Derivation of (14)

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Thank you for reading our paper. We provide interested reader with the proof of Eqn (14) as a supplemental information of our paper.

We define a function $F(x)$ as follows:

$$\begin{aligned} F(x) &\equiv \sum_{p=1,3,\cdots}^{\infty} a_p \left(x+\eta_{\mathrm{tx}}x^*\right)\left|x+\eta_{\mathrm{tx}}x^*\right|^{p-1} \\ &= \sum_{m=0}^{\infty} a_{2m+1}\left(x+\eta_{\mathrm{tx}}x^*\right)\left|x+\eta_{\mathrm{tx}}x^*\right|^{2m} \\ &= \sum_{m=0}^{\infty} a_{2m+1}F_{2m+1}(x), \end{aligned} \tag{1}$$

where the function $F_{2m+1}(x)$ is described as

$$F_{2m+1}(x) = \left(x+\eta_{\mathrm{tx}}x^*\right)\left|x+\eta_{\mathrm{tx}}x^*\right|^{2m} = \left(x+\eta_{\mathrm{tx}}x^*\right)^{m+1}\left(x^*+\eta_{\mathrm{tx}}^*x\right)^{m} \tag{2}$$

By using the binomial theorem, (2) can be rewritten as

$$\begin{aligned} F_{2m+1}(x) &= \left(\sum_{k_1=0}^{m+1}\binom{m+1}{k_1}x^{k_1}\left(\eta_{\mathrm{tx}}x^*\right)^{m+1-k_1}\right)\left(\sum_{k_2=0}^{m}\binom{m}{k_2}\left(\eta_{\mathrm{tx}}^*x\right)^{k_1}\left(x^*\right)^{m-k_2}\right) \\ &= \sum_{k_1=0}^{m+1}\sum_{k_2=0}^{m}\binom{m+1}{k_1}\binom{m}{k_2}x^{k_1}\left(\eta_{\mathrm{tx}}x^*\right)^{m+1-k_1}\left(\eta_{\mathrm{tx}}^*x\right)^{k_2}\left(x^*\right)^{m-k_2} \end{aligned} \tag{3}$$

On each term of double summation of (3), the sum of exponents of x and x^* is

$$k_1+(m+1-k_1)+k_2+(m-k_2) = 2m+1, \tag{4}$$

and each exponent of x and x^* ranges from 0 to $2m+1$. Thus, we can write the following expansion:

$$F_{2m+1}(x) = \sum_{q=0}^{2m+1} c_{2m+1-q,q}x^{2m+1-q}\left(x^*\right)^q, \tag{5}$$

where $c_{u,v}$ is a coefficient for $x^u\left(x^*\right)^v$. Thus, the function $F(x)$ can be written as

$$F(x) = \sum_{m=0}^{\infty}\sum_{q=0}^{2m+1} a_{2m+1}c_{2m+1-q,q}x^{2m+1-q}\left(x^*\right)^q. \tag{6}$$

To derive Eqn (16) of the original paper, we define an operator $[x^u\left(x^*\right)^v]f(x)$ that extract a coefficient from an arbitrary power series that have coefficients $d_{u,v}$ as follows:

$$[x^u\left(x^*\right)^v]\left(\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}d_{u,v}x^u\left(x^*\right)^v\right) \equiv d_{u,v}. \tag{7}$$

Also, on an arbitrary power series $f(x)$, we can write the following equation:

$$[x^u\left(x^*\right)^v]f(x) = [x^{u+u'}\left(x^*\right)^{v+v'}]\left(x^{u'}\left(x^*\right)^{v'}f(x)\right). \tag{8}$$

From (6), when $u<0$ or $v<0$, we obtain

$$[x^u\left(x^*\right)^v]F(x) = 0. \qquad (u<0 \;\vee\; v<0) \tag{9}$$

Also, from (5), we obtain

$$[x^u (x^*)^v]F_{2m+1}(x) = \begin{cases} c_{u,v} & (u+v=2m+1) \\ 0 & (\text{otherwise}) \end{cases} \tag{10}$$

Thus, when $u \geq 0$ and $v \geq 0$, we obtain

$$[x^u (x^*)^v]F(x) = \sum_{m=0}^{\infty} a_{2m+1} \left([x^u (x^*)^v]F_{2m+1}(x)\right) = a_{u+v}[x^u (x^*)^v]F_{u+v}(x) = a_{u+v}c_{u,v}, \tag{11}$$

and then $c_{u,v} = [x^u (x^*)^v]F_{u+v}(x)$. When $u+v=1$ (i.e. $(u,v) \in \{(1,0),(0,1)\}$), we can derive the following equaiton:

$$c_{u,v} = [x^u (x^*)^v]F_1(x) = [x^u (x^*)^v] \left(x + \eta_{\mathrm{tx}} x^*\right) = \begin{cases} 1, & (u=1 \;\vee\; v=0) \\ \eta_{\mathrm{tx}}. & (u=0 \;\vee\; v=1) \end{cases} \tag{12}$$

Also from (2), we can write the equation

$$\begin{aligned} c_{u,v} &= [x^u (x^*)^v]F_{u+v}(x) && (13) \\ &= [x^u (x^*)^v] \Big(\left|x + \eta_{\mathrm{tx}} x^*\right|^2 F_{u+v-2}(x) \Big) \\ &= [x^u (x^*)^v] \Big(\eta_{\mathrm{tx}}^* x^2 F_{u+v-2} + (1+|\eta_{\mathrm{tx}}|^2) x x^* F_{u+v-2} + \eta_{\mathrm{tx}} (x^*)^2 F_{u+v-2} \Big) \\ &= \eta_{\mathrm{tx}}^* [x^u (x^*)^v] \left(x^2 F_{u+v-2} \right) + (1+|\eta_{\mathrm{tx}}|^2)[x^u (x^*)^v] \left(x x^* F_{u+v-2} \right) + \eta_{\mathrm{tx}} [x^u (x^*)^v] \Big((x^*)^2 F_{u+v-2} \Big) && (14) \end{aligned}$$

because of the distributive property of the extraction operator of (7). By using (8) to each term of (14), we obtain the following equation:

$$\begin{aligned} c_{u,v} &= \eta_{\mathrm{tx}}^* [x^{u-2} (x^*)^v] \left(F_{u+v-2}\right) + (1+|\eta_{\mathrm{tx}}|^2)[x^{u-1} (x^*)^{v-1}] \left(F_{u+v-2}\right) + \eta_{\mathrm{tx}} [x^u (x^*)^{v-2}] \left(F_{u+v-2}\right) \\ &= \eta_{\mathrm{tx}}^* c_{u-2,v} + (1+|\eta_{\mathrm{tx}}|^2) c_{u-1,v-1} + \eta_{\mathrm{tx}} c_{u,v-2}. && (15) \end{aligned}$$

To summarize (6), (9), (12), and (15), Eqn (14) of our paper are derived.