Time series analysis of S&P 500 with other economic factors

Takahiro YANAGI
Tohoku University, department of Economics
B7EB1247

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1 Abstract

This paper studied the relationship between S&P 500 and other economic factors, especially GDP with Python, utilizing the VAR model, Granger causality, impulse response function, variance decomposition for time series analysis. Through examining both the yearly and monthly data of the U.S.A, it becomes evident that many economic factors influence each other as S&P 500 and GDP correlates with each other. Hence, theoretical models need developing, considering that stock indices can be a good proxy of Tobin's q and the capital growth rate. Last but not least, this study reveals that stock indices have different mechanisms for different periods with high possibility.

2 Introduction

This study's main issue is to validate the relationship between the stock index and other economic factors. The initial motivation is a personal interest in how to choose stocks to invest in. On top of that, an article reported the economic policy so-called "Abenomics" helped companies double their net profit[17], but it is vague whether people enjoy the benefit of the double growth. This article also leeds the author to examine the causal relationship between them. Soon after starting the research, this topic turned out to be essential for both society and academia.

To begin with, there are social needs for investments, especially in Japan. The problem behind the scene is the short of our pensions to survive after our retirements. Indeed, many companies started to adopt the defined-contribution pension instead of the defined-benefit one, and so the importance of individual investments is even greater. The Japanese government encourages people to invest by establishing systems like IDECO or NISA, on which the government does not impose a tax with some conditions. However, without understanding the relationship between stocks and economic factors, bureaucrats would wrongly design such systems. Hence, it is essential even for politicians to understand the relationship between stocks and economic indices to increase national wealth. Still, it is quite difficult for individuals to choose individual stocks. As a matter of fact, passive investments outperform active investments overall. [*1] Hence, individuals must first understand the relationship between such index prices and other economic factors, especially GDP, to invest appropriately. If such indices have strong correlations with GDP

[*1] https://www.ifa.com/articles/despite_brief_reprieve_2018_spiva_report_reveals_active_funds_fail_dent_indexing_lead_-_works/

growth, we invest in developing countries to achieve higher returns. In this respect, understanding the relationship between stock indices and economic factors is crucial for society.

Furthermore, many reputed economists have already discussed whether economic growth correlates with financial development, such as stock markets and banking systems. At this point, even the novel winners present different perspectives on their relations and hence concluded oppositely. In this light, understanding their relationship deeper is essential for anybody, especially for policymakers, to create better policies, as noted above.

While general studies try to uncover the linkage between the economic growth and the potential factors to drive its growth, this research focuses on the stock index and its correlation with other related economic indices, especially GDP. This is because this paper aims to have a deeper understanding of stock indices to invest better and let others make better decisions based on this research.

This paper reviews previous studies in the next chapter, and then examines time series methods in section 4. In section 5, this paper analyzes both yearly and monthly data of the U.S.A. Time series analysis of the data finds the possibility that stock markets have different mechanisms in different periods and the necessity of developing economic models. After all, this paper summarizes significant outcomes in the conclusion part as well as its limitations.

3 Previous studies

In this section, this study examines previous studies opposed to the idea and others favoring the theme.

3.1 The theoretical role of stock market

This paper explains the stock market's principal roles for economic development before checking conflicting opinions about the relation between stock markets and economic development.

First of all, Guglielmo Maria Caporale et al. (2004) [2] remark that, in principle, well-developed stock markets let consumers diverse their risk, increase their savings, allocate resources efficiently, and contribute to economic growth.

In detail, Ross(2003) [8] explains the role of the stock markets and other financial institutions well. According to him, the cost of such as acquiring information and putting contracts into effect initiates financial institutions' merge. All financial institutions provide a primary function to the markets, such as generating information and exerting corporate governance. Financial development is equal to that financial organizations improve the quality of their roles.

Here, this paper cites one of five primary financial functions to generate information in his paper, which Ross(2003) mentions. [8] It is usually quite expensive to investigate companies and markets itself by individual investors. If information costs were kept expensive, investors would not invest in those requiring high information costs, and markets would fail to raise capital for the most lucrative firms. Hence, there should be enough information on markets, firms, and related ones in order to hypothesize the most efficient companies gather capital. In such situations, financial intermediaries reduce information costs and ameliorate resource allocation. Through this process, financial intermediaries improve resource allocation. Besides, such financial intermediaries provide opportunities to produce new goods, introduce new production processes, and lead to innovations. Stock markets may also stimulate companies to disclose their information. Nevertheless, there does not appear a new model to explain the connection between market liquidity and information production and economic development strictly.

Now, it is clear that the financial role is quite essential to smooth markets and allocate resources. However, each bank provides such fundamental elements at different levels. This may trigger a controversial discussion on the importance of financial establishments.

3.2 Economists who are against the idea

This study picks up some economists who disagree with the concept in this part. Stiglitz(1985)[12] states that banks perform better than stock markets on allocating resources. Besides, stock markets harm macroeconomics's stability and long-term growth once stock markets negatively influence economics.

Moreover, Umar Bida Ndako(2010)[10] claims that even though banks and economics have a bilateral relationship, economics causes the growth of stock markets in one direction based on his research in Southern Africa.

Furthermore, Paola Bongini et al. (2017)[1] point out that the host countries decline their growth if foreign banks hold shares in domestic credits more even if banks have a bilateral relationship with economics. This finding opposes the common belief that foreign banks are the source of innovation and have a good side effect of increasing the bank system's efficiency as well as their competition.

In addition, Jung-Suk Yu et al.[14] conduct cross-section analysis to study the dynamics of economic development and stock market development more broadly. They reveal that the stock market development does not have any statistically significant relationship with GDP's growth rate. The growth rate of GDP has a statistically negative correlation with domestic credits provided by banking sectors.

3.3 Economists who are in favor of the concept

Despite the opponents' discussion, others admit its importance of financial development, especially stock markets' evolution, for economic development.

Guglielmo Maria Caporale et al. (2004) [2] point out that some previous studies conclude wrongly and result in a false conclusion because they do not consider the stock market's development as a variable, which results in false causal inferences. They carry out causal inference with financial variables commonly used and without variables usually ignored before. They discover that only two countries out of seven have the causal inference between the variables naturally adapted and economic development. Nonetheless, they reveal that five countries out of seven have relationships when they add variables often overlooked. Their study shows the importance of considering stock markets to examine the relationship between financial development and economic growth. They also note that financial liberalization realizes the most efficient capital allocation. If financial markets are composed of only banks, such markets fail to allocate capital efficiently due to the information asymmetry in raising capital by debt. Therefore, stock market development is an essential part of efficient resource allocation.

3.4 Summary and the position of this thesis

To put the previous discussion brief, the ideal financial institutions accelerate economic growth. Nonetheless, some scientists cast doubt on financial organizations' necessities, especially stock markets, while others accept its importance in the markets. In this sense, these previous studies do not fully cover the stock markets' mechanism and how other factors influence the stock markets.

Analyzing the U.S.A yearly data, one of this study's claims is that variables usually adopted may not be enough, and thus it would be better to include other economic factors. This study shows that S&P 500 and GDP have a close relationship, and S&P 500 can be a good proxy of Tobin's q. In line with this, this paper claims that researchers need to develop economic models in which variables such as stock markets are incorporated for further study.

4 Time Series Analysis and its essentials

This study adopts the VAR model to analyze variables and some methods utilizing the VAR model, such as impulse response functions. Therefore, in this section, this paper introduces the fundamental concepts of time series analysis, AR model, and VAR model, which this paper directly uses for the following section. This paper also explains other useful concepts for the following analysis, such as Granger causality, unit root, impulse response, variance decomposition, and cointegration. The following explanation is mainly based on Tsay(2002) and 沖本 (2010). [13] [15]

4.1 Stationarity and Autocovariance

First of all, stationarity and autocovariance are the fundamentals of time series analysis. Stationary means the function does not depend on time, and there are two types of stationary. These are strictly stationary and weak stationary. $\{r_t\}$ is strictly stationary if and only if $(r_t, .., r_{t_k})$ is identical to $(r_{t_1+t}, ..., r_{t_k+t_k})$ where k is any positive integer, where $(t_a, ..., t_k)$ is the set of size n with positive integers arbitrary chosen. This means $(r_{t_1}, ..., r_{t_k})$ is time-invariable. $\{r_t\}$ is weakly stationary if $E(r_t) = \mu$ where μ is constant, and $Cov(r_t, r_{t-\ell})$ is time-invariable over time where ℓ is any integer. Weakly stationary shows that the values of size T observations constantly vary at a certain level. If the first two moments are finite and strictly stationary, then r_t is weakly stationary, but not vice versa. Here, $\gamma_\ell = Cov(r_t, r_{t-\ell})$ is lag-ℓ autocovariance of r_t. The weakly stationary's property is $\gamma_0 = Var(r_t)$ and $\gamma_{-\ell} = \gamma_\ell$.

4.2 ACF and White noise

ACF stands for autocorrelation function, and ACF implies the correlation between current data and past data of one variable. Given a weakly stationary series $\{r_t\}$, the correlation between the current r_t and its past value $r_{t-\ell}$ is called lag-l autocorrelation of r_t, and it is defined as:

$$\rho_\ell = \frac{Cov(r_t, r_{t-\ell})}{\sqrt{Var(r_t)Var(r_{t-\ell})}} = \frac{Cov(r_t, r_{t-\ell})}{Var(r_t)} \quad = \frac{\gamma_\ell}{\gamma_0}$$

The series $\{r_t\}$ is weakly stationary, and so $Var(r_t) = Var(r_{t-\ell})$ and $\rho_0 = 1,\ \rho_\ell = \rho_{-\ell},\ -1 \leq \rho_\ell \leq 1$.

If ρ_ℓ is 0 for all $\ell > 0$, the series is not serially correlated.

lag - ℓ sample autocorrelation of time series r_t is written as:

$$\rho_\ell = \frac{\sum_{t=\ell+1}^{T}(r_t - \bar{t})(r_{t-\ell} - \bar{r})}{(\sum_{t=1}^{T} r_t - \bar{r})^2}$$

where $0 \leq \ell \leq T - 1$ and $\rho_0 = 1,\ \rho_\ell = \rho_{-\ell},\ -1 \leq \rho_\ell \leq 1$

if $\{r_t\}$ is the sequence of iid random variables with finite expectation and variance, the series r_t is **white noise**, mainly if $\{r_t\}$ follows $N(0, \sigma^2)$, the series is called **"Gaussian white noise."** Here, all the ACFs of the white noise series is 0

4.3 Portmanteau test

Before starting time series analysis, we want to confirm that the data is stationary to utilize the stationary property. The portmanteau test checks whether some autocorrelations of $\{r_t\}$ is 0 or not. One of the most famous portmanteau tests is the Box-Pierce test, defined as:

$$Q^*(m) = T \sum_{\ell=1}^{m} \hat{\rho}_\ell^2$$

In this test the null hypothesis is $H_0 : \rho_1 = ... = \rho_m = 0$ and the alternative hypothesis is $H_a : \rho_i \neq 0$ for some i in {1,..,m}.
Ljung-Box test improves $Q^*(m)$ of Box-Pierce test in finite space and is described as

$$Q(m) = T(T+2) \sum_{t=2}^{m} \frac{\hat{\rho}_\ell^2}{T - \ell}$$

The setting of m affects the Q(m) performance, and empirically $m \approx \ln(T)$ works well.

4.4 AR model

If lag-1 autocorrelation is statistically significant, r_{t-1} can be used to estimate r_t. That is equal to:

$$r_t = \phi_0 + \phi_1 r_{t-1} + a_t$$

where a_t is white noise series with mean 0 and variance σ_a^2. This equation is essentially the same as a simple regression in which r_t is the dependent variable, and r_{t-1} is the explanatory variable.
If a_t is white noise series, the conditional expectation and variance of r_t are as follows:

$$E(r_t | r_{t-1}) = \phi_0 + \phi_1 r_{t-1}, \ Var(r_t | r_{t_1}) = \sigma_a^2$$

The expectation and variance imply that r_t pivots around $\phi_0 + \phi_1 r_{t-1}$ with the variance σ_a^2.
Likewise, AR(p) model is described as:

$$r_t = \phi_0 + \phi_1 r_{t-1} + ... + \phi_p r_{t-p} + a_t$$

where p is a non-negative integer and $\{a_t\}$ is the white noise series as before. AR(p) model is the same as a multiple linear regression model with lagged values, $r_{t-1}, ..., r_{t-p}$.
Here, this paper introduces AR(1) model. If the series r_t is weakly stationary, $E(r_t)$ is μ, $Var(r_t)$ is γ_0, and $Cov(r_t, r_{t_j})$ is γ_j where μ and γ_0 are constant, and γ_j is the function of j.
Taking the expectation of (3), we obtain $E(r_t) = \phi_0 + \phi_1 E(r_{t-1})$. Given the property of white noise, $E(a_t) = 0$, this equation is equal to $\mu = \phi_0 + \phi_1 \mu$ and $\mu = \frac{\phi_0}{1-\phi_1}$ where $E(r_t) = E(r_{t-1}) = \mu$.
From here, it is assumed that μ exists if $\phi_1 \neq 1$, and μ is 0 only if $\phi_0 = 0$.

Using AR model's property the model is rewritten as follows:

$$\phi_0 = (1 - \phi_1)\mu$$
$$r_t - \mu = \phi_1(r_{t-1} - \mu) + a_t$$
$$r_t - \mu = a_t + \phi_1 a_{t-1} + \phi_2 a_{t-2} + ... = \sum_{t=0}^{\infty} \phi_1^i a_{t-i}$$

$Cov(r_{t-1}, a_t) = E(r_t - \mu)a_{t+1} = 0$ supposed that $\{a_t\}$ series is independent. The result implies that $\{r_{t-1}\}$ happened before t and a_t does not depend on the past information.
Similarly, $Var(r_t) = \frac{\sigma_a^2}{1-\phi_1^2}$ where $\phi_1^2 < 1$.
Therefore, the weakly stationary of AR(1) means $-1 < \phi_1^2 < 1$, and $\{r_t\}$'s mean and variance is finite as well supposing $r_t - \mu = \sum_{t=0}^{\infty} \phi_1^i a_{t-i}$.

Assume the series is weakly stationary, the expectation of AR(p) is obtained as the same procedure, and the expectation is $E(r_t) = \frac{\phi_0}{1-\phi_1-...-\phi_p}$ where $1 \neq \phi_1 + ... + \phi_p$.

4.4.1 ACF of AR model

We take the expectation of (5) multiplying a_t, using the independence of a_t and r_{t-1} as follows:

$$E[(a_t(r_t - \mu)] = E[a_t(r_{t-1} - \mu)] + E[a_t^2] = \sigma_a^2.$$

Then we take the covariance of (5) and the lag-ℓth value, and we obtain:

$$E[(r_t - \mu)(r_{t-\ell} - \mu)] = E[(\phi_1(r_{t-1} - \mu) + a_t)(r_{t-\ell} - \mu)]$$

and this results in $\gamma_\ell = \phi_1\gamma_\ell$ for $\ell > 0$ and $\phi_1\gamma_1 + \sigma_a^2 if \ell = 0$.

From (3), the variance of r_t is $\gamma_0 = \frac{\sigma^2}{1-\phi_1^2}$, and $\gamma_\ell = \phi_1 r_{\ell-1}$ for $\ell > 0$

Now, we can take the autocorrelation of lag-ℓth covariance: $\rho_\ell = \frac{\gamma_\ell}{\gamma_0} = \frac{\phi_1\gamma_{\ell-1}}{\gamma_0} = \phi_1\rho_{\ell-1}$ for $\ell \geq 0$, $\rho_0 = 1$, and this implies $\rho_\ell = \phi_1^\ell$, which means ACF of the AR(1) model starts from $\rho_0 = 1$, and exponentially decreases at ϕ_1 speed.

Next, consider the AR(2) model, the equation is

$$e_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t$$

We can derive the expectation and covariance r_t and $r_{t-\ell}$ in the AR(2) model as the same procedure. $E(r_t) = \mu + \phi_0 + \phi_1\mu + \phi_2\mu + a_t$, and it is equal to $E(r_t) = \mu = \frac{\phi_0}{1-\phi_1-\phi_2}$. The covariance of r_t and $r_{t-\ell}$ in AR(2) model is $E[(r_t - \mu)(r_{t-\ell} - \mu)] = \phi_1\gamma_{\ell-1} + \phi_2\gamma_{\ell-2} = \gamma_\ell$ for $\ell > 0$. This result is called the moment equation of a stationary AR(2) model. Dividing the result by γ_0, we obtain:

$$\rho_\ell = \phi_1\rho_{\ell-1} + \phi_2\rho_{\ell-2} \text{ for } \ell > 0$$

Therefore, the correlation of the series r_t in a stationary AR(2) model is $\rho_0 = 1$, $\rho_1 = \frac{\phi_1}{1-\phi_2}$ for $\ell = 1$, $\rho_\ell = \phi_1\rho_{\ell-1} + \phi_2\rho_{\ell-2}$ for $\ell \geq 2$. The result of (7) implies the second-order difference equation of a stationary AR(2) series: $(1 - \phi_1 B - \phi_2 B^2)\rho_\ell = 0$ Here, B is the back-shift operator and $B^i\rho_\ell = \rho_{\ell-i}$. The equation corresponding to the differential equation mentioned above is $x^2 - \phi_1 x - \phi_2 = 0$. The solution of this equation is the characteristic roots, that is, $x = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$. If the two characteristic roots are the real number, the second order difference equation can be factored as $(1 - \omega_1 B)(1 - \omega_2 B)$. We can allege the AR(2) model is build based on the AR(1) model. However, if $\phi_1^2 + 4\phi_2$ is negative, the characteristic roots are complex numbers and they are called a complex conjugate pair.

The stationary condition of the AR(2) model is the characteristic roots, which are less than 1, and the equation (7) indicates that the ACF of the AR(2) model converges to 0 as the lag-ℓ increases, and this property is the necessary condition of stationary time series. This condition is applied to the AR(p) model in the same way. The characteristic equation is $x^p - \phi_1 x^{p-1} + \phi_2 x^{p-2} + ... + \phi_p = 0$, and the series is stationary when all of the characteristic roots of this equation is less than 1. ACF of a stationary series AR(p) satisfies the difference equation $(1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^p)\rho_\ell = 0$ for $\ell > 0$.

4.5 The way to determine the order p of AR(p) model

To begin, PACF is useful to determine the order p of the AR(p) model, and we first consider AR models as follows:

$$
\begin{aligned}
r_t &= \phi_{0,1} + \phi_{1,1} r_{t-1} + e_{1t} \\
r_t &= \phi_{0,2} + \phi_{1,2} r_{t-1} + \phi_{2,2} r_{t-2} + e_{2t} \\
r_t &= \phi_{0,3} + \phi_{1,3} r_{t-1} + \phi_{2,3} r_{t-2} + \phi_{3,3} r_{t-3} + e_{2t} ....
\end{aligned}
$$

where $\phi_{0,j}$ is a constant term, $\phi_{i,j}$ is the coefficients of r_{t-i}, and $\{e_{j,t}\}$ is the error term of AR(j) model. These equations are forecasted by the least-squares method, and we can utilize a Partial-F test for this equation. In these equations, $\hat{\phi}_{1,1}$ is r_t's sample PACF, and $\hat{\phi}_{2,2}$ is the contribution of r_{t-2} to r_t, and $\hat{\phi}_{3,3}$ is the contribution of r_{t-3} to r_t $\hat{\phi}_{j,j}$ should be close to 0 for all j > p, and the sample PACF diminish at lag p.

Another way to choose the order p of AR(p) is by using information criteria such as Akaike Information Criterion, which is often abbreviated as AIC.

$$
AIC = \frac{-2}{T} \ln(\text{likelihood}) + \frac{2}{T}(\text{number of parameters}) \tag{1}
$$

where the likelihood function is evaluated at the maximum likelihood estimates, and T is the sample size. AIC's second term's role is to penalize models by the number of parameters to keep models simple, and AIC recommends the order p at which the model has the minimum AIC value.

4.5.1 Parameter Estimation

If we choose the order p of the AR model, the model is estimated by the least-squares method. The fitted model is

$$
\hat{r}_t = \hat{\phi}_0 + \hat{\phi}_1 r_{t-1} + ... + \hat{\phi}_p r_{t-p}
$$

and the associated residual is $\hat{a}_t = r_t - \hat{r}_t$. The series $\{a_t\}$ is residual series, by which we gain $\hat{\sigma}_a^2 = \frac{\sum_{t=p+1}^{T} \hat{a}_t^2}{T-2p-1}$.

4.6 Model checking

If estimated models are well fitted, their residual series behave like white noise series. ACF and Ljung-Box Test can validate whether residual series are close to white noise ones. Ljung- Box statistic Q(m) in AR(p) models asymptotically approaches a chi-square distribution with m - p degrees of freedom.

4.6.1 Forecasting

Forecasting is to hypothesize we are at the time h, and predict $r_{h+\ell}$ for $\ell \geq 1$. The time index h is called forecast origin, and $\hat{r}_h(\ell)$ is the forecast of $r_{t+\ell}$ estimated by the minimum squared loss function.The minimus squared loss function is

$$\hat{r}_h(\ell) : E[r_{h+\ell} - \hat{r}_h(\ell) \leq min_g E(r_{h+\ell} - g)^2$$

where g is the function of the information available at time h. We call $\hat{r}_h(\ell)$ the ℓ step ahead forecast of r_t at the time origin h.

The ℓ step ahead forecast is based on the minimum squared error loss function and conditional expectation of $r_{h+\ell}$ given $\{r_{h-i}\}_{i=0}^{\infty}$. That is

$$\hat{r}_h(\ell) = \phi_0 + \sum_{i=1}^{p} \phi_i \hat{r}_h(\ell - i)$$

where $\hat{r}_h(\ell) = r_{h+i}$ for $i \leq 0$. The ℓ step ahead forecast error is $e_h(\ell) = r_{h+\ell} - \hat{r}_h(\ell)$, which converges to 0 as ℓ goes up to ∞. Therefore, If the forecast is too far, the predicted value would be the unconditional mean of r_t, which is called mean reversion, and the variance of the forecasted error is the unconditional variance of r_t.

4.7 Weak stationary vector process

For the following explanations, this paper mainly relies on 沖本 (2010) [15] as well as Tsay(2002).

The expectation of the vector is the set of the expectation of each component.

$$E[y_t] = [E(y_{1t}), ..., E(y_{nt})]'$$

The k-th autocovariance matrix is defined as below.

$$Cov(y_t, t_{t-k}) = [Cov(y_{it}, y_{j,t-k})]_{ij} = \begin{vmatrix} Cov(t_{1t}, y_{1,t-k}) & Cov(y_{1t}, y_{2,t-k}) & \cdots & Cov(y_{1,t}, y_{n,t-k}) \\ Cov(t_{2t}, y_{1,t-k}) & Cov(y_{2t}, y_{2,t-k}) & \cdots & Cov(y_{2,t}, y_{n,t-k}) \\ \cdots & \cdots & \cdots & \cdots \\ Cov(t_{nt}, y_{1,t-k}) & Cov(y_{nt}, y_{2,t-k}) & \cdots & Cov(y_{n,t}, y_{n,t-k}) \end{vmatrix}$$

The (i,j) element of the k-th autocovariance matrix is the covariance of $y_{i,t}, y_{j,t-k}$. The diagonal elements of $n \times n$ matrix are the k-th autocovariance. The autocovariance matrix is the function of k. If k is zero, the autocovariance matrix is identified with the covariance matrix of the random variables and corresponds with the symmetric matrix. On the contrary, if k is not zero, the autocovariance matrix would not be a symmetric matrix.

In general, expectation and autocovariance is a function of time index t. If the expectation and autocovariance of a vector do not rely on time, the vector is assumed to be weak stationary. Hereafter, the paper posits that vectors are weak stationary, and denote μ means the expectation and Γ_k means the k-th autocovariance matrix.

Although, the autocovariance of one variable satisfies $\gamma_k = \gamma_{-k}$, the autocovariance of multivariable does not support $\Gamma_k = \Gamma_{-k}$. The correct relationship is $\Gamma_k = \Gamma'_{-k}$

Both of one variable and multivariable's autocovariance depend on their units, and so it is used the autocorrelation matrix that normalizes the autocovariance matrix as defined below.

$$\rho_k = Corr(y_t, y_{t-k}) = [Corr(y_{it}, y_{j,t-k})]_{ij}$$

The (i,j) component of a k-th autocorrelation matrix is the correlation coefficient of y_{it} and $y_{j,t-k}$. D is set as a diagonal matrix, whose diagonal element is the variance of y_t, that is, $D = diag(Var(y_1)), ..., Var(y_n))$, and the k-th autocorrelation matrix can be factored as:

$$\rho_k = D^{-1/2}\Gamma_k D^{-1/2}$$

The diagonal elements of an autocorrelation matrix are the autocorrelation of each variable, and $\rho_k = \rho'_{-k}$ is valid.

The white noise in a vector form is also noted. ϵ_t is called white noise vector if all the vector process ϵ_t at any time index t, complies with the condition:

$$E(\epsilon_t) = 0$$

$$E(\epsilon_t, \epsilon'_{t-k}) = \begin{cases} \sum & k = 0 \\ 0, & k \neq 0 \end{cases}$$

where $\sum$ is n x n positive definite matric and is not necessarily a diagonal matrix. In short, ϵ_t is not affected intertemporally, but each element of the ϵ_t can correlate with each other. Such white noise vectors are weak stationary and do not have autcovariance. $\epsilon_t \sim W.N.(\sum)$ signifies that ϵ_t is the white noise vector of the covariance matrix $\sum$.

4.8 VAR model

The VAR model is the generalization of the AR model in a vector form, and VAR(p) model is the regression of y_t on constant and its lag-p value as described below.

$$y_t = c + \Phi_1 y_{t-1} + ... + \Phi_p y_{t-p} + \epsilon_t,\ \epsilon_t \sim W.N.(\sum)$$

where c is $n \times 1$ vector, and Φ_i is $n \times n$ coefficient matrix.

VAR(p) model with n variables has n regression equations and each equation has np + 1 coefficients, and n(np+1) parameters, and the $\sum$ of the ϵ_t has n(n+1)/2 parameters. VAR(p) model with n variables has n(np+1) + n(n+1)/2 parameters in total.

The condition of stationary for VAR model is the every solution of the AR characteristic equation written as

$$|I_n - \Phi_1 B - ... - \Phi_p B^p| = 0$$

is larger than one where I_n is a $n \times n$ identity matrix. $|B| > 1$ means the absolute value of eigenvalue of each Φ_i (i = 1,...p) is larger than 1.

The expectation of VAR model is

$$\mu = E(y_t) = (I_n - \Phi_1 - ... - \Phi_p)^{-1}c$$

and the autocovariance is obtained by

$$\Gamma_k = \Phi_1 \Gamma_{k-1} + ... + \Phi_p \Gamma_{k-p}$$

.

The VAR model does not contain other variables at the same time index, and thus the VAR model is not a simultaneous equation model. However, the equations correlate with each other through their error terms, and such models are called seemingly unrelated regression models.

In general, such SUR models need to consider the correlation of their error terms, but each equation of the VAR model consists of the same explanatory variable. In this situation, the estimated coefficients of each equation by OLS are the best linear unbiased estimator. If ϵ_t follows a multivariate normal distribution, the estimated coefficients of each equation by OLS correspond with the maximum likelihood estimator. Hence, a VAR model is estimated by OLS. The order of a VAR model is often estimated by information criterion such as AIC setting the maximum order p_{max}. However, most macroeconomic data slowly influence each other, while AIC tends to set the lower order, and the order is set based on personal experience.

In the following three subsections, the paper introduces three useful tools for VAR analysis, that is, Granger causality, Impulse response function, and Variance decomposition.

4.9 Granger causality

One of the aims of analyzing economics is to understand the causality between economic factors. In this process, the Granger causality helps to judge the causality between variables. The definition of the Granger causality is to compare the forecast based on the past values of the variable x itself with the forecast based on the past values of the variable x and y, and if the latter prediction has a smaller MSE, the Granger causality from y_t x_t exists. That is to say, if y does not affect x, all the coefficients of y in the regression equation of x are zero. To test the Granger causality is to use F-test or Chi-square test with the null hypothesis, which assumes the coefficients of y are zero. In this respect, the VAR model with cointegrated variables would nullify the Granger causality test. It should be paid attention that the Granger causality is not the causality itself, but the direction of the Granger causality, and thus the causality can be different because the Granger causality only considers the current and past data but future data.

4.10 Impulse response function

Although the Granger causality is useful to examine the relationship between variables, the Granger causality does not evaluate how much variables have an influence on each other. Impulse response function and variance decomposition mentioned in the next section can compensate for this demerit.

Assume the VAR(p) model with n variables:

$$y_t = c + \Phi_1 t_{t-1} + ... + \Phi_p y_{t-p} + \epsilon_t,\ \epsilon \sim W.N.(\sum)$$

where the $\sum$ is not a diagonal matrix.

The unorthogonalized impulse response is defined as the change of $y_{i,t+k}$ by a shock of the error term, $\epsilon_{j,t}$ of $y_{j,t}$, and the unorthogonalized impulse response function takes the unorthogonalized impulse response as the function of k. Mathematically, the unorthogonalized impulse response function is calculated as below.

$$IRF_{ij}(k) = \frac{\partial y_{i,t+k}}{\partial \epsilon_{jt}} \ k = 0, 1, 2, ...$$

Other unorthogonalized impulse response function is estimated by repeatedly calculating in the same way. The problem of this method is the impulse response function does not take the correlation of error terms into account but cause a shock to only one error term.

The orthogonalized impulse response function overcomes this problem by splitting error terms into correlated parts and uncorrelated units. Nevertheless, it is usually impossible to chop up error terms as mentioned, and need some hypotheses. The orthogonalized impulse error terms posit that error terms can be divided into correlated elements and uncorrelated ones by LU decomposition of the variance-covariance matrix. The uncorrelated ones are named the orthogonalized error terms. The orthogonalized impulse response function assumes the impulse response as the function of time. In general, the impulse response function means the orthogonalized one. The variance-covariance matrix $\sum$ of the error terms ϵ is positive definite and can be decomposed as $\sum = ADA'$ where A is the lower triangular matrix having one at its diagonal elements and D is the diagonal matrix, and $u_t = A^{-1}\epsilon_t$ is the definition of the orthogonalized error terms and presumed to describe the change of each variable. The impulse response function is to calculate

$$IRF_{ij}(k) = \frac{\partial y_{i,t+k}}{\partial u_{jt}} \ k = 0, 1, 2, ...$$

Unorthogonalized error terms and orthogonalized error tems have different initial values and hence do not match each other unless a variable given a shock does not affect other variables. From the definition of the orthogonalized error terms: $u_t = A^{-1}\epsilon_t$, error terms can be described by the product of orthogonalized error terms and lower triangular matrices. Hence, ϵ_{1t} is u_{1t}, and ϵ_{2t} is the linear combination of u_{1t} and u_{2t}, and in general ϵ_{kt} is the linear combination of $u_{1t}, u_{2t}, ..., u_{kt}$. Such a structure is called recursive structure meaning variables having more exogeneity are placed from the left. If k is smaller than m, u_{kt} has an effect on ϵ_{mt} while ϵ_{kt} is not influenced by u_{mt}. Therefore, variables should be placed to make this assumption sense. The order of variables depends on the type of data. It is not easy to choose the order. The recursive structure has a strong hypothesis to forbid the effect of variables placed on the right side to other variables placed on the left side for one period. When placing variables, this hypothesis should be paid attention to. The order of variables does not change the coefficients and inference of the variance-covariance matrix. The sequence determines how to calculate orthogonalized error terms and alter the outputs of impulse response function and variance decomposition.

4.11 Variance decomposition

Variance decomposition also examines the dynamic relationship of variables. Variance decomposition evaluates how much the variables' error terms contribute to the MSE of the variables' prediction using orthogonalized error terms. That means variance decomposition adopts the hypothesis of the recursive structure.

Relative variance contribution, RVC in short, is the contribution ratio of $u_{j,t+1}, u_{j,t+2}, ...u_{j,t+k}$ to the MSE of the y_i's forecast for k periods ahead, and written as $RVC_{ij}(k)$. Variance decompostiion is to calculate $RVC_{ij}(k)$ for each variable. Forecast error variance decomposition is the other name of variance decomposition.

Mathematically, $RVC_{ij}(k)$ is calculated by

$$RVC_{ij}(k) = \frac{y_j's \text{ contribution to } MSE(\hat{y}_{i,t+k|t})}{MSE(\hat{y}_{i,t+k|t})}$$

and variance decomposition calculates $RVC_{ij}(k)$ as a function of k.

y_i's forecast for k period ahead in VAR model with n variables is the linear combination of $u_{t+k}, u_{t+k-1}, ..., u_{t+1}$. That is

$$\hat{e}_{i,t+k|t} = \sum_{h=1}^{k} \omega^i_{1,t+h} u_{1,t+h} + ... + \sum_{h=1}^{k} \omega^i_{n,t+h} u_{n,t+h}$$

The forecast MSE is

$$\begin{aligned}
MSE(y_{i,t+k|t}) &= E(\sum_{h=1}^{k} \omega^i_{1,t+h} u_{1,t+h} + ... + \sum_{h=1}^{k} \omega^i_{n,t+h} u_{n,t+h})^2 \\
&= \sum_{h=1}^{k} (\omega^i_{1,t+h})^2 E(u^2_{1,t+h}) + ... + \sum_{h=1}^{k} (\omega^i_{n,t+h})^2 E(u^2_{n,t+h}) \\
&= \sigma_1^2 \sum_{h=1}^{k} (\omega^i_{1,t+h})^2 + ... + \sigma_n^2 \sum_{h=1}^{k} (\omega^i_{n,t+h})^2 \\
&= \sum_{l=1}^{n} \sigma_l^2 \sum_{h=1}^{k} (\omega^i_{l,t+h})^2
\end{aligned}$$

where $\sigma_l^2 = E(u_{lt}^2)$.

The contribution of the variable j is $\sigma_j^2 \sum_{h=1}^{k} (\omega^i_{j,t+h})^2$,and $RVC_{ij}(k)$ is computed as below.

$$RVC_{ij}(k) = \frac{\sigma_j^2 \sum_{h=1}^{k} (\omega^i_{j,t+h})^2}{\sum_{l=1}^{n} \sigma_l^2 \sum_{h=1}^{k} (\omega^i_{l,t+h})^2}$$

It should be noticed that $RVC_{ij}(k)$ requires y_i's forecast errors to be the linear combination of $u_{t+k}, u_{t+k-1}, ..., u_{t+1}$.

4.12 Unit root and Augmented Dickey Fuller Test

In this section, this paper also refers to some web sites [4] [11]. Now, the importance of stationarity to analyze time series data is evident. The Augmented Dickey-Fuller test, called the ADF test for short, is frequently adopted to test whether the data is a unit root or not.

Before examining what the ADF test is, this thesis describes unit roots. Unit roots include stochastic processes or random walks. It would be better to given consideration to the fact unit roots are nonstationary, not vice versa.

The simplest example, AR(1) unit root process, is shown below.

$$Y_t = \phi_0 + Y_{t-1} + \epsilon_t$$

Y_t is added ϕ_0 at each period, and the expectation of Y_t increases by ϕ_0 given the mean of ϵ_t is zero. On top of that, the variance also depends on time, as below.

$$var(Y_t) = var(\epsilon_1 + \epsilon_2 + \epsilon_3 + ... + \epsilon_t)$$

Unit root raises concerns such as permanent effects, spurious correlation, and invalid inference.

Therefore, it is essential to understand tools that can test whether data contains unit roots or not. For this reason, the ADF test is utilized.

The ADF test uses equations written below and tries to verify $\gamma = 0$.

$$\Delta y_t = \alpha + \beta t + \gamma y_{t-1} + \phi_1 \Delta Y_{t-1} + \phi_2 \Delta Y_{t-2} + ... + \phi_p \Delta Y_{t-p} + \epsilon_t$$

The test's null hypothesis is that the data has a unit root, and the alternative hypothesis is that the data is stationary. However, it is not correct that failing to reject a null hypothesis means the series is nonstationary. In the next section, the paper delves into the problems which unit roots cause.

4.13 Spurious regression

Spurious regression is a critical issue in time series analysis. One reason to cause spurious regression is ignoring the fact that explanatory variables and dependent variables are unit roots.

Given two unit roots, y_t and x_t, which are irrelevant to each other, the regression of y_t on x_t shows a high correlation, that is spurious regression. One way to avoid the problem is to include lag variables of the explanatory and dependent variables into regression equations. On this point, VAR is adopted broadly. However, some tests like Granger causality tests are invalid under a VAR model with unit roots. Another way to evade spurious regression is to difference a time series and make it stationary. Differencing time series data too many times results in losing information about data, so checking whether data is stationary is essential.

Provided two unit root variables, x_t and y_t, the error term ϵ_t of the regression of y_t on x_t can be unit root or stationary. If ϵ_t is unit root, the regression is spurious. If ϵ_t is stationary, x_t and y_t are said to be cointegrated.

To verify whether regression is spurious is to check whether ϵ_t is unit root. Yet, tested ϵ_t is unknown, and estimated $\hat{\epsilon}_t$ is used for a test called Engle-Granger cointegration test.

Cointegration also raises a problem that differenced series fail to predict precisely, apart from spurious regression. Cointegrated variables need another specific model. In the next subsection, the paper explains cointegration in detail.

4.14 Cointegration

Before defining cointegration, this study considers the linear combination of time series. The combination of a unit root process and a stationary process is a unit root process as well as that of two stationary processes becomes stationary. Still, it is unclear what the linear combination of unit roots would be. If the linear combination of unit roots gets stationary, there is cointegration between unit roots. There are two unit root processes, x_t and y_t. If there are any a and b such that $ax_t + by_t$ gets stationary, x_t and y_t is said to be cointegrated or x_t and y_t is referred to having a relationship of cointegration. In general, if y_t has a unit root and there is a such that $a'y_t$ is stationary, y_t has a relationship of cointegration. In other words, if y_t is cointegrated, (a, b) is referred to as a cointegrating vector. Though, such cointegrating vectors are not always unique. It is natural to impose cointegrating vectors on standardization condition such as $a = 1$ or $\sqrt{a'a} = 1$. Furthermore, some systems have more than one cointegration, and n variables can have n-1 cointegration at most. The number of cointegration is known as cointegration rank.

It is hard to predict unit root process separately, but cointegrated variables having unit roots are stationary and can be predicted more precisely. Therefore, it is crucial to consider cointegration between variables in the analysis.

This paper omits specific clarification of cointegration tests, which are an advanced topic.

5 The empirical analysis

In this part, this paper analyzes the yearly and monthly data of the U.S.A due to data availability, using the methods mentioned above. Readers can refer to the author's Github for the detailed procedure, the entire data sets, and the Python codes. [*2]

The analysis procedure is:

1. Confirm the Granger causality between the variables of used data.
2. Choose variables based on the Granger causality as well as their orders.
3. Modify data to analyze and split data into test one and train one.
4. Fit the VAR model to the data and estimate the period of test data.
5. Reviewing impulse response and variance decomposition of the model.
6. Evaluate and compare the model and its estimates with others.

As Pantelis Kalaitzidakis and George Korniotis (2000) [6] claim, correct specification comes first so that a model can capture all the systematic information by the systematic part instead of its error term. In this light, this paper also conducts various pairs of variables to correctly specify which factors impact economic development and the stock index. This paper mainly uses pairs of cointegrated variables to avoid misspecification. Nonetheless, even if there are specific findings, there is no development without a theoretical model. In this connection, this study also tries to match the findings to theoretical models at the end of the analysis.

5.1 The case of the U.S.A with the yearly data : S&P500

Through the subsection, the paper examines the yearly data of the U.S.A. Most of the data used here is obtained from the data bank of the world bank. [*3] Other data is gained from the United States Patent and Trademark Office [*4], the Damodaran Online Home Page [*5], and the American Association for the Advancement of Science. [*6]

5.1.1 the main variables

There are many variables, and some of them are very similar such as the growth rate of savings and the total amount of savings. Therefore, this subsubsection explains limited variables. Readers can check

[*2] Here is the author's Github: https://github.com/Takahiro-YANAGI/FYP-StockEconomics.git
[*3] https://databank.worldbank.org/source/world-development-indicators
[*4] https://www.uspto.gov/learning-and-resources/electronic-data-products/historical-patent-data-files
[*5] http://people.stern.nyu.edu/adamodar/New_Home_Page/datafile/wacc.htm
[*6] https://www.aaas.org/programs/r-d-budget-and-policy/historical-trends-federal-rd

the entire variables in the Appendix A as well as the writer's Github mentioned before.
S&P 500: S&P 500
S&P 500(G%): The percentage growth rate of S&P500
GDP: GDP (current US$ /billion)
GDP(G%): The percentage growth rate of GDP
RGDP: Real GDP
GDPIF(G%): The percentage growth rate of GDP adjusted by the inflation rate
ST(G%): the percentage growth rate of real total savings
BCP(%): the ratio of the domestic credit to the private sector by banks to GDP
TRADE(T): Total amount of TRADE
TRADE(%): the ratio of trade to GDP
R&D(G%): the percentage growth of the total R&D
GCE(%): the ratio of government final consumption expenditure
GCFC(G%): Gross fixed capital formation (annual % growth)
POP(TM): the total population(million)
POP(1564M): Population between the ages of 15 and 64(million)
WP(M): the number of working people(million)
WP(G%): the percentage growth rate of working people
INF: Inflation, GDP deflator (annual %)

It is worth considering that VAR models with too many variables often fail to compute because similar variables make the matrix not positive definite, and so it is better to avoid using similar variables of the same type.

5.1.2 Time series analysis with the variables used in the other papers

In this section, this paper examines the relationship between the variables used in other papers. [10] [14]

First of all, this paper tests the Granger causality between the variables. The variables are tested twice because some can fail to reject the null hypothesis of unit root existence even though they are differenced once. Hence, the paper conducts the test twice for the variables differenced only once if necessary, and the variables differenced until they get stationary. On the contrary to the author's consideration, these two tests returns the same output except that the pair of $GCE(\%)1\_y$ and $BM(\%)\_1\_x$ shows "False" under the test, in which the variables are differenced at most once. On this point, Figure 1 of the Granger causality test, differencing the variables at most once is enough.

[図 1 about here.]

This figure's interpretation is that $variable\_x$ Granger causes $variable\_y$, and the number added after variables like $GDS(\%)\_1\_y$ shows how many times the variables are differenced. In addition, the figure shows "False" if a p-value is larger than 5% for readability.

It is clear that all the variables Granger cause each other with 95% statistical significance, excluding the pair of $GCE(\%)1\_y$ and $BM(\%)\_1\_x$.

As the next step, this paper checks impulse response functions and variance decompositions for the pairs of cointegrated variables to avoid misspecification.

The chosen variables are $[GDS(\%), GCE(\%), GDP(G\%), S\&P500]$, which are cointegrated with 99% statistical significance.

[図 2 about here.]

[図 3 about here.]

The impulse response and the variance decomposition of Figures 2 and 3 tell that a shock in the GDP growth rate has both positive and negative effects on S&P 500. Likewise, the effect of a shock in S&P 500 on the GDP growth rate seems unstable and changes its direction. The GDP growth rate occupies a small portion of the variance decomposition of S&P 500, and vice versa. S&P appears to be gradually significant in the variance decomposition of the GDP growth.

Therefore, the relationship between the stock index and economic development is not apparent enough at this point. Given the previous discussion, it would be better to add other economic factors, and thus another set of variables are analyzed in the next part.

5.1.3 VAR analysis with the different pairs of the variables

Expanding the data set with various elements helps to understand the relationship between economic factors and the stock index.

To begin with, the paper checks the Granger causality of the main variables used in this part.

[図 4 about here.]

It is evident that all but the pair of $[ISS\_1\_y, ST(G\%)\_0]$ Granger cause each other from Figure 4, and this result is not changed even if variables are differenced until they become stationary.

These variables are tested whether they have cointegrated relationships. The pairs of ["$POP(TM)$", "$ST(G\%)$", "$TRADE(T)$","GDP","$S\&P500$"], ["$POP(1564M)$","$ST(G\%)$","$TRADE(T)$","GDP", "$S\&P500$"], and ["$WP(M)$","$ST(G\%)$","$TRADE(T)$","GDP","$S\&P500$"] are cointegrated with 99% statistical significance. The VAR models with these pairs exhibit an implication.

[図 5 about here.]

[図 6 about here.]

Figures 5 and 6 make the difference between these pairs apparent, in that the model with the number of working people forecasts better than that with the total population.

[図 7 about here.]

[図 8 about here.]

A shock in the number of working people positively affects the GDP while one in the total population sticks to zero for several periods as the two impulse responses of Figures 7 and 8.

[図 9 about here.]

[図 10 about here.]

Moreover, the variance decompositions of Figures 9 and 10 tell the number of working people has a more significant portion of the GDP's variance decomposition than the total population.

[図 11 about here.]

It should also be emphasized that considering the unemployment rate raises the difference between these pairs because the model with the population between age 15 and 64 does not forecast better than that with the working people as Figure 11.

Another illustration uses the two pairs of [$"WP(M)", "ST(G\%)", "TRADE(\%)", "GDP", "S\&P500"$] and [$"WP(M)", "ST(G\%)", "TRADE(\%)", "RGDP", "S\&P500"$]. Both of them are cointegrated at the level of 95% statistical significance.

[図 12 about here.]

Figures 12 is the impulse response of [$"WP(M)", "ST(G\%)", "TRADE(\%)", "GDP", "S\&P500"$]. It is questionable why the impulse response of Figure 12 from S&P 500 to GDP initially shows a negative effect.

The theoretical background of this phenomenon is that S&P 500 and Tobin' s q moves quite similarly as Christis Hassapis and Sarantis Kalyvitis(2002) claim. [5] Their study points out that there is a negative relationship between output growth and future stock price returns. On top of that, they also claim that real stock prices are a good proxy of q over long periods. Tobin' s q is the shadow price of capital, and the rise in stock prices as a proxy of the q let people predict the rise in investment and adjustment cost, by which GDP may initially fall a little, but stock markets accelerate the output growth in total. In contrast, as the output growth increases, q and investment increments raise adjustment costs, which manages the initial jump in real stock prices returns to the price at the economic equilibrium. The economy realizes higher growth and real stock prices in the end.

In this connection, if only the stock index receives a shock, the stock index' s rise enables others to predict the rise of capital price, and the rise in the stock index may drag down GDP for a short while, as Figure 12. By contrast, the positive shock in GDP causes higher adjustment costs and negatively impacts the S&P 500 as time goes by, as shown in Figure 12.

The theoretical consequence is also true for the pair of [$"WP(M)", "ST(G\%)", "TRADE(\%)", "RGDP", "S\&P500"$].

[図 13 about here.]

Figure 13 exhibits the impulse response of [$"WP(M)", "ST(G\%)", "TRADE(\%)", "RGDP", "S\&P500"$]. What is to be noted about the diagram is that a shock in the stock index gives positive effects from the inception, and the shock in the real GDP has a less negative effect on S&P 500.

Given the theoretical explanation and empirical evidence demonstrated above, it is natural to conclude that S&P 500 and GDP are closely tied together and affect each other, and economic growth can be associated with various factors. Still, it remains unclear why these components correlate with each other. Without a theoretical background, empirical findings would be fragile, and the above explanation of Tobin's q also implies the necessity of theoretical development for further study. Therefore, the article investigates whether theoretical models can work in practice in the next section.

5.1.4 the validity of macroeconomic theory with VAR model

Though there are many models for economic development, this paper starts the analysis from the neoclassical Cobb-Douglas production function: $Y = K^{\alpha}(AL)^{1-\alpha}$ $0 < \alpha < 1$ where Y denotes output, K does capital, L means labor, A shows the level of technology, and α means the income share of capital. The paper reviewed the textbook written by 二神 and 堀 (2017) [16] for the following calculation. The log form of this function is

$$\begin{aligned} \ln Y_t &= \ln A_t^{1-\alpha} K_t^{\alpha} L_t^{1-\alpha} \\ &= \ln B_t + \alpha \ln K_t + (1-\alpha) \ln L_t \end{aligned}$$

where B_t is set to denote $A_t^{1-\alpha}$. Using this equation, Y_{t+1} minus Y_t is calculated as follows.

$$\begin{aligned} \ln Y_{t+1} - \ln Y_t &= \ln B_{t+1} - \ln B_t + \alpha[\ln K_{t+1} - \ln K_t] + (1-\alpha)[\ln L_{t+1} - \ln L_t] \\ \ln \frac{Y_{t+1}}{Y_t} &= \ln \frac{B_{t+1}}{B_t} + \alpha \ln \frac{K_{t+1}}{K_t} + (1-\alpha) \ln \frac{L_{t+1}}{L_t} \end{aligned}$$

Given that $\ln(1+x) \approx x$, the above equation can be modified as:

$$g_t^Y = g_t^B + \alpha g_t^K + (1-\alpha) g_t^L$$

where g_t^Y denotes the growth rate of GDP, g_t^B is the growth rate of total factor productivity (TFP), g_t^K means the growth rate of capital, and g_t^L designates the growth rate of the number of labor.

The alternative of the $[g_t^Y, g_t^B, g_t^K, g_t^L]$ is the pair of ["INF","$WP(G\%)$","$R\&D(G\%)$","$GFCF(G\%)$", "$GDP(G\%)$"] to analyze with VAR model. This is because the pair of ["$WP(G\%)$","$R\&D(G\%)$", "$GFCF(G\%)$", "$GDP(G\%)$"] is not cointegrated with 95% statistical significance unless the inflation rate is attached to the pair. As Mairesse and Benoit [9] adopts $R\&D$ capital stock as a proxy of productivity, this paper takes the growth rate of $R\&D$ as a proxy of the growth rate of the total factor productivity g_t^B.

[図 14 about here.]

[図 15 about here.]

Remarkably, both of "$S\&P500(G\%)$" and "$GFCF(G\%)$" showed a vary similar impulse response as Figures 14 and 15.

[図 16 about here.]

[図 17 about here.]

Besides, the pair of ["INF","$WP(G\%)$","$R\&D(G\%)$","$S\&P500(G\%)$","$GDP(G\%)$"] outperforms the pair of ["INF","$WP(G\%)$","$R\&D(G\%)$","$GFCF(G\%)$","$GDP(G\%)$"] for forecasting the growth rate of GDP in terms of correlation and root mean square error as Figures 16 and 17. Therefore, it is possible that S&P 500 takes the place of the growth rate of fixed capital.

Another economic development model is Romer's AK model, for which the paper also refers to 二神

and 堀 (2017).[16]The AK model is calculated as:

$$Y_t = C_t K_t^{\alpha} L_t^{1-\alpha}$$
$$C_t = A(\frac{K_t}{L_t})^{1-\alpha}$$

where A is a positive constant, which posits that increasing the capital per capita improves productivity, and other characters denote the same as before.

The equation can be rewritten as follows by substituting C_t into the first equation.

$$Y_t = AK_t$$

Provided that the inflation rate is essential to establish cointegrated variables, the paper adopts the pair of ["$R\&D(G\%)$,""$GFCF(G\%)$,""$GDPIF(G\%)$"], and analyze them. Here "GDPIF(G%)" is the growth rate of GDP adjusted by the inflation rate.

As a result, these variables are cointegrated with 99% statistical significance as the pair of ["$R\&D(G\%)$,""$S\&P500(G\%)$,"$GDPIF(G\%)$"] is cointegrated with 95% statistical significance.

The result of the VAR analysis with the pair was not satisfactory.

[図 18 about here.]

As Figure 18 exhibits, $GFCF(G\%)$ and $R\&D$ occupies a specific portion of $GDPIF(G\%)$ in the variance decomposition, but not vice versa. This result is not altered even with the pair of ["$R\&D(G\%)$,""$S\&P500(G\%)$,"$GDPIF(G\%)$"]. That is to say; the model cannot explain how $GFCF(G\%)$ and $R\&D$ are influenced by other factors, including "$GDPIF(G\%)$". Therefore, other economic models should be studied further in the future.

One of the related precious researches is the work of Pantelis Kalaitzidakis and George Korniotis(2000). [7] They also conduct another empirical research using the VAR model to verify whether the augmented Solow model well explains G7 countries. Their paper states that the augmented Solow model generally holds. In addition, the output per capita, the saving rate, and the population's growth rate are cointegrated in most countries except the UK and Canada. However, they do not clearly describe the exogeneity of the investment rate and population growth rate in their paper, and they assume that technological development and depreciation rates are 0.05 constantly.

Future researches can incorporate the stock index as a proxy of Tobin's q into a model. According to Christis Hassapis and Sarantis Kalyvitis(2002) [5], the growth of capital is calculated by Tobin's q, adjustment cost, and the depreciation rate.

$$g_Y = \frac{(q-1)}{\phi} - \delta$$
$$\delta = \frac{(q-1)}{\phi} - g_Y$$

This equation shed light on the way to incorporate Tobin's q and stock indices as a good proxy for Tobin's q into the economic model through the depreciation rate. The further study of economic models would be of high value to reveal the relationship between stock markets, financial sectors, and economic development.

5.2 The case of the U.S.A with the montly data : S&P500

Another way to scrutinize the relationship between stock indices and other economic components is to use another frequency data like monthly data. As a next step, this study tries to analyze the monthly data of the U.S.A. In this respect, this study tries to reveal the relationship between variables using the U.S.A's monthly data in this chapter. The data is available at FRED-MD. *7 The data set contains over 100 variables for over half a century, and so this study sets the period from November 1st, 2011, to December 1st, 2019, in order to handle the extensive data set without the possibility of a structural break. This paper does not describe all the variables but only the used ones because FRED-MD provides a detailed description at its site. *8 Besides, this paper follows the direction of FRED-MD on how to make variables stationary.

Although the paper tries to choose the alternatives of the pairs that work well with the yearly data, it is not easy to find variables with a one-to-one relationship because macroeconomic variables such as GDP are affected by many other elements. In this light, variables need to be carefully selected.

The chosen pair is ["$S\&P500$","$CLAIMSx$","$CMRMTSPLx$","$INDPRO$","$TB3SMFFM$"]. IP Index, "$INDPRO$" can be a proxy of GDP and evaluate the economy's scale. It can be assumed that Total Consumer Loans and Leases Outstanding, "$DTCTHFNM$" goes up when companies accelerate their growth.

This hypothesis is based on Ross(2003)[8], which introduces the idea that evolving banking systems increase the portion of companies growing at the level requiring other's capital, and his paper concludes that the portion of companies growing faster than the companies with their own capital and short term debt, positively correlates with the liquidity of stock markets and the banking system's scale. Such companies growing faster than others increase if a country has a high turnover ratio and bank assets, and bank assets are equal to companies' debt. On this point, this paper takes "$DTCTHFNM$" as a proxy of economic growth.

Initial Claims, "$CLAIMSx$" shows how many people lose their jobs and correlate with the unemployment rate. 3-Month Treasury C Minus Effective Federal Funds Rate, "$TB3SMFFM$" is close to the inflation rate.

[図 19 about here.]

[図 20 about here.]

The pair of these variables do not show the Granger causality between each other as Figure 19, let alone the cointegrated relationship, and the pair predict future values poorly, as Figure 20. This may result from two reasons. One is that selected variables are not appropriate, and the other is that the stock index has a different mechanism in a short span.

Investigating the date further, this study finds pairs of cointegrated variables such as ["$TOTRESNS$", "$COMPAPFFx$", "$CLAIMSx$","$INDPRO$","$S\&P500$"]. Again, they forecast poorly.

[図 21 about here.]

*7 https://research.stlouisfed.org/econ/mccracken/fred-databases/

*8 https://s3.amazonaws.com/files.fred.stlouisfed.org/fred-md/Appendix_Tables_Update.pdf

It can be checked that the variance of S&P 500 is mainly affected by itself by examining the variance decomposition of Figure 21. "$CLAIMSx$" is supposed to have a close relationship with the stock index given that the number of working people occupies a large portion of the stock index's variance. On the contrary to this idea, surprisingly, financial activities such as "$TOTRESNS$" and "$COMPAPFFx$" have more impact on the stock index than the initial claims, "$CLAIMSx$" does. Given this outcome, there is a reason to believe that the stock market is driven by different components in different periods.

In summary, even if macroeconomic variables affect each other in the long term, monthly data may fail to capture it. In this connection, it is implied that the stock index has different mechanisms for the short term and a longer one.

In fact, Chow and Lawler (2003) [3] claim that the efficient market hypothesis is correct for the weekly rate of stock return in both Shanghai and New York markets. Besides, they also state that monthly data would fail to capture the finer or high-frequency movements, and it is better to use the weekly observations of the rate of return and its volatility for research.

In line with this, the stock indices mechanism in a short period should consider the data frequency carefully, and it is better to differentiate the stock market mechanisms according to their length.

6 Conclusion

This paper starts from the importance of revealing the relationship between the stock index and economic development, explaining the social needs of investments, and the controversial discussion in academics.

After reviewing the time series analysis methods, this paper analyzes the yearly and monthly data of the U.S.A and obtains three primary outcomes stated below.

First, the pair of variables using the number of working people instead of the total population forecasts future values more precisely. In this connection, this paper succeeds in providing another evidence to support the theory that the rise in the stock index accelerates economic development.

Besides, Romer's AK model was verified by adding inflation rate to the pair of the variables, while the reason inflation rate helps to make the variables cointegrated should be studied further. On top of that, this study notes that Tobin's q and stock indices can be applied to other economic models.

Furthermore, this paper also analyzes the monthly data of the U.S.A and finds that different components drive stock indices in different periods quite likely. For the following research, it is worth noting that data frequency can alter the stock indices' behavior.

One of the limitations of this study is data availability, and a continuous study of this topic across countries would be desired. Still, there is a significant issue left unsolved. That is how to choose variables. The reason people have not reached an agreement on this topic is partly because of the arbitrariness of choosing variables, given that similar pairs of variables can display different results. On this account, it is strongly recommended for future researches to take into consideration variable selection procedures such as lasso and ridge.

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図目次

	GDS(%)_1_x	GCE(%)_1_x	GDP(G%)_0_x	S&P500_1_x	GDP_1_x	RGDP_1_x	GDPP(%)_1_x	TRADE(%)_1_x	RIR(%)_1_x	TR_1_x	BM(%)_1_x
GDS(%)_1_y	1.0	0.0000	0.0000	0.0	0.0	0.0	0.0	0.0	0.0373	0.0	0
GCE(%)_1_y	0.0	1.0000	0.0000	0.0	0.0	0.0	0.0	0.0	0.0000	0.0	False
GDP(G%)_0_y	0.0	0.0000	1.0000	0.0	0.0	0.0	0.0	0.0	0.0000	0.0	0
S&P500_1_y	0.0	0.0000	0.0000	1.0	0.0	0.0	0.0	0.0	0.0000	0.0	0
GDP_1_y	0.0	0.0000	0.0000	0.0	1.0	0.0	0.0	0.0	0.0000	0.0	0
RGDP_1_y	0.0	0.0000	0.0001	0.0	0.0	1.0	0.0	0.0	0.0000	0.0	0
GDPP(%)_1_y	0.0	0.0000	0.0000	0.0	0.0	0.0	1.0	0.0	0.0000	0.0	0
TRADE(%)_1_y	0.0	0.0000	0.0000	0.0	0.0	0.0	0.0	1.0	0.0001	0.0	0
RIR(%)_1_y	0.0	0.0000	0.0000	0.0	0.0	0.0	0.0	0.0	1.0000	0.0	0.0088
TR_1_y	0.0	0.0000	0.0017	0.0	0.0	0.0	0.0	0.0	0.0000	1.0	0
BM(%)_1_y	0.0	0.0004	0.0000	0.0	0.0	0.0	0.0	0.0	0.0000	0.0	1

図 1: Granger causality with the variables differenced at most once

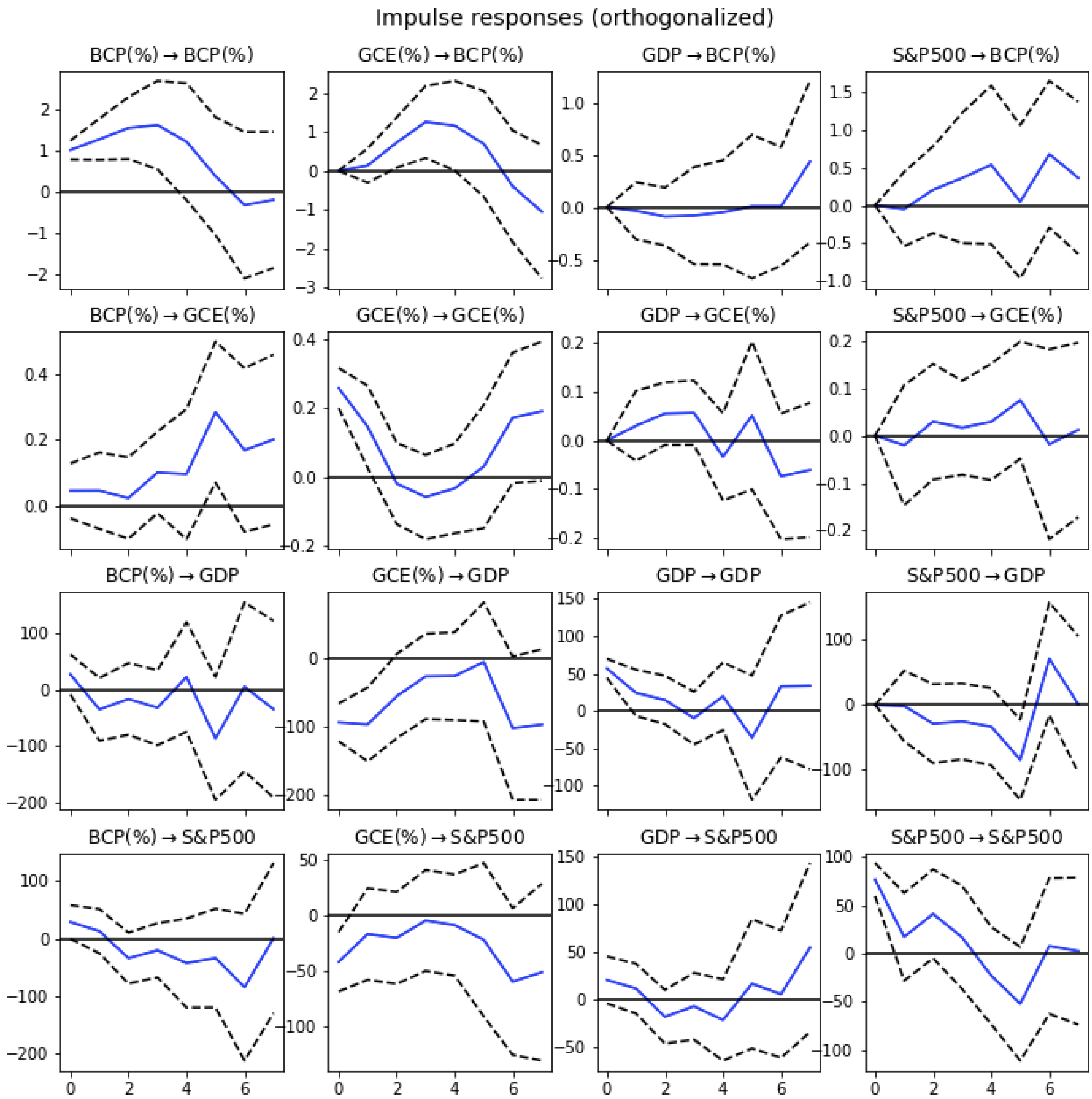

図 2: Impulse Response: [BCP(%), GCE(%),GDP,S&P500]

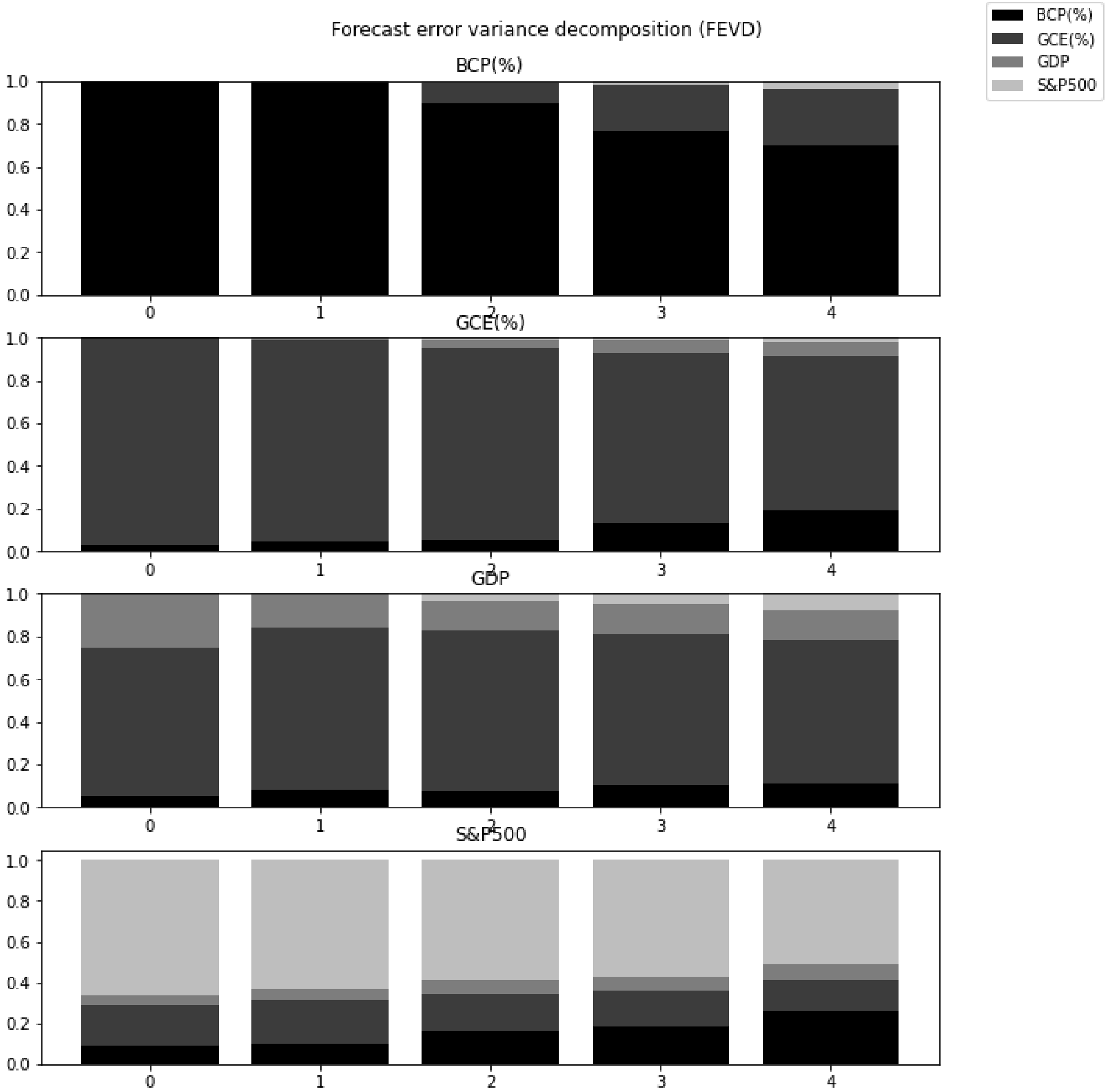

図 3: Variance Decomposition: [BCP(%), GCE(%),GDP,S&P500]

	S&P500_1_x	ST(G%)_0_x	TORADE(T)_1_x	GDP_1_x	RGDP_1_x	GDP(G%)_0_x	WP(M)_1_x	R&D(T)_1_x	ISS_1_x
S&P500_1_y	1.0	0.0192	0.0171	0.0075	0.0008	0.0057	0.0055	0.0000	0.0004
ST(G%)_0_y	0.0	1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003
TORADE(T)_1_y	0.0	0.0046	1.0000	0.0000	0.0000	0.0147	0.0012	0.0054	0.0000
GDP_1_y	0.0	0	0.0000	1.0000	0.0015	0.0000	0.0000	0.0000	0.0000
RGDP_1_y	0.0	0	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0032
GDP(G%)_0_y	0.0	0	0.0000	0.0000	0.0000	1.0000	0.0014	0.0000	0.0094
WP(M)_1_y	0.0	0	0.0000	0.0000	0.0000	0.0403	1.0000	0.0000	0.0001
R&D(T)_1_y	0.0	0	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000
ISS_1_y	0.0	False	0.0000	0.0000	0.0026	0.0036	0.0005	0.0025	1.0000

図 4: Granger causality with the variables differenced at most once

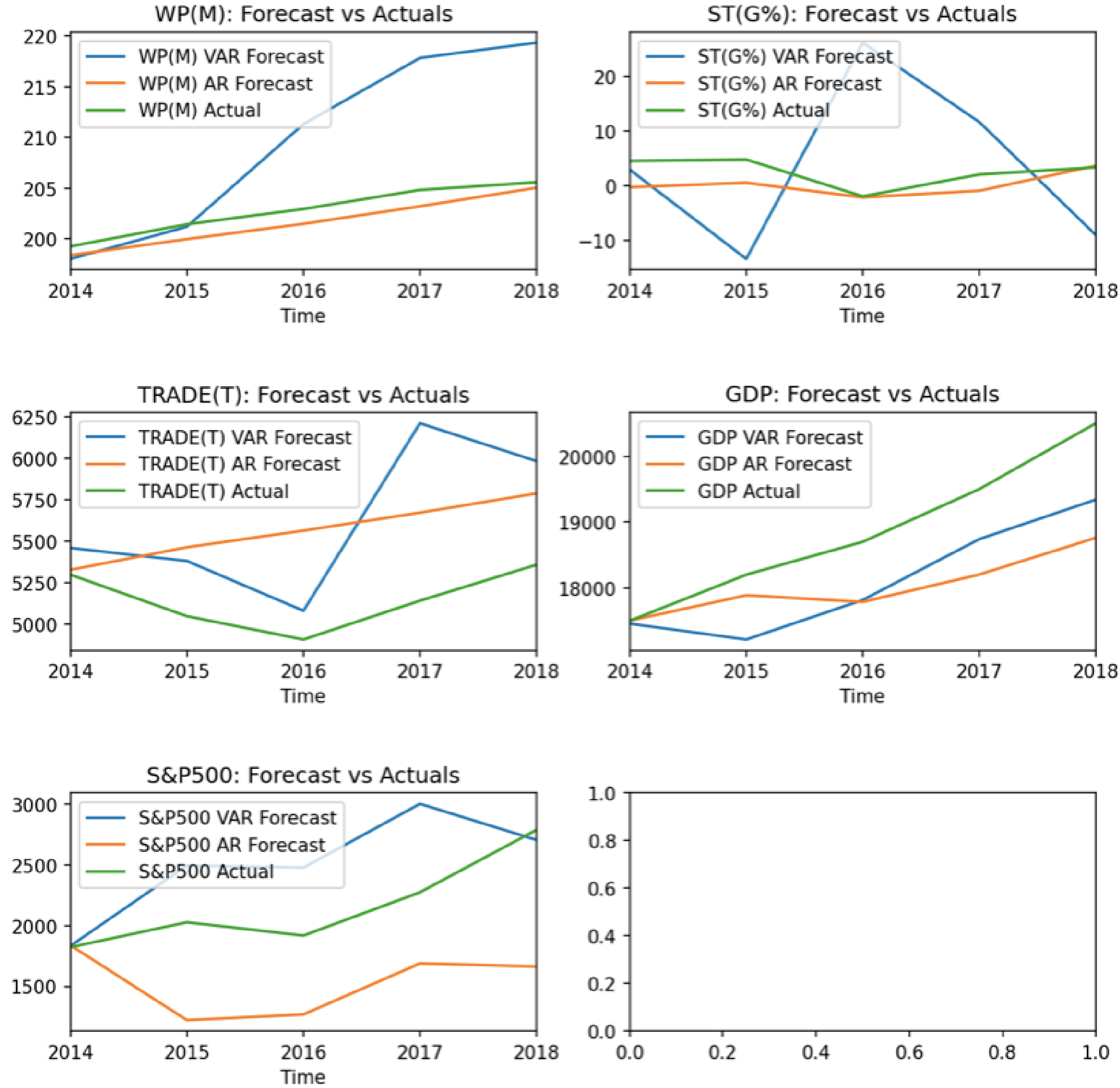

図 5: Forecast of the variables: ["$WP(M)$","$ST(G\%)$","$TRADE(T)$","GDP","$S\&P500$"]

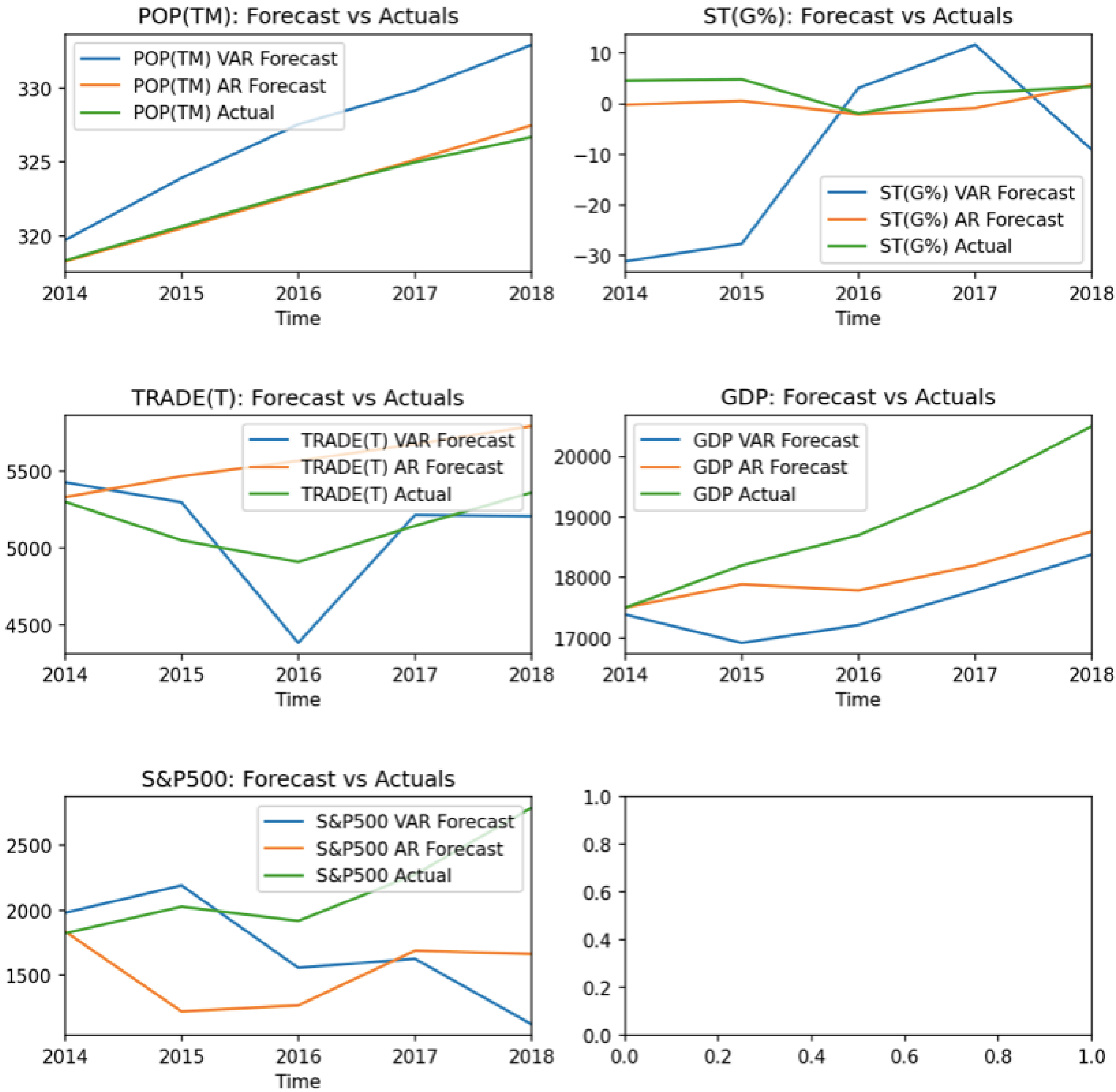

図 6: Forecast of the variables: $["POP(TM)","ST(G\%)","TRADE(T)","GDP","S\&P500"]$

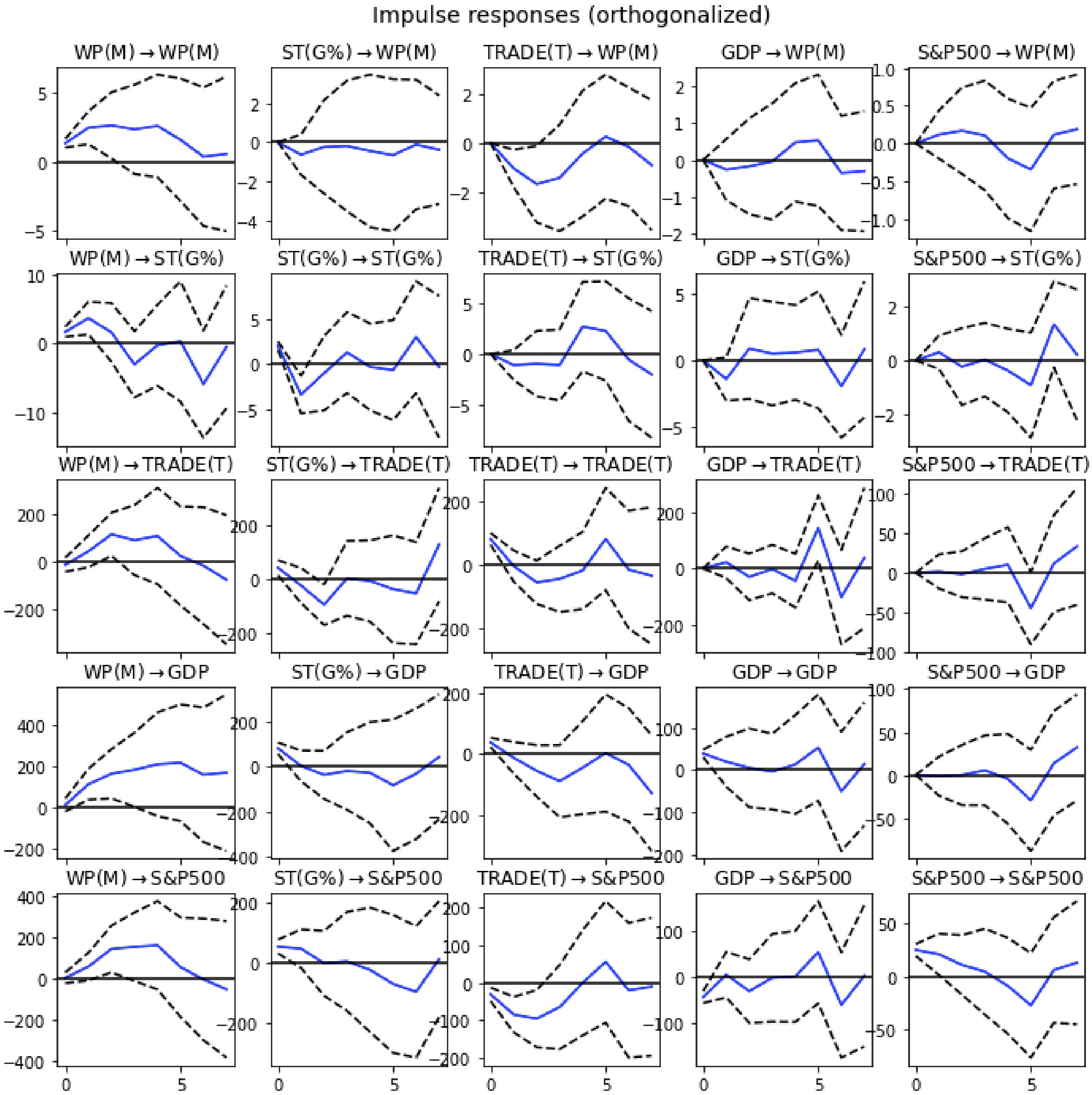

図 7: Impulse responses: $["WP(M)","ST(G\%)","TRADE(T)","GDP","S\&P500"]$

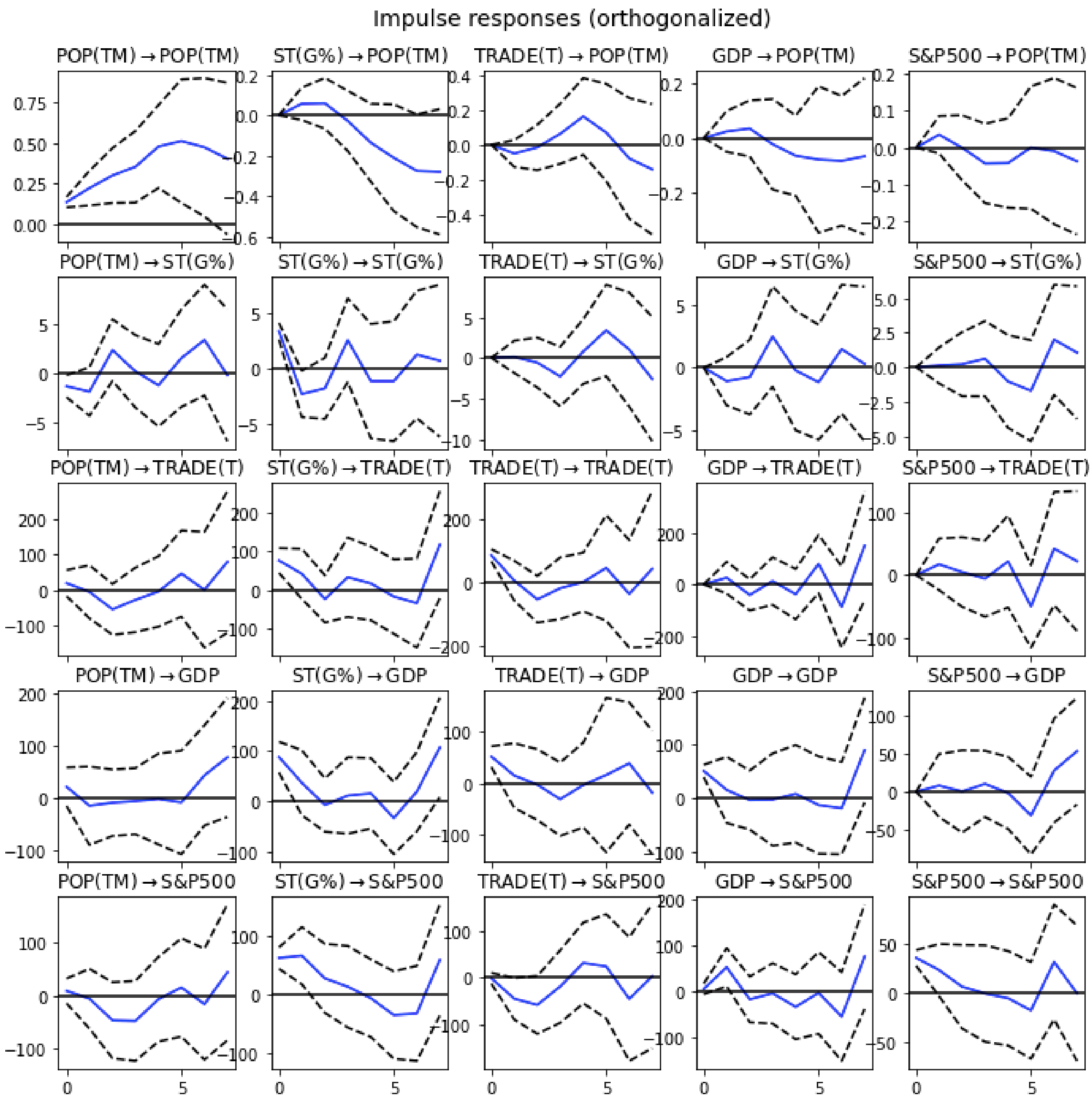

図 8: Impulse response: ["$POP(TM)$","$ST(G\%)$","$TRADE(T)$","GDP","$S\&P500$"]

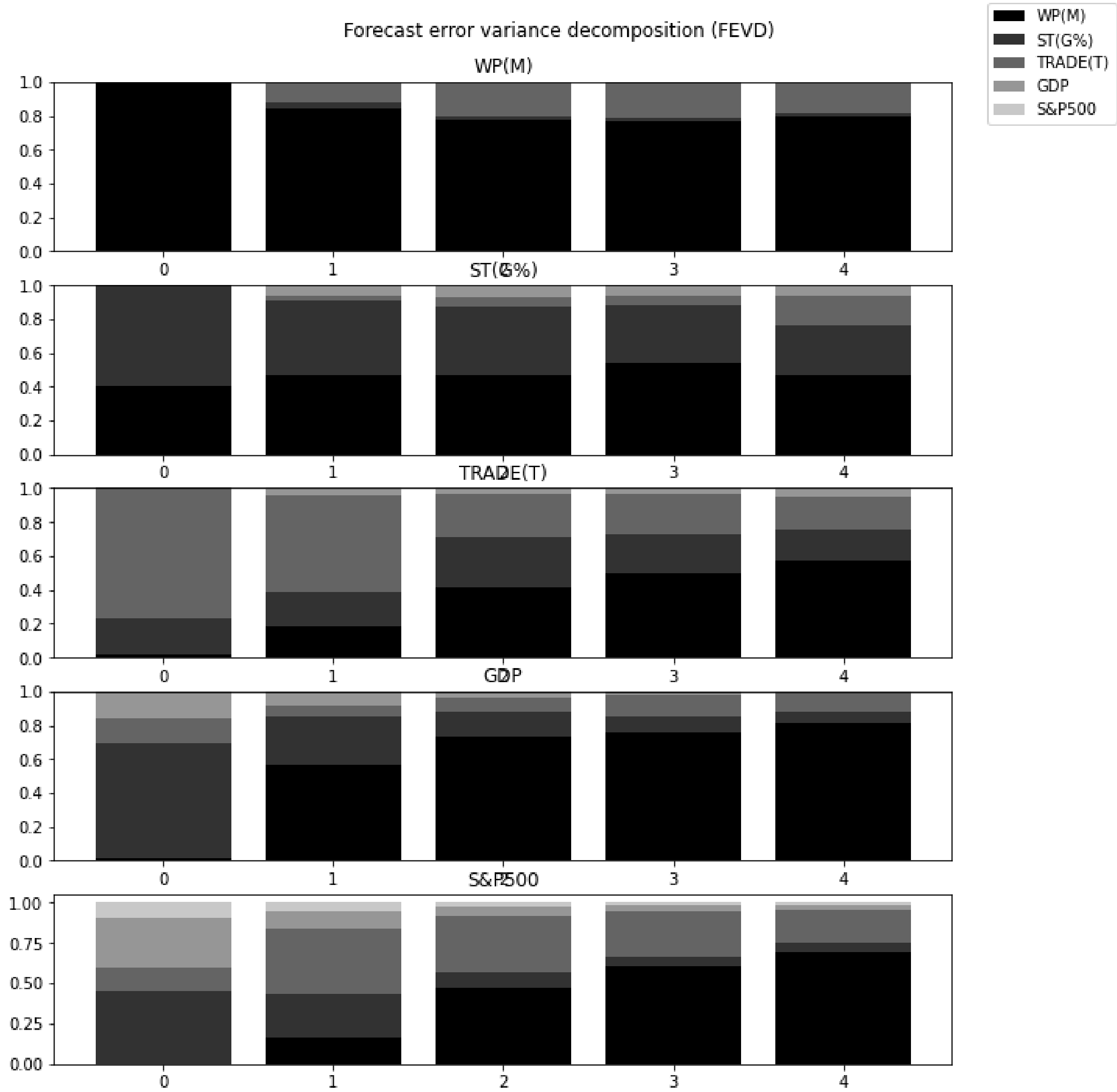

図 9: Variance Decomposition of the variables: $["WP(M)", "ST(G\%)", "TRADE(T)", "GDP", "S\&P500"]$

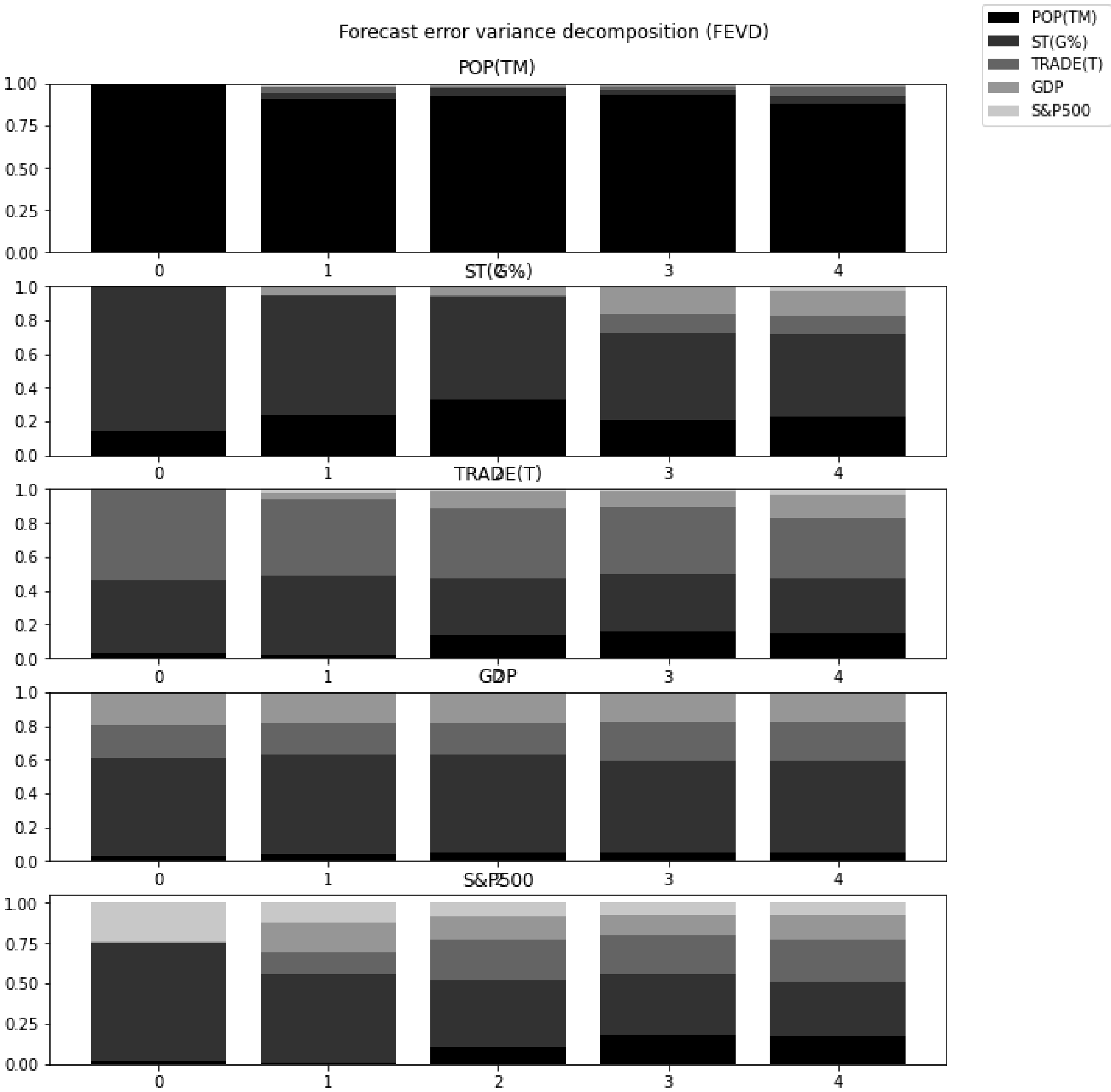

図 10: Forecast of the variables: [”$POP(TM)$”,”$ST(G\%)$”,”$TRADE(T)$”,”GDP”,”$S\&P500$”]

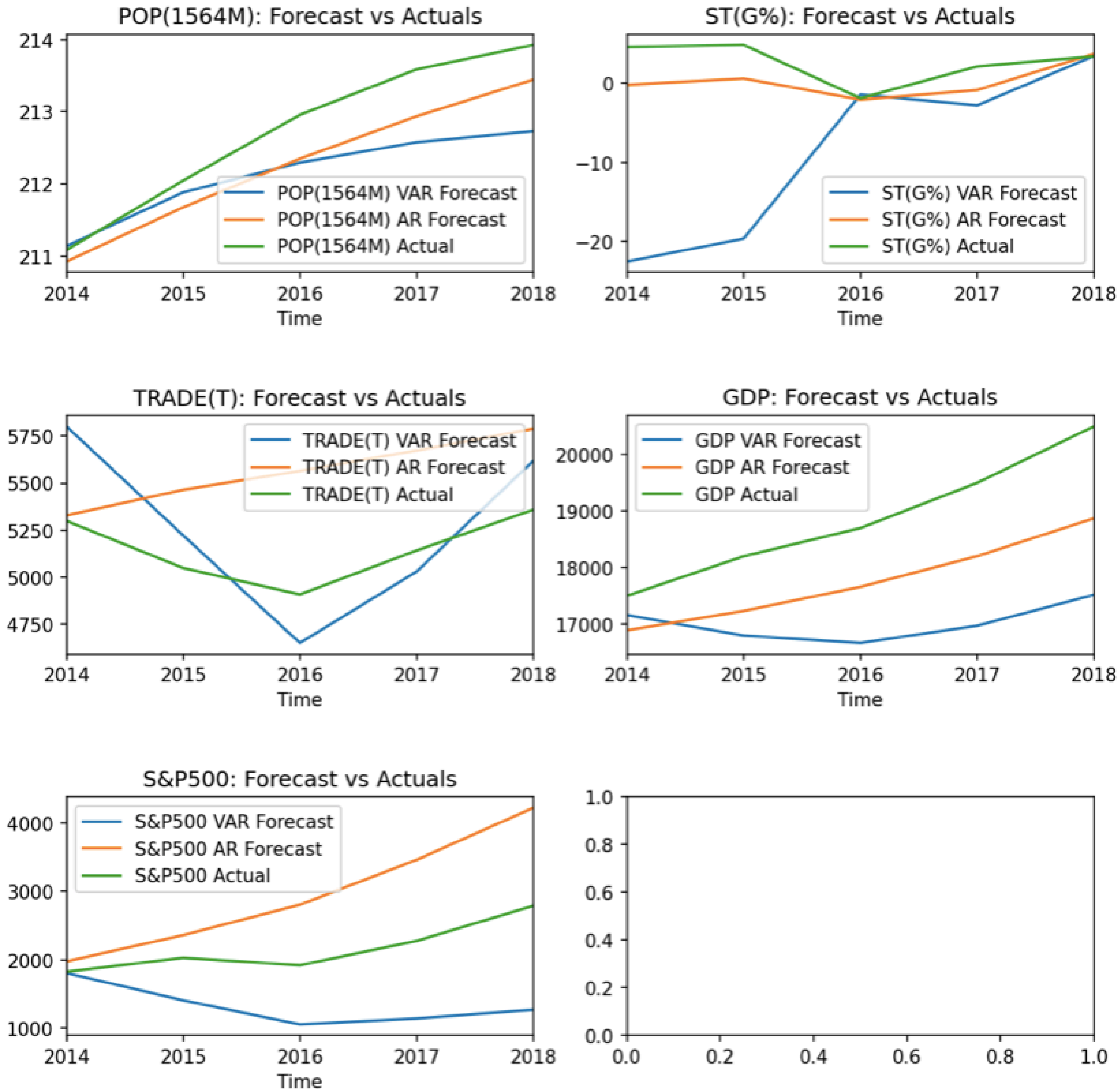

図 11: Forecast: ["$POP(1564M)$","$ST(G\%)$","$TRADE(T)$","GDP","$S\&P500$"]

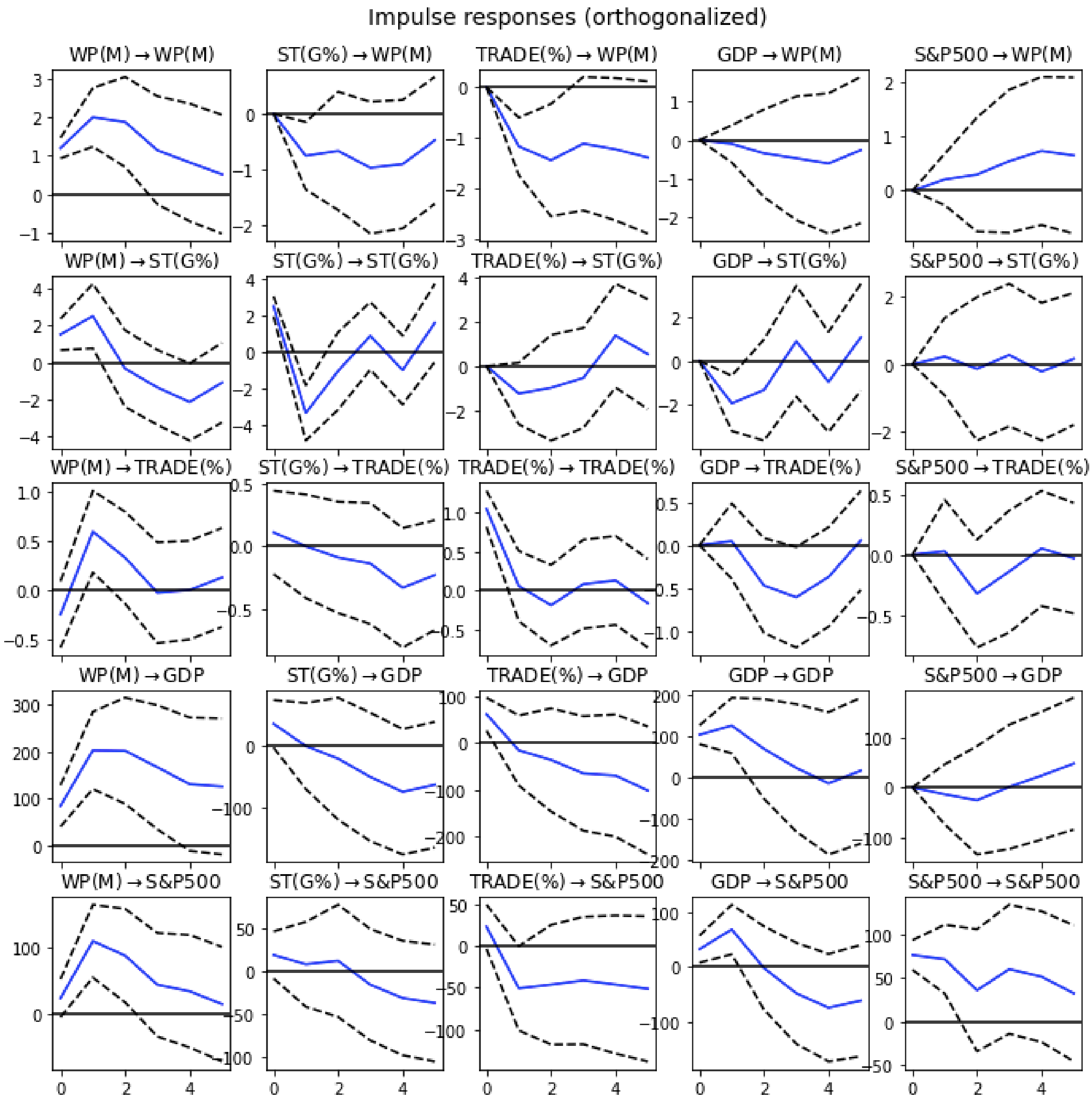

図 12: Impulse Response: [$"WP(M)", "ST(G\%)", "TRADE(\%)", "GDP", "S\&P500"$]

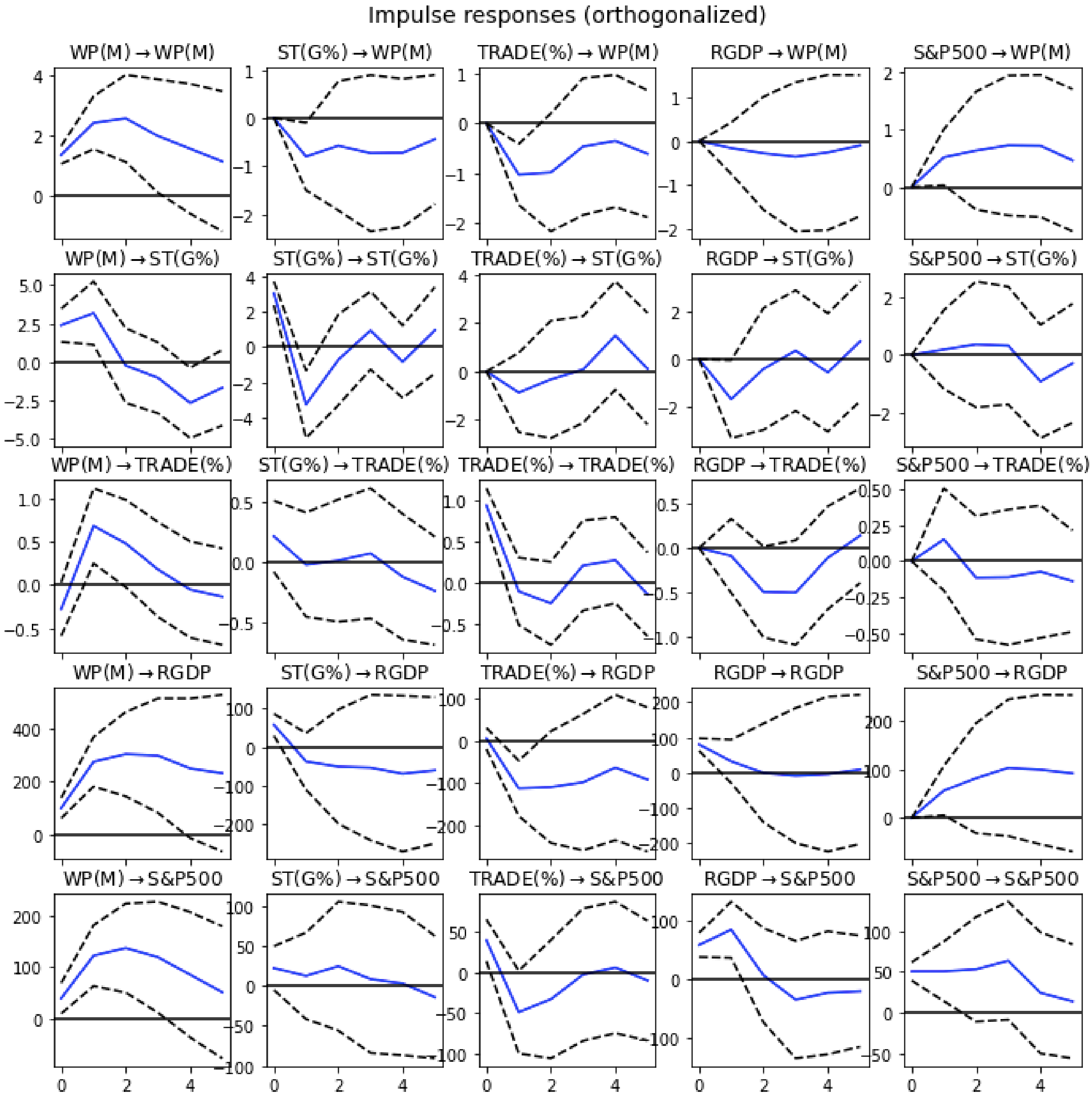

図 13: Impulse Response: $["WP(M)", "ST(G\%)", "TRADE(\%)", "RGDP", "S\&P500"]$

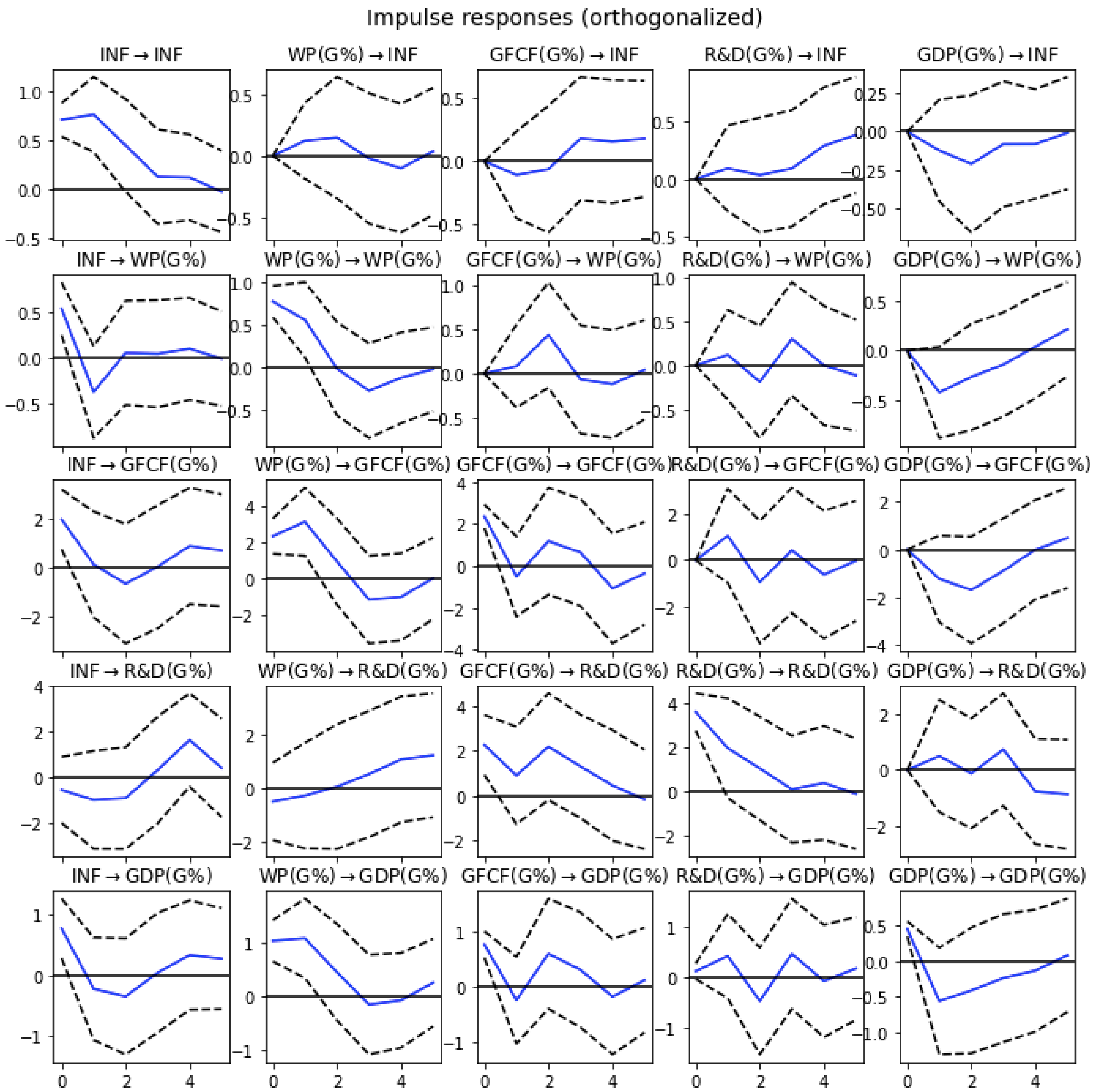

図 14: Impulse Response: ["INF","$WP(G\%)$","$R\&D(G\%)$","$GFCF(G\%)$","$GDP(G\%)$"]

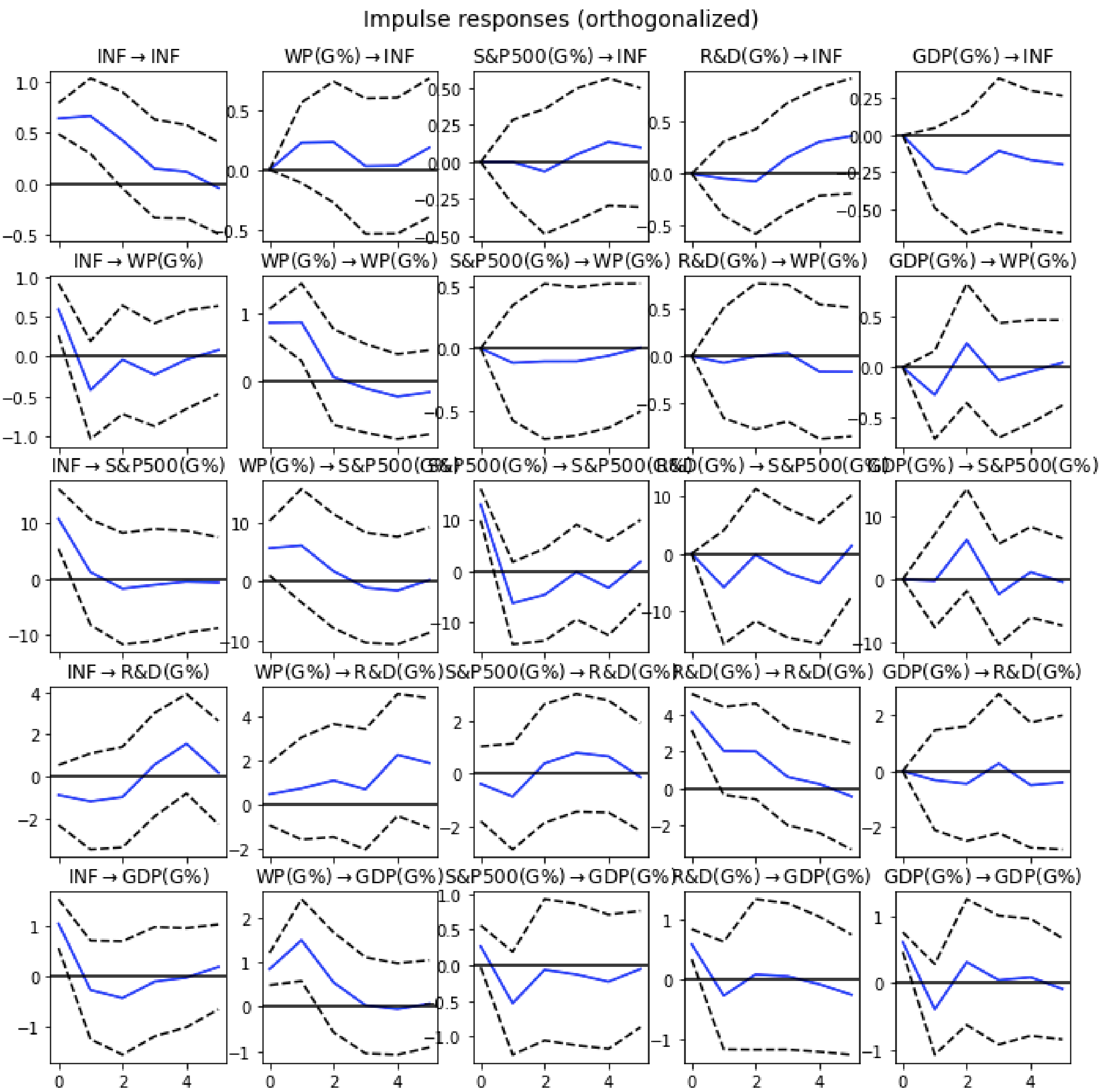

図 15: Impulse Response: [”INF”,”$WP(G\%)$”,”$R\&D(G\%)$”,”$S\&P500(G\%)$”,”$GDP(G\%)$”]

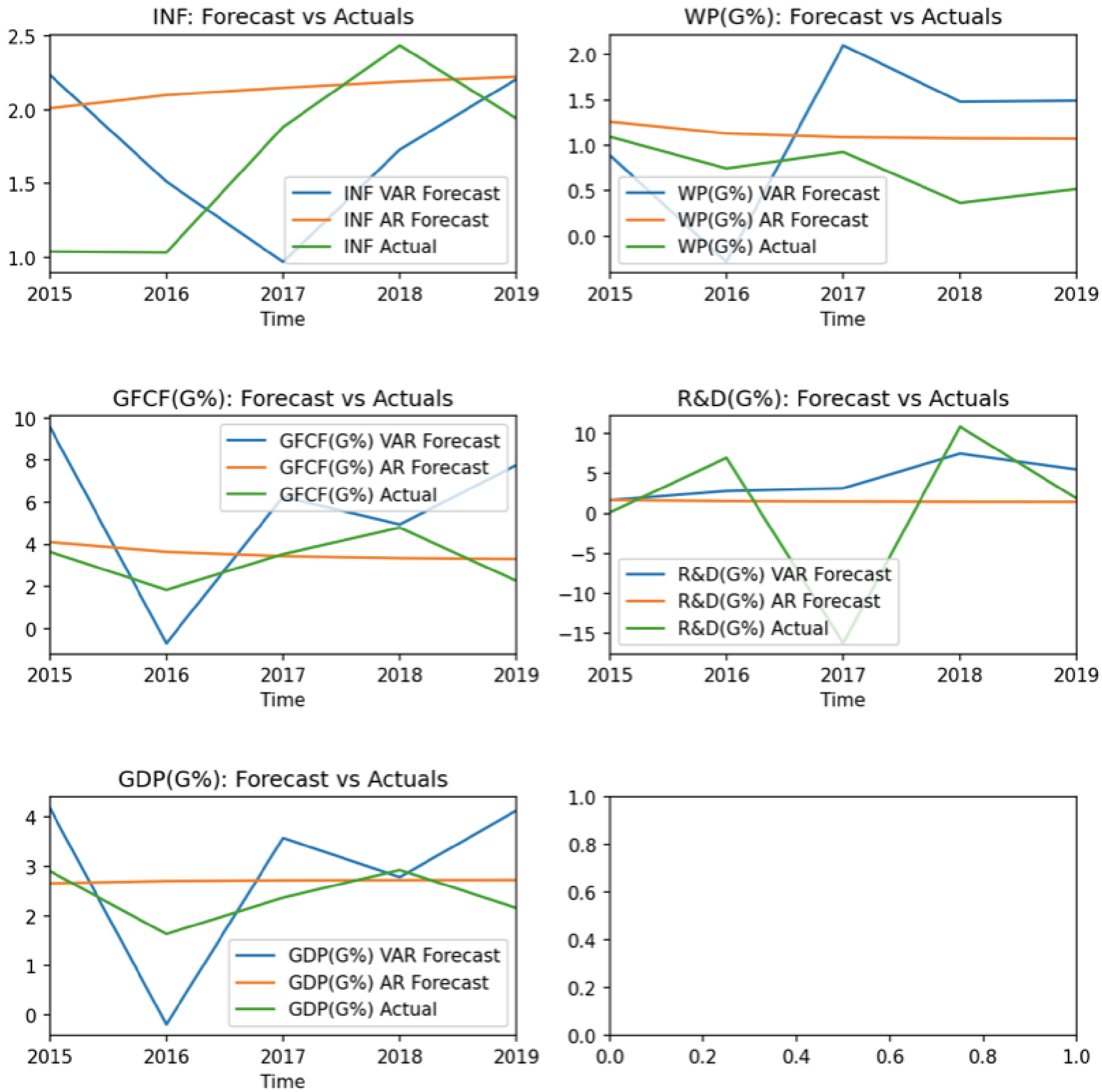

図 16: Forecast: $["INF","WP(G\%)","R\&D(G\%)","GFCF(G\%)","GDP(G\%)"]$

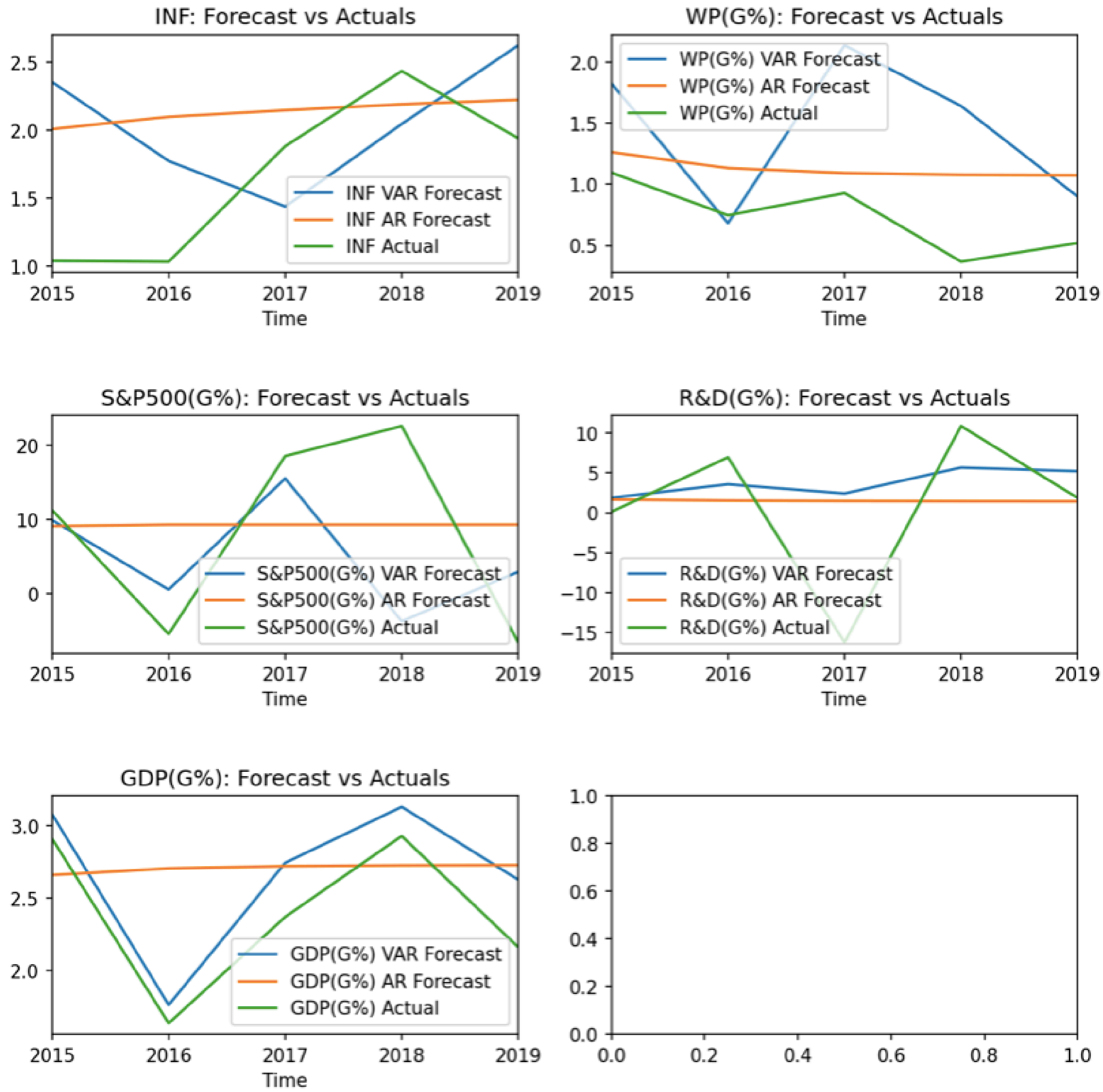

図 17: Forecast: ["INF","$WP(G\%)$","$R\&D(G\%)$","$S\&P500(G\%)$","$GDP(G\%)$"]

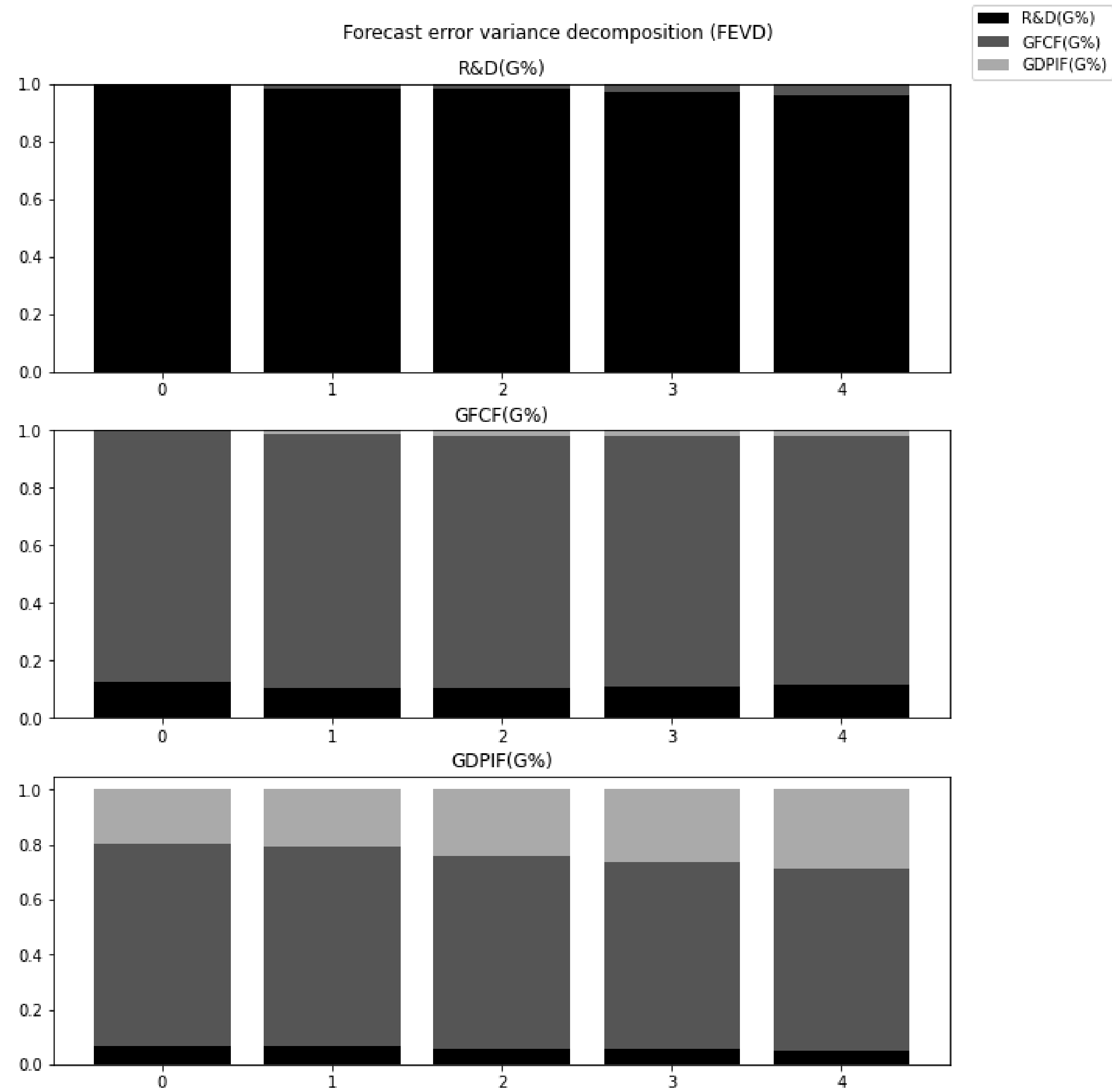

図 18: VD: [”$R\&D(G\%),$””$GFCF(G\%),$””$GDPIF(G\%)$”]

	TOTRESNS_x	COMPAPFFx_x	CMRMTSPLx_x	CLAIMSx_x	INDPRO_x	TB3SMFFM_x
TOTRESNS_y	1	0.0128	False	False	False	False
COMPAPFFx_y	False	1	False	False	0.0221	0.007
CMRMTSPLx_y	0.0002	False	1	False	0.044	False
CLAIMSx_y	False	False	False	1	0.0432	False
INDPRO_y	False	False	0.0018	0.0495	1	False
TB3SMFFM_y	False	0.0039	0.0244	False	0.0063	1

図 19: Granger causality of FRED-MD with selected variables

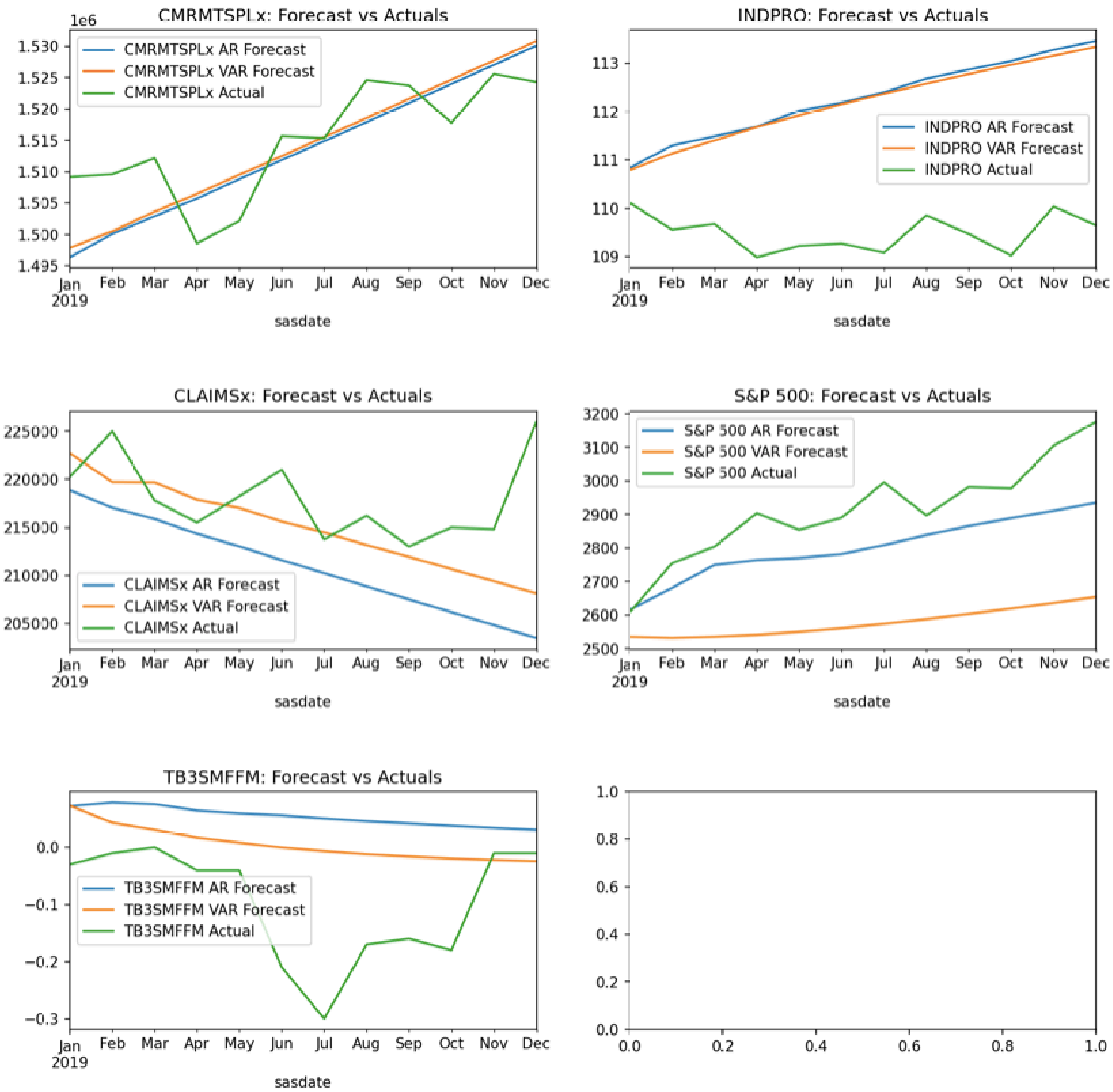

図 20: Forecast: [$"S\&P500"$,$"CLAIMSx"$,$"CMRMTSPLx"$,$"INDPRO"$,$"TB3SMFFM"$]

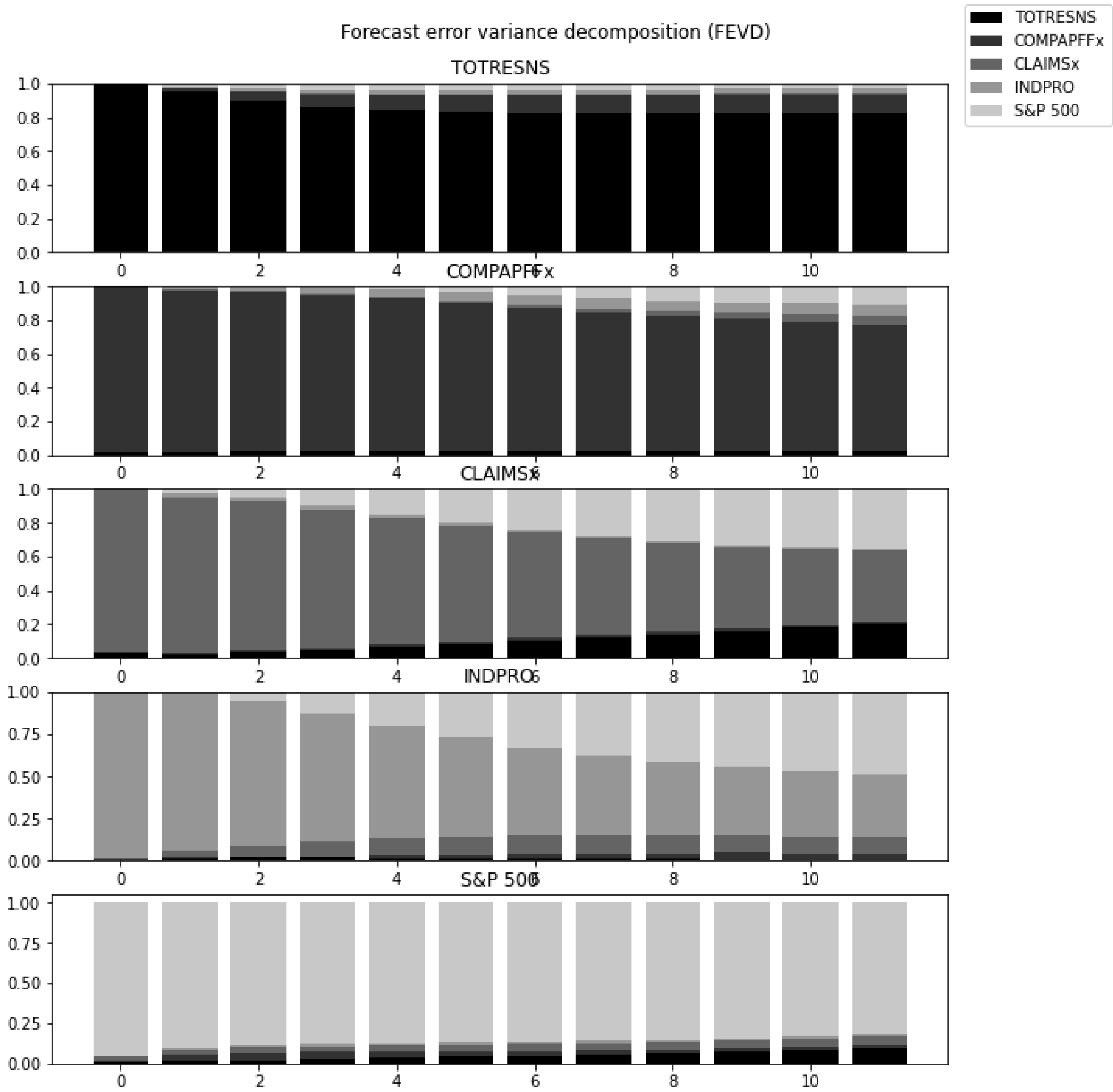

図 21: VD:["$TOTRESNS$","$COMPAPFFx$","$CLAIMSx$","$INDPRO$","$S\&P500$"]

Appendix A The name of variables

Yearly data of the U.S.A (S is for the stock market variables, B is for the bank variables, and E is for the economic variables.)		
Name	Type	Full-name of the variables
S&P500	S	S&P 500
S&P500(G%)	S	The percentage growth rate of S&P500
RIR(%)	E	Real interest rate
FCE	E	Final consumption expenditure (current US$/billion)
GDP	E	GDP (current US$/billion)
GDPDEF	E	GDP deflator (base year is 2005)
INF	E	Inflation, GDP deflator (annual %)
RGDP	E	Real GDP
GDPIF	E	GDP adjusted by the inflation rate
GDP(G%)	E	The percentage growth rate of GDP
GDP(G%P)	E	The percentage growth rate of GPD per capita
GDPIF(%)	E	The percentage growth rate of GDP adjusted by the inflation rate
GCE(%)	E	the ratio of government final consumption expenditure
RGFI	E	Real government final expenditure
RGFI(G%)	E	General government final consumption expenditure (annual % growth)
GDS(%)	B	the ratio of domestic savings to GDP. GDS is expected to contribute to financial development
RTS	B	Real Total Savings
ST(G%)	B	the percentage growth rate of real total savings
BCP(%)	B	the ratio of the domestic credit to private sector by banks to GDP
GDPP(%)	B	the ratio of the private sector's domestic credit to GDP
PSDC	B	the total amount of private sector's domestic credit
PSDC(G%)	B	the percentage growth rate of private sector's domestic credit
TRADE(%)	E	the ratio of trade to GDP

TRADE(T)	E	Total amount of TRADE
TRADE(G%)	E	the percentage growth rate of TRADE(T)
BM(%)	B	Broad money
BM(T)	B	the total amount of the broad money
BM(G%)	B	the percentage growth rate of the broad money
POP(TM)	E	the total population(million)
POP(1564M)	E	Population between the ages of 15 and 64(million)
POP(G%)	E	the percentage growth rate of total population
UNE(%)	E	Unemployment(%)
WP(M)	E	the number of working people(million)
WP(G%)	E	the percentage growth rate of working people
R&D(total)	E	the total R&D including both of defense and non-defense (billion)
R&D(G%)	E	the percentage growth of the total R&D
ISS	E	the number of the issued patents during the year
ISS(G%)	E	the percentage growth rate of the number of the issued patents
GCF(G%)	E	Gross capital formation (annual % growth)
GFCF(G%)	E	Gross fixed capital formation (annual % growth)
dummy	E	the dummy variable for the years: 1974, 1978, 2009
Turnover_Ratio	S	Turnover ratio is computed as the total value of traded stocks divided by market capitalization, and evaluate the market's liquidity.
SMT	S	the ratio of stock market total value traded to GDP. SMT zeroes in on how the market is active
Monthly data of the U.S.A: further information can be obtained at FRED-MD		
INDPRO		IP Index
DTCTHFNM		Total Consumer Loans and Leases Outstanding
CLAIMSx		Initial Claims
TB3SMFFM		3-Month Treasury C Minus Effective Federal Funds Rate
TOTRESNS		Total Reserves of Depository Institutions
COMPAPFFx		3-Month Commercial Paper Minus Effective Federal Funds Rate