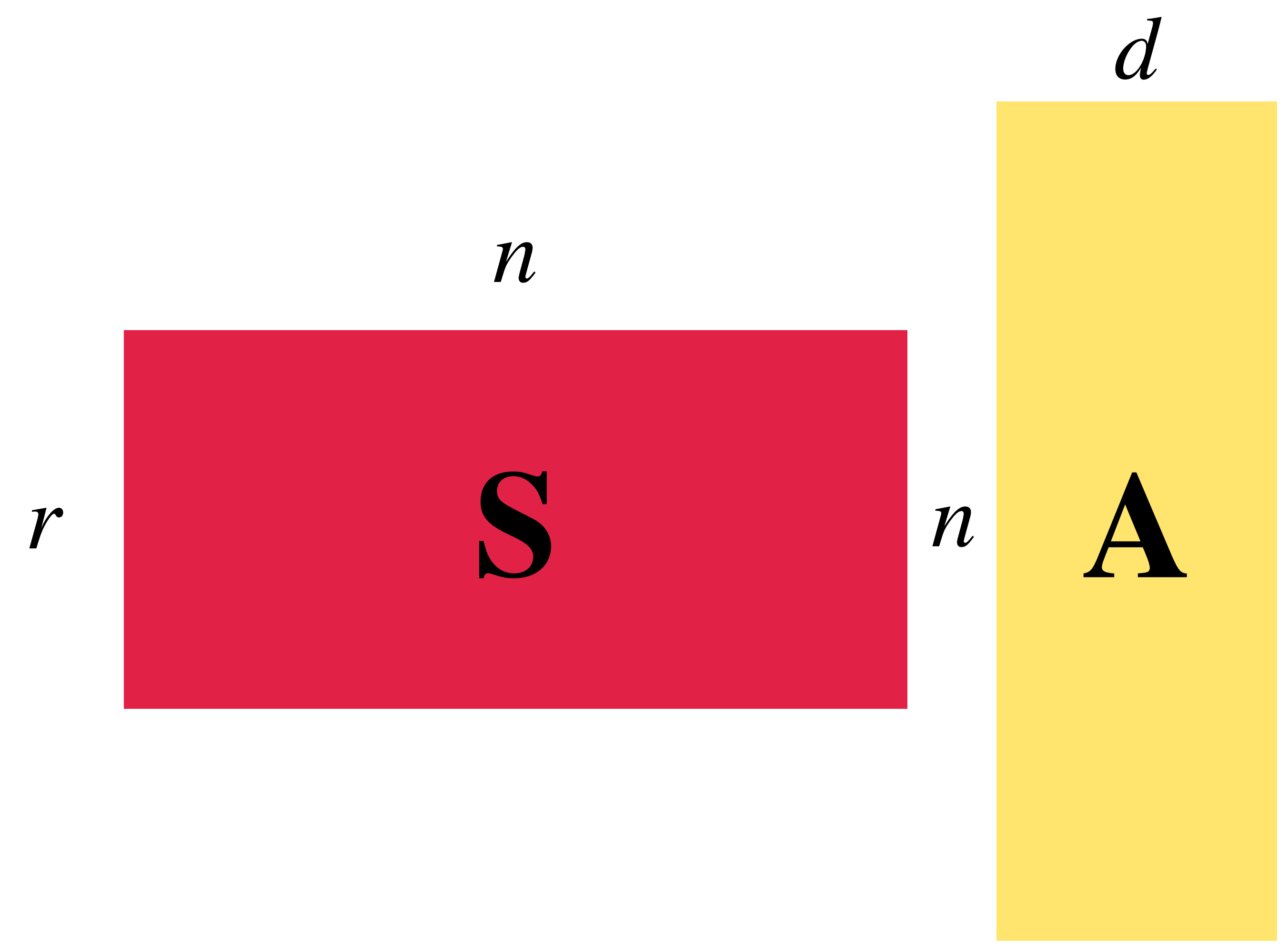

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Matrix Approximation

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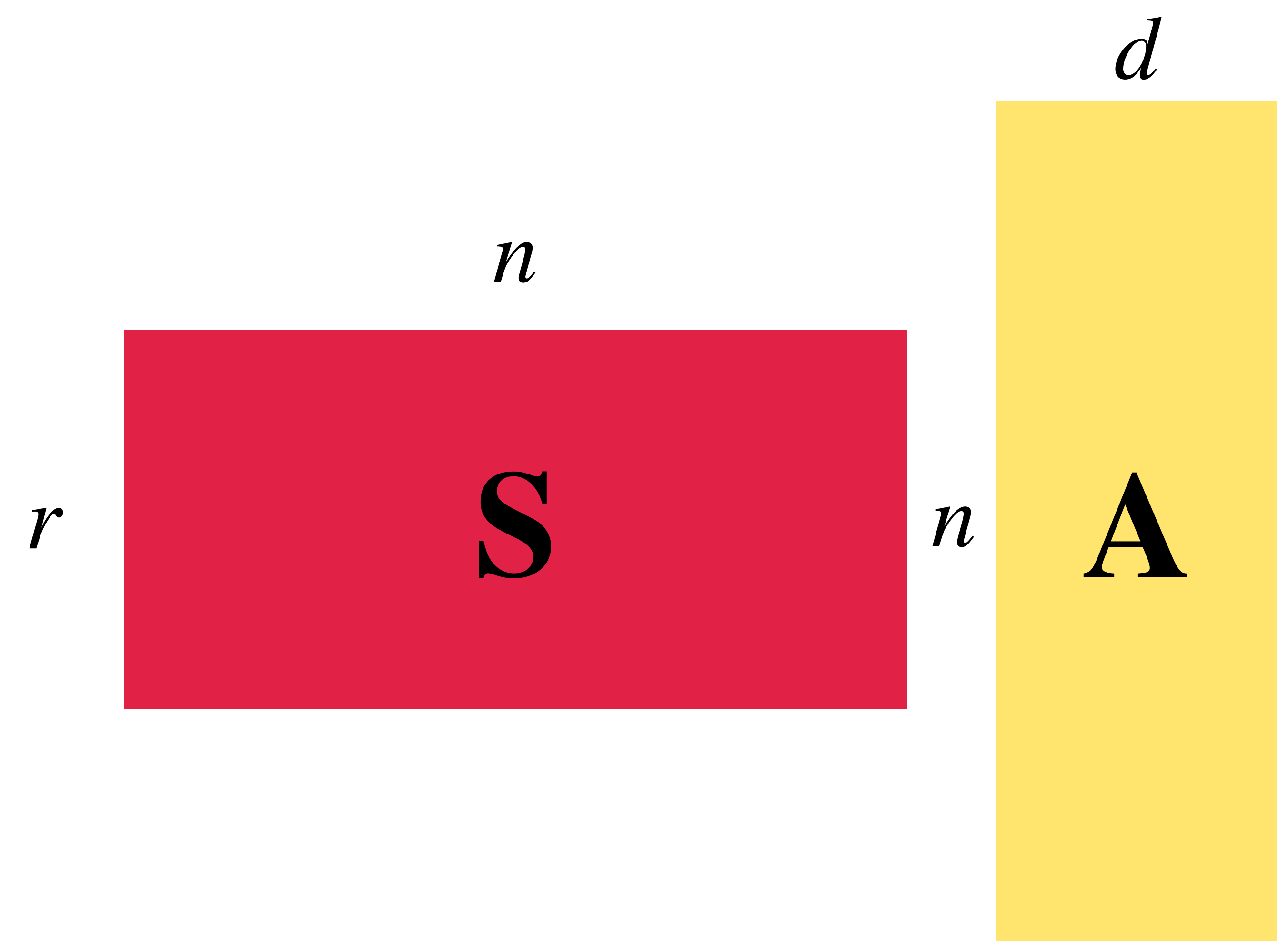

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Matrix Approximation

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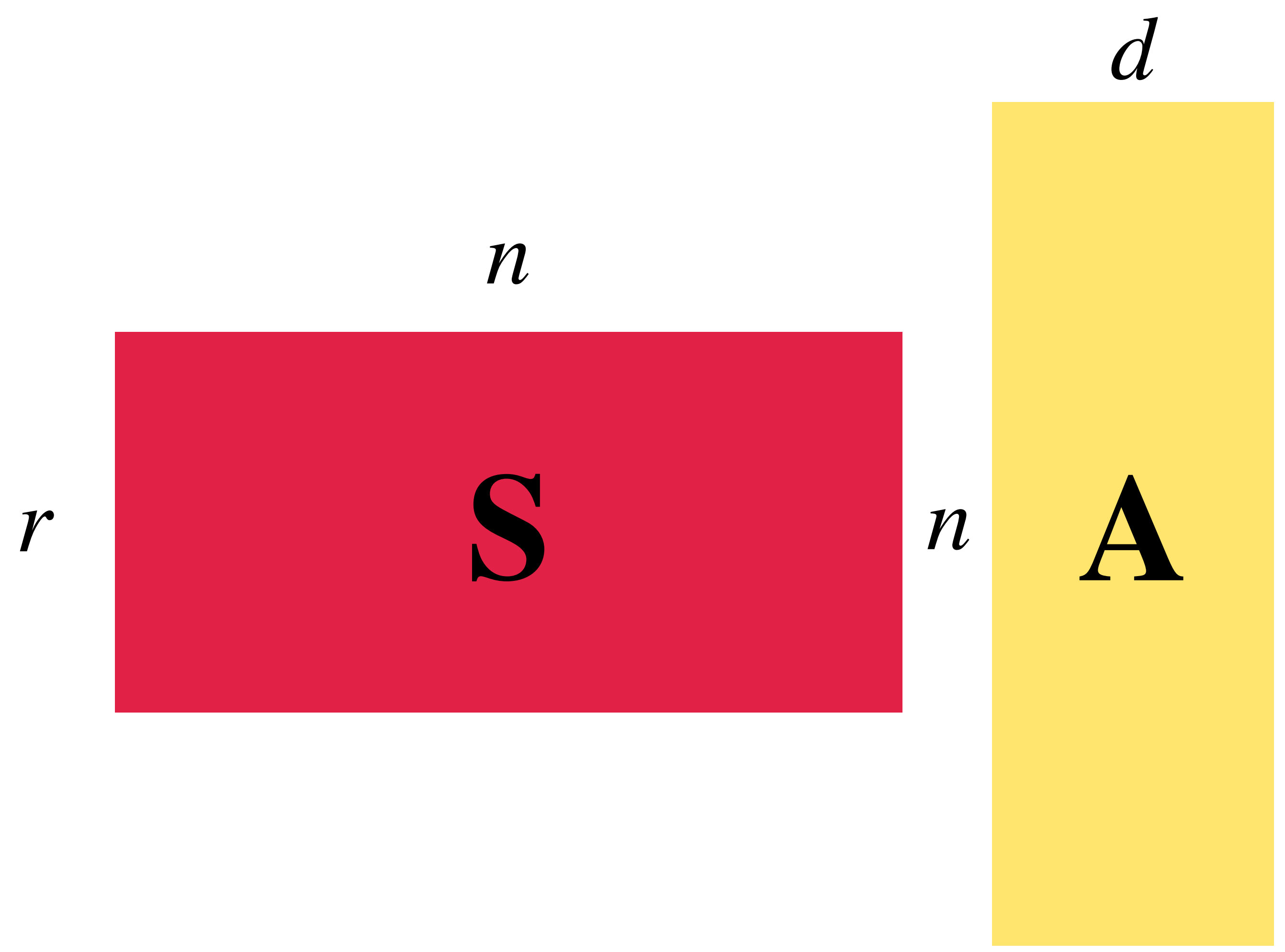

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Matrix Approximation

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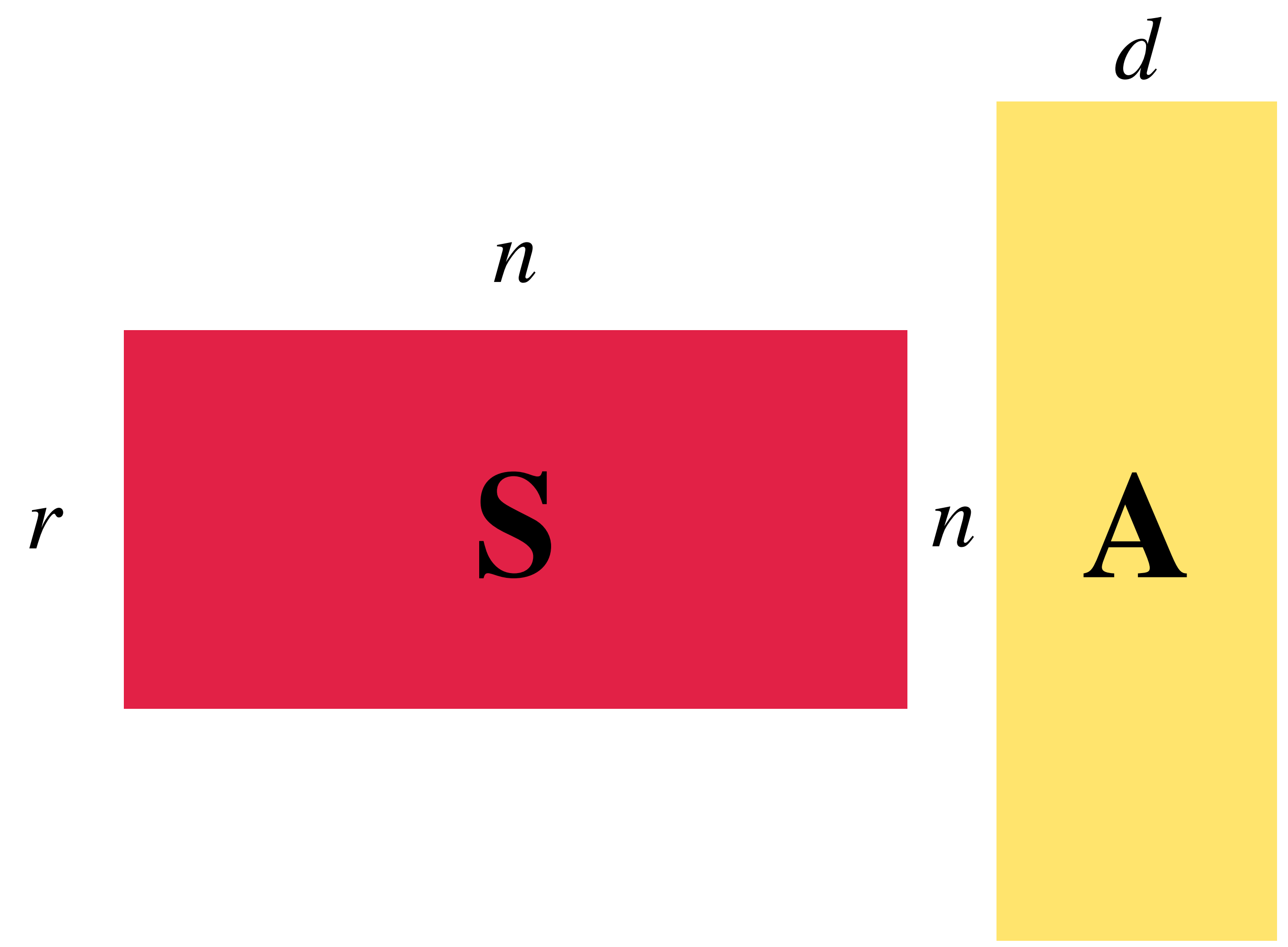

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Matrix Approximation

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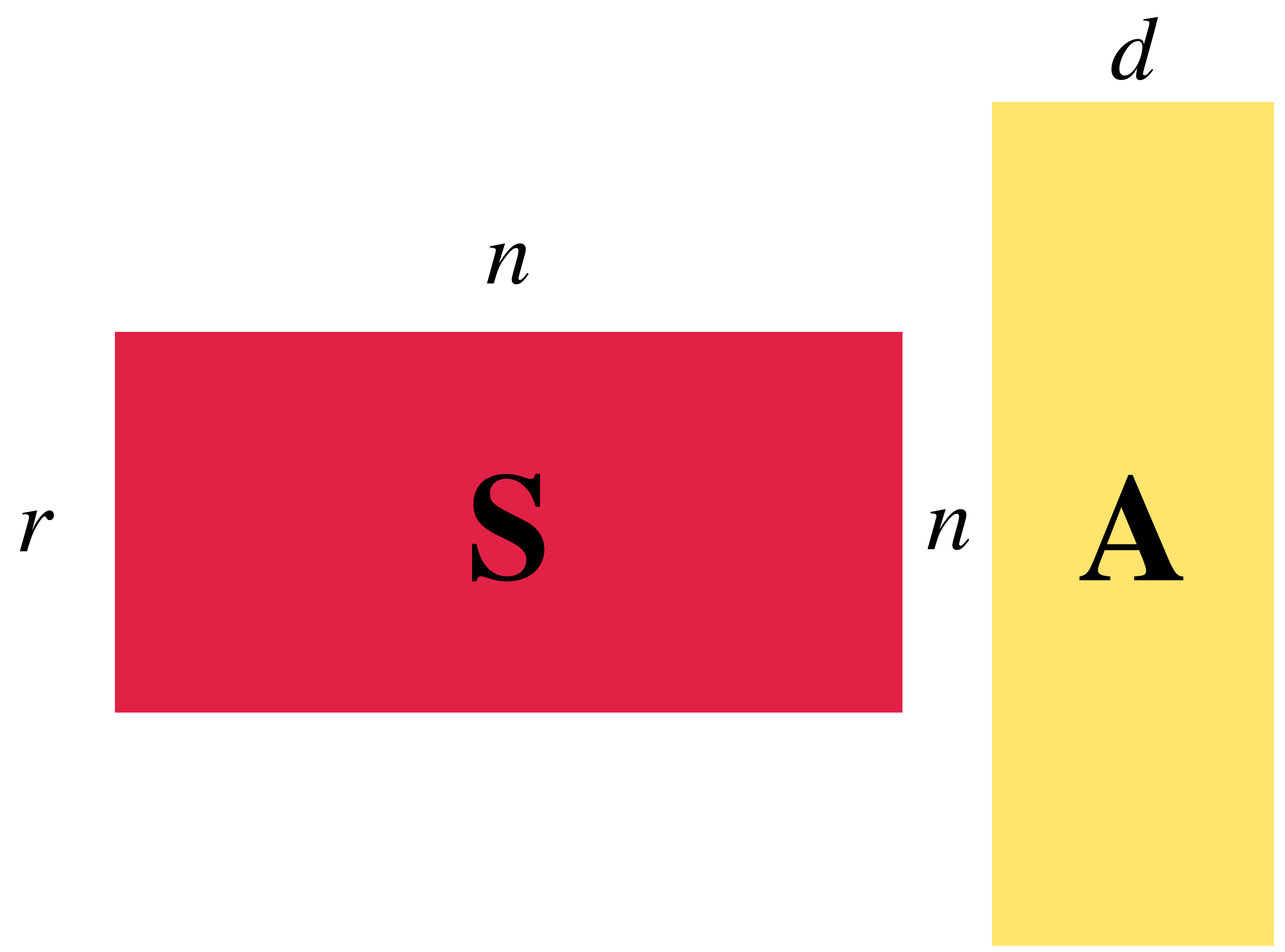

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- Let $\mathbf{U}$ be an ~~orthonormal basis~~ well-conditioned basis for $\mathbf{A}$
  - ~~$\|\mathbf{U}\|_F \leq d^{1/2}$ (with equality)~~ $\|\mathbf{U}\|_{p,p} \leq \alpha$ entrywise ℓ_p norm
  - $\|\mathbf{x}\|_2 \leq \|\mathbf{U}\mathbf{x}\|_2$ for every $\mathbf{x} \in \mathbb{R}^d$ (with equality)

Matrix Approximation

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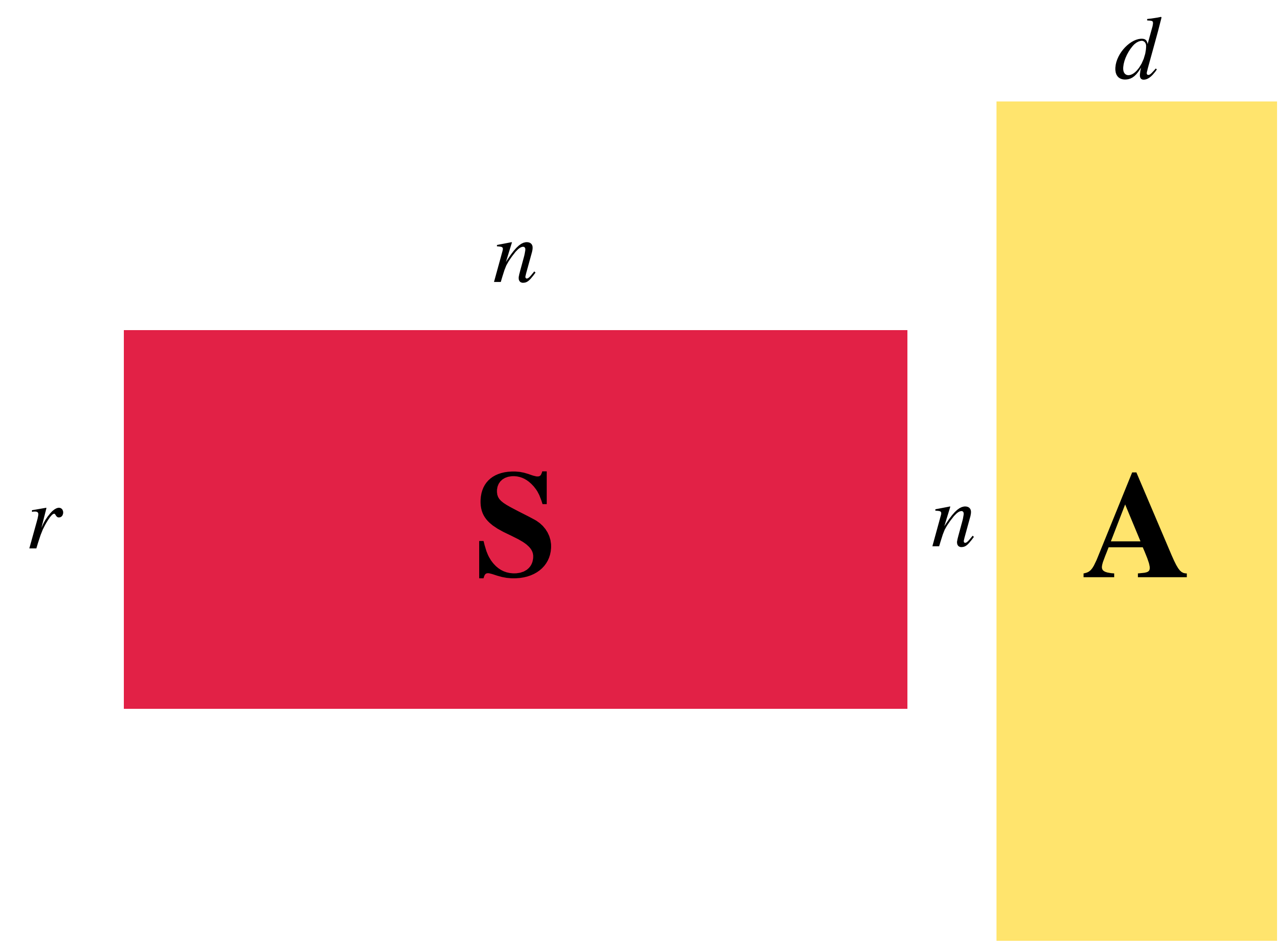

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$\approx$ orthonormal bases for ℓ_p norms

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  - $\cancel{\|\mathbf{U}\|_F \leq d^{1/2}}$ ~~(with equality)~~
    $\|\mathbf{U}\|_{p,p} \leq \alpha$ entrywise ℓ_p norm
  - $\cancel{\|\mathbf{x}\|_2 \leq \|\mathbf{U}\mathbf{x}\|_2}$ ~~for every~~ $\cancel{\mathbf{x} \in \mathbb{R}^d}$ ~~(with equality)~~

    $\|\mathbf{x}\|_q \leq \|\mathbf{U}\mathbf{x}\|_p$ for every $\mathbf{x} \in \mathbb{R}^d$

Matrix Approximation

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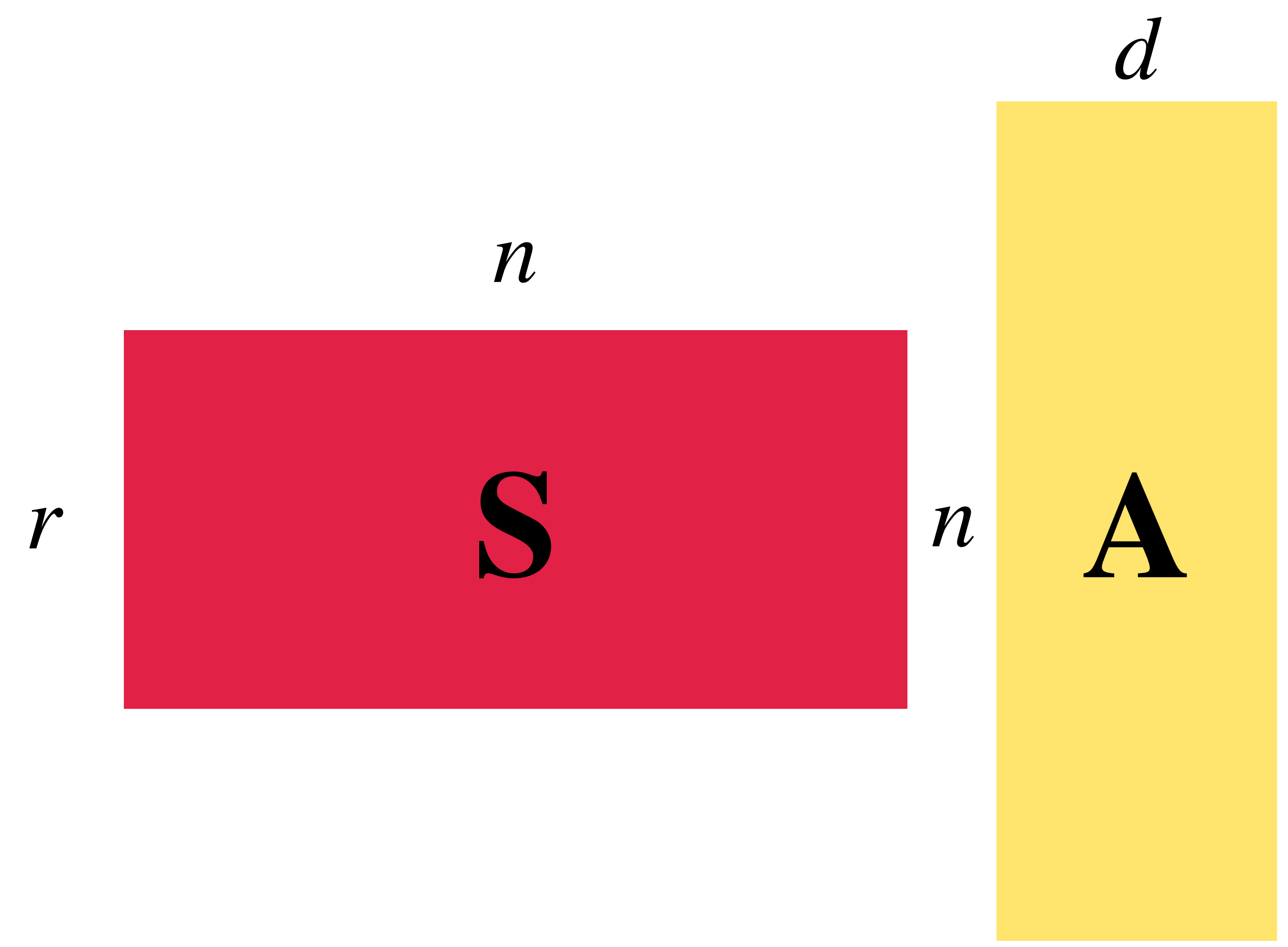

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Hölder conjugate, $\frac{1}{p} + \frac{1}{q} = 1$

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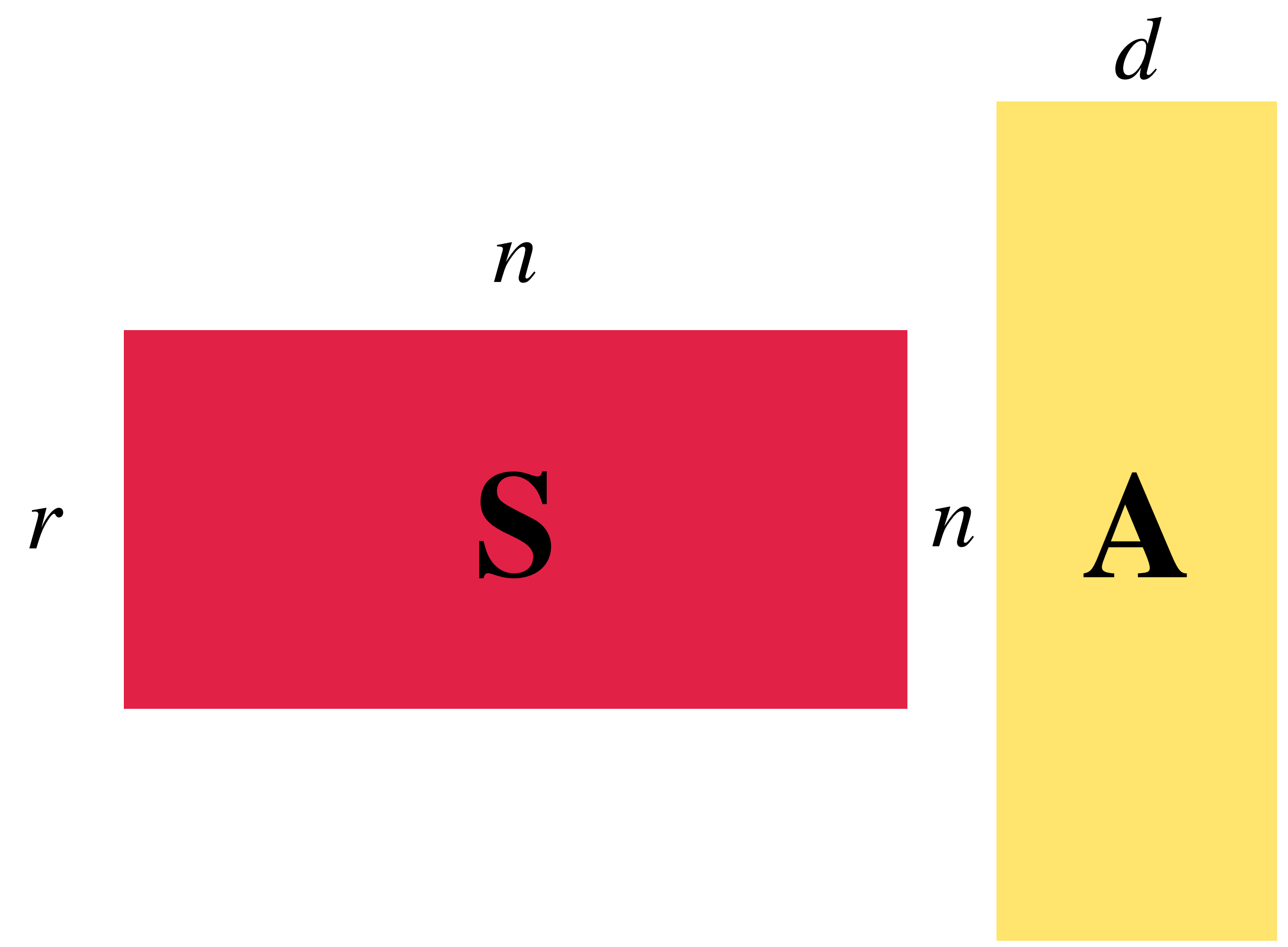

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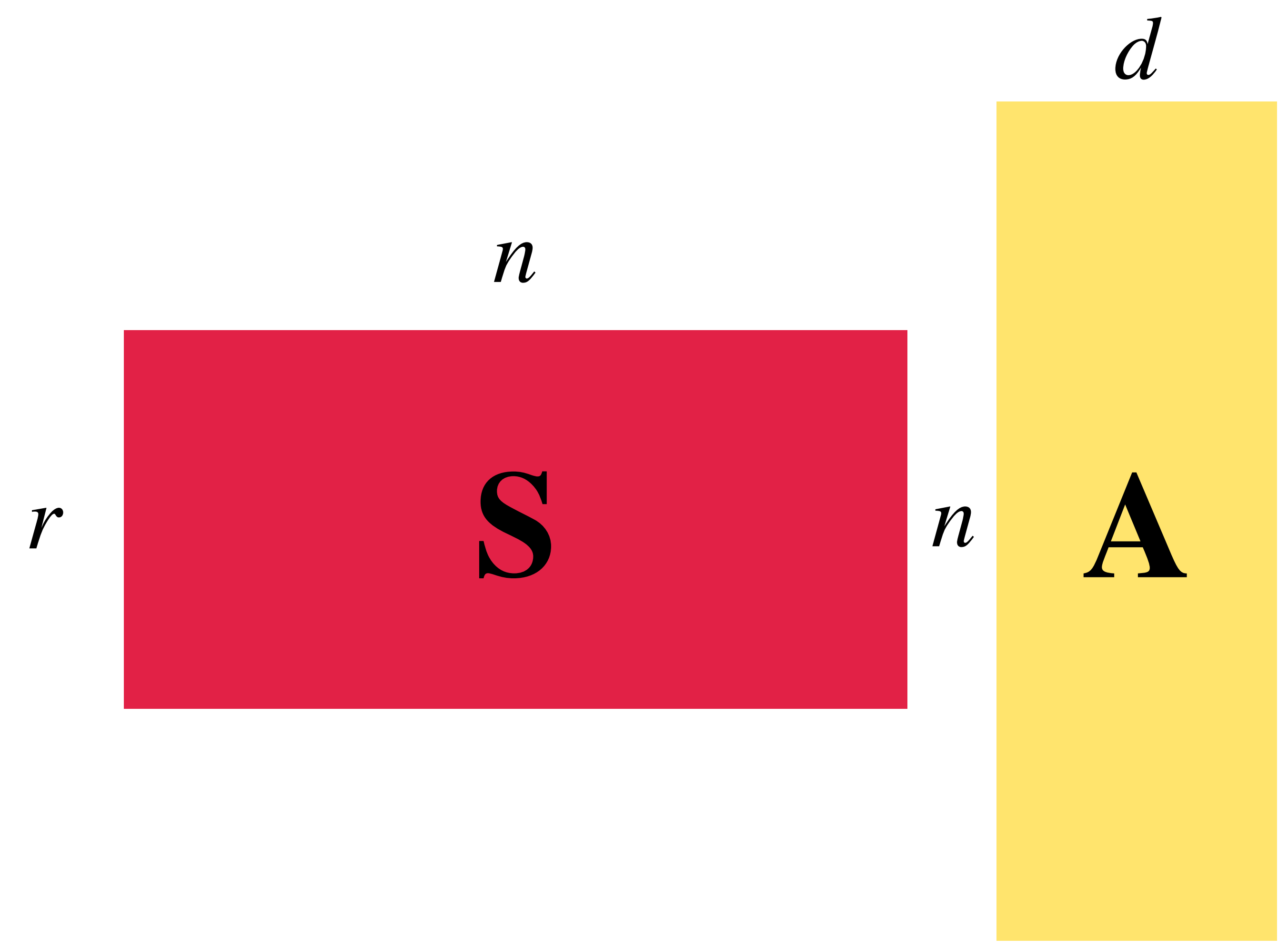
Theorem (Auerbach 1930). For any $\mathbf{A}$, there is $\mathbf{U}$ with $\alpha = d$.

Matrix Approximation

Oblivious ℓ_p Subspace Embeddings



Fact. Oblivious ℓ_p subspace embeddings reduce to constructing **well-conditioned bases** for subspaces

≈ orthonormal bases for ℓ_p norms

- Let $\mathbf{U}$ be an ~~orthonormal basis~~ for $\mathbf{A}$
  well-conditioned basis
  - ~~$\|\mathbf{U}\|_F \leq d^{1/2}$ (with equality)~~
    $\|\mathbf{U}\|_{p,p} \leq \alpha$ entrywise ℓ_p norm
  - ~~$\|\mathbf{x}\|_2 \leq \|\mathbf{U}\mathbf{x}\|_2$ for every $\mathbf{x} \in \mathbb{R}^d$ (with equality)~~
    $\|\mathbf{x}\|_q \leq \|\mathbf{U}\mathbf{x}\|_p$ for every $\mathbf{x} \in \mathbb{R}^d$

Hölder conjugate, $\frac{1}{p} + \frac{1}{q} = 1$

Theorem (Auerbach 1930). For any $\mathbf{A}$, there is $\mathbf{U}$ with $\alpha = d$.

Conjecture. For any $\mathbf{A}$, there is $\mathbf{U}$ with $\alpha = d^{1/p}$.

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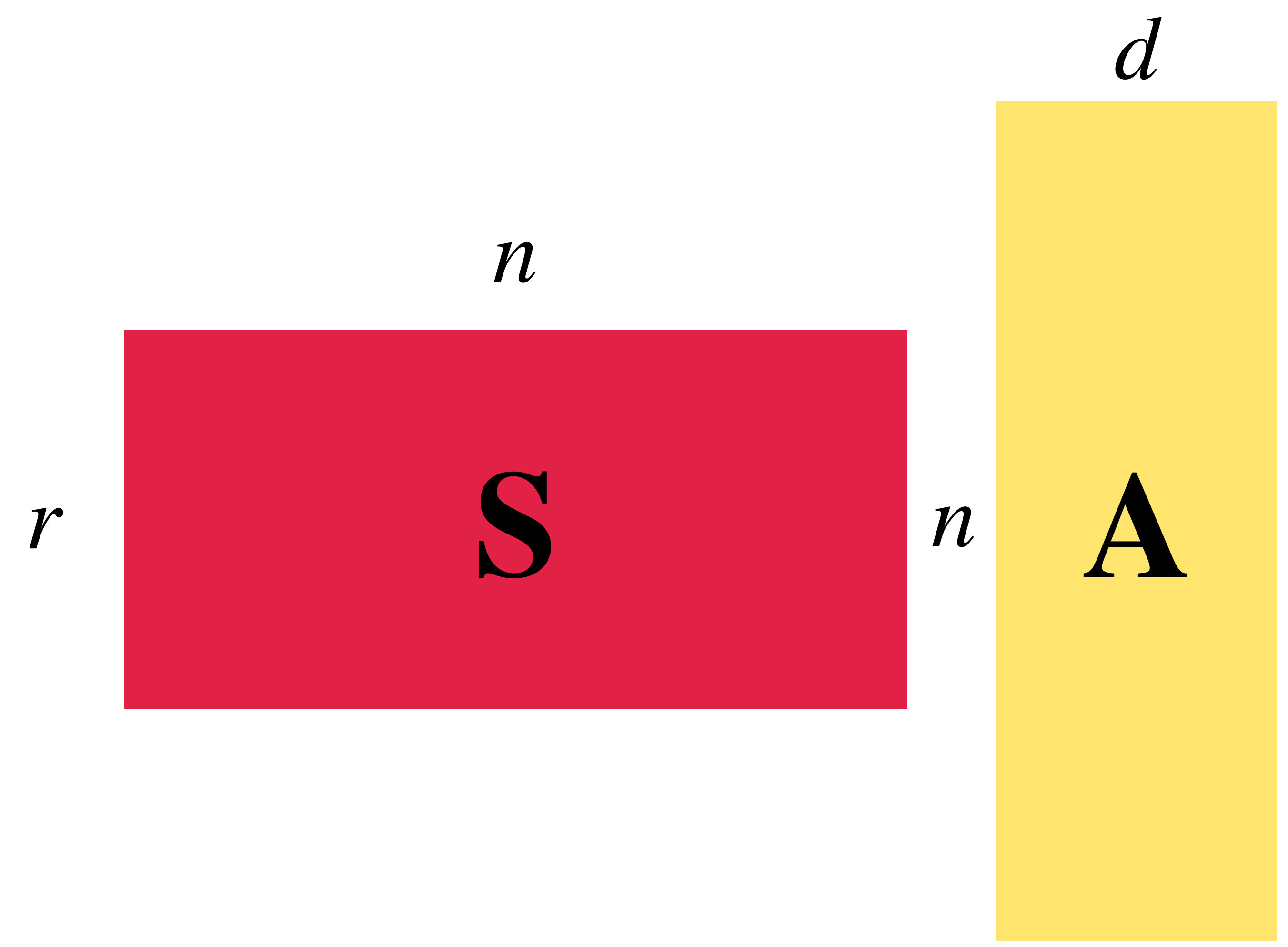

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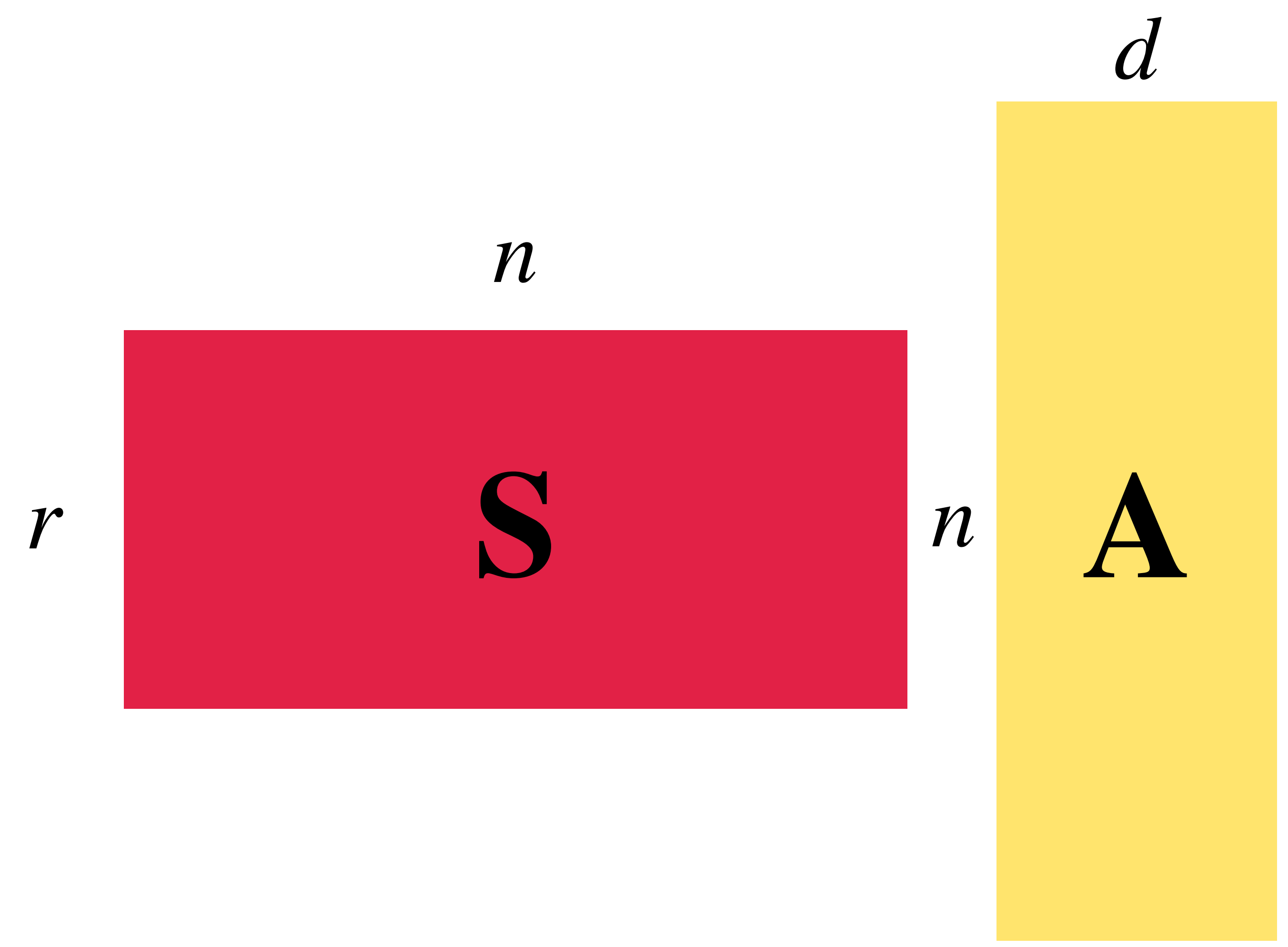

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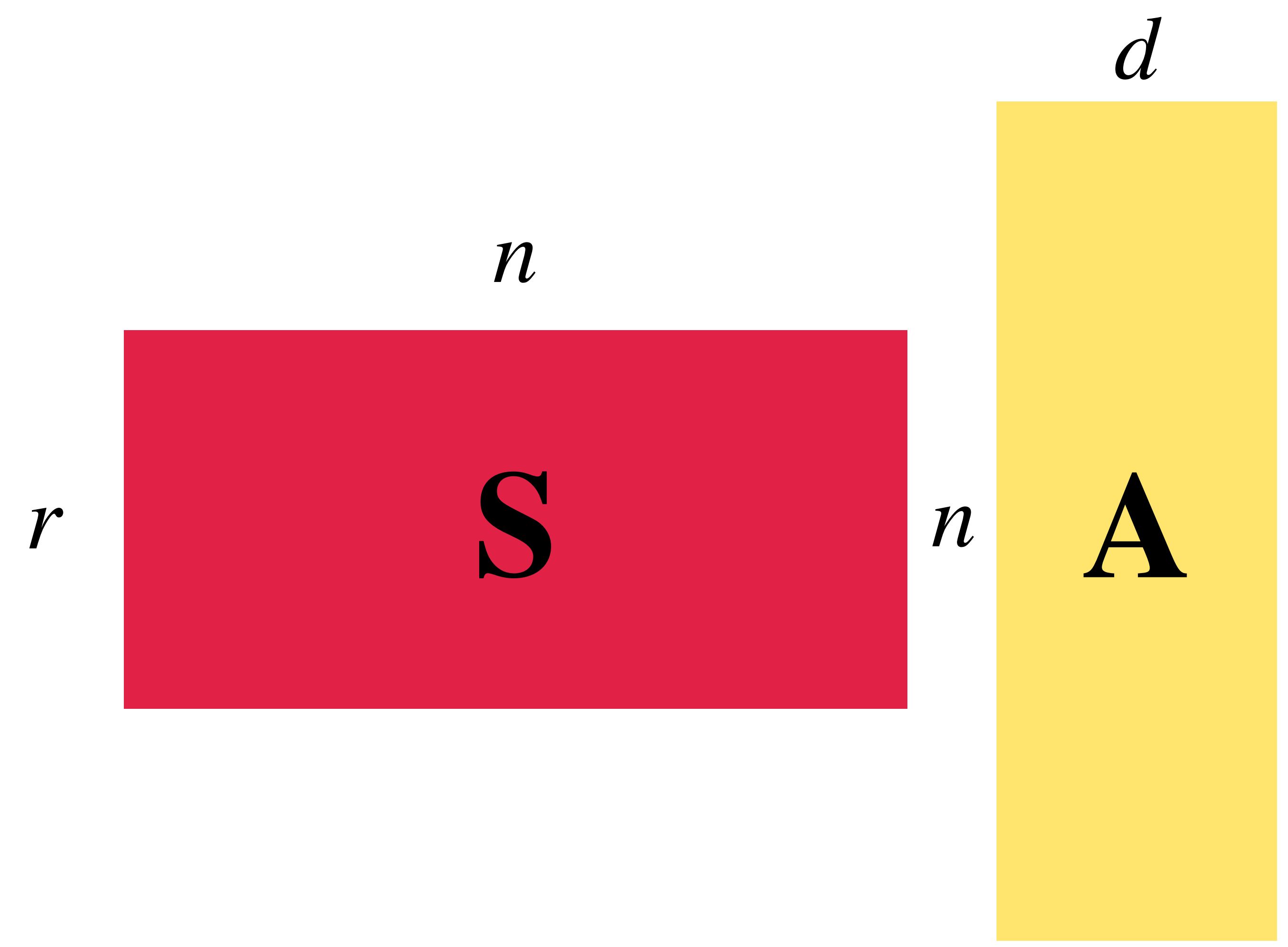

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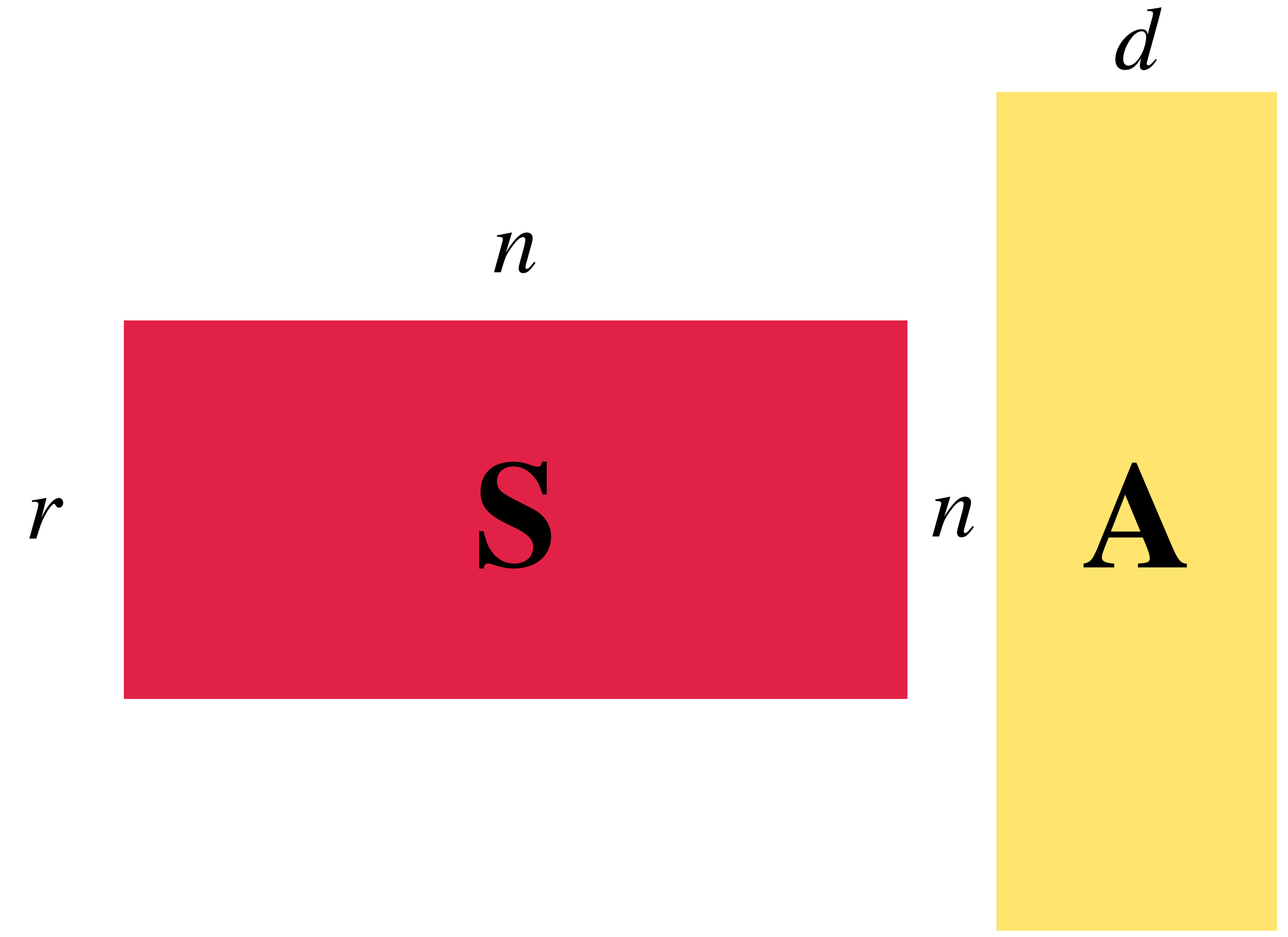

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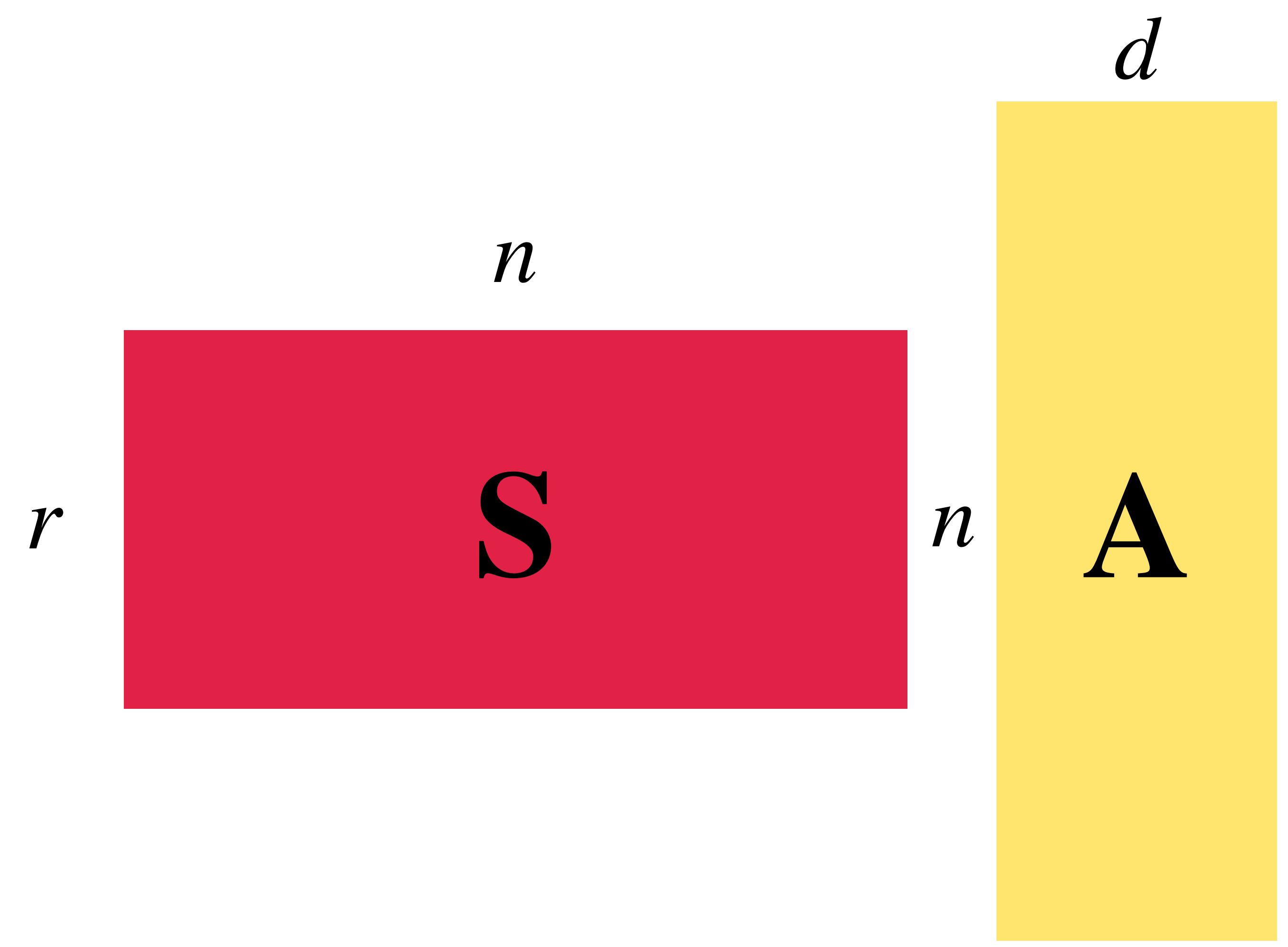

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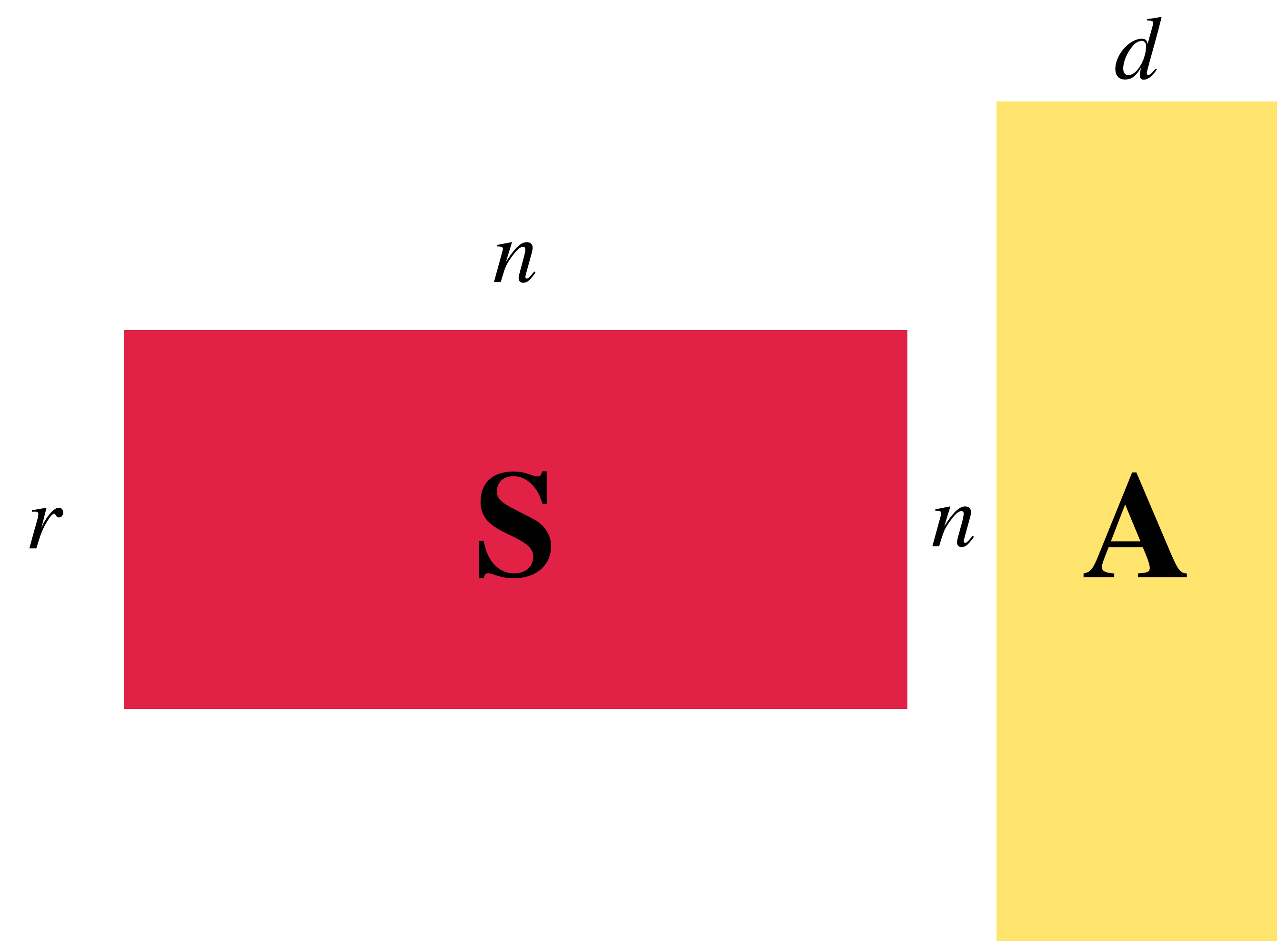

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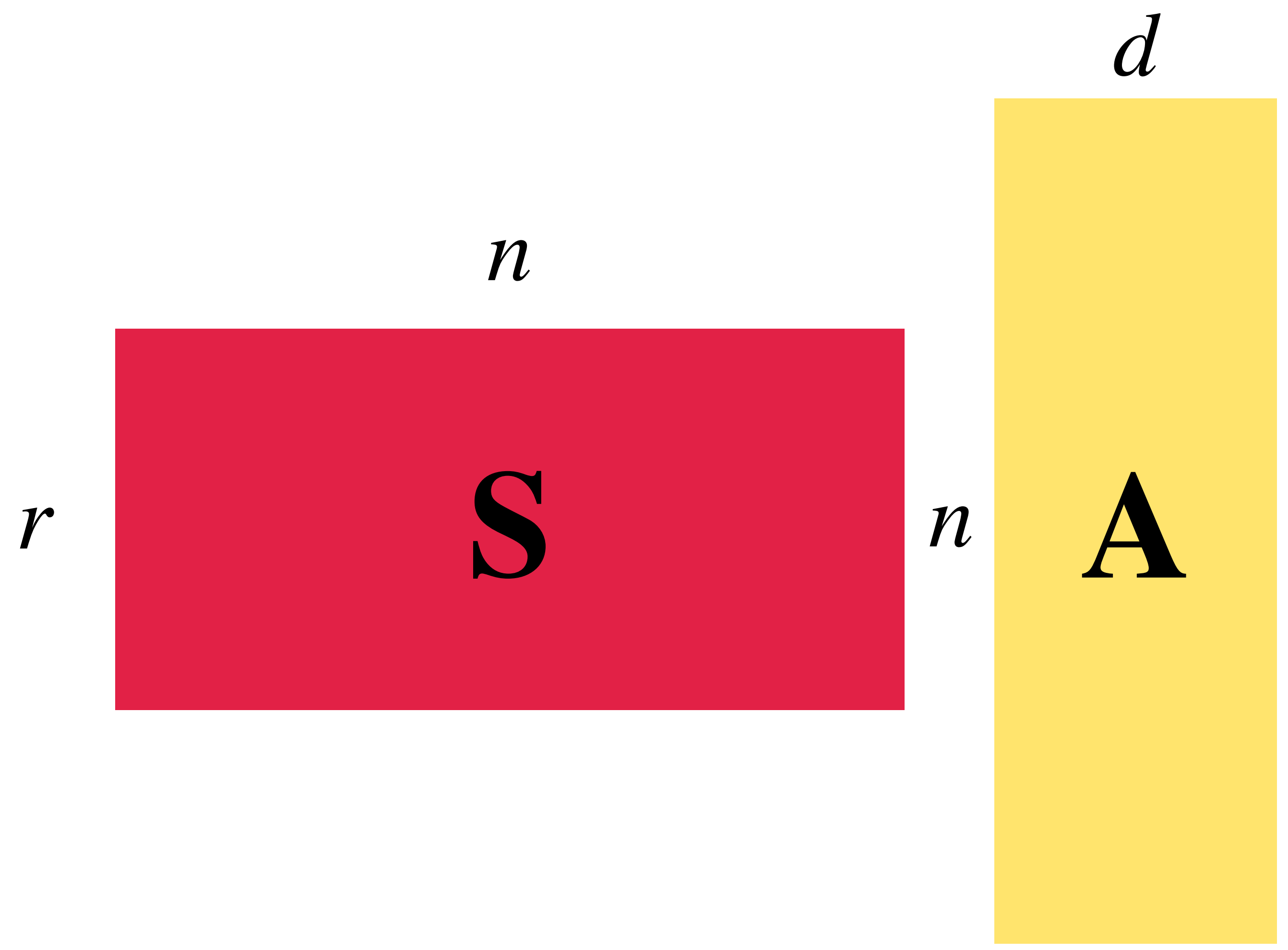

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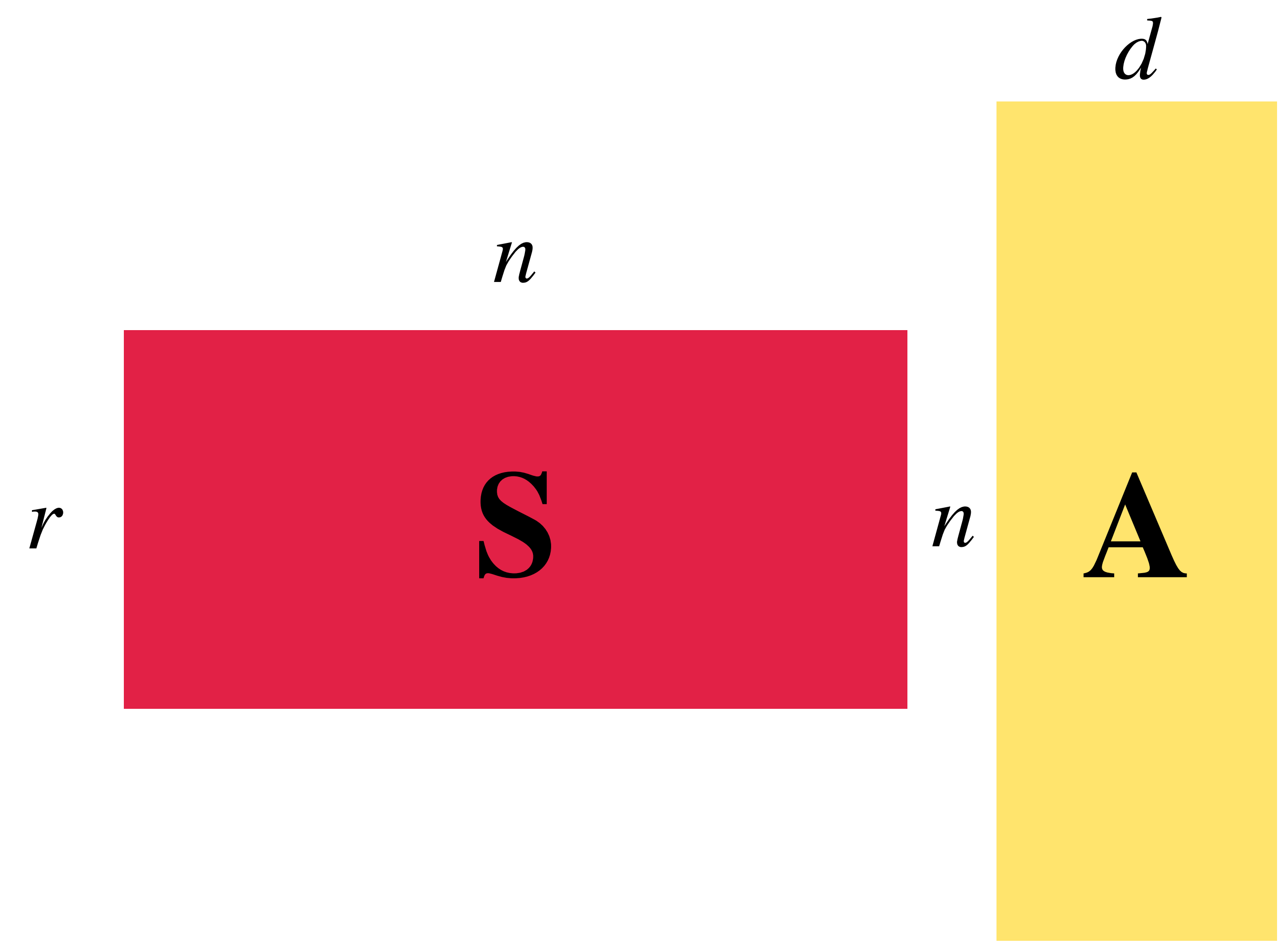

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≈ as good as showing $\alpha = s^{1/p}$

Matrix Approximation

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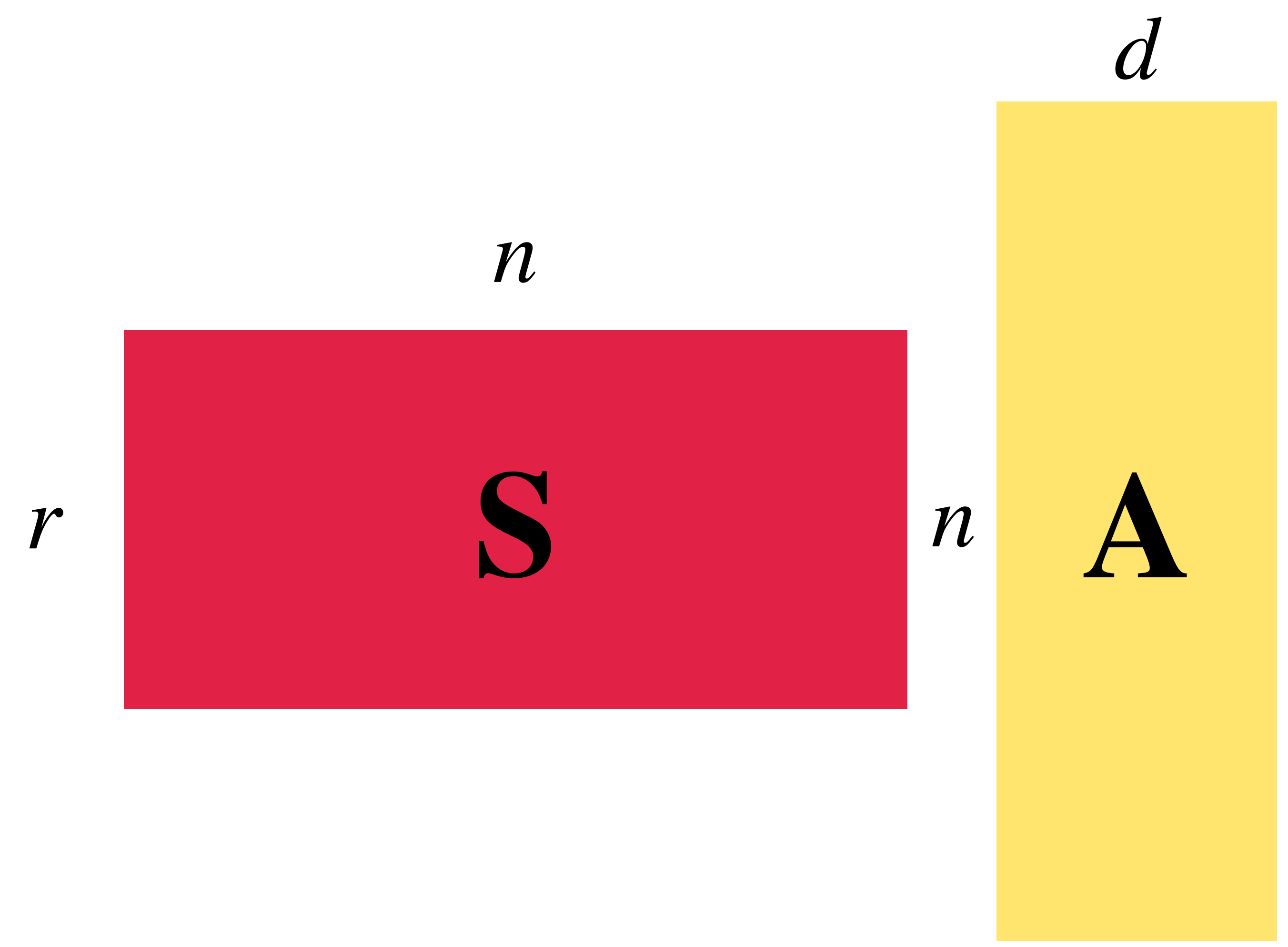

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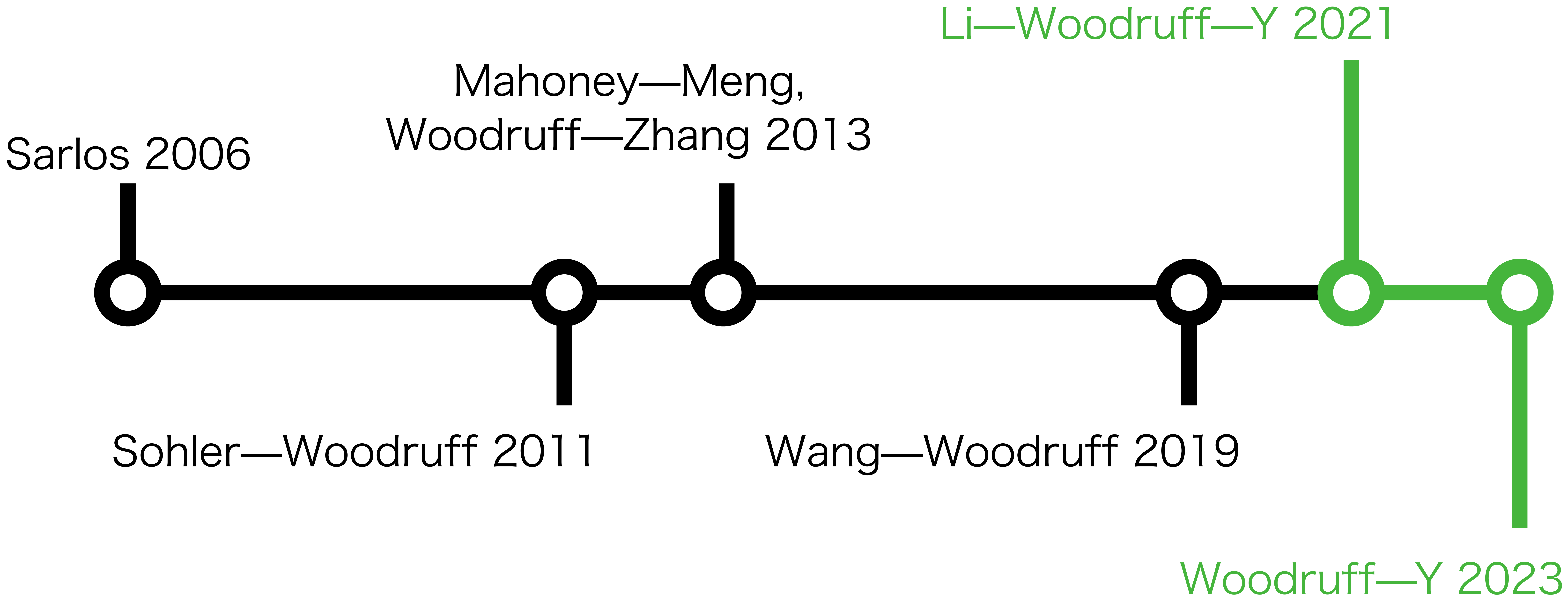
Theorem (Woodruff—Y 2023). There are oblivious ℓ_p subspace embeddings with $r = \tilde{O}(d)$ and $\kappa = \tilde{O}(d^{1/p})$, which is **nearly optimal**.

Matrix Approximation

Oblivious ℓ_p Subspace Embeddings



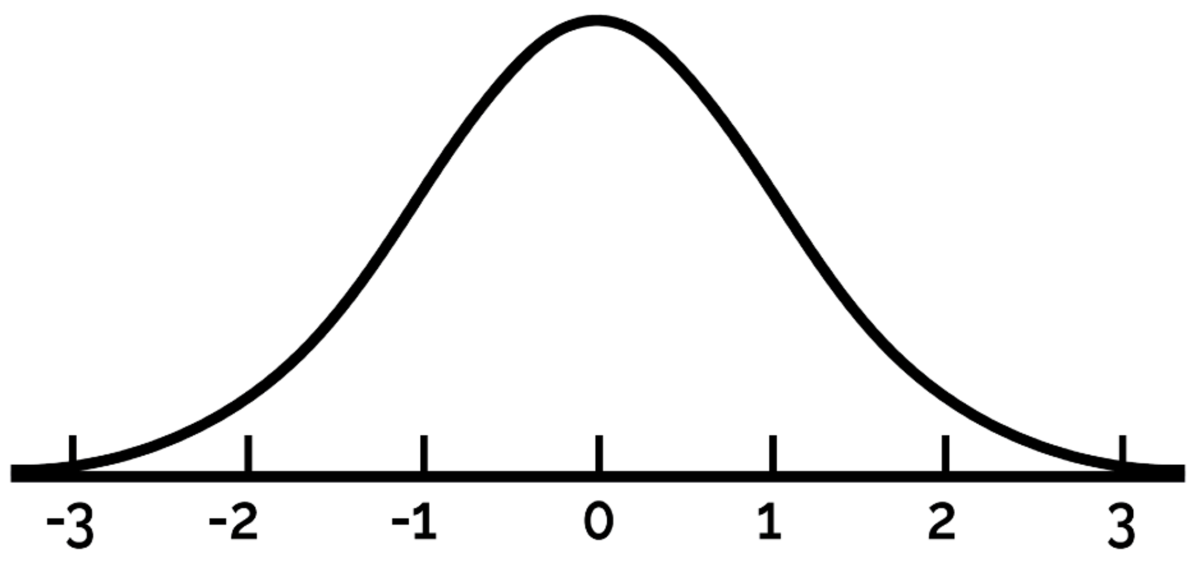
Subspace Embeddings and Linear Regression

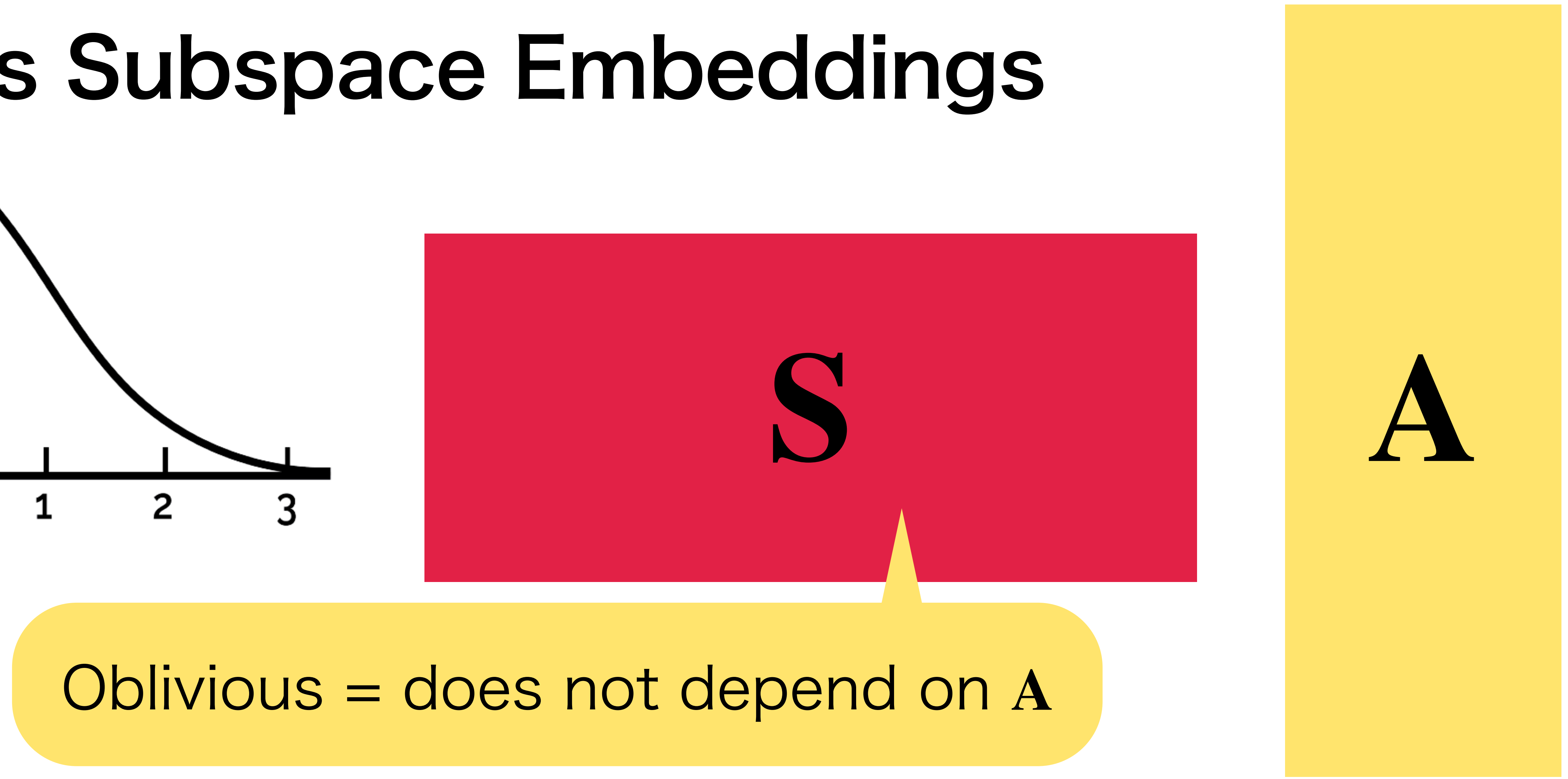
- Oblivious ℓ_p subspace embeddings: high distortion and low distortion
- $\boxed{\text{Non-oblivious subspace embeddings: } \ell_p \text{ Lewis weight sampling, general losses}}$
- Applications: active learning, streaming computational geometry, low rank approximation

Matrix Approximation

Non-oblivious Subspace Embeddings

Oblivious





Matrix Approximation

Non-oblivious Subspace Embeddings

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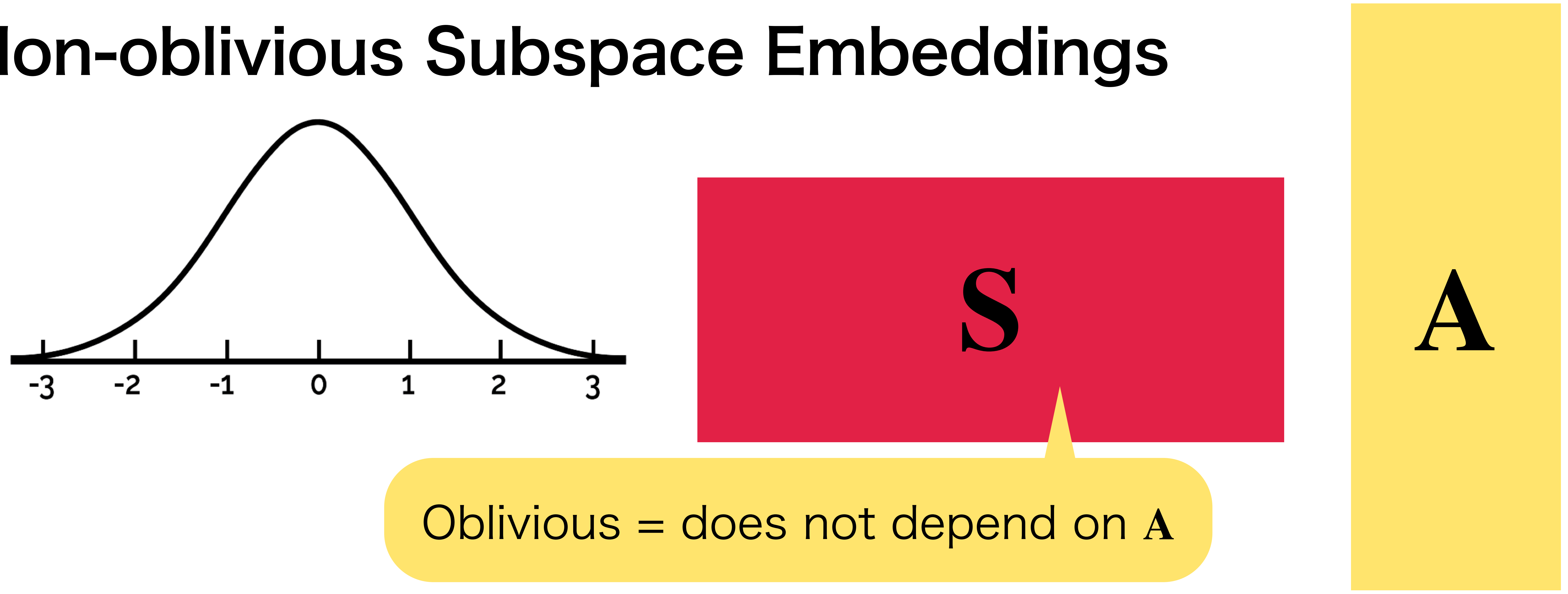

Non-oblivious/ Sampling

Matrix Approximation

Non-oblivious Subspace Embeddings

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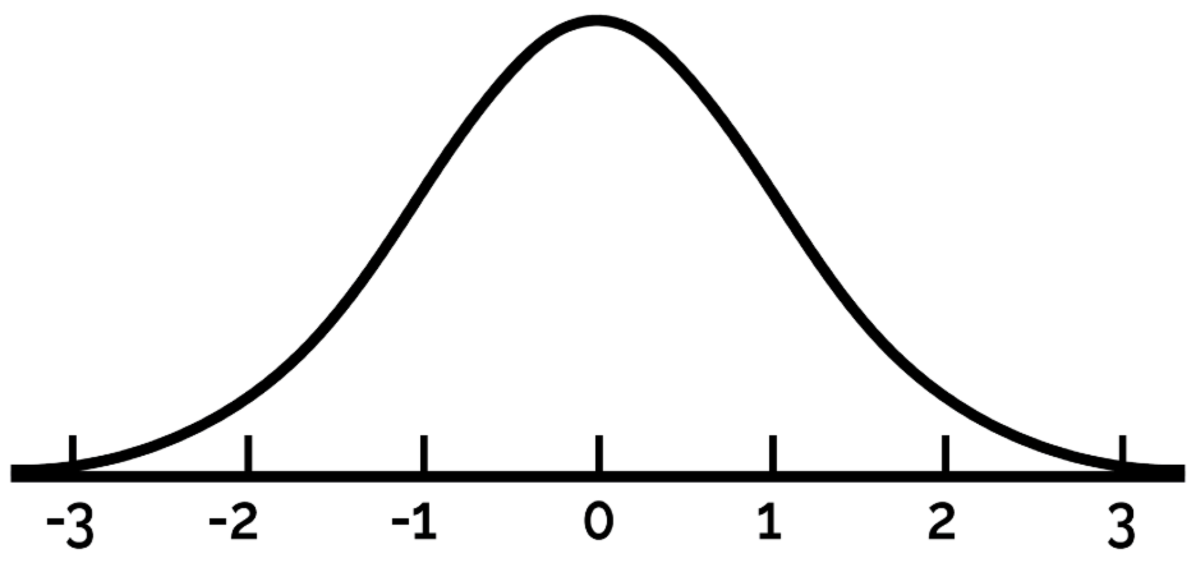


$\mathbf{S}$

$\mathbf{A}$

Oblivious = does not depend on $\mathbf{A}$

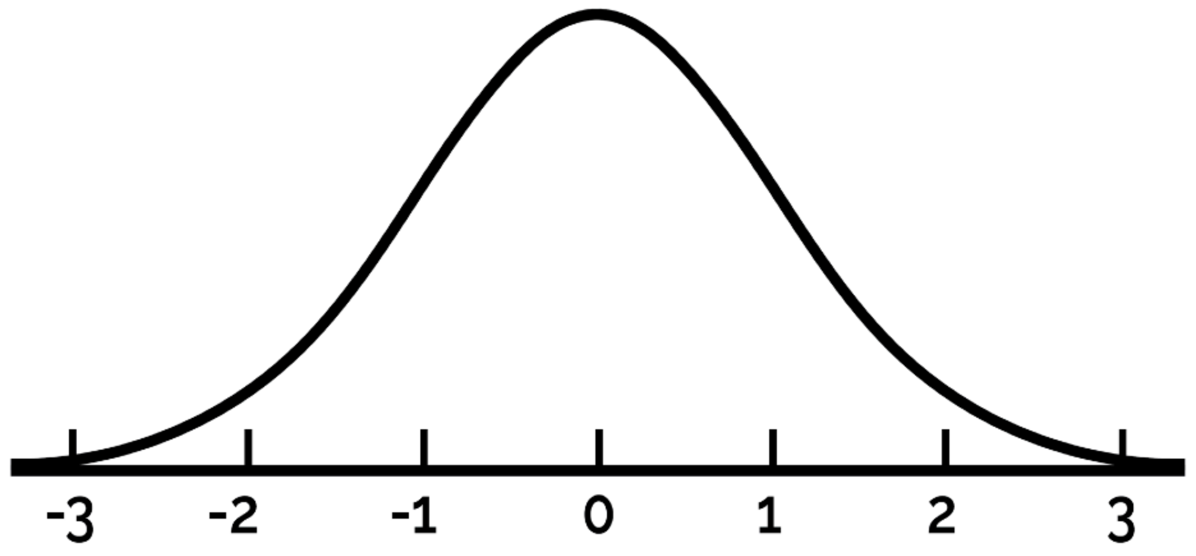
Non-oblivious/ Sampling

$\mathbf{A}$

Matrix Approximation

Non-oblivious Subspace Embeddings

Oblivious



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Non-oblivious/ Sampling

Step 1. Compute "importance scores" for the rows of $\mathbf{A}$

q_1

q_2

$\vdots$

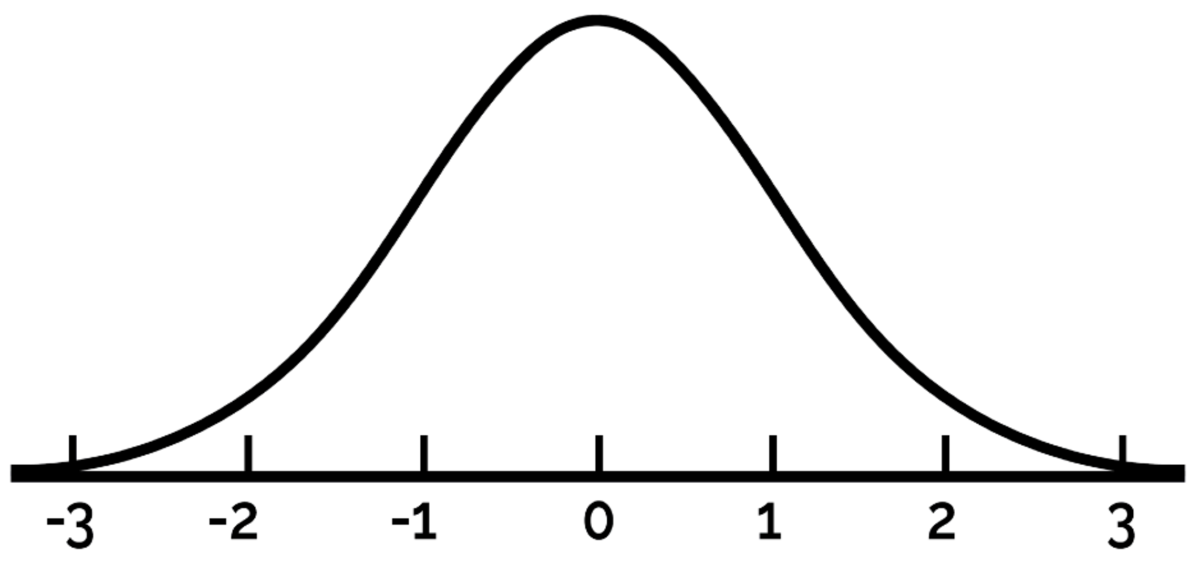
q_n

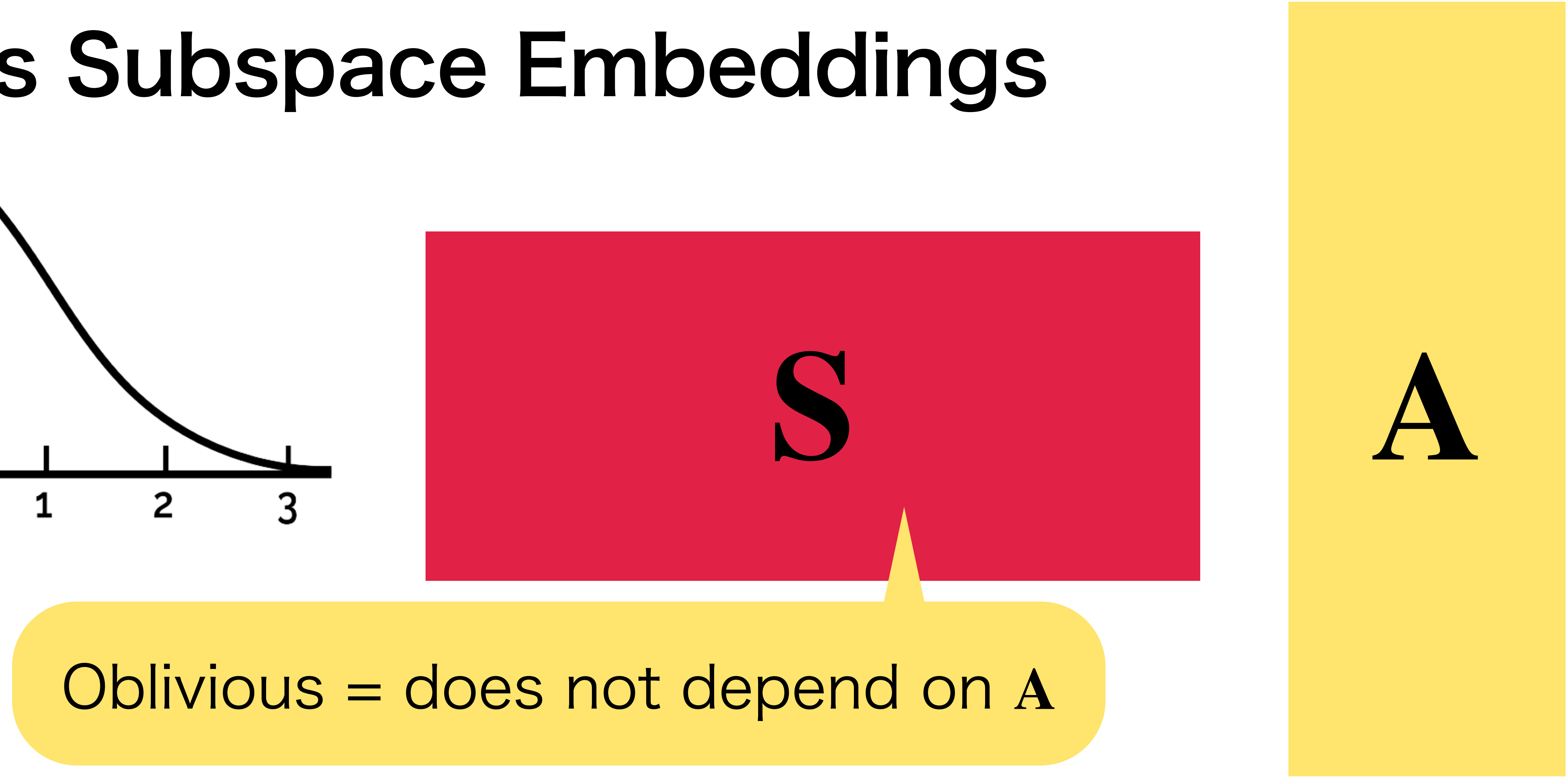
$\mathbf{A}$

Matrix Approximation

Non-oblivious Subspace Embeddings

Oblivious





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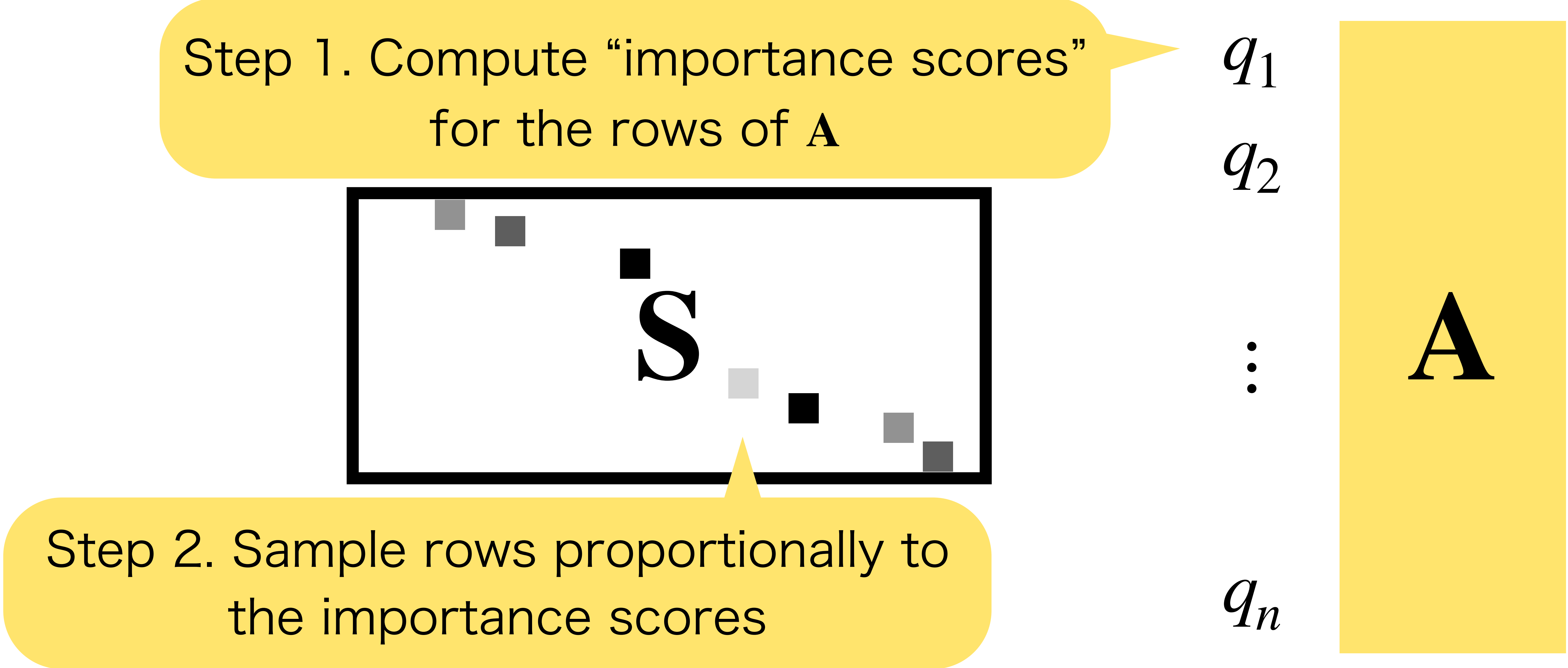
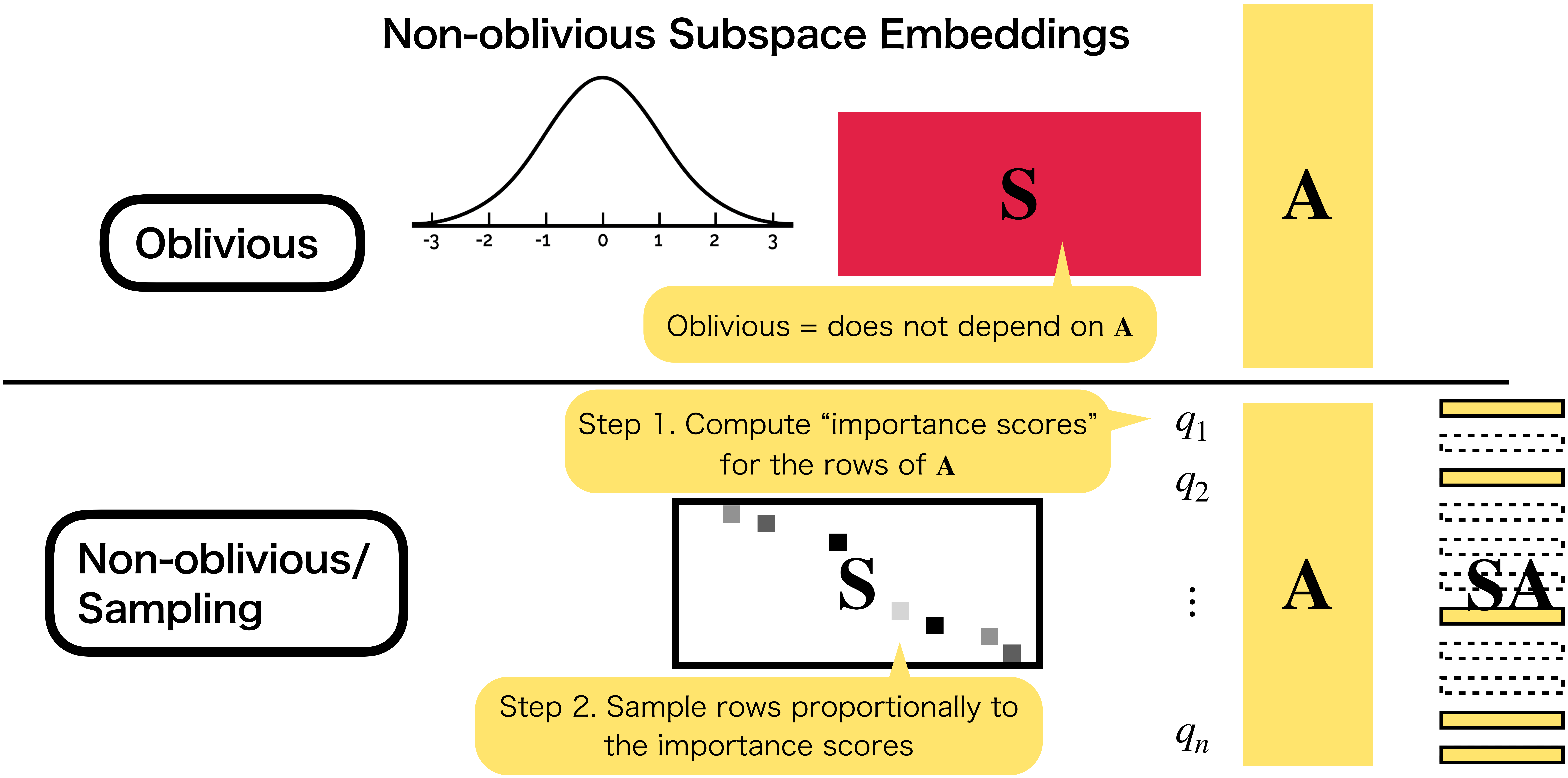


Matrix Approximation

Non-oblivious Subspace Embeddings



Matrix Approximation

Non-oblivious Subspace Embeddings

Theorem (Leverage score sampling). For any $\mathbf{A} \in \mathbb{R}^{n \times d}$, there are probabilities $q_1, q_2, \ldots q_n$ that sample $r = \tilde{O}(\varepsilon^{-2} d)$ rows of $\mathbf{A}$ that forms an ℓ_2 subspace embedding with distortion $\kappa = (1 + \varepsilon)$, with probability 99%.

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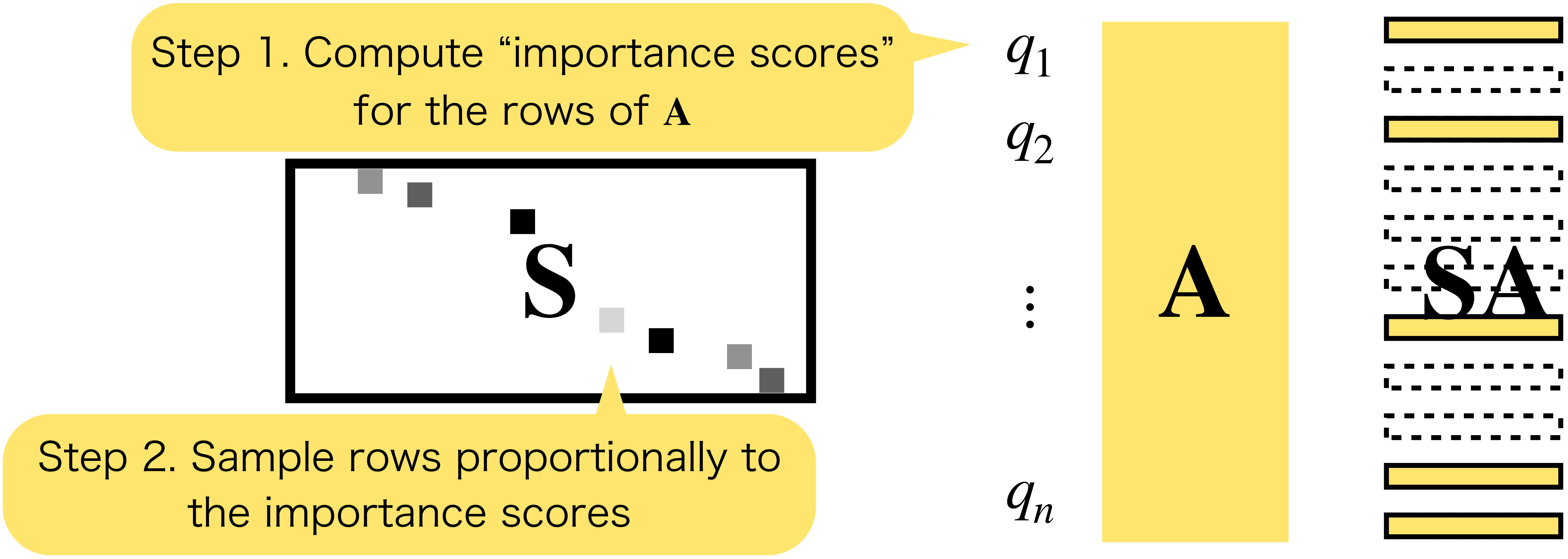

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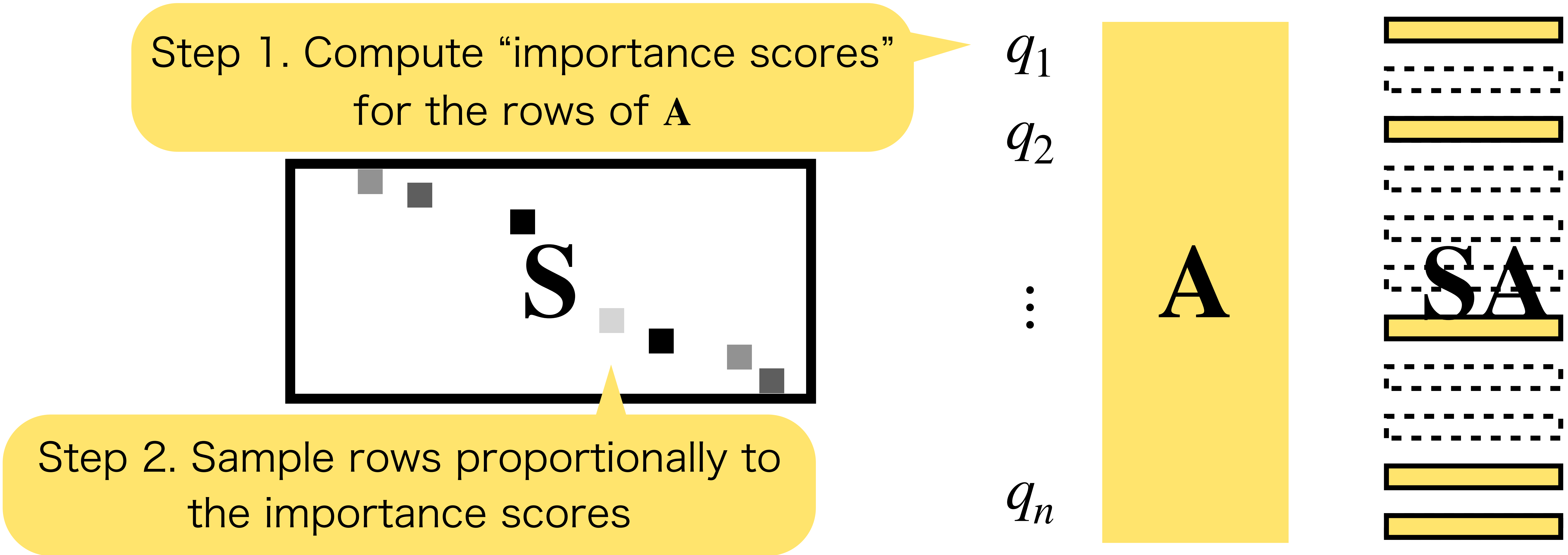

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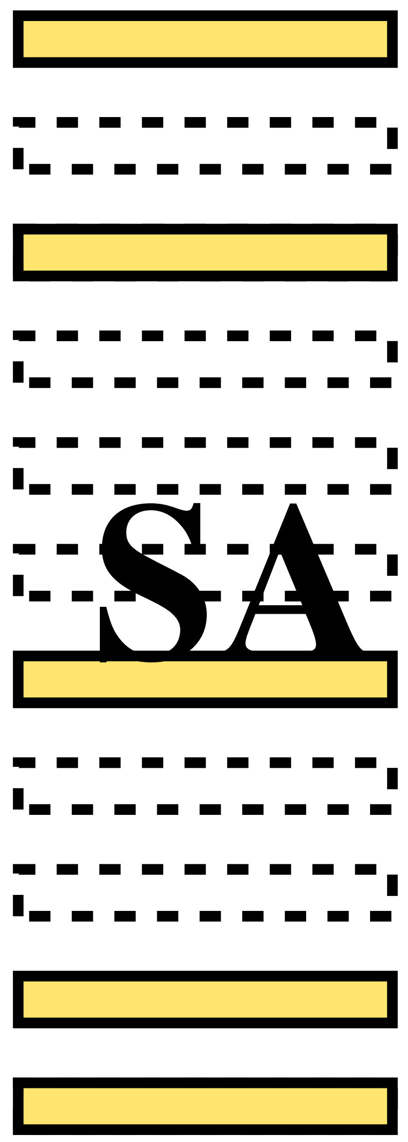

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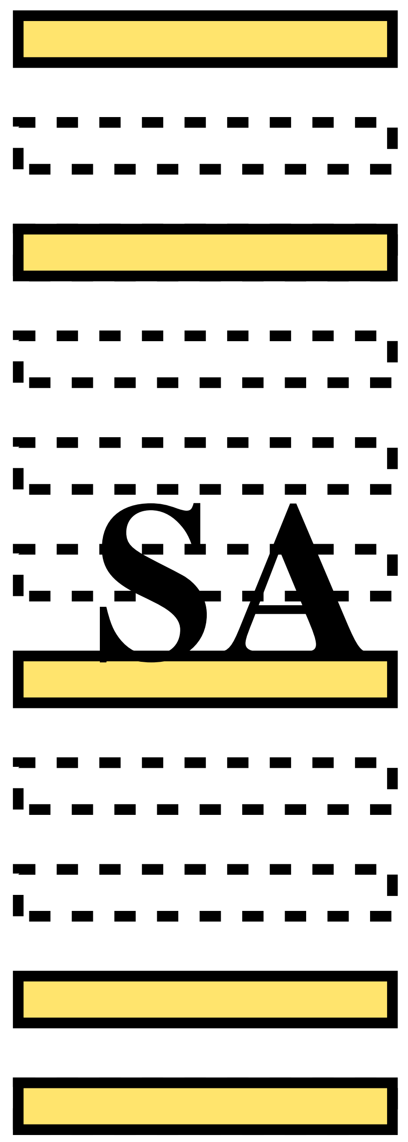

Matrix Approximation

Non-oblivious Subspace Embeddings

Definition (Leverage scores). For $\mathbf{A} \in \mathbb{R}^{n \times d}$ and $i \in [n]$, the ith leverage score is

$$\tau_i(\mathbf{A}) = \sup_{\mathbf{A}\mathbf{x} \neq 0} \frac{\langle \mathbf{a}_i, \mathbf{x} \rangle^2}{\|\mathbf{A}\mathbf{x}\|_2^2}$$

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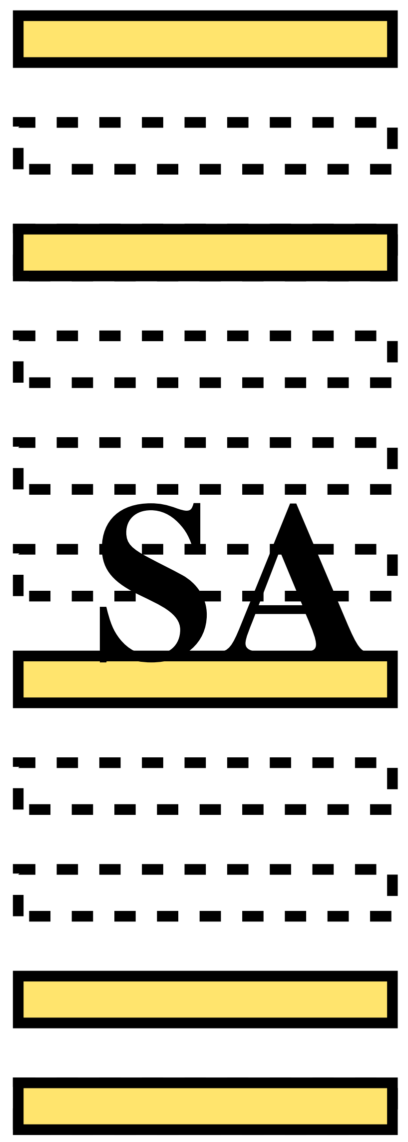
ℓ_2 norm of $\mathbf{Ax}$



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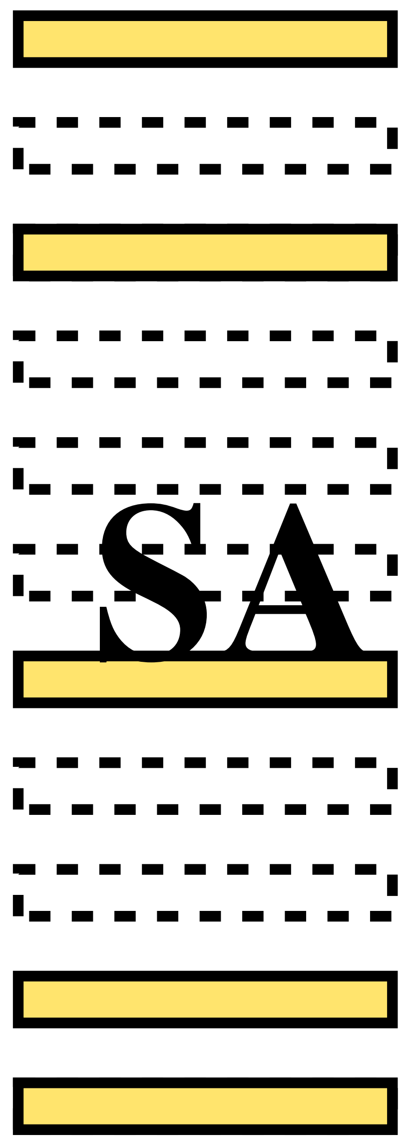
“Importance score”: largest fraction of ℓ_2 norm occupied by the ith coordinate



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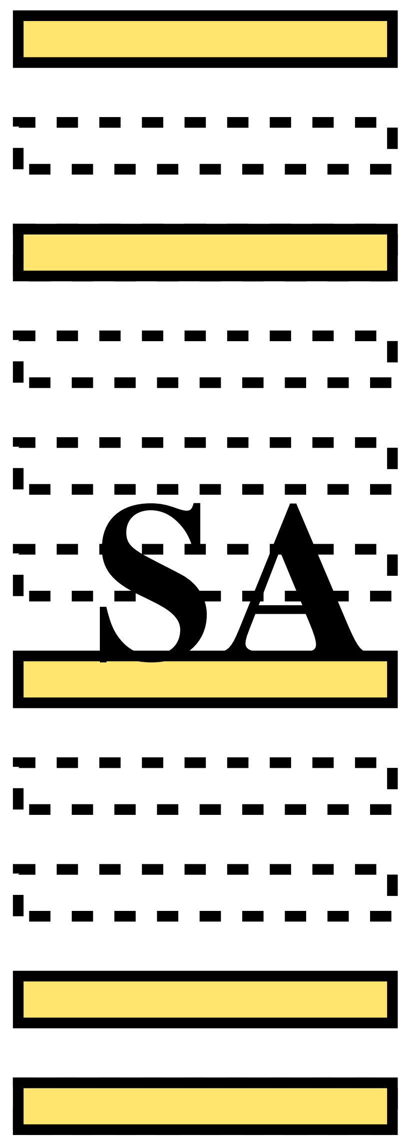

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  - ℓ_p sensitivity scores (Langberg—Schulman 2010)

Matrix Approximation

Non-oblivious Subspace Embeddings

Definition (Leverage scores). For $\mathbf{A} \in \mathbb{R}^{n \times d}$ and $i \in [n]$, the ith leverage score is

$$\tau_i(\mathbf{A}) = \sup_{\mathbf{A}\mathbf{x} \neq 0} \frac{\langle \mathbf{a}_i, \mathbf{x} \rangle^2}{\|\mathbf{A}\mathbf{x}\|_2^2}$$

Contribution of the ith coordinate

ℓ_2 norm of $\mathbf{Ax}$

“Importance score”: largest fraction of ℓ_2 norm occupied by the ith coordinate

- Many generalizations:
  - ℓ_p sensitivity scores (Langberg—Schulman 2010)
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Matrix Approximation

Non-oblivious Subspace Embeddings

Theorem (Lewis weight sampling, Cohen—Peng 2015). For any $\mathbf{A} \in \mathbb{R}^{n \times d}$, there are probabilities $q_1, q_2, \ldots q_n$ that sample

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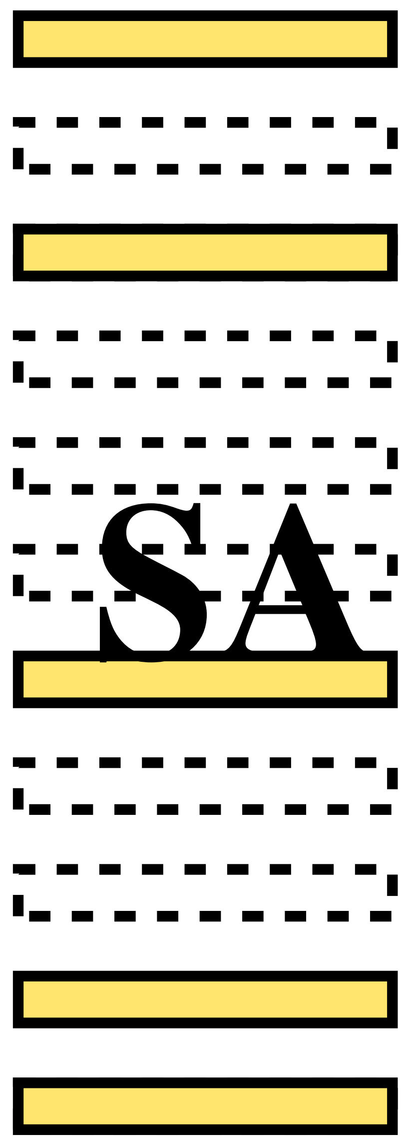
rows of $\mathbf{A}$ that forms an ℓ_p subspace embedding with distortion $\kappa = (1 + \varepsilon)$, with probability 99%.



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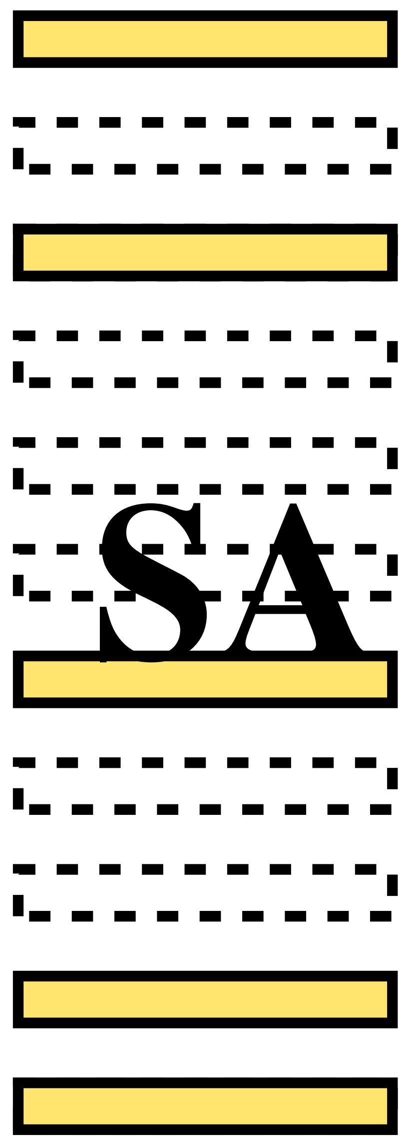

Matrix Approximation

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Matrix Approximation

Non-oblivious Subspace Embeddings

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Matrix Approximation

Non-oblivious Subspace Embeddings

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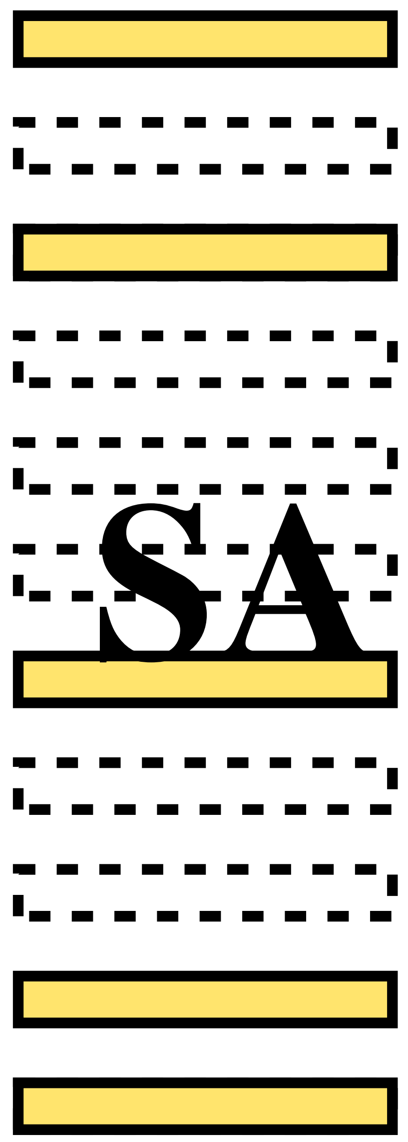

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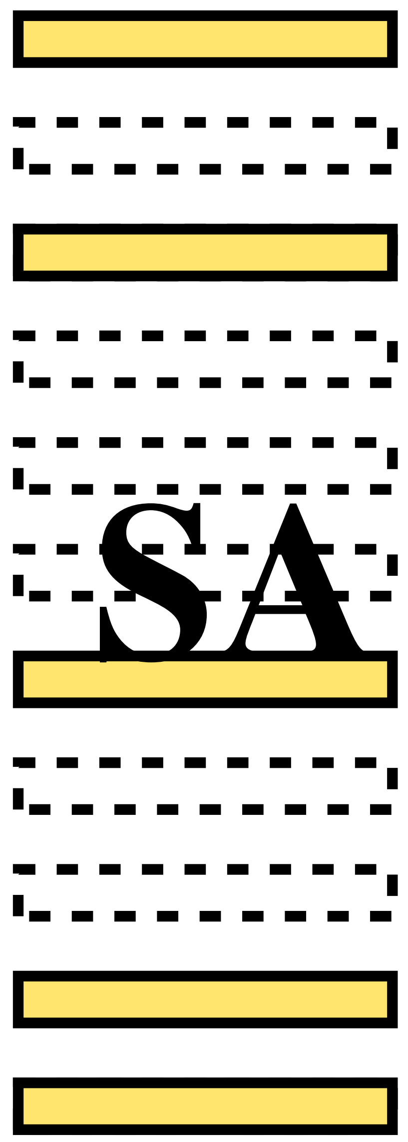

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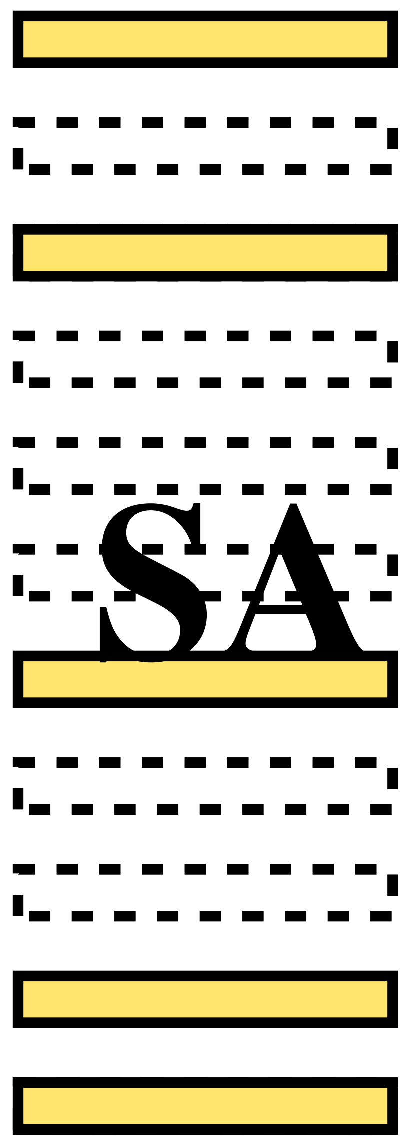

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ε^{-2} should be possible here!

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Matrix Approximation

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Woodruff—Y 2023

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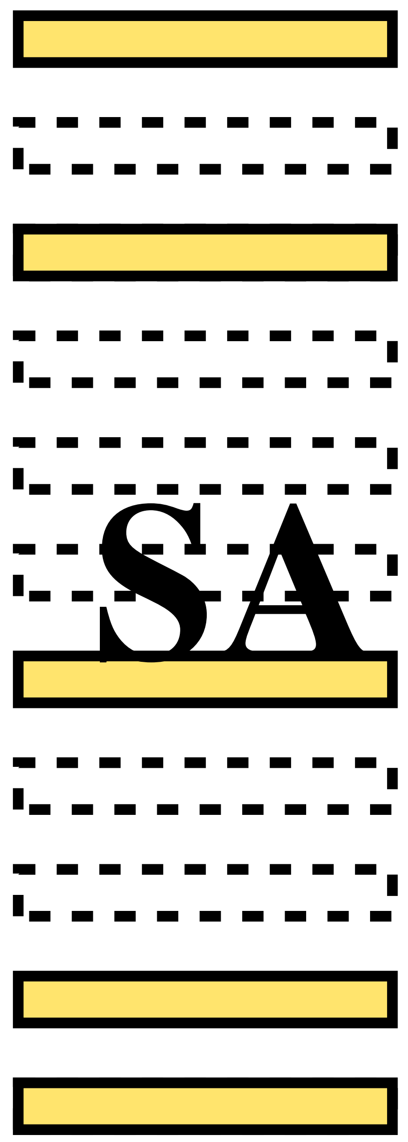

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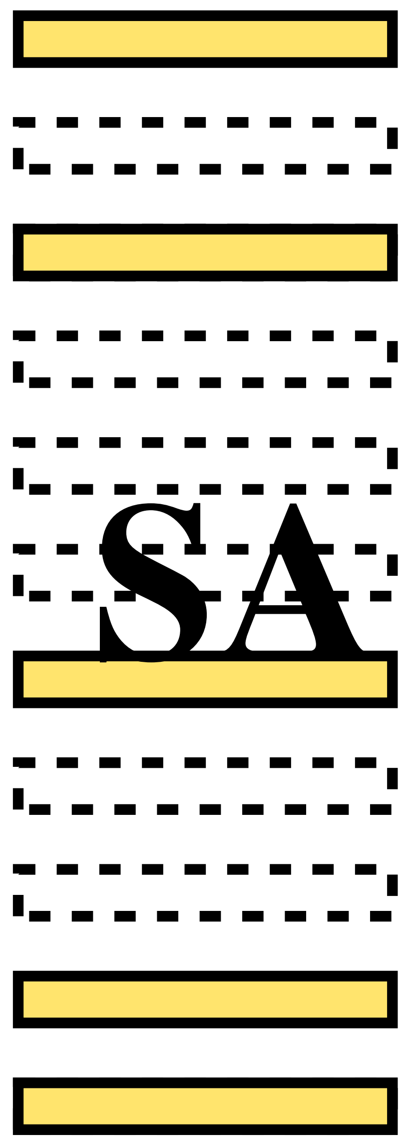

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These bounds also hold in the streaming setting (Woodruff—Y 2023)

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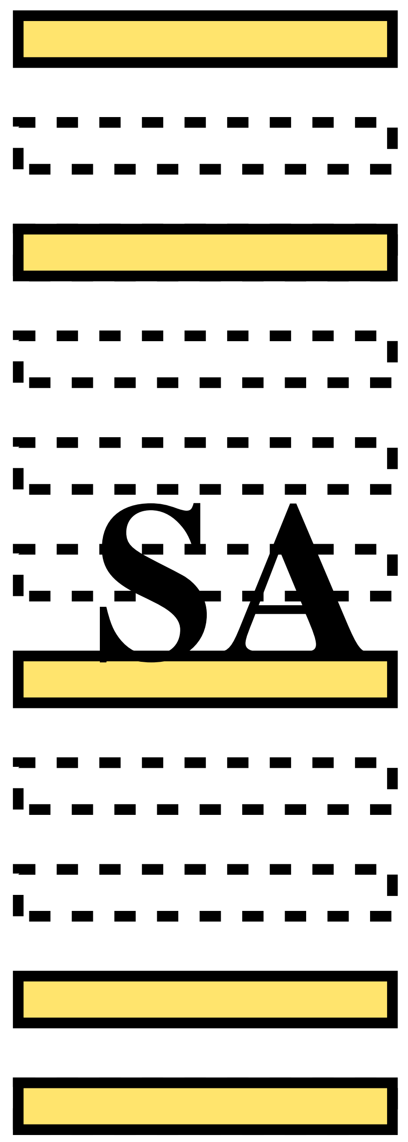

Matrix Approximation

Non-oblivious Subspace Embeddings

Question 3. Are there similar results for other loss functions?

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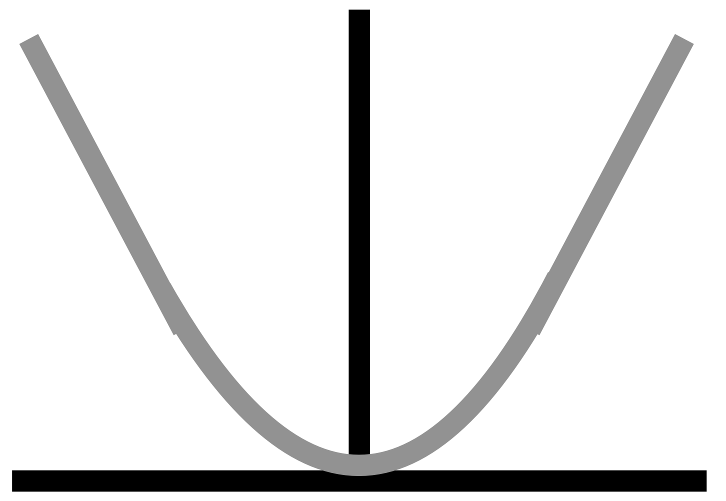
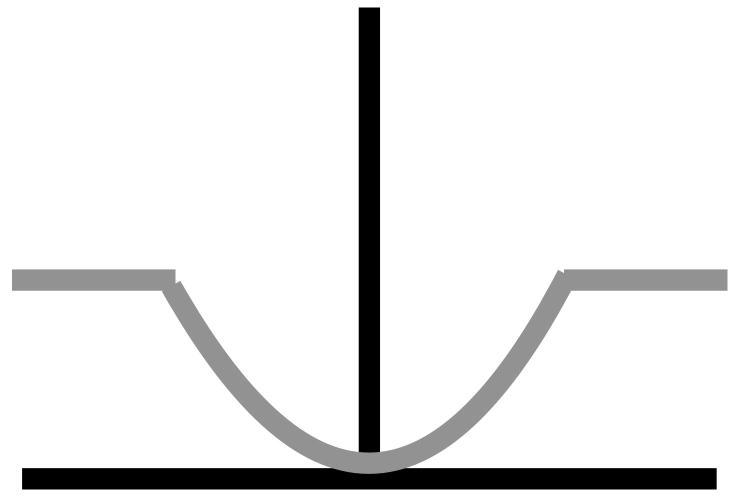
  - Huber/Tukey loss for robust statistics





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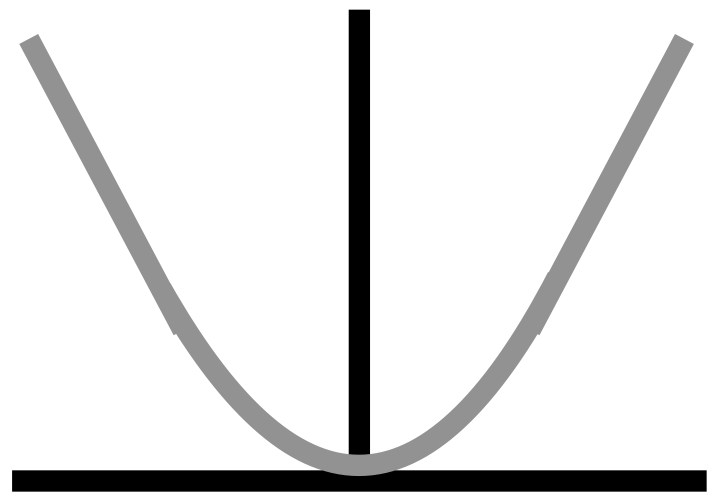

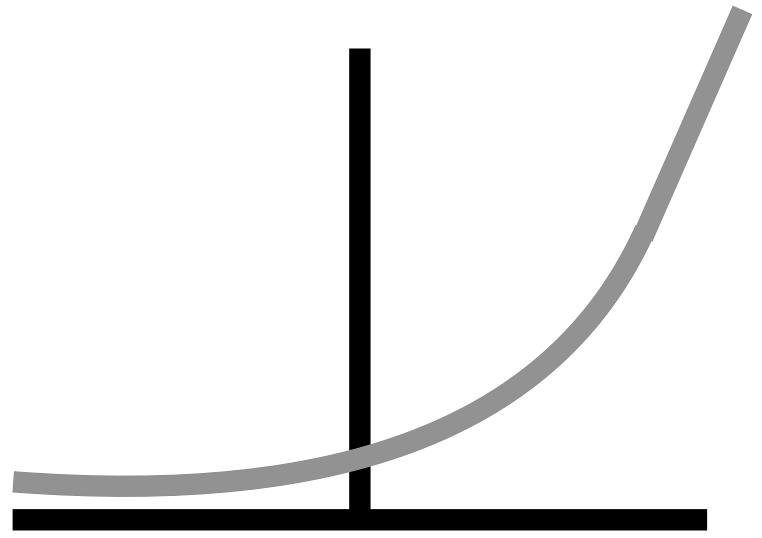

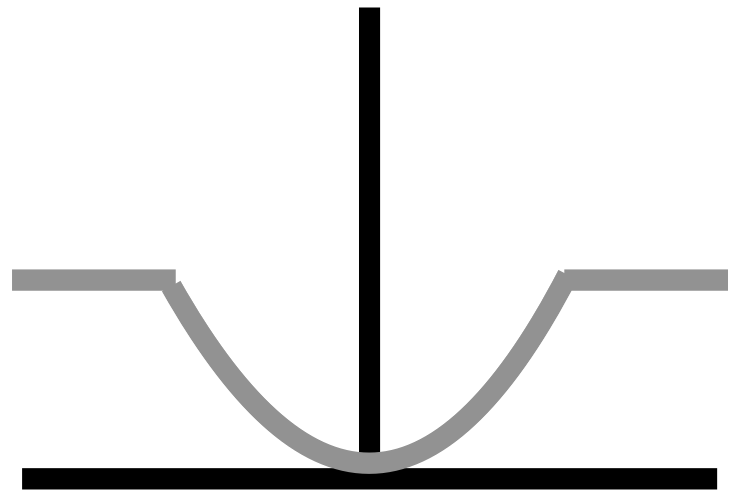

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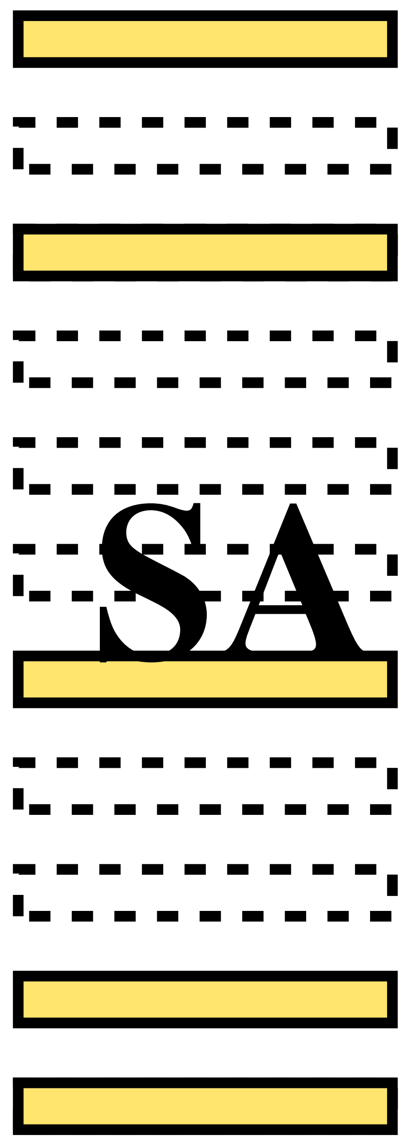

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Musco—Musco—Woodruff—Y 2022

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Musco—Musco—Woodruff—Y 2022

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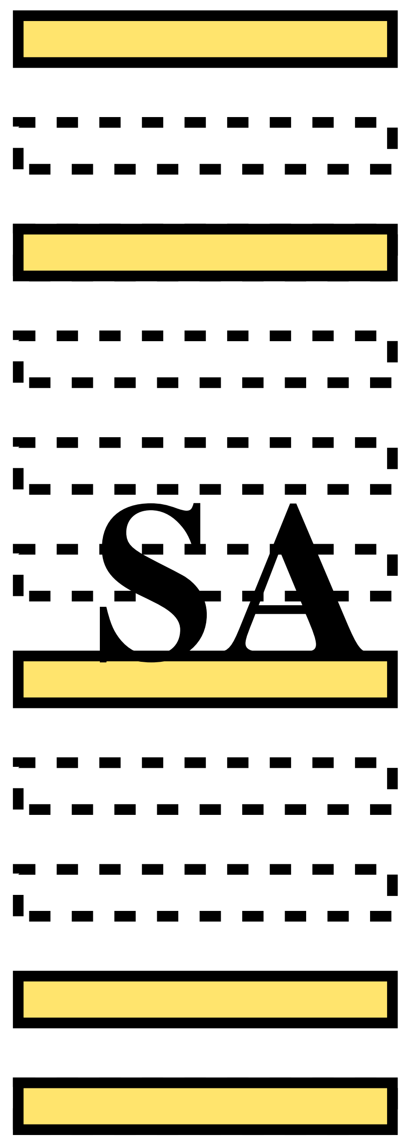

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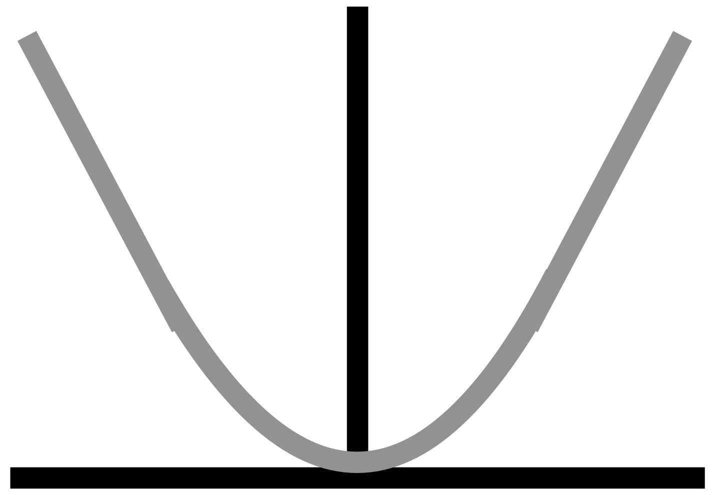

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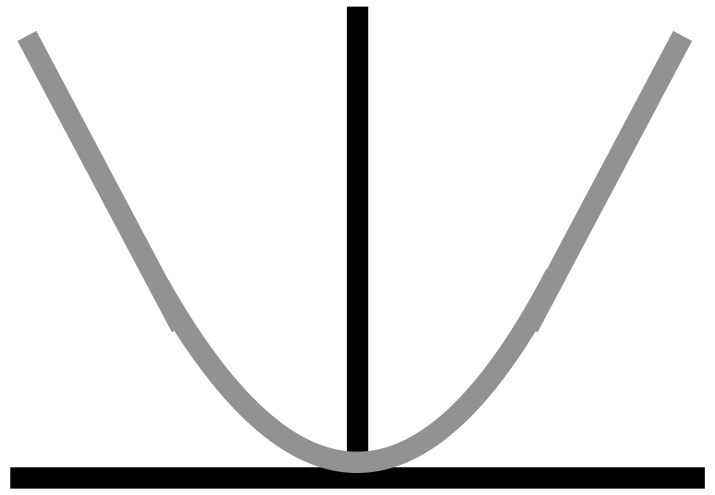

$r = \mathrm{poly}(\varepsilon^{-1}) d^{4-2\sqrt{2}} \approx d^{1.172}$

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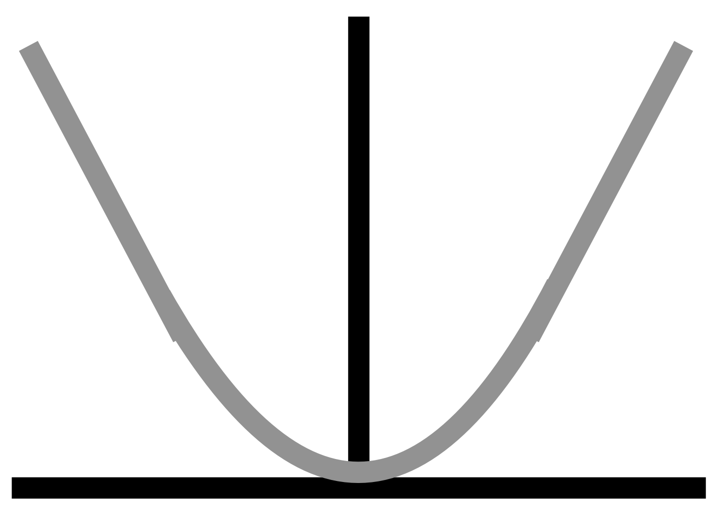

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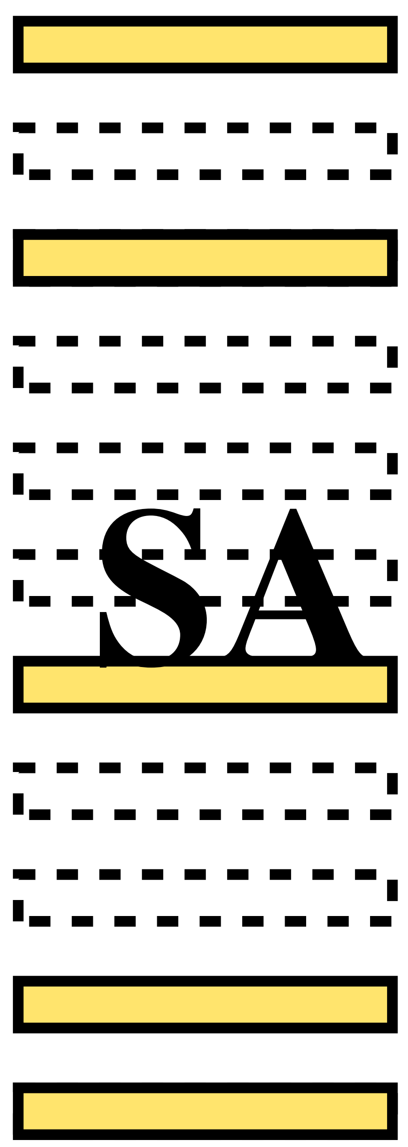
I am very interested in improving this to d



Subspace Embeddings and Linear Regression

- Oblivious ℓ_p subspace embeddings: high distortion and low distortion
- Non-oblivious subspace embeddings: ℓ_p Lewis weight sampling, general losses

$\boxed{\text{Applications: active learning, streaming computational geometry, low rank approximation}}$

Matrix Approximation

Active ℓ_p Linear Regression

$$\min_{\mathbf{x} \in \mathbb{R}^d} \| \mathbf{A}\mathbf{x} - \mathbf{b} \|_p^p$$

Matrix Approximation

Active ℓ_p Linear Regression

$$\min_{\mathbf{x}\in\mathbb{R}^d} \|\mathbf{Ax}-\mathbf{b}\|_p^p$$

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- Goal: **minimize the number of label entries** that are read

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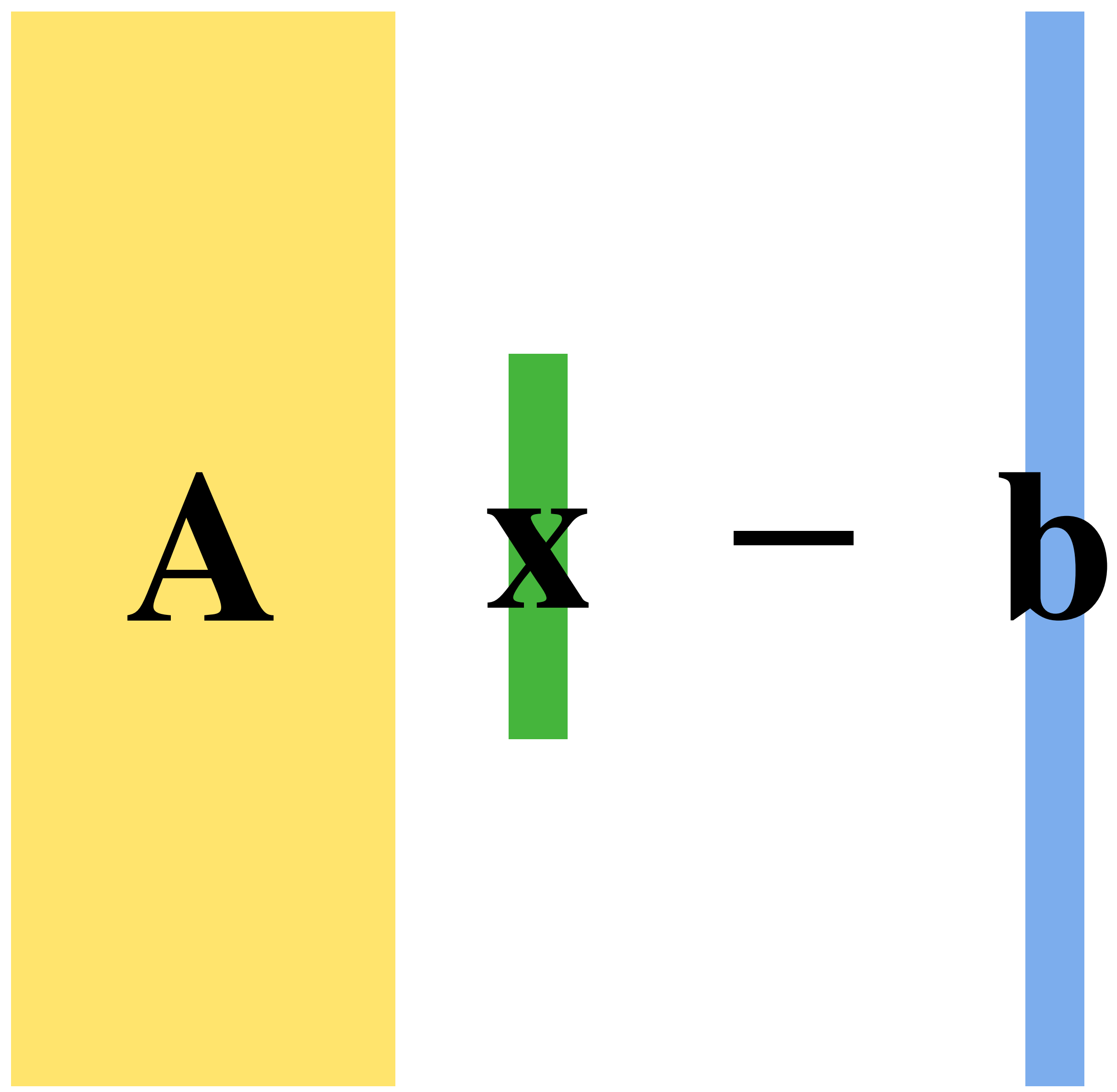

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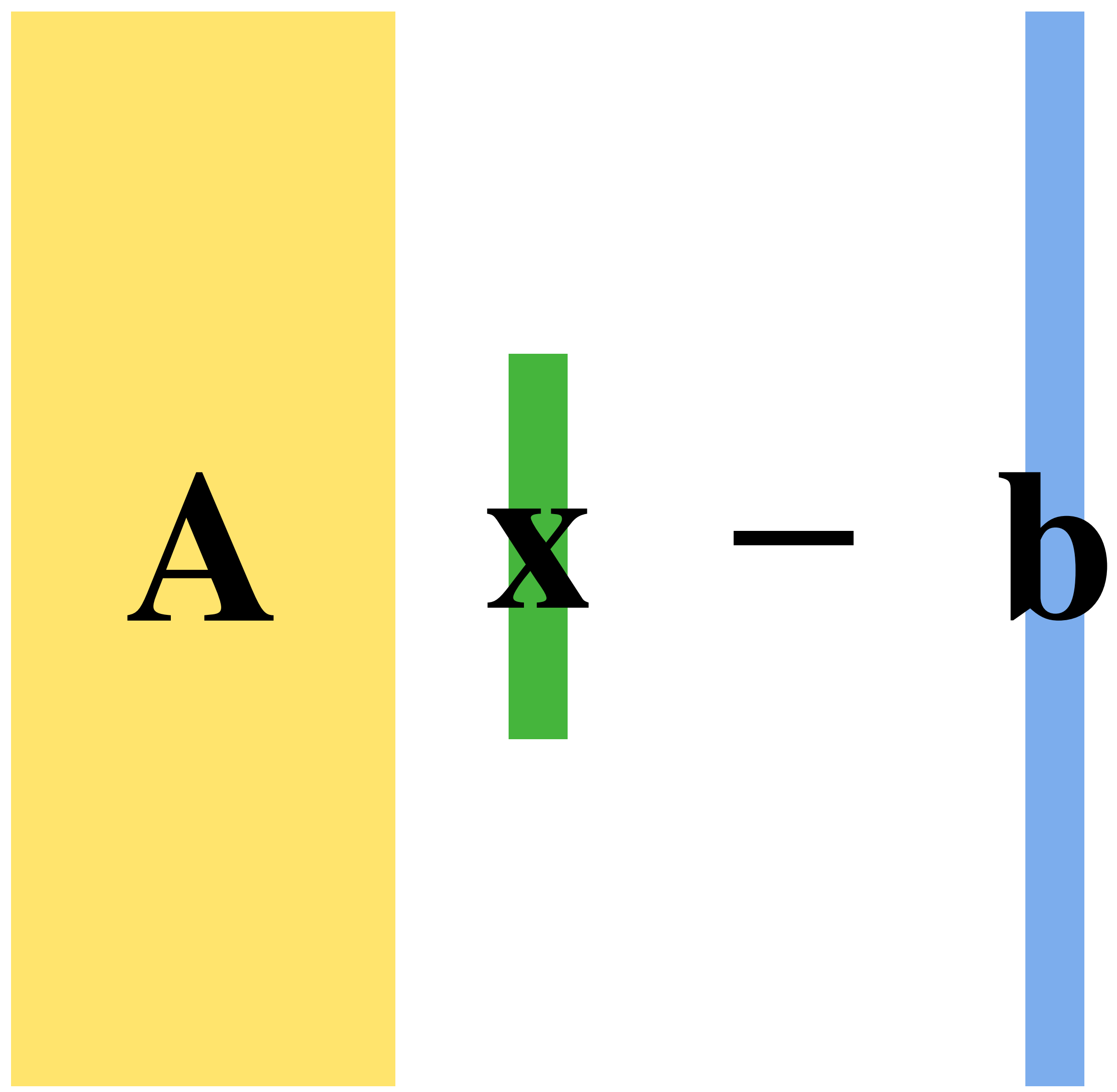
The algorithm has full access to $\mathbf{A}$



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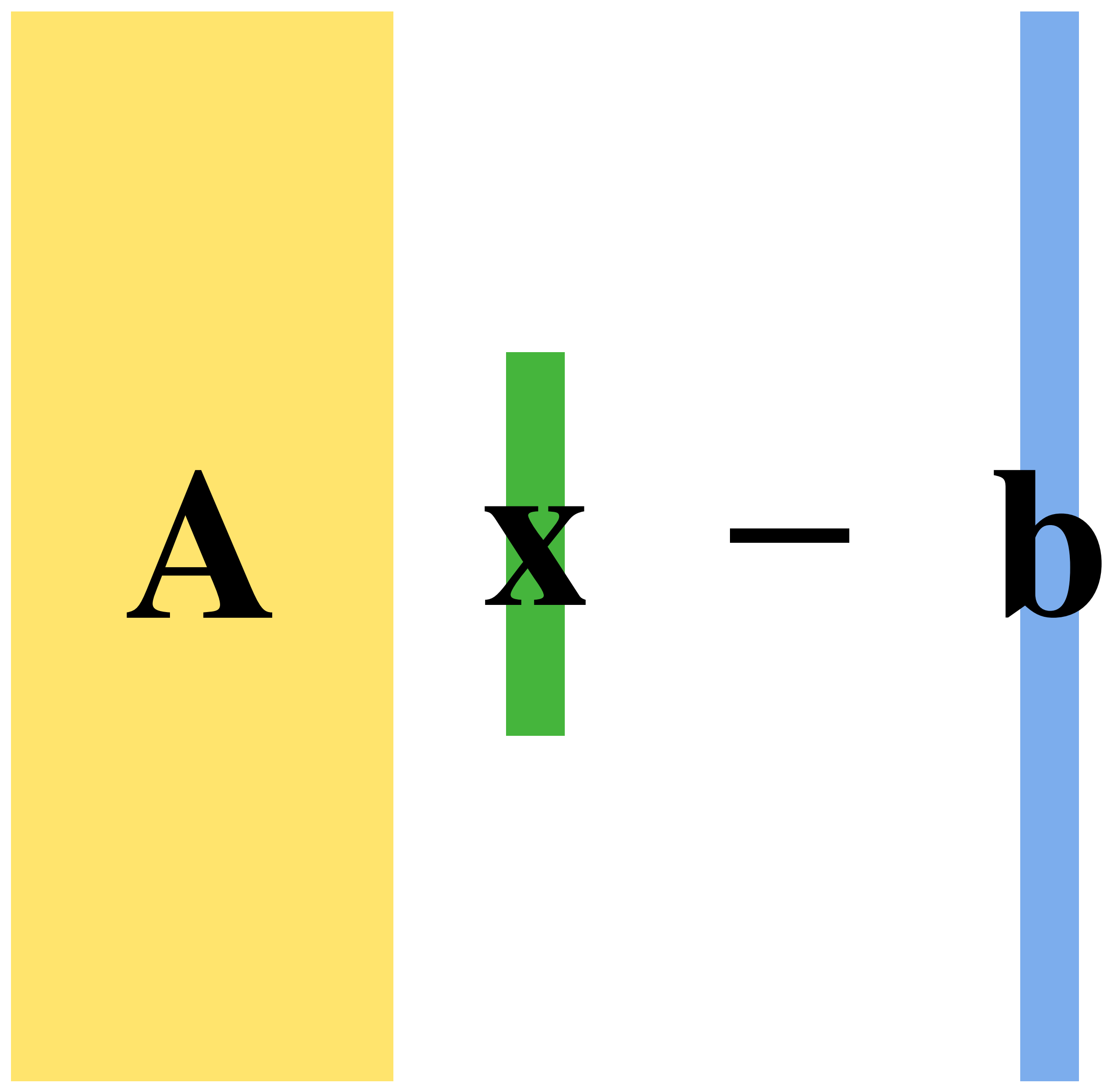

$\mathbf{b}$ vector is hidden, and the algorithm has query access to it

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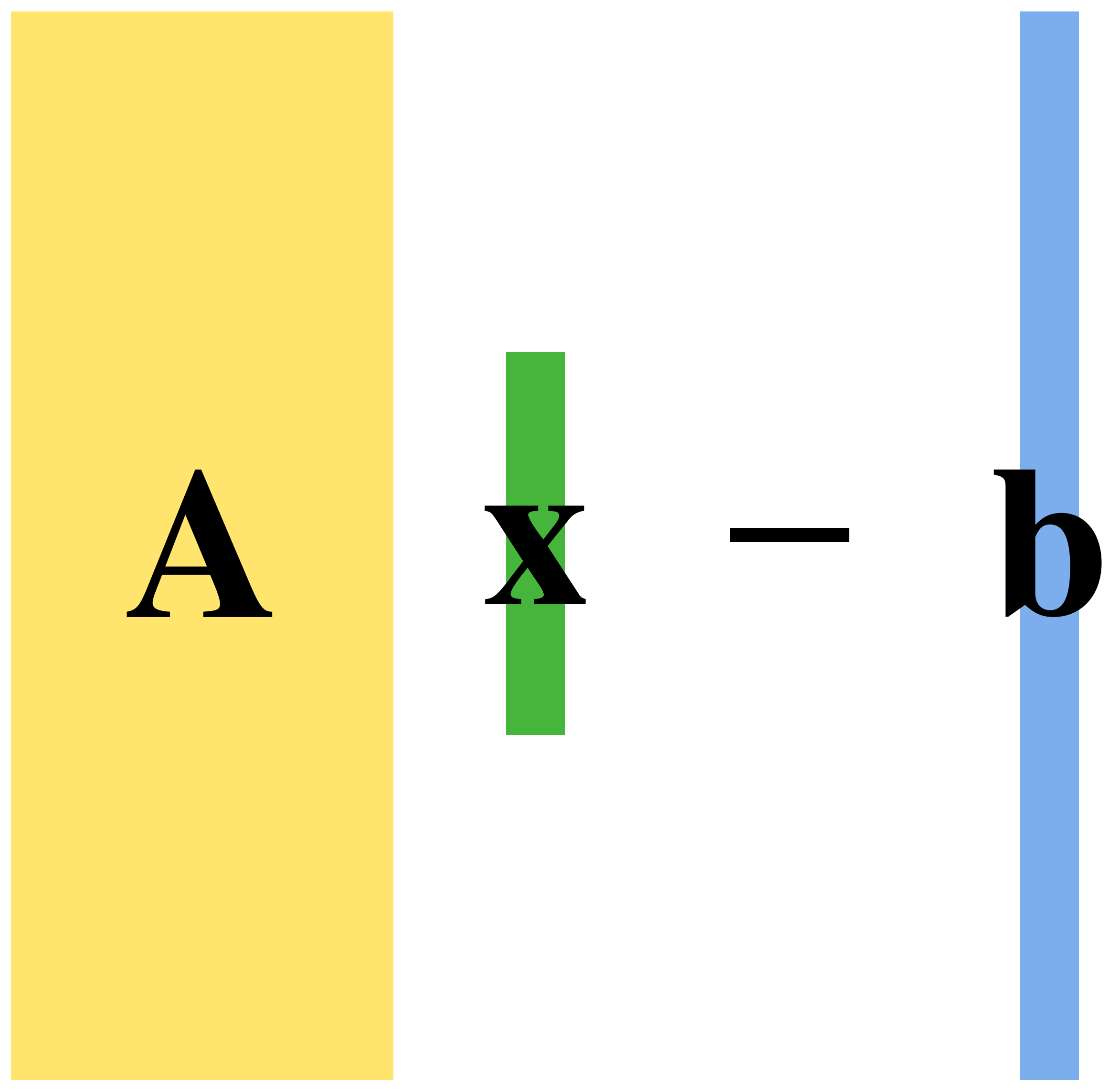

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Question. How many entries of $\mathbf{b}$ need to be read?

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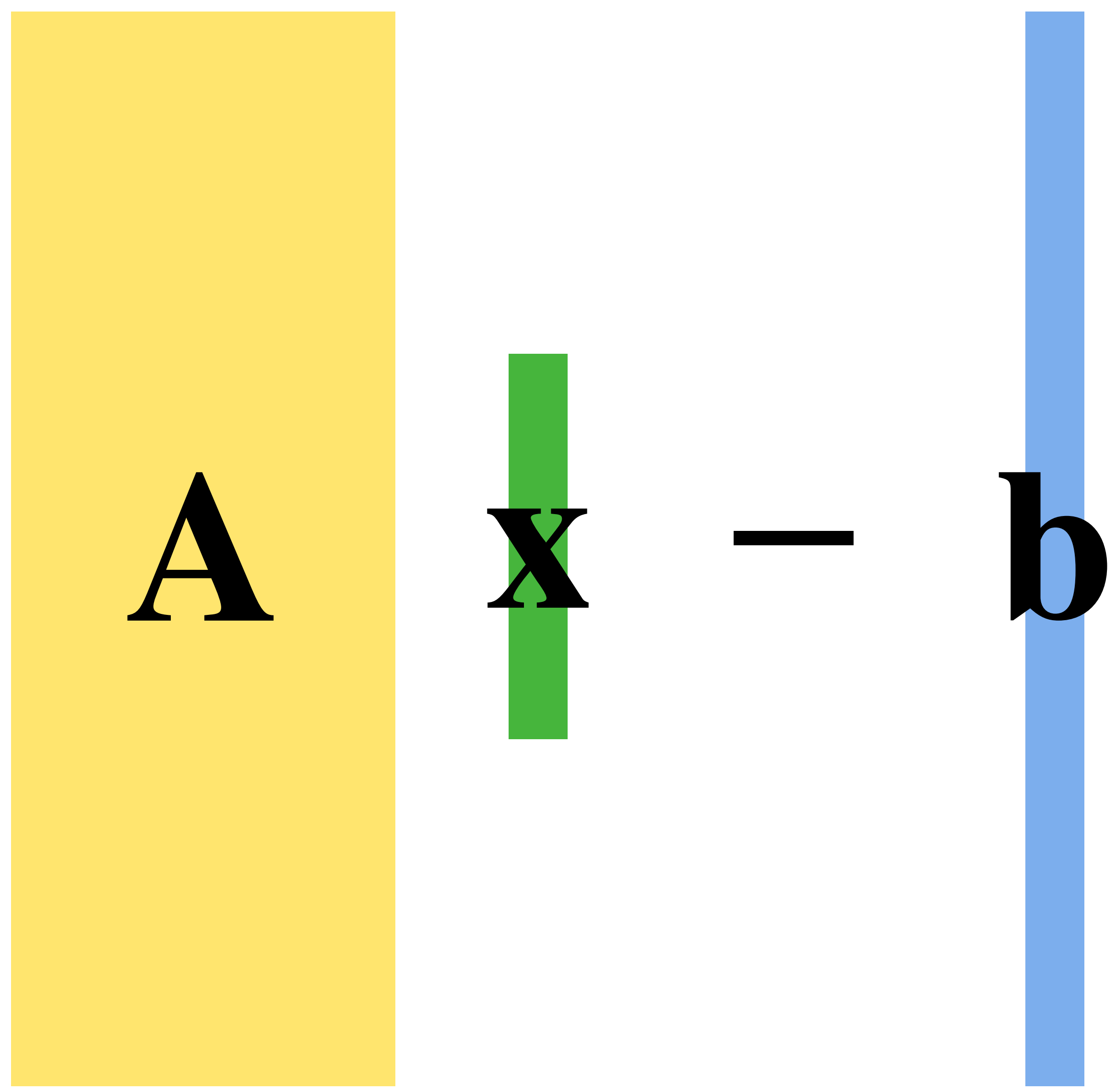

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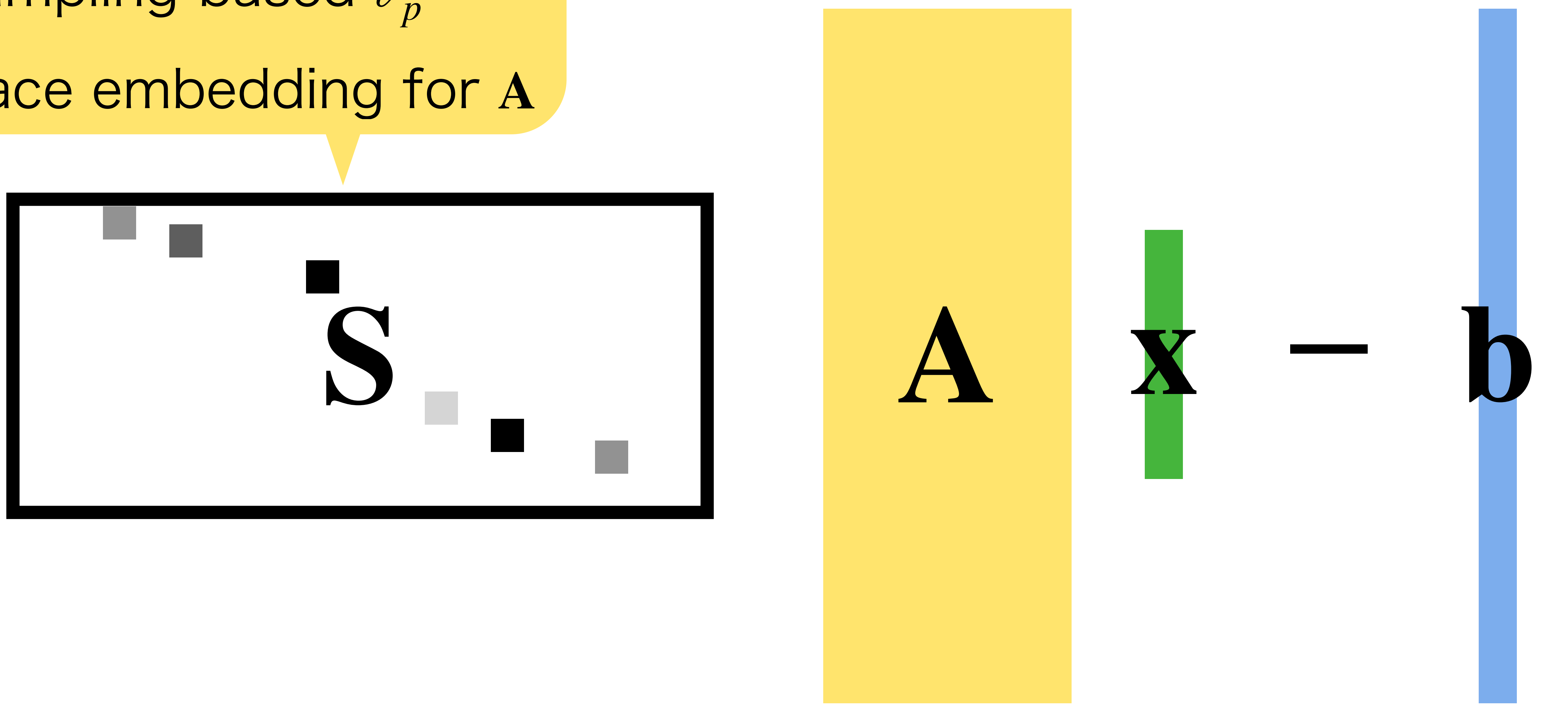
Sampling-based ℓ_p subspace embedding for $\mathbf{A}$



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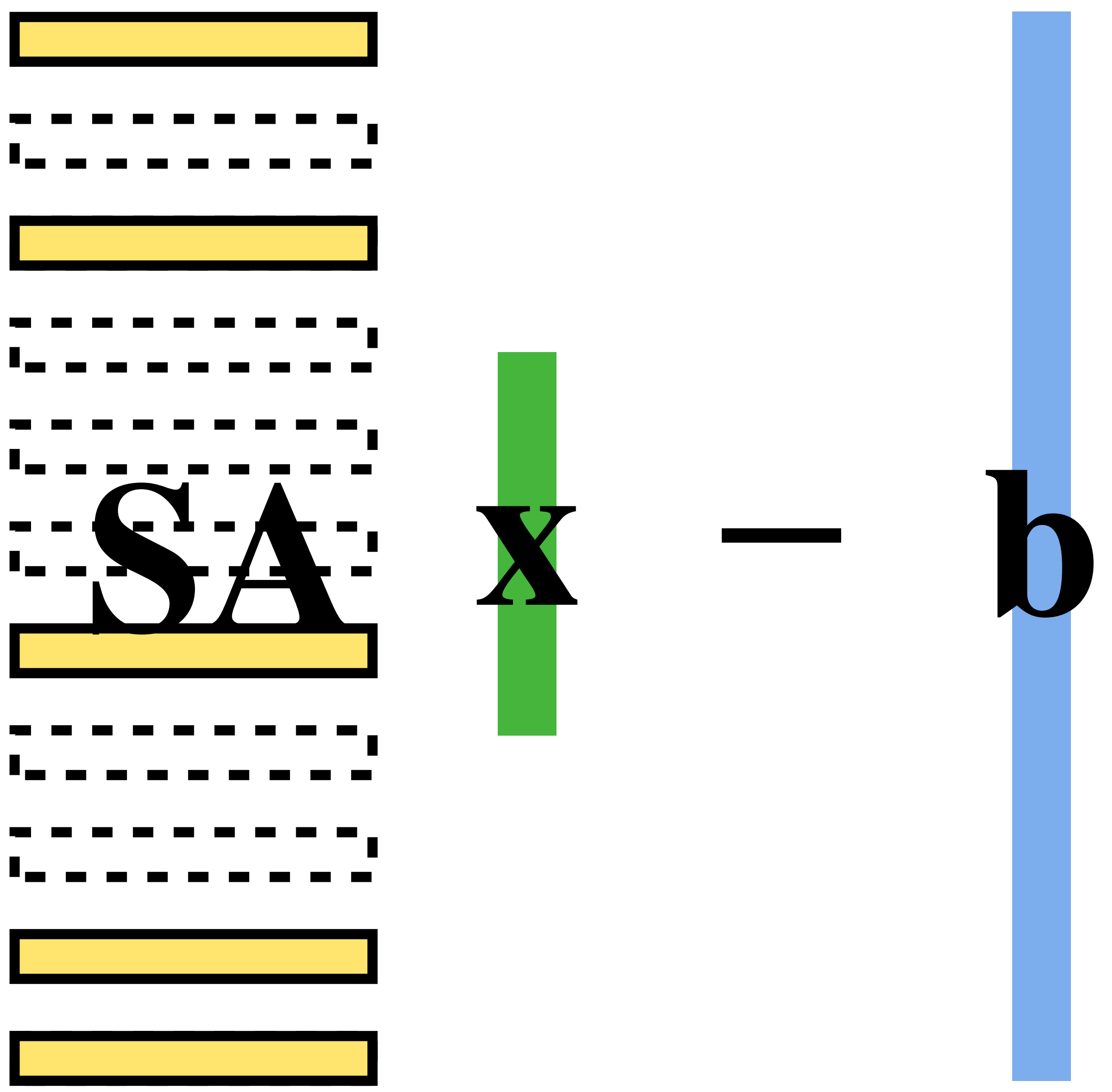

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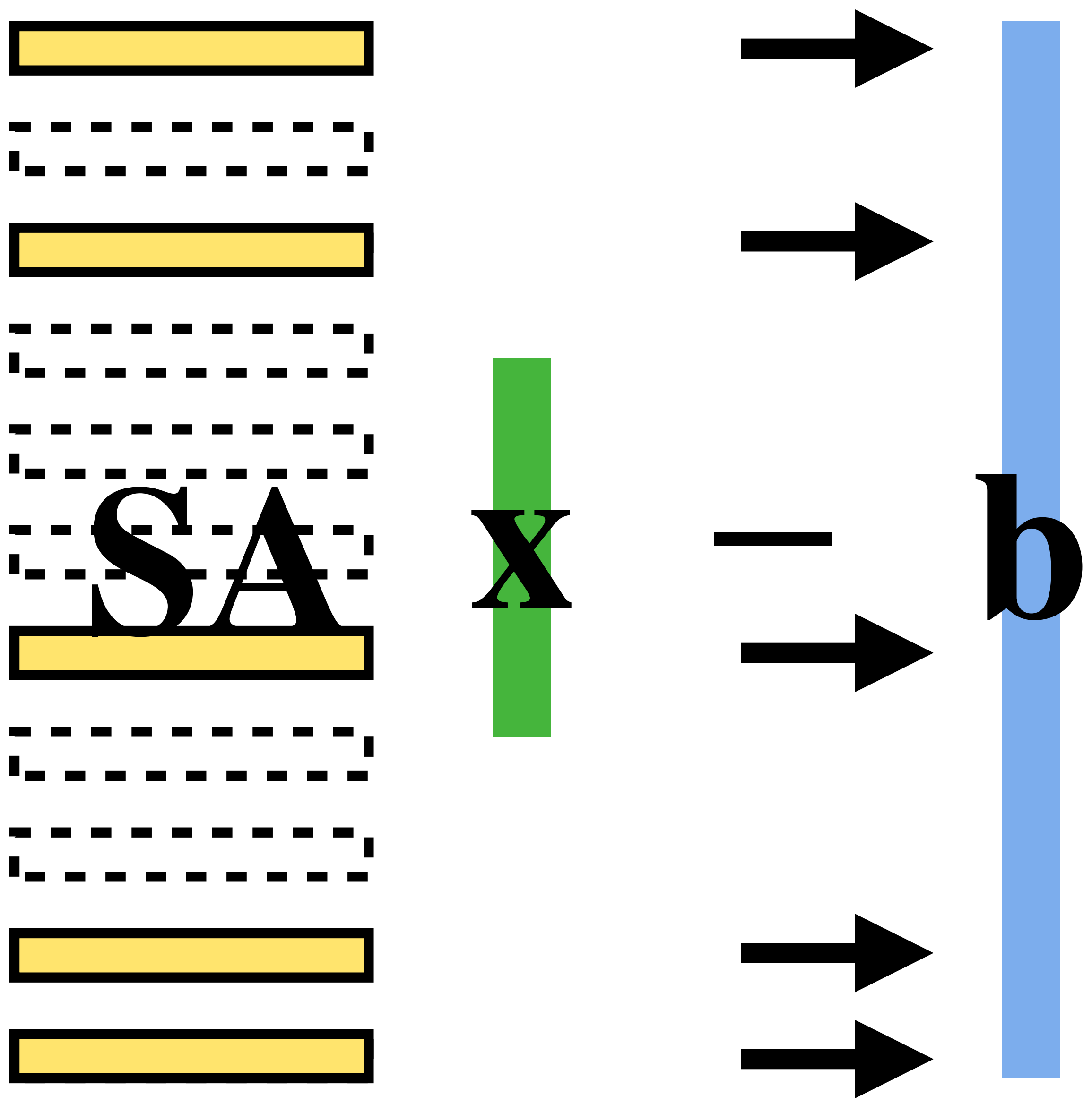

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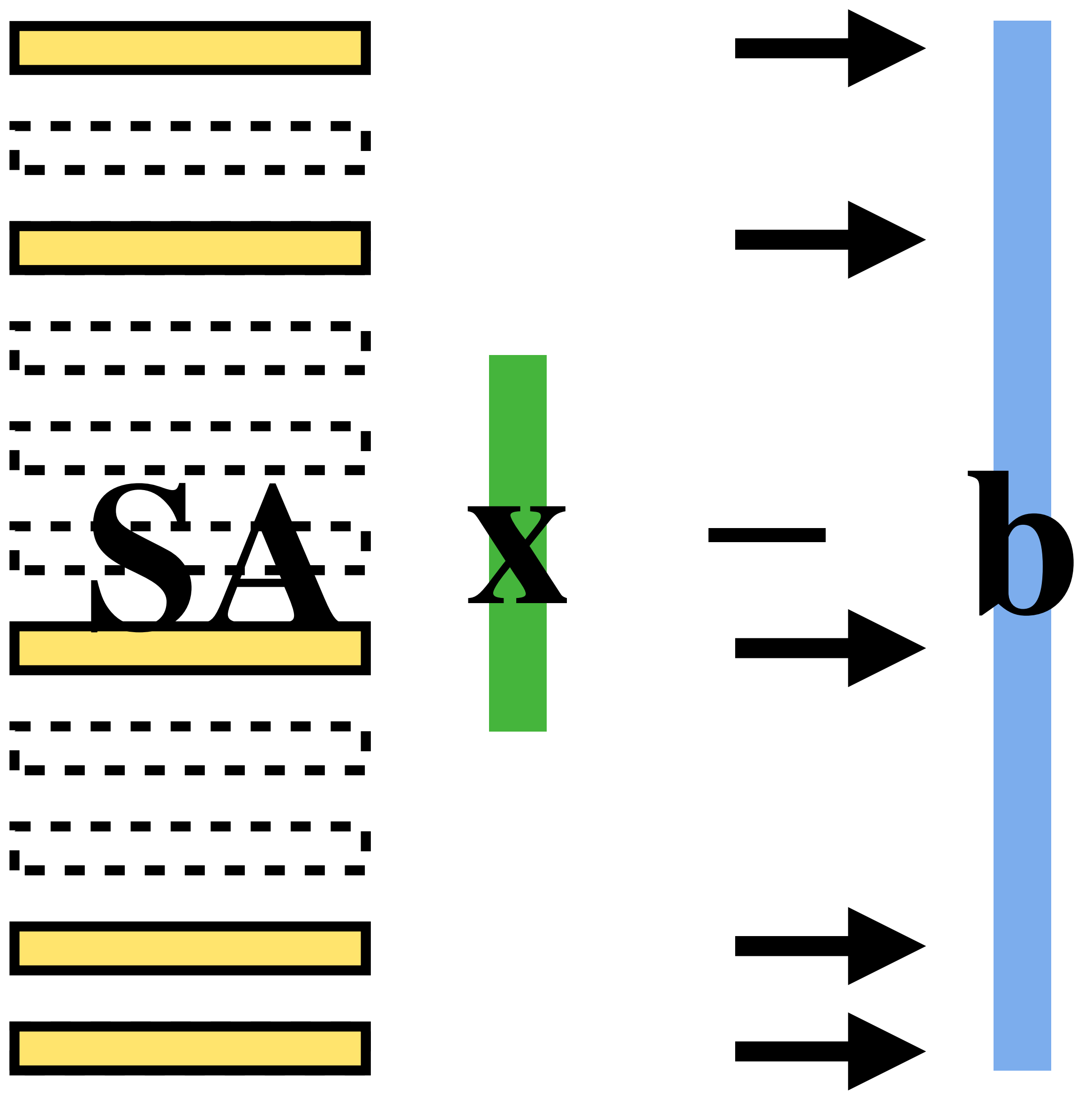

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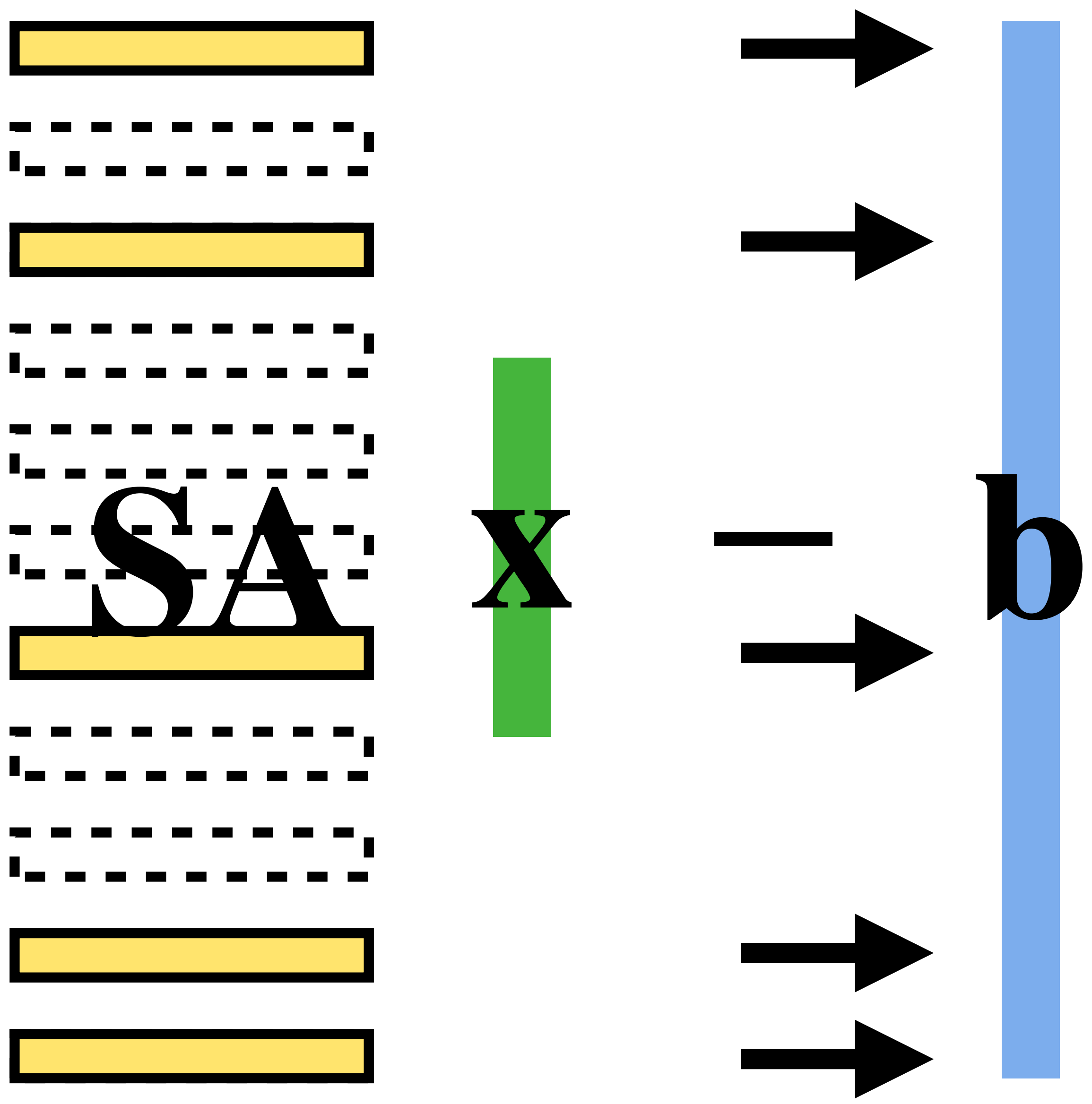


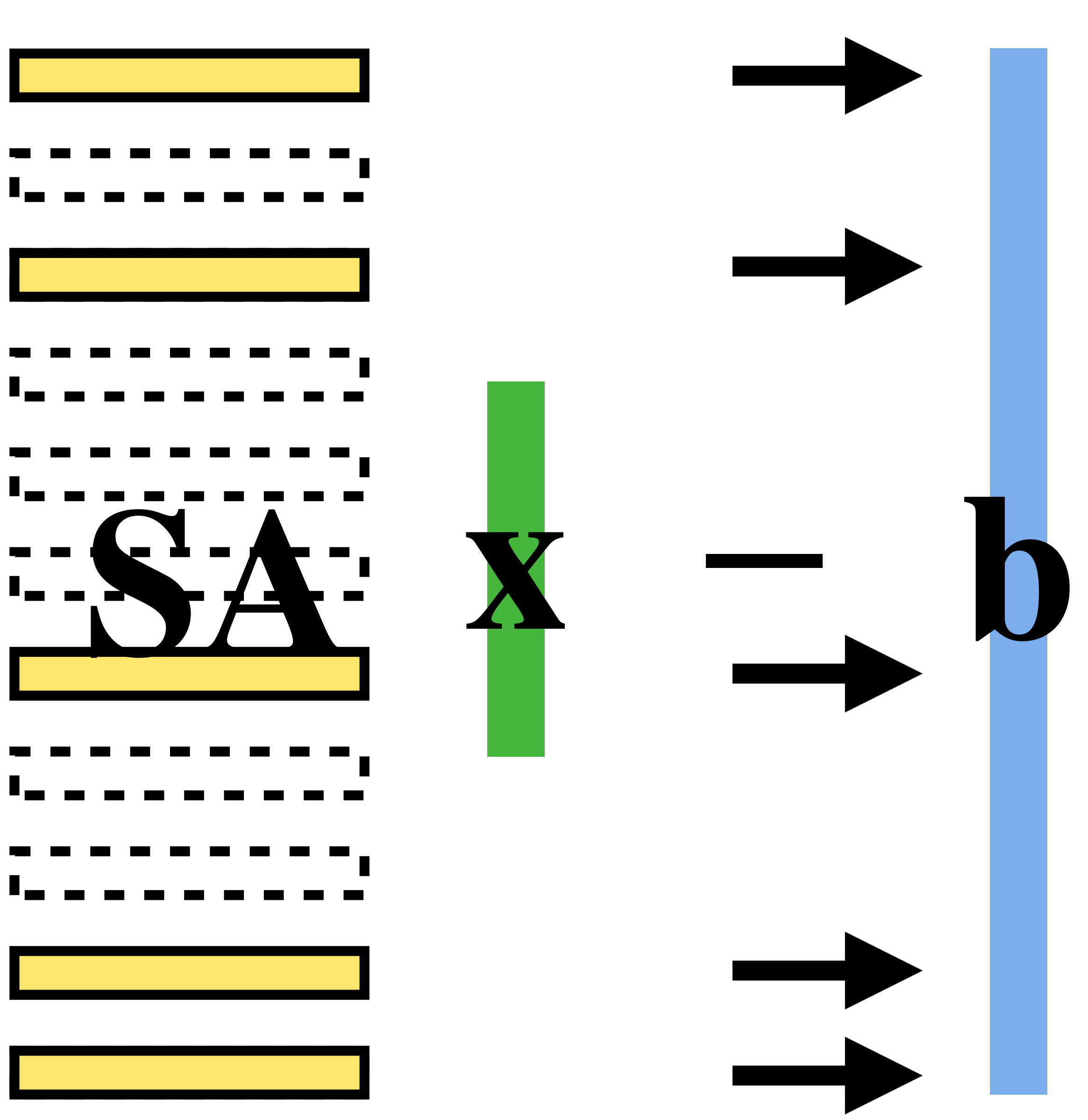
- $p = 2$: $\Theta(\varepsilon^{-1}d)$ (Chen—Price 2019)

Question. How many entries of $\mathbf{b}$ need to be read?

Matrix Approximation

Active ℓ_p Linear Regression

$$\min_{\mathbf{x}\in\mathbb{R}^d} \| \mathbf{Ax} - \mathbf{b} \|_p^p$$



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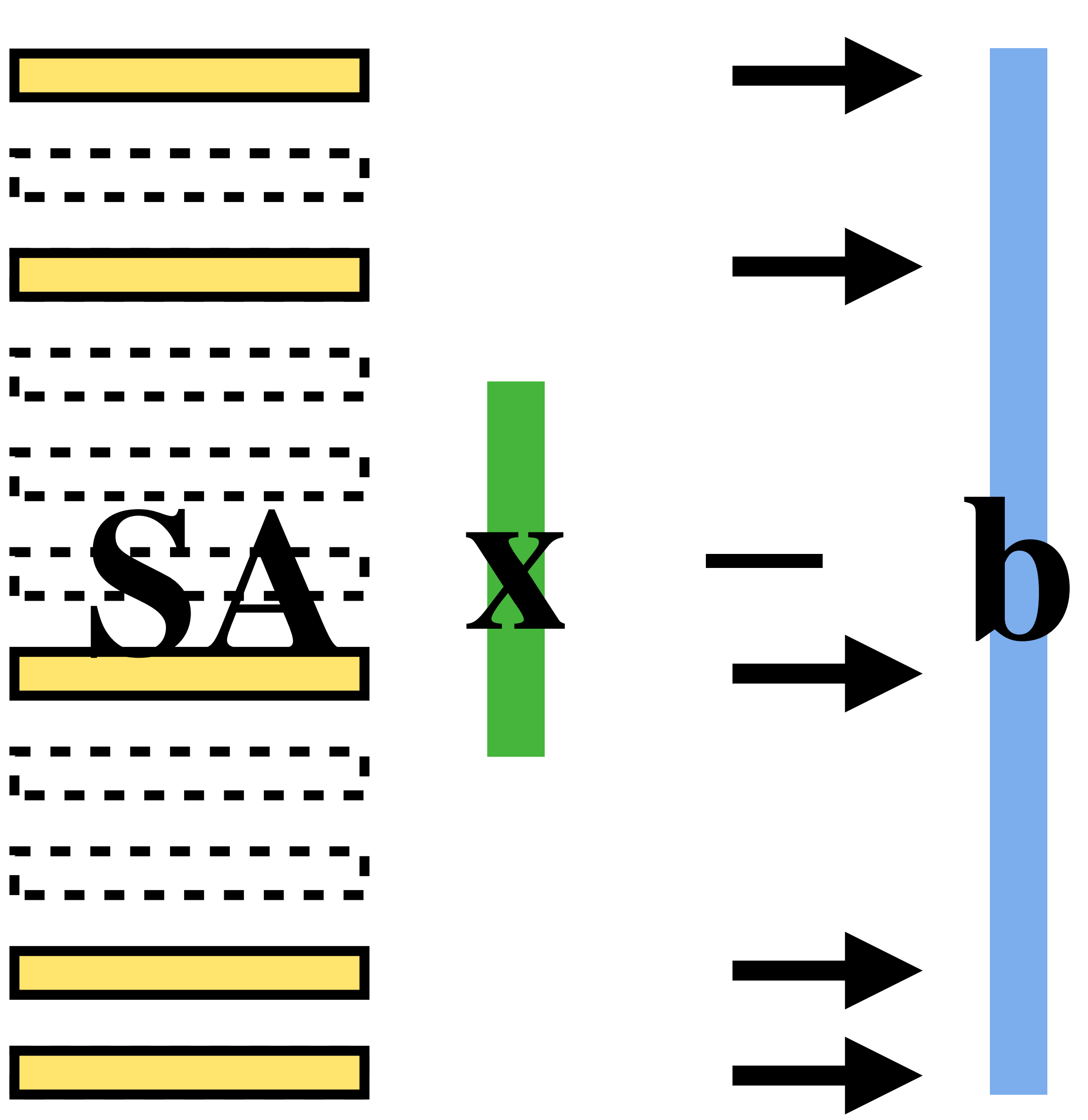

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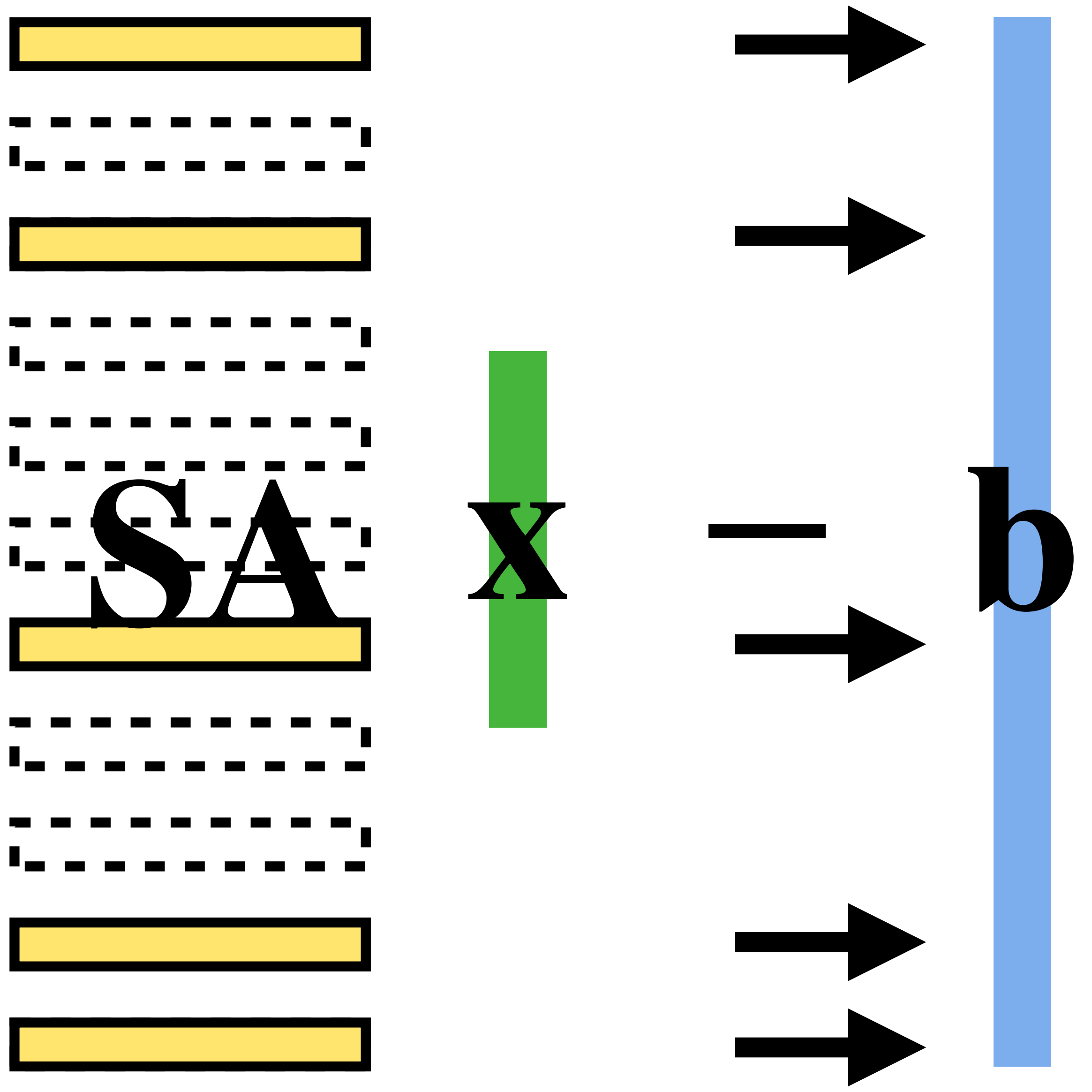

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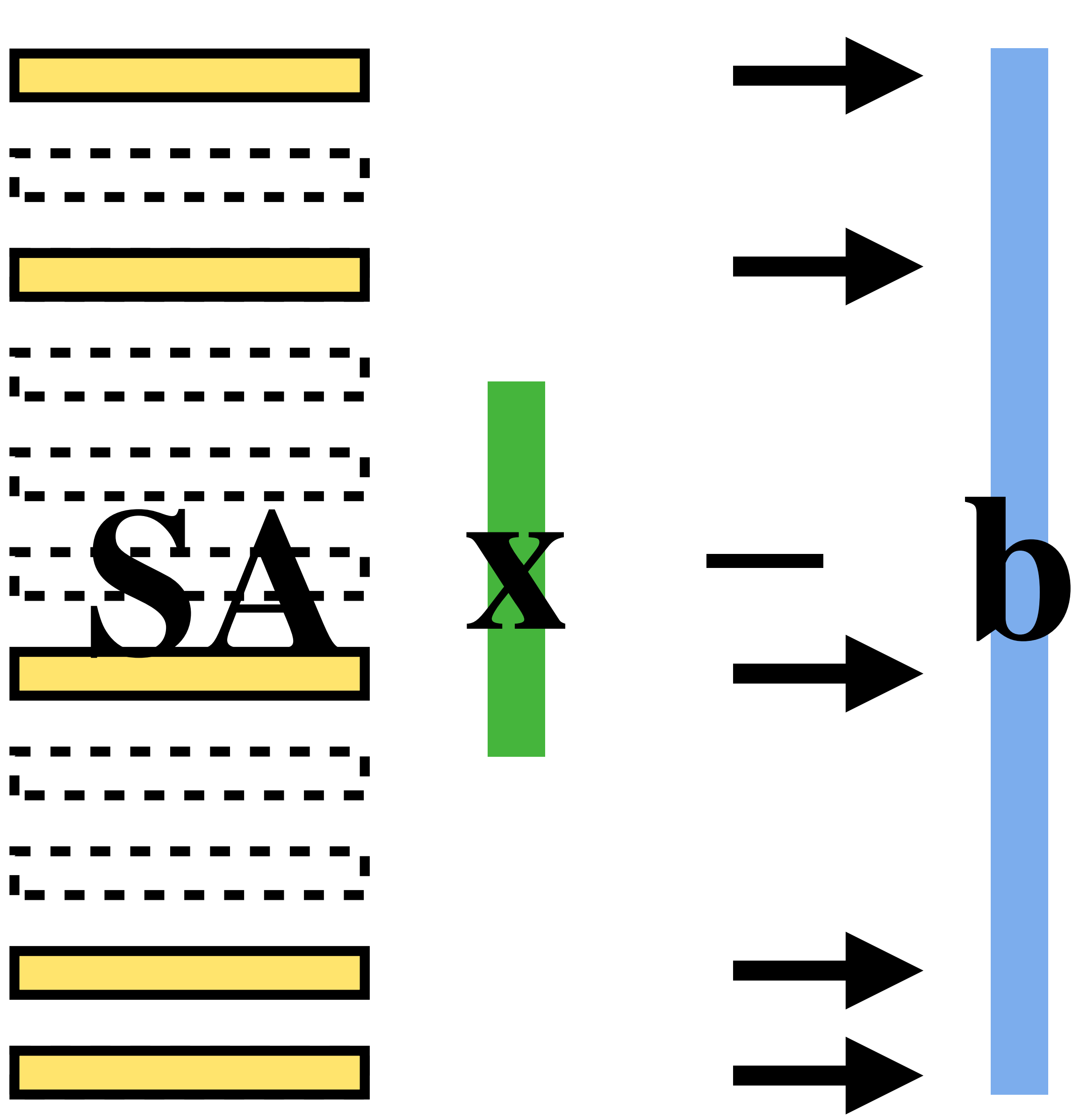

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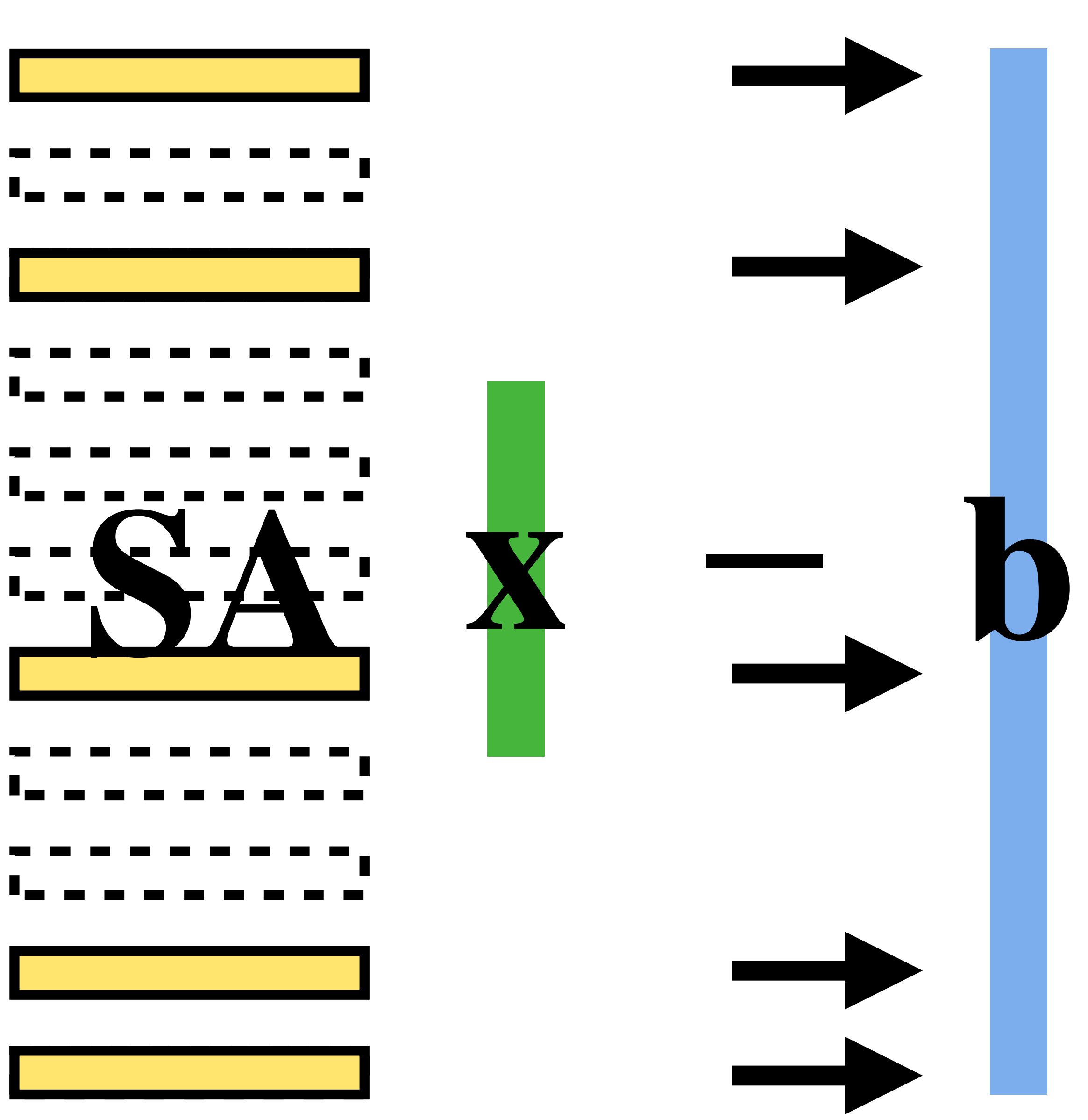

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Matrix Approximation

Streaming Löwner—John Ellipsoids

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Input: symmetric polytope with
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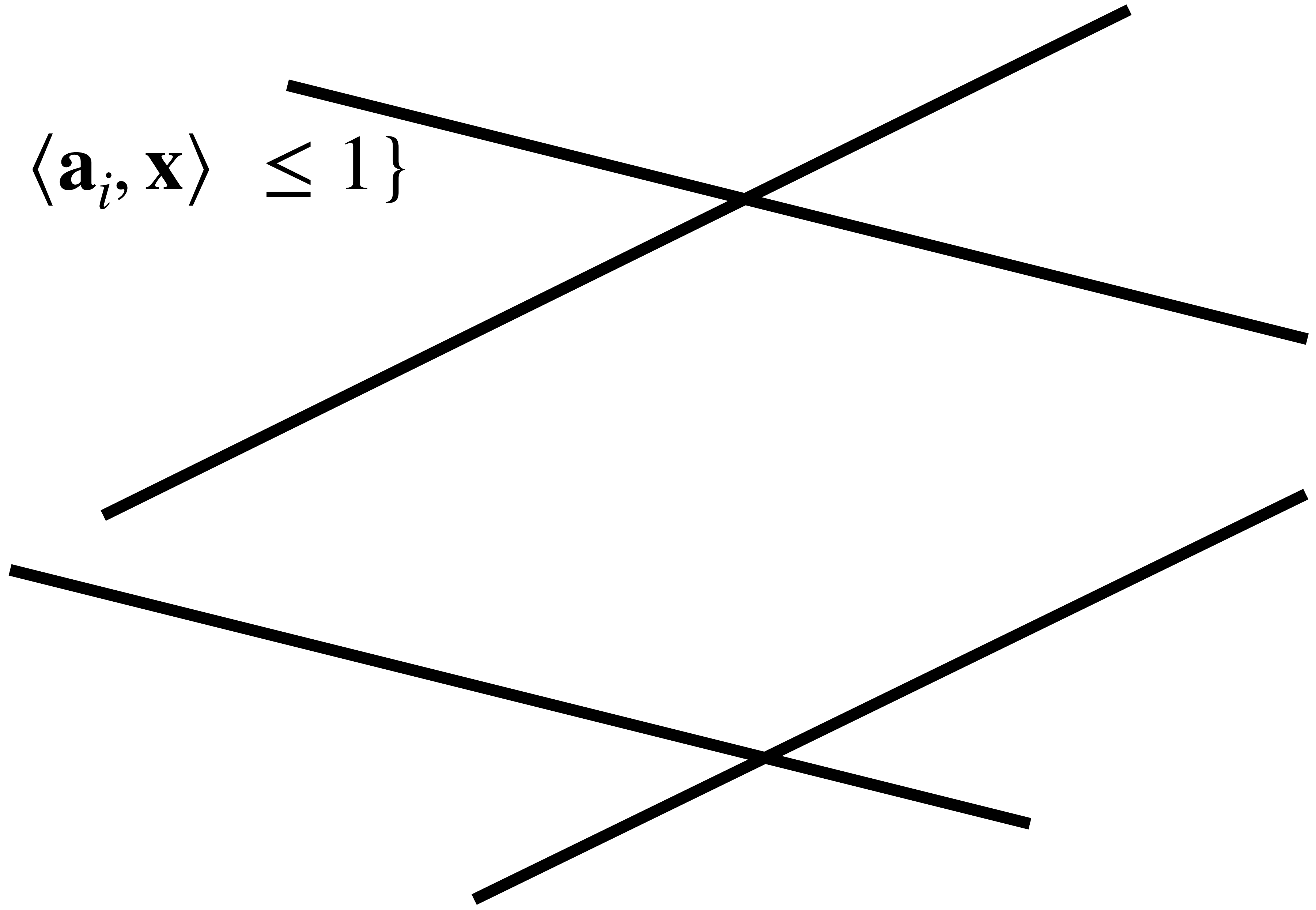

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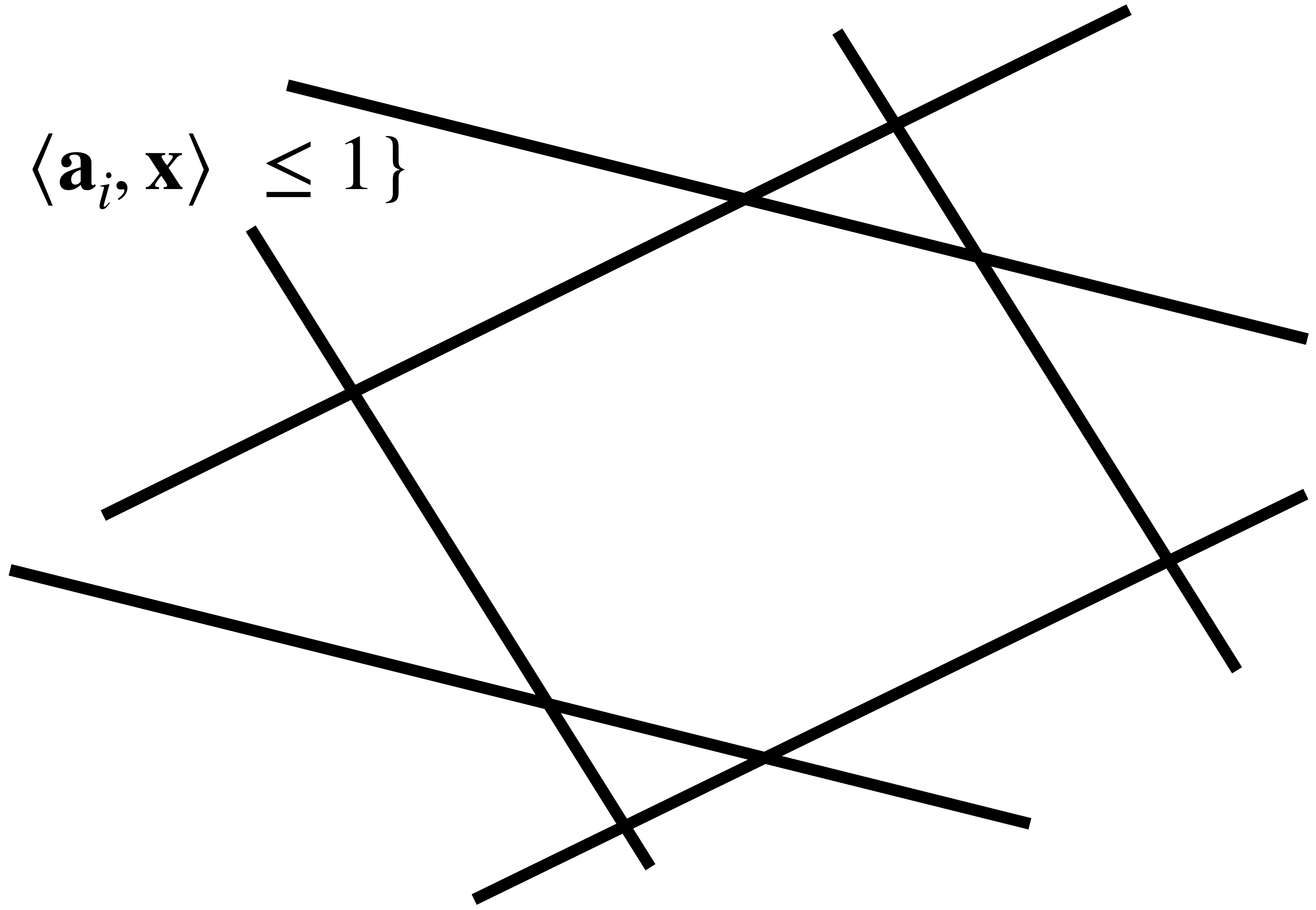

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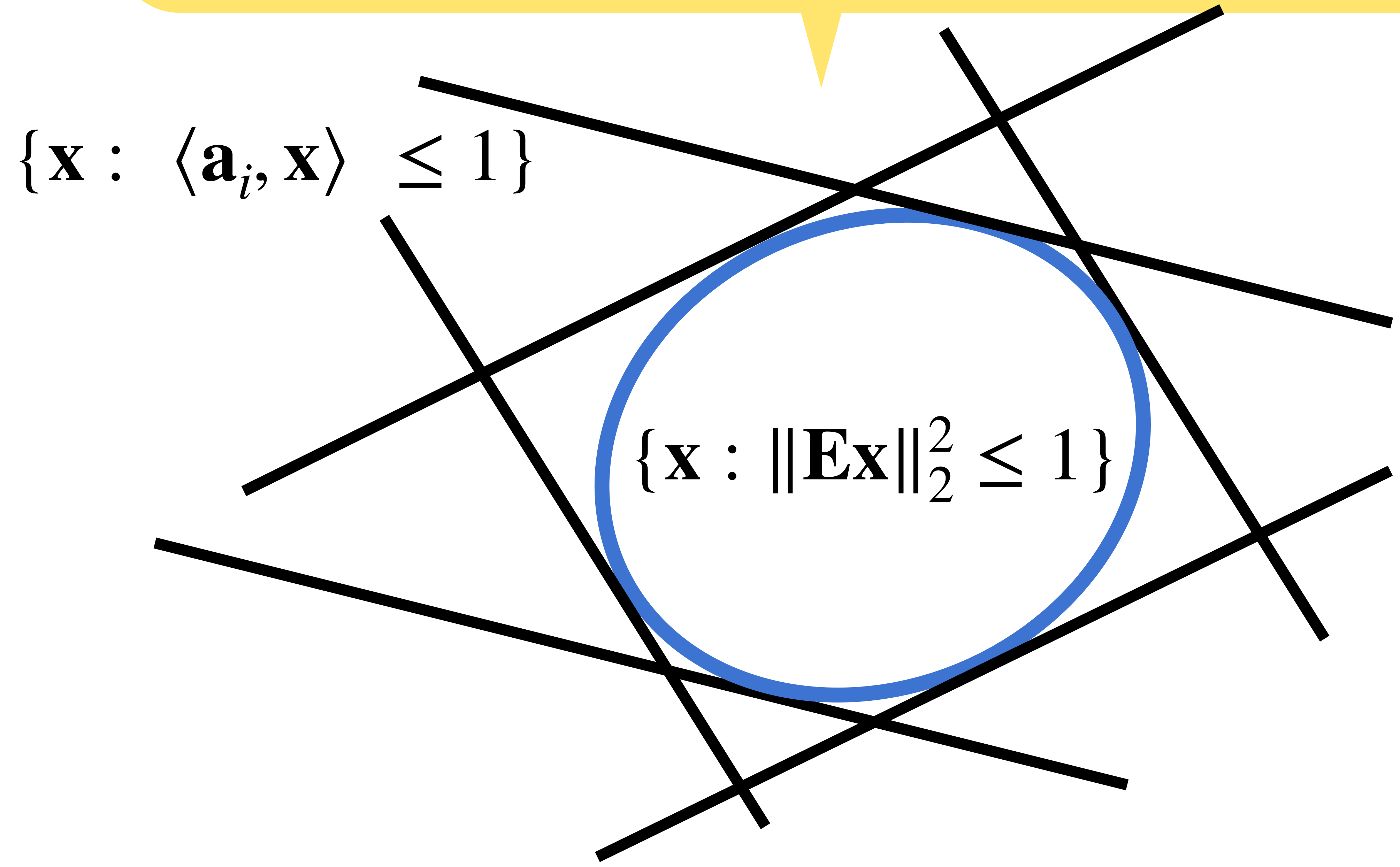

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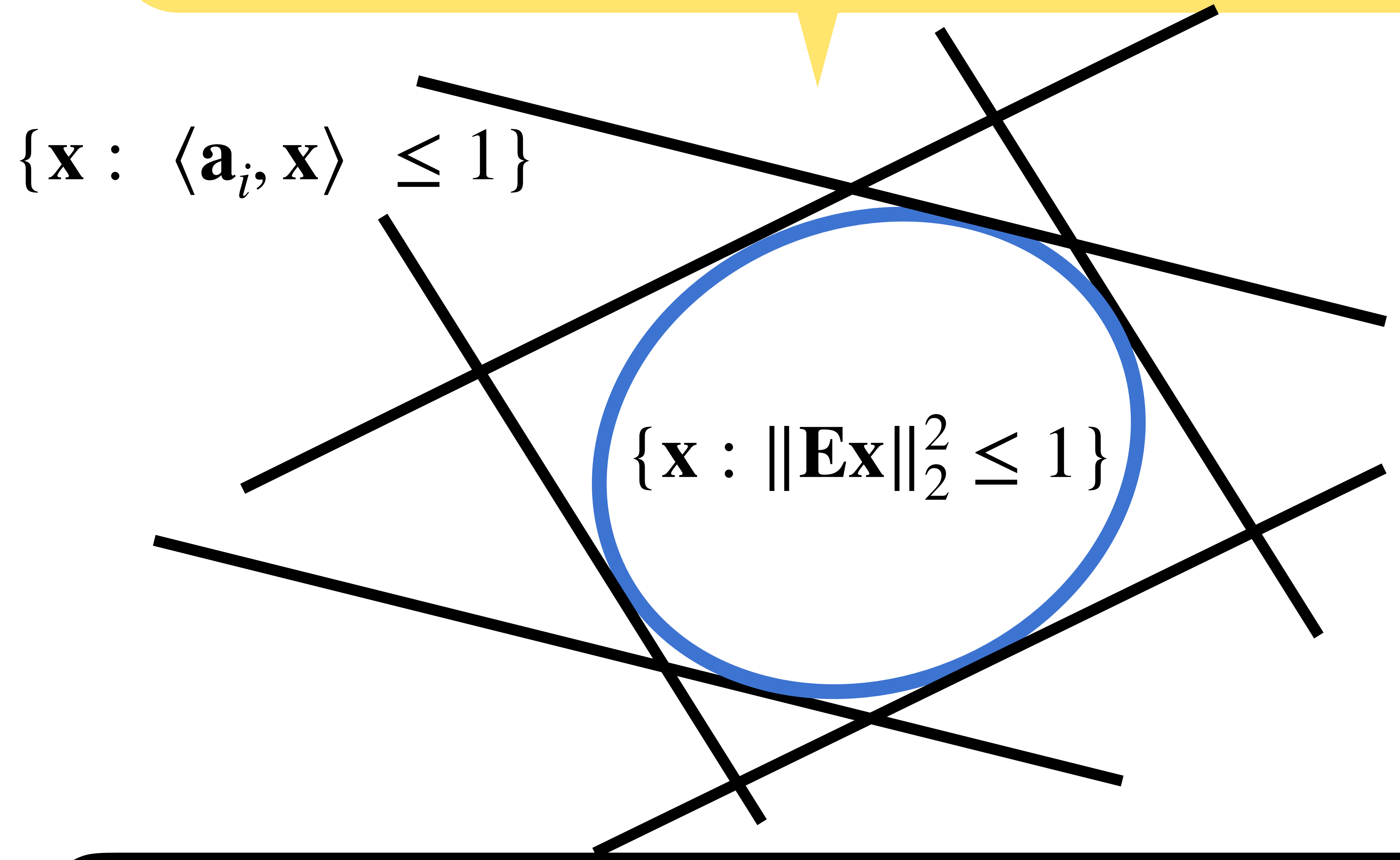

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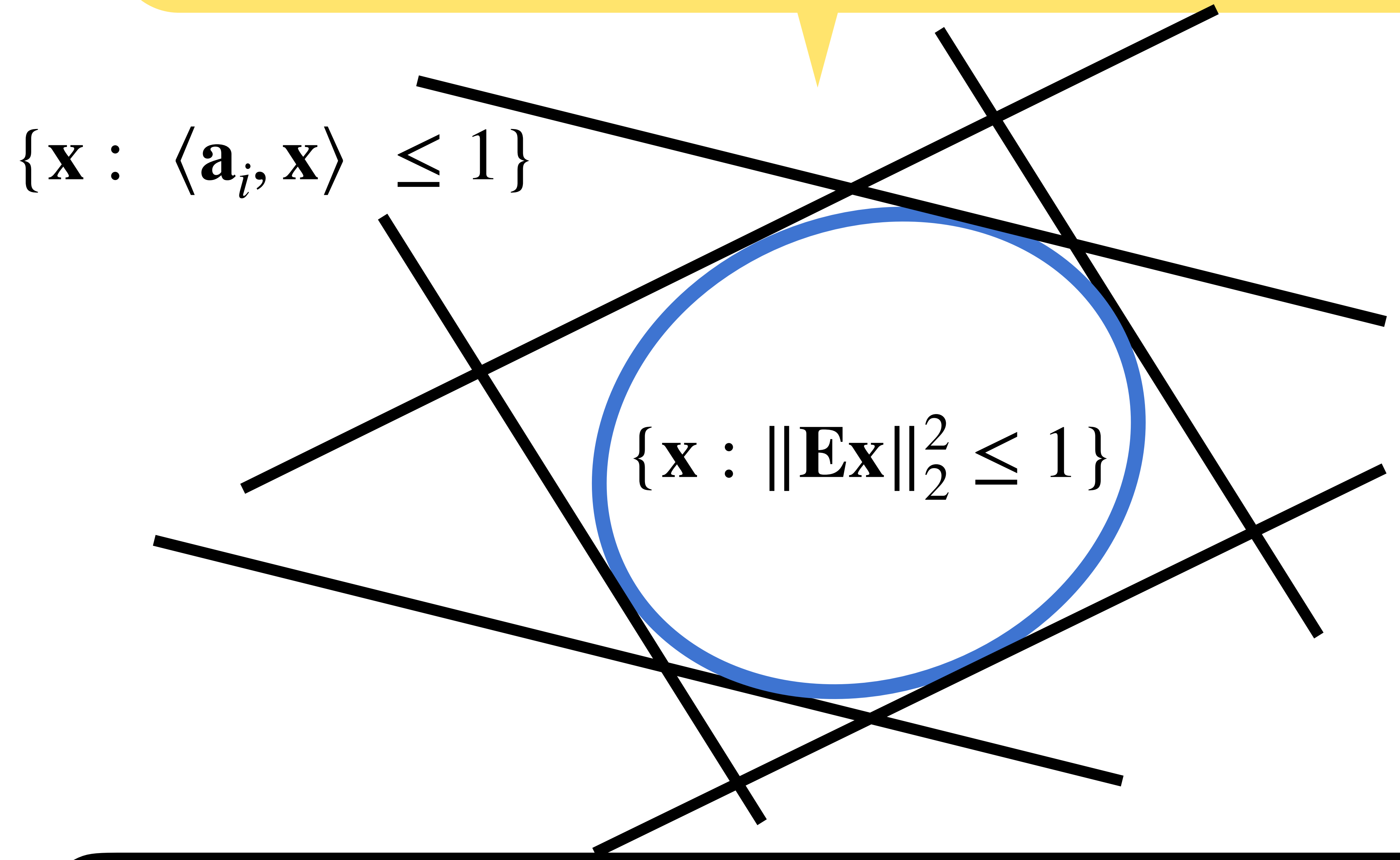

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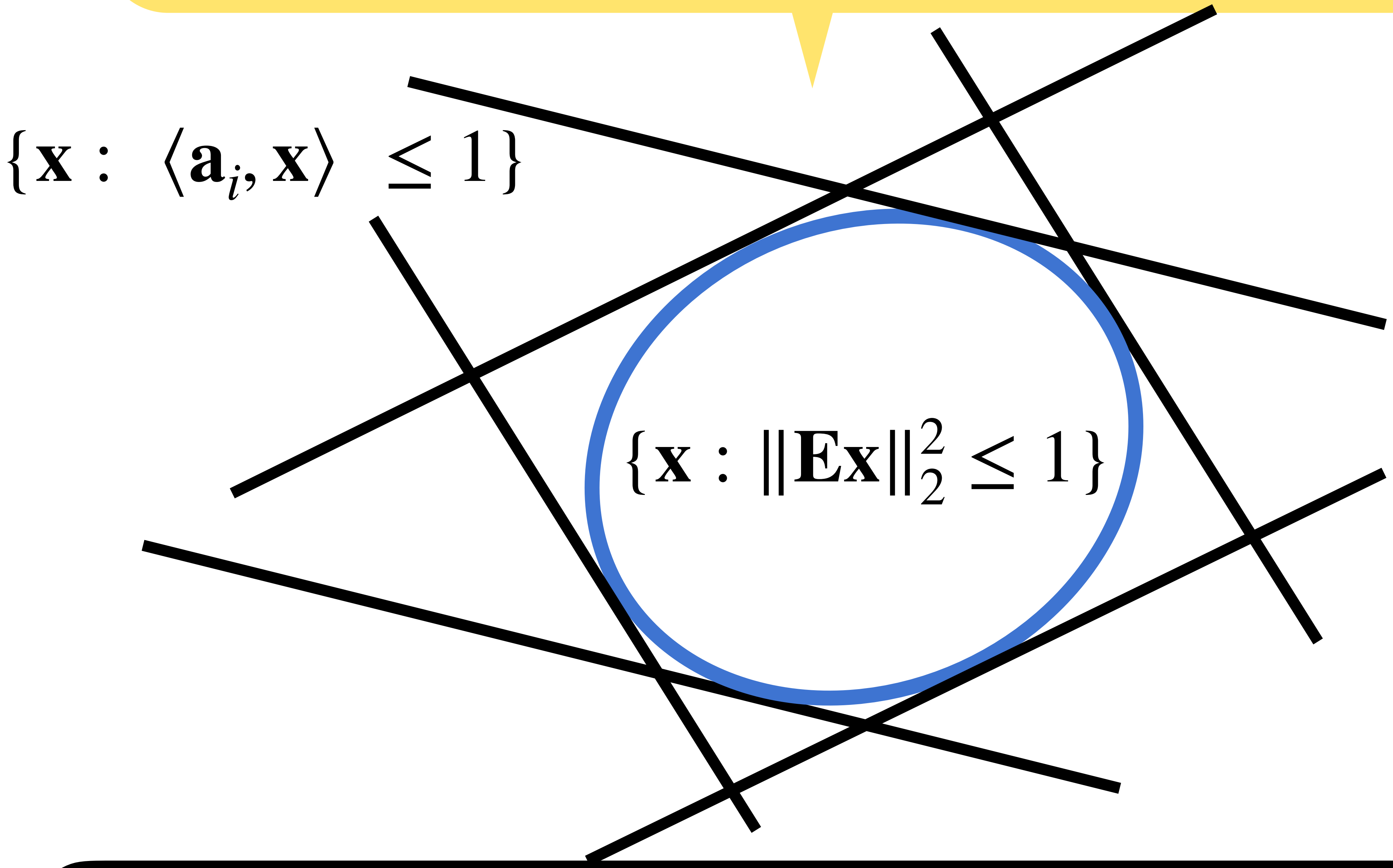

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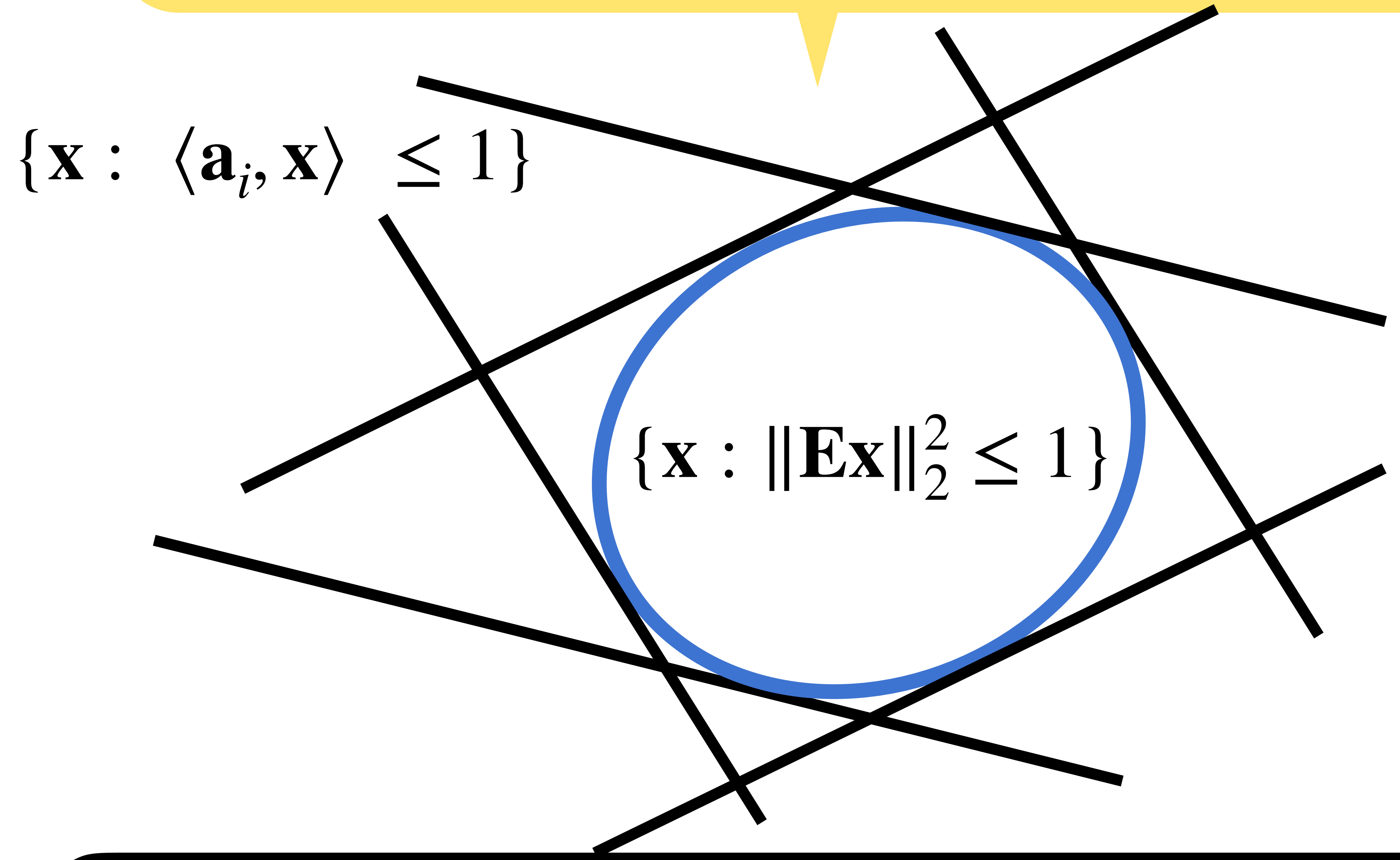

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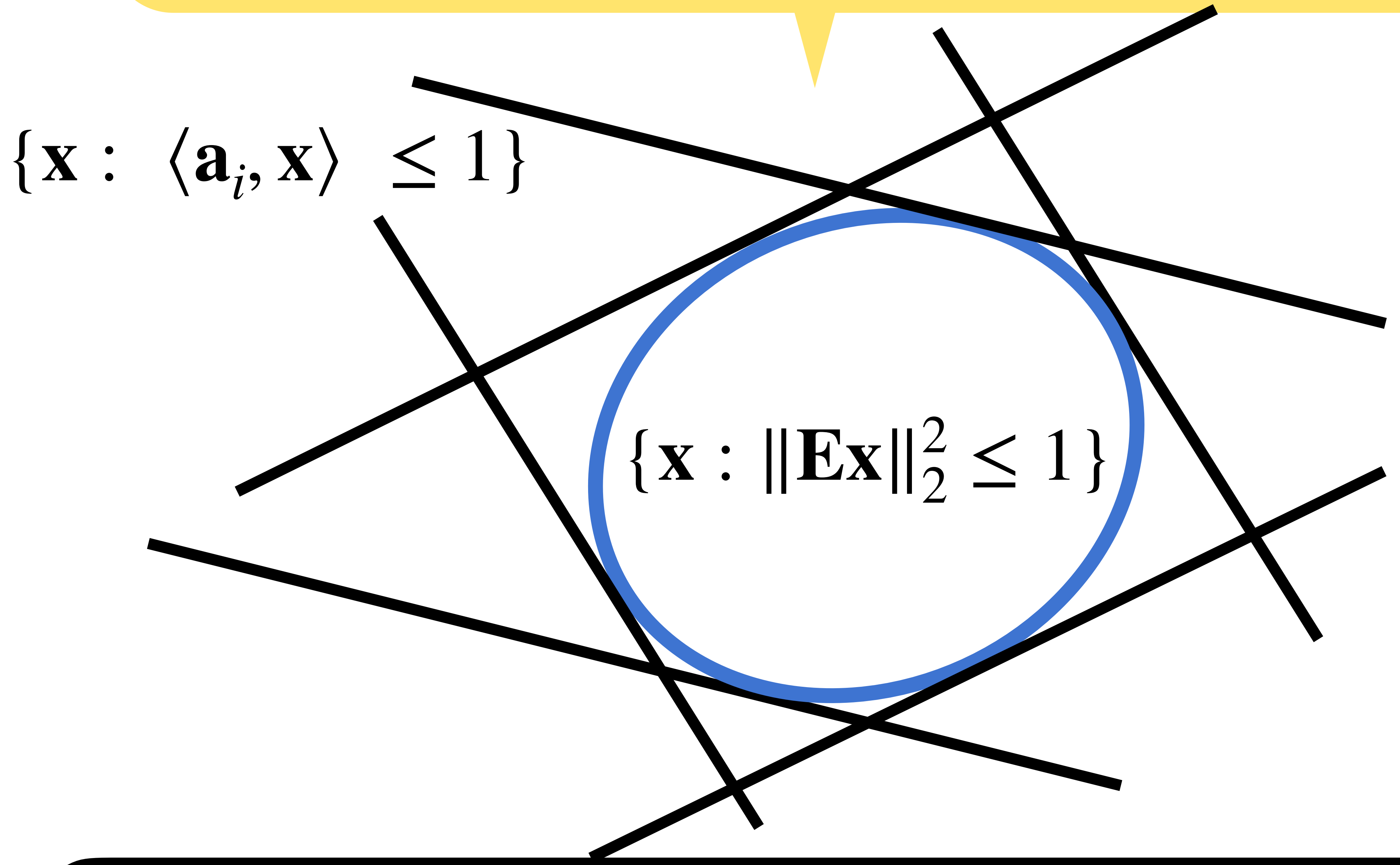

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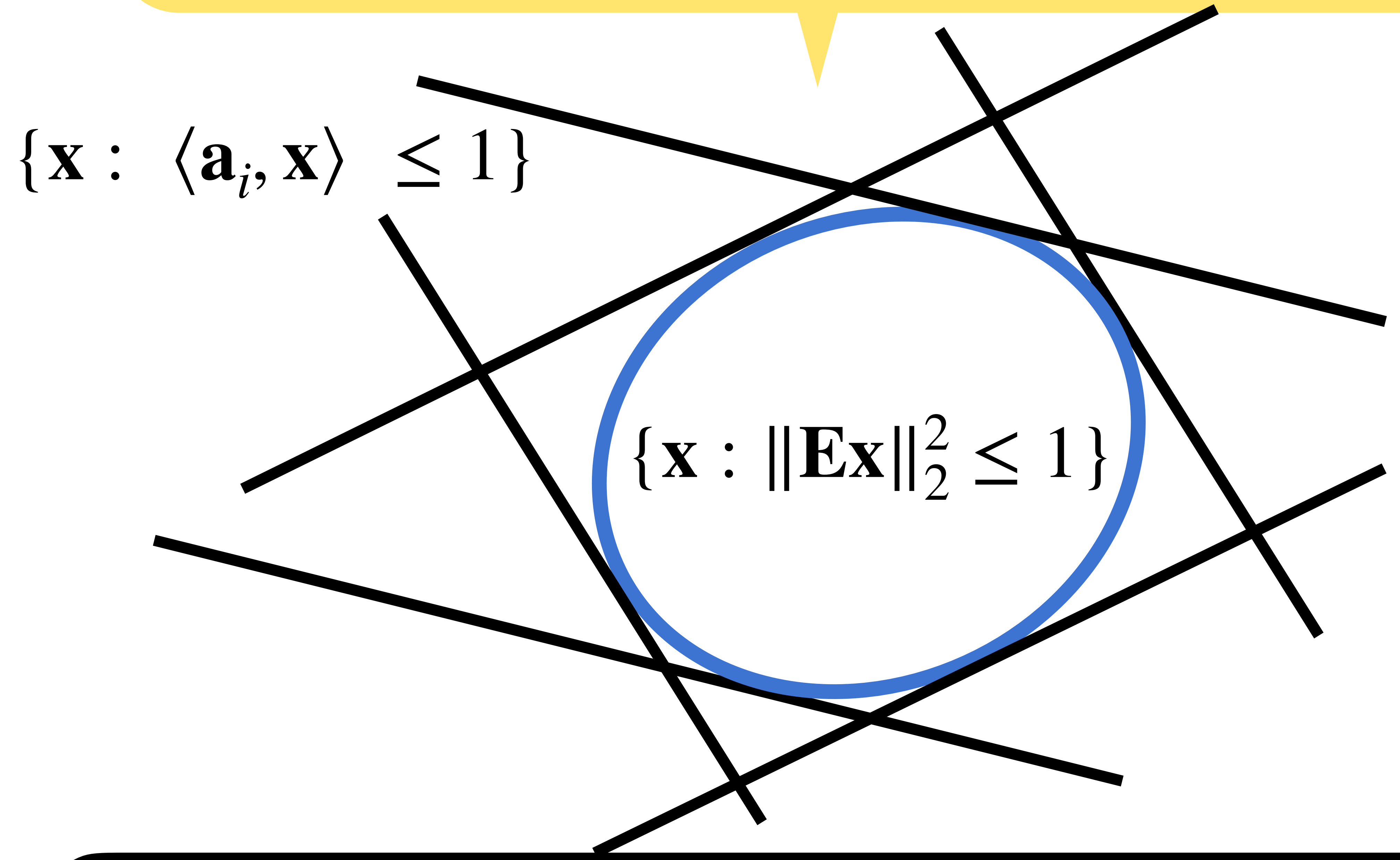

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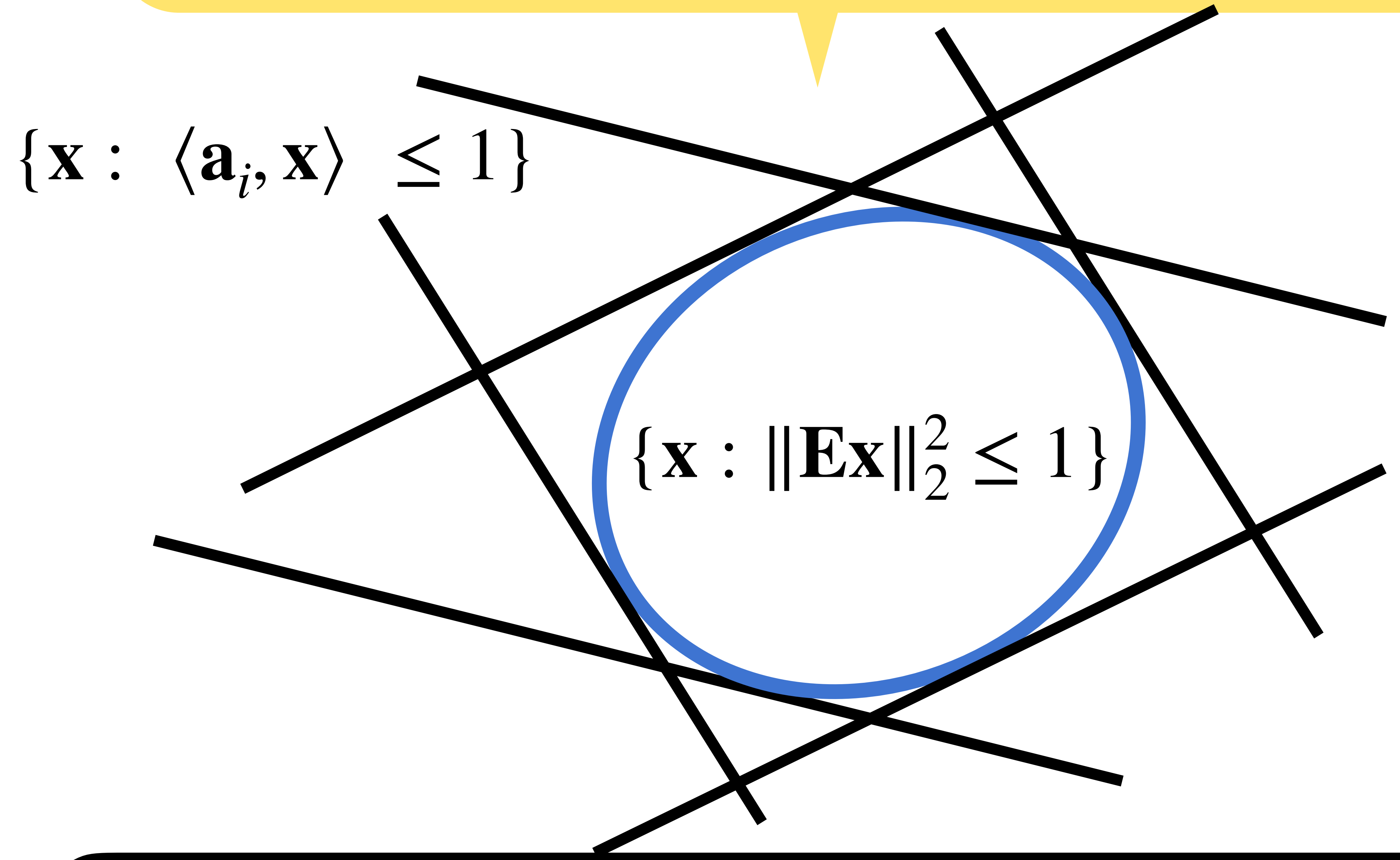

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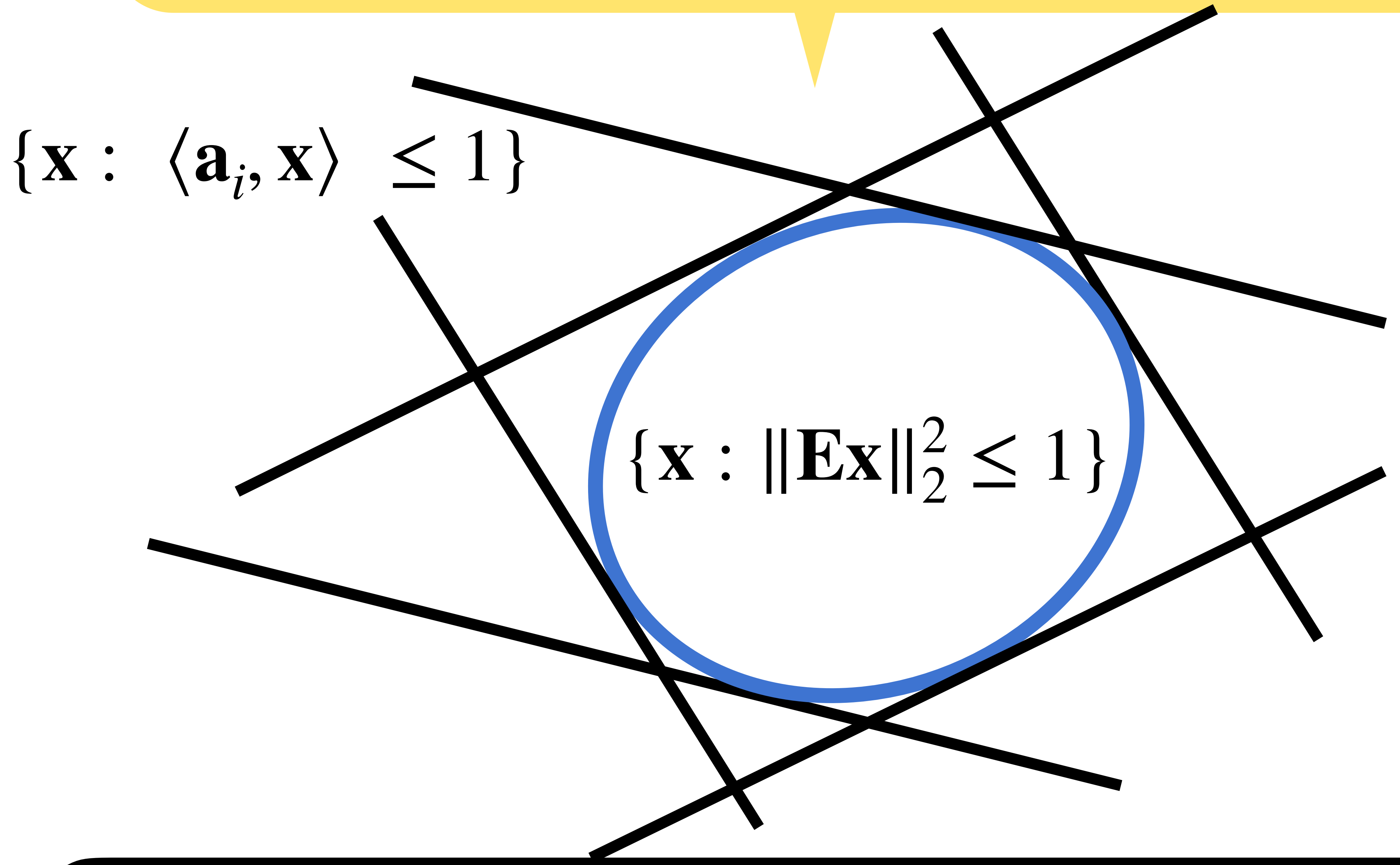

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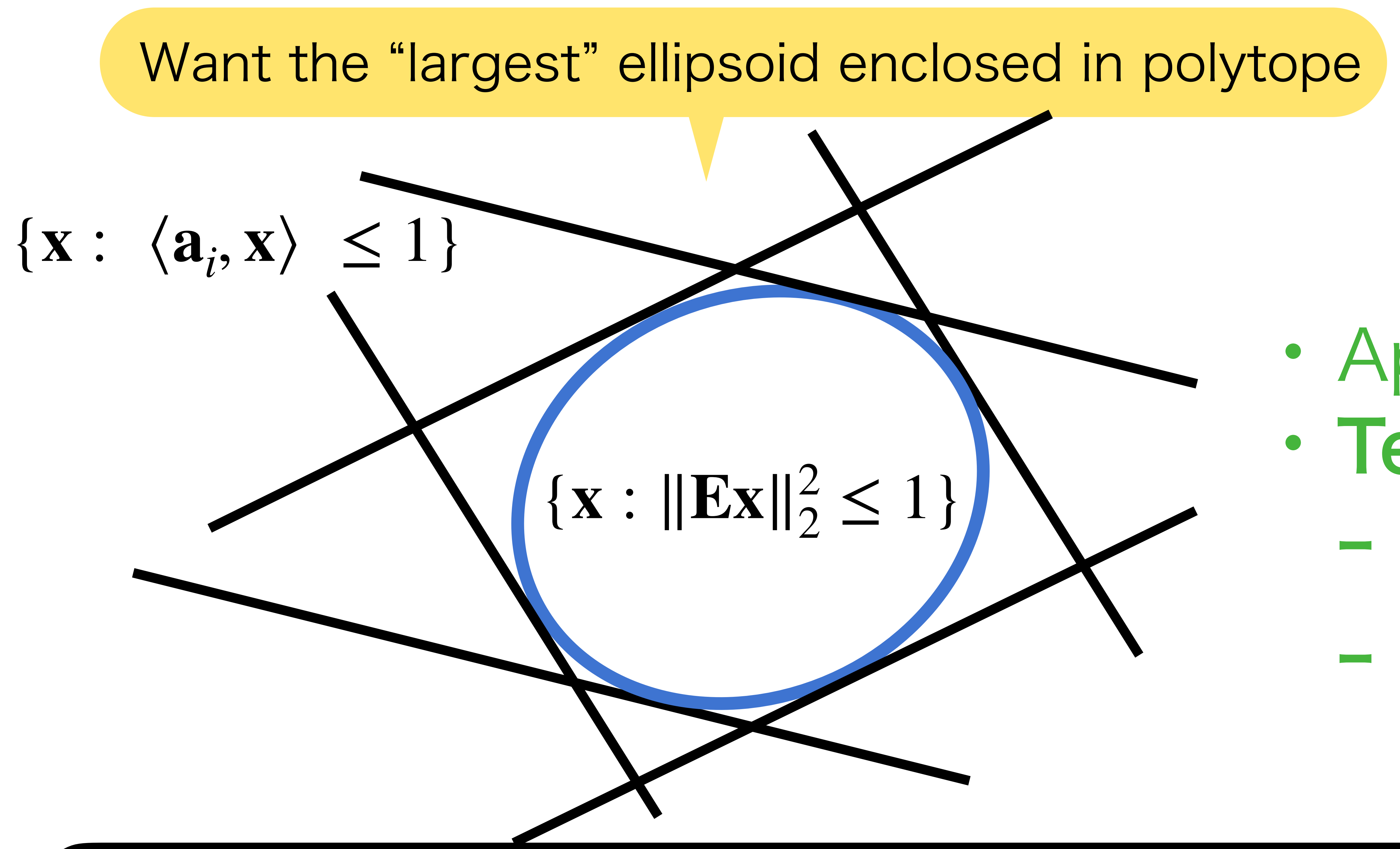

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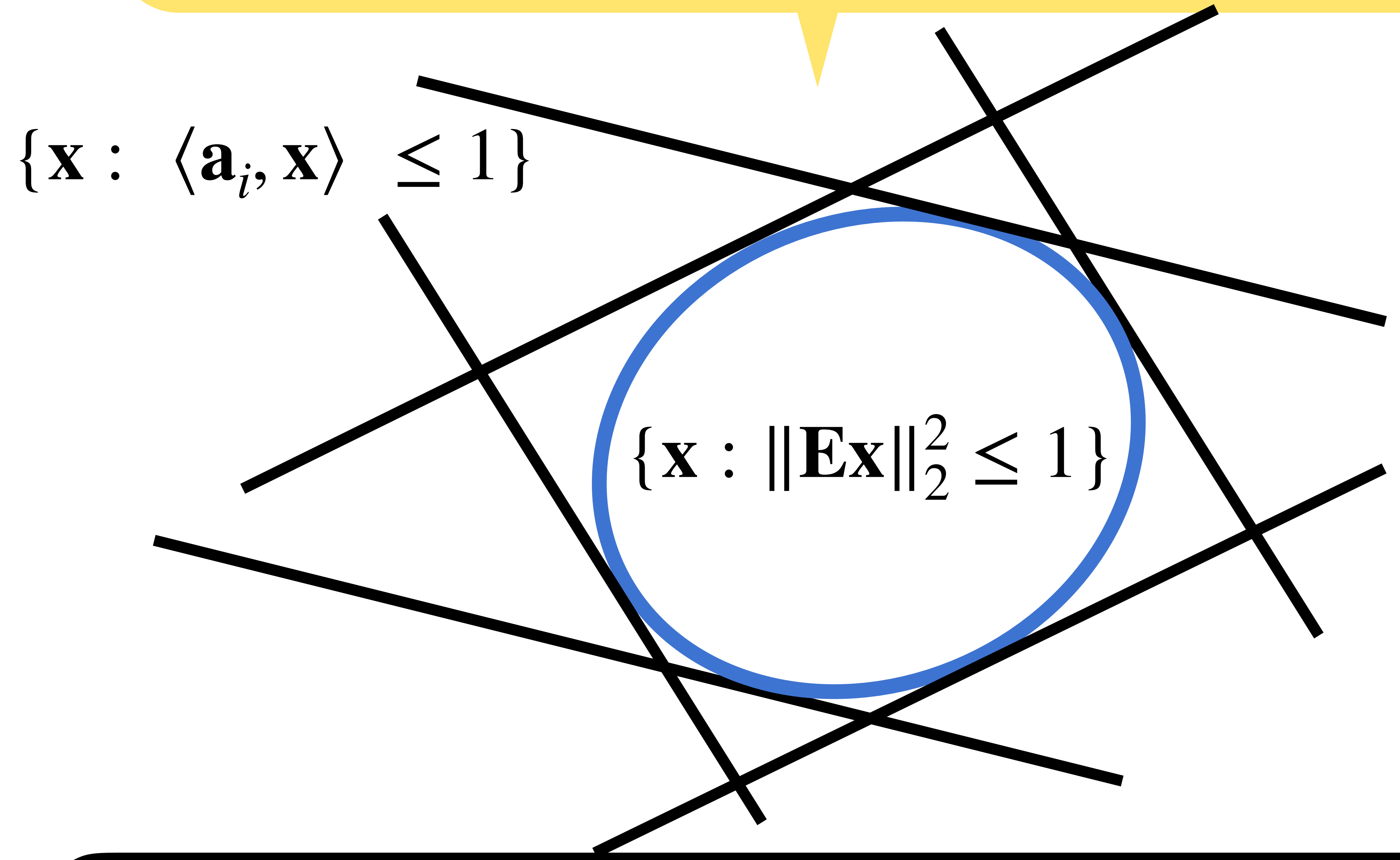

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Leverage scores: $\tau_i(\mathbf{A}) = \sup_{\mathbf{A}\mathbf{x} \neq 0} \frac{\langle \mathbf{a}_i, \mathbf{x} \rangle^2}{\|\mathbf{A}\mathbf{x}\|_2^2}$

Matrix Approximation

Low Rank Approximation with General Losses

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$$\min_{\mathbf{U}\in\mathbb{R}^{n\times k},\mathbf{V}\in\mathbb{R}^{k\times d}} \| \mathbf{A} - \mathbf{UV} \|_F^2$$

Matrix Approximation

Low Rank Approximation with General Losses

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  - This framework gives the best known algorithms for this problem!

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$$\| \mathbf{A} - \hat{\mathbf{U}}\hat{\mathbf{V}} \|_g \leq \kappa \min_{\mathbf{U}\in\mathbb{R}^{n\times k},\mathbf{V}\in\mathbb{R}^{k\times d}} \| \mathbf{A} - \mathbf{U}\mathbf{V} \|_g$$

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Subset of $\approx k$ columns of $\mathbf{A}$

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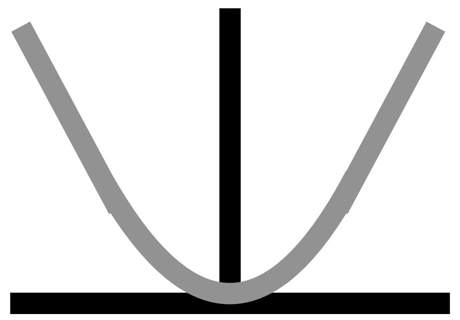
- Huber: $\kappa \approx k^2$ (Song—Woodruff—Zhong 2019) 

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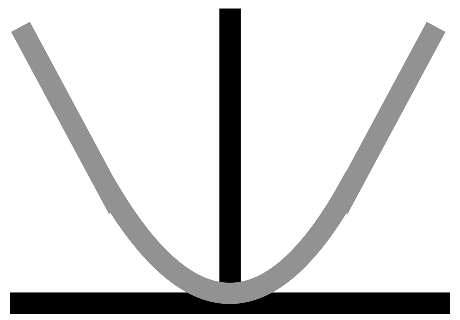
$$\| \mathbf{A}-\hat{\mathbf{U}}\hat{\mathbf{V}} \|_g \le \kappa \min_{\mathbf{U}\in\mathbb{R}^{n\times k},\mathbf{V}\in\mathbb{R}^{k\times d}} \| \mathbf{A}-\mathbf{U}\mathbf{V} \|_g$$

Subset of $\approx k$ columns of $\mathbf{A}$

Approximation factor κ

- ℓ_p, $p < 2$: $\kappa \approx k^{1/p-1/2}$ (Mahankali—Woodruff 2021)
- $\ell_p, p > 2$: $\kappa \approx k^{1-1/p}$ (Dan—Wang—Zhang—Zhou—Ravikumar 2019)

  $\kappa \approx k^{1/2-1/p}$ (Woodruff—Y 2023)
- Huber: $\kappa \approx k^2$ (Song—Woodruff—Zhong 2019)



Matrix Approximation

Low Rank Approximation with General Losses

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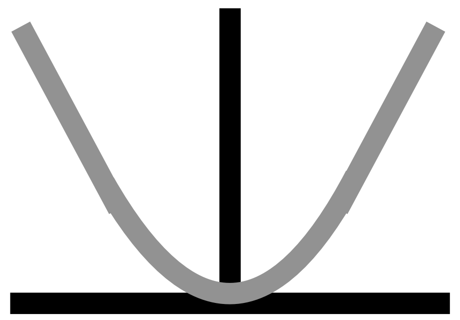

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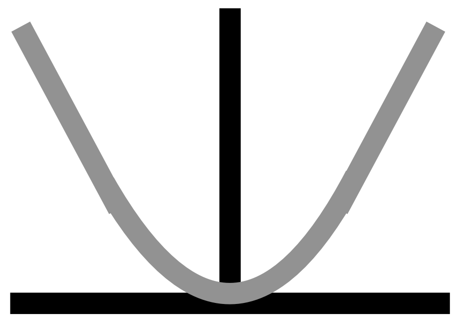

Techniques: well-conditioned spanning sets, ℓ_p Lewis weights, ...

Matrix Approximation

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  - **Applications**: we apply algorithmic techniques from matrix approximation to solve fundamental problems in machine learning and computational geometry

Matrix Approximation

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Matrix Approximation

Other works

- Subspace Embeddings + Applications:
  - **Sharper Bounds for ℓ_p Sensitivity Sampling [preprint]**
  - New Subset Selection Algorithms for Low Rank Approximation: Offline and Online [STOC'23]
  - Online Lewis Weight Sampling [SODA'23]
  - High-Dimensional Geometric Streaming in Polynomial Space [FOCS'22]
  - Active Linear Regression for ℓ_p Norms and Beyond [FOCS'22]
  - Exponentially Improved Dimensionality Reduction for ℓ_1: Subspace Embeddings and Independence Testing [COLT'21]
- Low Rank Approximation
  - New Subset Selection Algorithms for Low Rank Approximation: Offline and Online [STOC'23]
  - **Improved Algorithms for Low Rank Approximation from Sparsity [SODA'22]**
- **Sequential Attention for Feature Selection [ICLR'23]**