Overview of Optimization

Topics

- Optimization
- Objective function
- Decision Variables
- Solutions for optimization problem
- Decision space
- Constraints or restrictions
- State variables
- Local and Global optima
- Near Optimal solutions
- Simulations

Optimization

- Engineers are commonly confronted with the tasks of designing and operating systems to meet or surpass specified goals while meeting numerous constraints imposed on the design and operation.
- Optimization is the organized search for such designs and operating modes.
- It determines the set of actions or elements that must be implemented to achieve optimized systems.
- In the simplest case, optimization seeks the maximum or minimum value of an objective function corresponding to variables defined in a feasible range or space.

Optimization (Contd…)

More generally, **optimization** is the search of the **set of variables** that produces the **best values** of one or more objective functions while complying with multiple constraints.

A single-objective optimization model embodies several mathematical expressions including an objective function and constraints as follows:

$$\text{Optimize } f(X), \quad X = (x_1, x_2, \ldots, x_i, \ldots, x_N)$$

$$g_j(X) < b_j, \quad j = 1, 2, \ldots, m$$

$$x_i^{(L)} \leq x_i \leq x_i^{(U)}, \quad i = 1, 2, \ldots, N$$

in which $f(X)$ = the objective function; X = a set of decision variables x_i that constitutes a possible solution to the optimization problem; x_i = i^{th} decision variable; N = the number of decision variables that determines the dimension of the optimization problem; $g_j(X)$ = j^{th} constraint; b_j = constant of the jth constraint; m = the total number of constraints; $x_i^{(L)}$ = the lower bound of the i^{th} decision variable; and $x_i^{(U)}$ = the upper bound of the i^{th} decision variable.

Objective Function

The objective function constitutes the goal of an optimization problem.

That goal could be maximized or minimized by choosing variables, or decision variables, that satisfy all problem constraints.

The desirability of a set of variables as a possible solution to an optimization problem is measured by the value of objective function corresponding to a set of variables.

Objective Function (CONTD…)

Some of the algorithms with optimization problems that involve maximizing the objective function.

Others do so with optimization problems that minimize the objective function.

A maximization (or minimization) problem can be readily converted, if desired, to a minimization (or maximization) problem by multiplying its objective function by -1.

Decision Variables

- The decision variables determine the **value of the objective function**.
- In each optimization problem we search for the **decision variables** that yield the **best value** of the objective function or **optimum**.
- In some optimization problems the decision variables range between an upper bound and a lower bound. This type of decision variables forms a **continuous decision space**.
- On the other hand, there are optimization problems in which the decision variables are discrete. **Discrete decision variables** refer to variables that take specific values between an upper bound and a lower bound.

Decision Variables (CONTD…)

- Optimization problems involving continuous decision variables are called **continuous problems**, and those problems defined in terms of discrete decision variables are known as **discrete problems**.

- There are, furthermore, optimization problems that may involve discrete and continuous variables. One such example would be an optimization involving the decision of whether or not to build a facility at a certain location and, if so, what its capacity ought to be. The siting variable is of the binary type (0 or 1), whereas its capacity is a real, continuous variable. This type of optimization problem is said to be of **mixed type.**

Decision Space

- The set of decision variables that satisfy the constraints of an optimization problem is called the **feasible decision space**.
- In an N-dimensional problem, each possible solution is an N-vector variable with N elements.
- Each element of this vector is a decision variable.
- Optimization algorithms search for a point (i.e., a vector of decision variables) or points (i.e., more than one vector of decision variables) in the decision space that optimizes the objective function.

Constraints or Restrictions

- Each optimization problem may have two types of constraints.
- Some constraints **directly restrict the possible value** of the decision variables, such as a decision variable x being a positive real number, $x > 0$.
- Another form of constraint is written **in terms of formulas**, such as when two decision variables x1 and x2 are restricted to the space $x1 + x2 \leq b$.
- The goal of an optimization problem is to find an **optimal feasible solution that satisfies all the constraints** and yields the best value of the objective function among all feasible solutions.

Figure 1 depicts a constrained two-dimensional decision space with infeasible and feasible spaces.

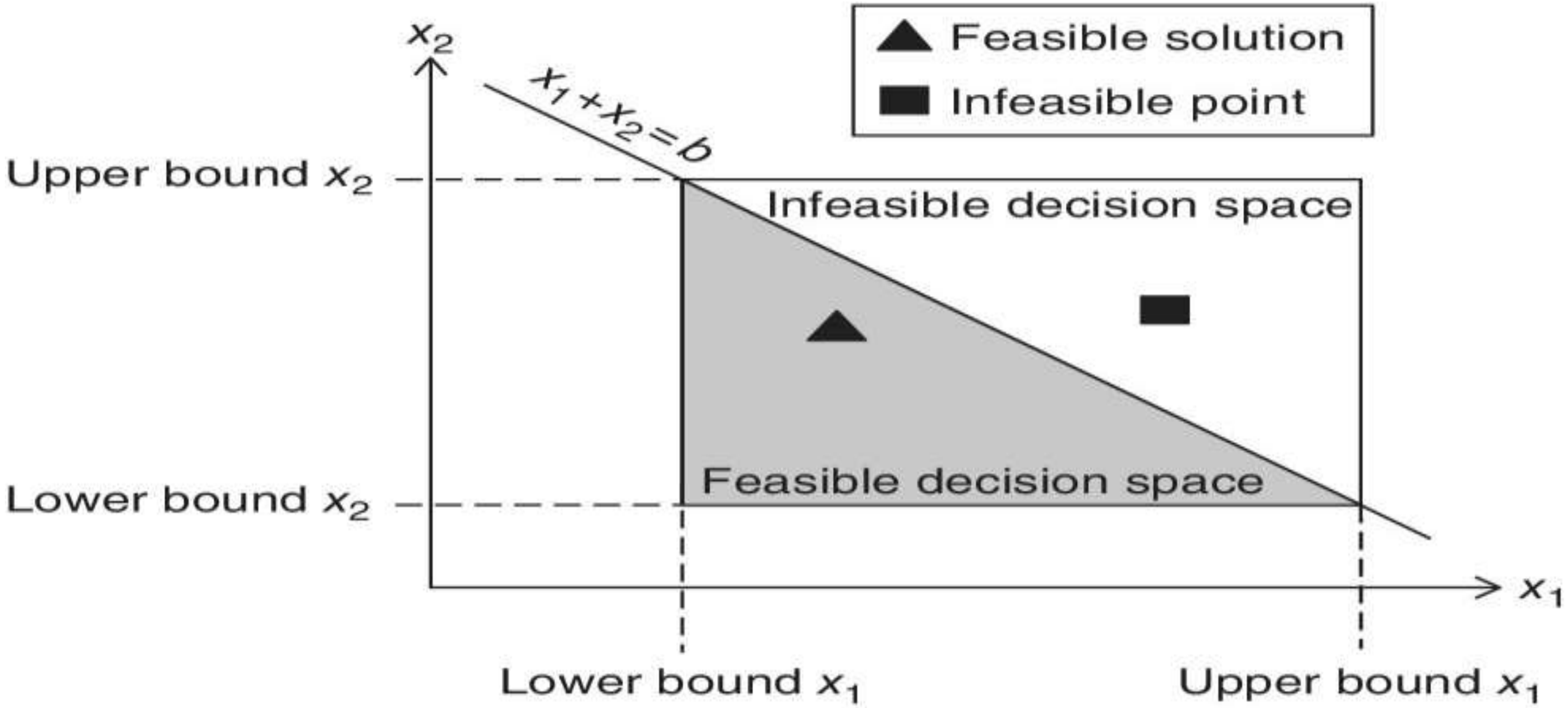


Figure. 1 : Decision space of a constrained two-dimensional optimization problem.

The set of all feasible solutions constitute the feasible decision space, and the infeasible decision space is made up of all the infeasible decision variables. Evidently, the optimal solution must be in the feasible space.

State variables

- State variables are dependent variables whose values change as the decision variables change their values.
- State variables are important in engineering problems because they describe the system being modeled and the objective function and constraints are evaluated employing their values.
- As an example, consider an optimization problem whose objective is to maximize hydropower generation by operating a reservoir.

State variables

- The **decision variable** is the amount of daily water release passing through turbines.
- The **state variable** is the amount of water stored in the reservoir, which is affected by the water released through turbines according to an equation of water balance that also involves water inflow to the reservoir, evaporation from the reservoir, water diversions or imports to the reservoir, water released from the reservoir bypassing turbines, and other water fluxes that change the amount of reservoir storage.

Local Optima

A local optimum refers to a solution that has the best objective function in its neighborhood

In a one-dimensional optimization problem, a feasible decision variable X* is a local optimum of a maximization problem if the following condition holds:

$$f(X^*) \geq f(X), \quad X^* - \varepsilon \leq X \leq X^* + \varepsilon$$

In a minimization problem the local-optimum condition becomes

$$f(X^*) \leq f(X), \quad X^* - \varepsilon \leq X \leq X^* + \varepsilon$$

where X^* = a local optimum and ε = limited length in the neighborhood about the local optimum. A local optimum is limited to a neighborhood of the decision space, and it might not be the best solution over the entire decision space.

Global Optima

A global optimum is the best solution in the decision space.

Some optimization problems may have more than one—in fact, an infinite number of global optima. These situations arise commonly in **linear programming problems.**

Global Optima

A one-dimensional optimization problem with decision variable X and objective function f(X) the value X* is a global optimum of a maximization problem for any decision variable X the following is true:

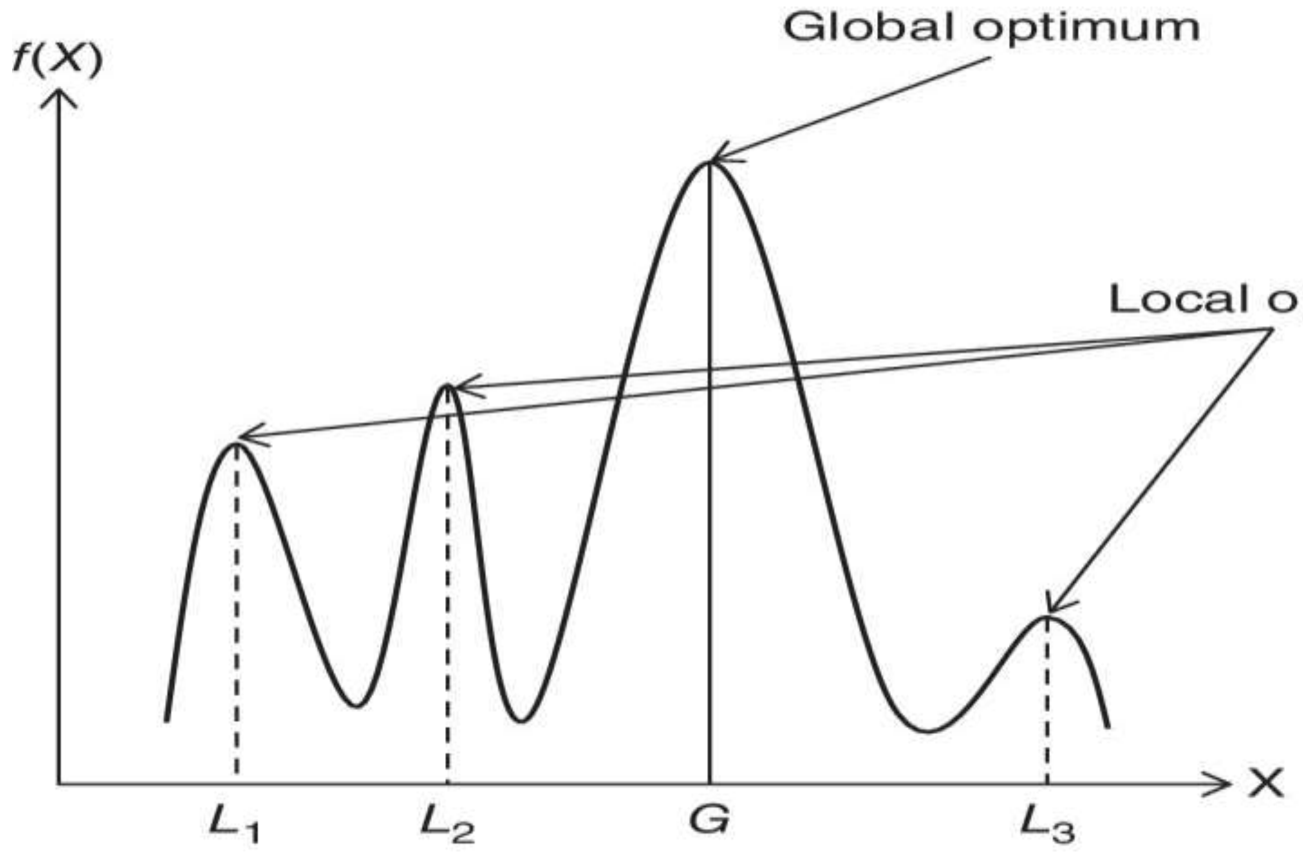
$$f(X^*) \geq f(X)$$

In a minimization problem we would have

$$f(X^*) \leq f(X)$$

Local and Global Optima

Figure 2 illustrates global and local optima for a one-dimensional maximization problem.



L_1, L_2, and L_3 in *Figure 2* are local optima, and G denotes the global optimum with the largest value of the objective function.

Types of Optimization Problems

Based on Objective:

- **Maximization**: Increase profit, efficiency, performance.
- **Minimization**: Reduce cost, time, error.

Based on Variables:

- **Single-variable Optimization**: Only one decision variable.
- **Multivariable Optimization**: More than one variable.

Based on Constraints:

- **Constrained Optimization**: With limits (budget, resources).
- **Unconstrained Optimization**: No restrictions.

Based on Nature of Functions:

- **Linear Optimization**: Both objective and constraints are linear.
- **Non-linear Optimization**: At least one is non-linear.

Optimization in Machine Learning

In ML, optimization is used to:

- Train models by minimizing a **loss function**.
- Tune **hyperparameters** for best performance.
- Select **features** for better accuracy and less complexity.

Example:

Gradient Descent is an optimization algorithm used to minimize the loss in neural networks.

General Steps in Optimization

- Define the problem clearly.
- Identify the decision variables.
- Set the objective function.
- Identify constraints.
- Select an optimization method.
- Solve and interpret the result.

Formulation of Linear Programming Problems (LPP)

What is Linear Programming?

Linear Programming (LP) is a **mathematical optimization technique** used to find the **best outcome** (maximum or minimum) of a problem, **where both the objective function and the constraints are linear**.

It is widely used in **resource allocation**, **production planning**, **transportation**, and **machine learning preprocessing**.

Components of an LPP

Every LPP has **three main components:**

1. **Decision Variables** ($x_1, x_2, \ldots, x_n$)
   - Represent the choices available to decision makers.
   - Example: Number of units of product A and B to produce.
2. **Objective Function**
   - A linear equation representing the goal.
   - Example:

$$Z = c_1x_1 + c_2x_2$$

   where Z is profit or cost, and c_1, c_2 are coefficients.

3. **Constraints**
   - Linear inequalities or equalities that restrict decision variables.
   - Example:

$$a_{11}x_1 + a_{12}x_2 \leq b_1$$

$$x_1, x_2 \geq 0$$

MATHEMATICAL FORMULATION OF LPP

The mathematical formulation of linear programming problem (LPP) is described in the following steps:

1. Identify the decision variables of the problem.
2. Express the objective function, which is to be optimised, i.e., maximised or minimised, as a linear function of the decision variables.
3. Identify the limited available resources, i.e., the constraints and express them as linear inequalities or equalities in terms of decision variables.
4. Since negative values of the decision variables do not have any valid physical interpretation, introduce non-negative restrictions.

Example 1:

A small scale industry manufactures two products P and Q which are processed in a machine shop and assembly shop. Product P requires 2 hours of work in a machine shop and 4 hours of work in the assembly shop to manufacture while product Q requires 3 hours of work in machine shop and 2 hours of work in assembly shop. In one day, the industry cannot use more than 16 hours of machine shop and 22 hours of assembly shop. It earns a profit of ₹3 per unit of product P and ₹4 per unit of product Q. Give the mathematical formulation of the problem so as to maximise profit.

Solution:

Let x and y be the number of units of product P and Q, which are to be produced.

Here, x and y are the decision variables. Suppose Z is the profit function.

Since one unit of product P and one unit of product Q gives the profit of ₹ 3 and ₹ 4, respectively, the objective function is

$$\textbf{Maximise } Z = 3x + 4y$$

The requirement and availability in hours of each of the shops for manufacturing the products are tabulated as follows:

	Machine Shop	Assembly Shop	Profit
Product P	2 hours	4 hours	$3 per unit
Product Q	3 hours	2 hours	$4 per unit

Ava

Total hours of machine shop required for both types of product = $2x + 3y$

Total hours of assembly shop required for both types of product = $4x + 2y$

Hence, the constraints as per the limited available resources are:

$$2x + 3y \leq 16$$

and $$4x + 2y \leq 22$$

Since the number of units produced for both P and Q cannot be negative, the non-negative restrictions are:

$$x \geq 0,\ y \geq 0$$

Thus, the mathematical formulation of the given problem is

Maximise $\quad Z = 3x + 4y$

subject to the constraints

$$2x + 3y \leq 16$$

$$4x + 2y \leq 22$$

and non-negative restrictions

$$x \geq 0, \quad y \geq 0$$

Try to formulate the following problem mathematically

A company produces two types of items P and Q that require gold and silver. Each unit of type P requires 4g silver and 1g gold while that of Q requires 1g silver and 3g gold. The company can produce 8g silver and 9g gold. Suppose each unit of type P brings a profit of $44 and that of type Q, $55.

Give the mathematical formulation for the problem to determine the number of units of each type that the company should produce to maximise the profit.

Solving LPP using Graphical Method

Steps of Graphical Solution

1. Change the inequality to equality → objective function and the constraints.
2. Plot the non-negativity constraints
3. Find the X and Y intercept of each constraint equation
4. Plot the constraints on X-Y plane by connecting X and Y intercept.
5. Identify the feasible solution space.
6. Find all the corner points of the feasible solution space
7. Among the corner points select the one that optimize the objective functions→(optimal solution).

Solving LPP using Graphical Method

Maximize $Z = 12x_1+16x_2$

Subjet to

$10\ x_1+20\ x_2 \leq 120$

$8\ x_1+8\ x_2 \leq 80$

x_1 and $x_2 \geq 0$

Solving LPP using Graphical Method

Maximise $Z = 6X + 3Y$

subject to the constraints

$$2X + 5Y \le 120$$
$$4X + 2Y \le 80$$
$$X \ge 0, Y \ge 0$$

Maximise $z = 3x_1 + 2x_2$

subject to the constraints

$$x_1 - x_2 \le 1$$
$$x_1 + x_2 \ge 3$$
$$x_1 \ge 0, x_2 \ge 0$$

Maximise $Z = x_1 + x_2$

subject to the constraints

$$x_1 + x_2 \le 1$$
$$-3x_1 + x_2 \ge 3$$
$$x_1 \ge 0, x_2 \ge 0$$

A company manufactures two products X and Y, each of which requires three types of processing. The length of time for processing each unit and the profit per unit are given in the following table:

	Product X (hr/unit)	Product Y (hr/unit)	Available capacity per day (hr)
Process I	12	12	840
Process II	3	6	300
Process III	8	4	480
Profit per unit (`)	5	7	

How many units of each product should the company manufacture per day in order to maximise profit?

Company produces soft drinks and has a contract requiring that a minimum of 80 units of chemical A and 60 units of chemical B go into each bottle of the drink. The chemicals are available in a prepared mix from two different suppliers. The supplier x_1 has a mix of 4 units of A and 2 units of B that costs \$ 10, and the supplier x_2 has a mix of 1 unit of A and 1 unit of B that costs \$ 4. How many mixes from the company x_1 and company x_2 should the company purchase to honour contract requirement and yet minimise cost?

Simplex Method

The graphical solution method is not a practical algorithm for most problems because the number of vertices for a linear program grows very fast as the number of variables and constraints increase. Dantzig developed a practical algorithm based on row reduction from linear algebra. The first step is to add more variables into the standard maximization linear programming problem to make all the inequalities of the form $x_i \geq 0$ for some variables x_i.

1.3.1. Slack Variables

Definition. For a resource constraint, a *slack variable* can be set equal to the amount of unused resource: The inequality $a_{i1}x_1 + \cdots + a_{in}x_n \le b_i$ with slack variable s_i becomes

$$a_{i1}x_1 + \cdots + a_{in}x_n + s_i = b_i \qquad \text{with } s_i \ge 0.$$

Example 1.3. The production linear program given earlier has only resource constraints:

$$\begin{array}{ll} \text{Maximize:} & 8x_1 + 6x_2, \\ \text{Subject to:} & x_1 + x_2 \le 2, \\ & 5x_1 + 10x_2 \le 16, \\ & 2x_1 + x_2 \le 3, \\ & x_1 \ge 0, \quad x_2 \ge 0. \end{array}$$

When the slack variables are included the equations become

$$\begin{aligned} x_1 + x_2 + s_1 &= 2 \\ 5x_1 + 10x_2 + s_2 &= 16 \\ 2x_1 + x_2 + s_3 &= 3. \end{aligned}$$

Solving LPP using Simplex Method

Maximize $Z = 12x_1+16x_2$

Subjet to

$10\ x_1+20\ x_2 \leq 120$

$8\ x_1+8\ x_2 \leq 80$

x_1 and $x_2 \geq 0$

Soln.

$$\text{Max } Z = 12x_1 + 16x_2 + 0S_1 + 0S_2$$

Subject to

$$10x_1 + 20x_2 + S_1 = 120 \quad \text{——— (1)}$$

$$8x_1 + 82 + S_2 = 80 \quad \text{——— (2)}$$

$$x_1, x_2, S_1 \text{ and } S_2 \geq 0$$

Initial simplex Table

CB_i	C_j	12	16	0	0		
	Basic variable	x_1	x_2	S_1	S_2	Solution	Ratio
0	S_1	10	20	1	0	120	$120/20 = 6$
0	S_2	8	8	0	1	80	$80/8 = 10$
	z_j	0	0	0	0	0	
$C_j - z_j$		12	16	0	0		

Optimality condition:

For Max:

all $C_j - z_j \leq 0$

For Min:

all $C_j - z_j \geq 0$

$$z_j = \sum_{i=1}^{2} (CB_i)(a_{ij})$$

Iteration - I

CB_i	C_j	12	16	0	0		
	B.V	x_1	x_2	S_1	S_2	Solution	Ratio
16	x_2	$1/2$	1	$1/20$	0	6	$6/(1/2) = 12$
0	S_2	4	0	$-2/5$	1	32	$32/4 = 8$
	Z_j	8	16	$4/5$	0		
$C_j - Z_j$		4	0	$-4/5$	0		

ew Value = Old Value − $\dfrac{\text{Corr. Key Column Value} \times \text{Corr. Key}}{\text{Key Element}}$

Iteration - II

CB_i	C_j	12	16	0	0	
	B.V	x_1	x_2	S_1	S_2	Solution
16	x_2	0	1	$1/10$	$-1/8$	2
12	x_1	1	0	$-1/10$	$1/4$	8
	Z_j	12	16	$2/5$	1	128
	$C_j - Z_j$	0	0	$-2/5$	-1	

Solve the following LPP using Simplex method

Minimize $Z = 2x_1 - 3x_2 + 6x_3$

Subject to

$$3x_1 - x_2 + 2x_3 \leq 7$$

$$2x_1 + 4x_2 \geq -12$$

$$-4x_1 + 3x_2 + 8x_3 \leq 10$$

$$x_1, x_2 \text{ and } x_3 \geq 0$$

$$
\begin{aligned}
3x_1 - x_2 + 2x_3 &\leq 7 \\
-2x_1 - 4x_2 &\leq 12 \\
-4x_1 + 3x_2 + 8x_3 &\leq 10
\end{aligned}
$$

In simplex method all the RHS values should be positive. Thus, multiply by (-1) both sides of second constraint, then solve using simplex minimization.

Standard form.

$$\text{Min } Z = 2x_1 - 3x_2 + 6x_3 + 0S_1 + 0S_2 + 0S_3$$

Subject to

$$3x_1 - x_2 + 2x_3 + S_1 = 7$$

$$-2x_1 - 4x_2 + S_2 = 12$$

$$-4x_1 + 3x_2 + 8x_3 + S_3 = 10$$

Initial Table

	C_j	2	-3	6	0	0	0	Soln.	Ratio
C_{B_i}	B.V	x_1	x_2	x_3	S_1	S_2	S_3		
0	S_1	3	-1	2	1	0	0	7	$7/-1 = -7$
0	S_2	-2	-4	0	0	1	0	12	$12/-4 = -3$
0	S_3	-4	$\boxed{3}$	8	0	0	1	10	$10/3 = 10/3$
	Z_j	0	0	0	0	0	0	0	
	$C_j - Z_j$	2	-3	6	0	0	0		

Iteration 1

	C_j	2	-3	6	0	0	0		
CB_i	B.V	x_1	x_2	x_3	S_1	S_2	S_3	Solu.	Ratio
0	S_1	$5/3$	0	$14/3$	1	0	$1/3$	$31/3$	$\frac{31}{\cancel{3}}\times\frac{\cancel{3}}{5}=\frac{31}{5}$
0	S_2	$-22/3$	0	$32/3$	0	1	$4/3$	$76/3$	$\frac{76}{\cancel{3}}\times\frac{\cancel{3}}{-22}=\frac{-76}{22}=\frac{-38}{11}$
-3	x_2	$-4/3$	1	$8/3$	0	0	$1/3$	$10/3$	$\frac{10}{\cancel{3}}\times\frac{\cancel{3}}{-4}=\frac{-10}{4}=\frac{-5}{2}$
	$Z_j =$	4	-3	-8	0	0	-1	-10	
	$C_j - Z_j =$	-2	0	14	0	0	1		

For New row:

Old Value − [Corr. Key Column Value × Corr. new value]

Iteration 2

	C_j	2	-3	6	0	0	0	
CBi	B.V	x_1	x_2	x_3	S_1	S_2	S_3	Solu.
2	x_1	1	0	$14/5$	$3/5$	0	$1/5$	$31/5$
0	S_2	0	0	$156/5$	$22/5$	1	$14/5$	$354/5$
-3	x_2	0	1	$32/5$	$4/5$	0	$3/5$	$58/5$
	Z_j =	2	-3	$\frac{-68}{5}$	$\frac{-6}{5}$	0	$-7/5$	$\frac{-112}{5}$
	$C_j - Z_j$ =	0	0	$\frac{98}{5}$	$\frac{6}{5}$	0	$7/5$	