Introduction to Numerical Methods and Application Homework Bonus

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1. (15%)

Two-point Gauss-Legendre formula has integral interval [-1, 1], but $f_1(x)$ and $f_2(x)$ have integral interval [0, 0.8]. [-1, 1] to [0, 0.8], so $x = 0.4\hat{x} + 0.4, dx = 0.4d(\hat{x})$.

$$\int_0^{0.8} f(x)dx = \int_{-1}^{1} f(\hat{x})0.4d(\hat{x}) = \frac{2}{5}\left(f(\hat{x} = \frac{1}{\sqrt{3}}) + f(\hat{x} = \frac{1}{\sqrt{3}}\right)$$

$$= \frac{2}{5}\left(f(\frac{6+2\sqrt{3}}{15}) + f(\frac{6-2\sqrt{3}}{15}\right)$$

$$\hat{x} = \pm\frac{1}{\sqrt{3}}, x = \frac{2}{5}\left(\pm\frac{1}{\sqrt{3}}\right) + \frac{2}{5} = \frac{6 \pm 2\sqrt{3}}{15}$$

Then use program to calculate the result below:

```
f1 Integrating Result:
Ground Truth              :43.14667
Approximate Result        :43.14667
Error Rate                :0.000%
f2 Integrating Result:
Ground Truth              :-15.83573
Approximate Result        :-14.19733
Error Rate                :10.346%
```

Figure 1 Approximate Result and Error Rate of Gauss-Legendre.

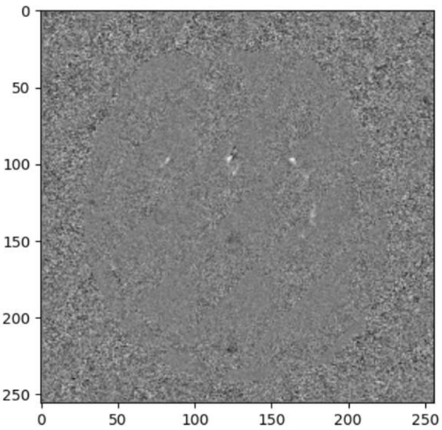
And $f_1^{(4)}(x) = 0, f_2^{(4)}(x) = -21600$, so error rate of f1 Integrate is 0%.

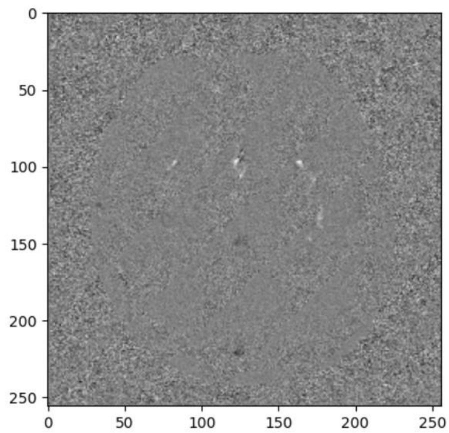
2. (未完成)

a. (15%) 對 mag 進行空間積分，得到每一個時間點 SSS 區域的面積。但是我不知道如何從 pc 的資料當中抓取 SSS 面積的流量。

```
[89.75272383 89.7969037  89.67562787 89.8178399  89.38477574 89.82545125
 90.00968563 90.1980204  90.05014807 90.17693985 90.36503673 90.63492941
 90.79598471 90.67000115 90.59076891 90.78611599 90.13603693 90.3187919
 90.08032584 90.11897785 89.89463039 89.65138057 89.97013458 89.83522239
 90.02733572 89.5210553  89.8158989  89.71908273 90.00158847 89.98202207
 89.76684085 89.84625029 89.64911014 89.73434473 90.05022239 89.79376822
 90.17627095 89.99549128 89.98544766 90.39079144 89.92015137 89.82827953
 89.90290978 89.79393939 89.88559573 89.78184272 89.62127812 89.85349274
 89.72345027 90.02061481 89.85282211]
```

b. (15%) 將 pc 中的每個像素重新排列成以時間為單位，對時間進行積分。左側的圖片是使用 Simpson 方法；右側是使用 Trapezoidal 方法。





3. (15%)

$$f(x_{i+1}) = f(x) + f'(x)h + \frac{f^{(2)}(x)}{2!}h^2 + \frac{f^{(3)}(x)}{3!}h^3 + O(h^4)\text{--------(1)}$$

$$f(x_{i-1}) = f(x) - f'(x)h + \frac{f^{(2)}(x)}{2!}h^2 - \frac{f^{(3)}(x)}{3!}h^3 + O(h^4)\text{--------(2)}$$

by (1) – (2) with $O(h^2)$

$$f'(x) = \frac{f(x_{i+1})-f(x_{i-1})}{2h} + O(h^2)\text{-----------------------(3)}$$

by (1) – (2) with $O(h^4)$

$$f'(x) = \frac{f(x_{i+1})-f(x_{i-1})}{2h} - \frac{1}{2h}\frac{f^{(3)}(x)}{3!}h^3 + O(h^4)\text{-------(4)}$$

by (1) +(2)

$$f^{(2)} = \frac{f(x_{i+1})-2f(x)+f(x_{i-1})}{h^2} + O(h^4)\text{------------------(5)}$$

by substituting (3) and (5) into $f^{(3)}(x)$

$$f^{(3)} = \frac{d}{dx}\frac{d^2f}{dx} = \frac{1}{2h}\left(\frac{f(x)-2f(x_{i-1})+f(x_{i-2})}{h^2} - \frac{f(x_{i+2})-2f(x_{i+1})+f(x)}{h^2}\right) + O(h^4)\text{----(6)}$$

Finally, by substituting (6) into (4)

$$f'(x) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} - \frac{1}{2h}\frac{f^{(3)}(x)}{3!}h^3$$

$$= \frac{1}{12h}(6f(x_{i+1}) - 6f(x_{i-1}) + f(x_{i-2}) - 2f(x_{i-1})$$

$$+ 2f(x_{i+1}) - f(x_{i-1}))$$

$$= \frac{1}{12h}(-f(x_{i+2}) + 8f(x_{i-1}) - 8f(x_{i-1}) + f(x_{i-1})) + O(h^4)$$

4. (20%)

velocity is to find $f'(x)$ and acceleration is to find $f^{(2)}$. For forward finite-difference method approximating $f'(x), f^{(2)}$ and centered finite-difference method approximating $f'(x)$, formula is show below with derived in class:

$$f'(t_i) = \frac{-f(t_{i+2}) + 4f(t_{i+1}) - 3\mathrm{f}(t_i)}{2h} + O(h^2)\ (forward)$$

$$= \frac{-f(t_{i+2}) + 8f(t_{i+1}) - 8\mathrm{f}(t_{i-1}) + \mathrm{f}(t_{i-2})}{12h} + O(h^4)(centered)$$

$$f^{(2)}(t_i) = \frac{-f(t_{i+3}) + 4f(t_{i+2}) - 5\mathrm{f}(t_{i+1}) + 2\mathrm{f}(t_i)}{h^2} + O(h^2)\ (forward)$$

Now derive $f^{(2)}(t_i)$ for centered method. Consider Taylor series for first five items.

$$\mathrm{f}(\mathrm{x}_{i-2}) = \mathrm{f}(\mathrm{x}) - \mathrm{f}'(\mathrm{x})\mathrm{h} + 4\frac{\mathrm{f}^{(2)}(\mathrm{x})}{2!}\mathrm{h}^2 - 9\frac{\mathrm{f}^{(3)}(\mathrm{x})}{3!}\mathrm{h}^3 + 16\frac{\mathrm{f}^{(4)}(\mathrm{x})}{4!}\mathrm{h}^4 + \mathrm{O}(\mathrm{h}^5)\text{------(1)}$$

$$f(x_{i-1}) = f(x) - f'(x)h + \frac{f^{(2)}(x)}{2!}h^2 - \frac{f^{(3)}(x)}{3!}h^3 + \frac{f^{(4)}(x)}{4!}h^4 + O(h^5) \text{-----------------(2)}$$

$$f(x_{i+1}) = f(x) + f'(x)h + \frac{f^{(2)}(x)}{2!}h^2 + \frac{f^{(3)}(x)}{3!}h^3 + \frac{f^{(4)}(x)}{4!}h^4 + O(h^5) \text{-------------(3)}$$

$$f(x_{i+2}) = f(x) + f'(x)h + 4\frac{f^{(2)}(x)}{2!}h^2 + 9\frac{f^{(3)}(x)}{3!}h^3 + 16\frac{f^{(4)}(x)}{4!}h^4 + O(h^5) \text{-----(4)}$$

Let terms $f'(x), f^{(3)}(x)\ and\ f^{(4)}(x)$ in $a * f(x_{i-2}) + b * f(x_{i-1}) + c * f(x_{i-1}) + d * f(x_{i+2})$ equal to 0, we can write following equations.

$$\begin{bmatrix} -1 & -1 & 1 & 1 \\ -9 & -1 & 1 & 9 \\ 16 & 1 & 1 & 16 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By solving the equations, we can find a: b: c: d = 1: -16: -16: 1, so

$$f(x_{i-2}) - 16f(x_{i-1}) - 16f(x_{i+1}) + f(x_{i+2}) = -30f(x_i) - 24\frac{f^{(2)}(x_i)}{2!}h^2$$

Finally centered finite-difference method,

$$f^{(2)}(t_i) = \frac{-f(t_{i+2}) + 16f(t_{i+1}) - 30f(t_i) + 16\mathrm{f}(t_{i-1}) - \mathrm{f}(t_{i-2})}{h^2} + O(h^4)$$

With above formula, we can calculate velocity acceleration with program.

```
By Forward finite-difference method
velocity        :0.575
acceleration    :-0.075
By Centered finite-difference method
velocity        :0.542
acceleration    :-0.042
```

Figure 2 Velocity and Acceleration with Forward and Centered finite-difference method