Honors Linear Algebra – Midterm 2 Solutions

Wednesday, April 11 2018

Multiple choice

1. For each of the three matrices, the columns are orthogonal to one another, but in the matrix in C), the columns do not have unit magnitude, so the matrix in C) is not orthogonal. The correct answer is **answer C**.

2. Using a cofactor expansion along the first row, we readily find $D = -3(5 \times 6 - 1 \times 2) = -3 \times 28 = -84$. The correct answer is **answer E**.

3. Here, we are looking for $\hat{\mathbf{x}} = (c_1, c_2)$ such that $||A\hat{\mathbf{x}} - \mathbf{b}||$ is minimized, with

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} \quad , \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

We solve this problem with the usual least square method:

$$A^T A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \quad \Rightarrow \quad (A^T A)^{-1} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

We therefore find

$$\hat{\mathbf{x}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

The correct answer is **answer C**.

4. The differential equations can be written as

$$\frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

with

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$$

The eigenvalues of A are solutions of

$$\begin{vmatrix} -1-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0 \quad \Leftrightarrow \quad (1+\lambda)(2-\lambda) + 4 = 0 \quad \Leftrightarrow \quad \lambda^2 - \lambda - 6 = 0$$

Hence A has two eigenvalues:$\lambda_1 = -2$ and $\lambda_2 = 3$.

The row reduced echelon form of $A - \lambda_1 I$ is

$$R_1 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

so $x_1 = (-2, 1)$ is an eigenvector for the eigenvalue λ_1.

The row reduced echelon form of $A - \lambda_2 I$ is

$$R_2 = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

so $x_2 = (1, 2)$ is an eigenvector for the eigenvalue λ_2.

We conclude that the general solution of the differential equation has the form

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$$

with c_1 and c_2 constants. The correct answer is **Answer C**.

5. A direction vector for the line is $\mathbf{a} = (1, -2, 1)$. Hence, the matrix for the projection is

$$P = \frac{1}{6} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

The correct answer is **Answer E**.

True or False

1. Any projection matrix can be written as $P = A(A^TA)^{-1}A^T$. $P^2 = A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T = A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T = A(A^TA)^{-1}A^T = P$.

   The statement is **True**.

2. If $P = A(A^TA)^{-1}A^T$, then $P^T = (A^T)^T((A^TA)^{-1})^TA^T = A(A^TA)^{-1}A^T = P$. So the statement is **True**.

3. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A + B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

   We have $\det A = 1$, $\det B = 0$, and $\det(A + B) = 2$, so $\det(A + B) \neq \det A + \det B$. The statement is **False**.

4. Let λ be an eigenvalue of AB, with eigenvector $\mathbf{x}$: $AB\mathbf{x} = \lambda\mathbf{x}$. Then $BA(B\mathbf{x}) = \lambda(B\mathbf{x})$. So λ is an eigenvalue of BA, with eigenvector $B\mathbf{x}$.

   Let μ be an eigenvalue of BA, with eigenvector $\mathbf{y}$: $BA\mathbf{y} = \mu\mathbf{y}$. Then $AB(A\mathbf{y}) = \lambda(A\mathbf{y})$. So μ is an eigenvalue of AB, with eigenvector $A\mathbf{y}$.

   The statement is **True**.

5. Let A and B be two orthogonal matrices: $A^TA = I$ and $B^TB = I$. Let us see if AB is orthogonal as well: $(AB)^TAB = B^TA^TAB = B^TB = I$. The statement is **True**.

Problem 1

A basis for the plane $x - 2y - z = 0$ is a basis for the nullspace of the matrix $A = [1\ -2\ -1]$. The matrix is already in row reduced echelon form, so a basis for the nullspace is readily found from the special solutions: $\mathbf{u} = (2, 1, 0)$ and $\mathbf{v} = (1, 0, 1)$.

$\mathbf{u}$ and $\mathbf{v}$ are not orthogonal: $\mathbf{u}^T\mathbf{v} = 2$. We construct an orthogonal basis from $\mathbf{u}$ and $\mathbf{v}$ with the Gram-Schmidt process. Let $\mathbf{U} = \mathbf{u}$. An orthogonal vector to $\mathbf{U}$ is

$$\mathbf{V} = \mathbf{v} - \frac{\mathbf{U}^T\mathbf{v}}{||\mathbf{U}||^2}\mathbf{U}$$

We have

$$\mathbf{U}^T\mathbf{v} = 2\ \ ,\ \ ||\mathbf{U}||^2 = 5$$

so

$$\mathbf{V} = (1, 0, 1) - \frac{2}{5}(2, 1, 0) = (\frac{1}{5}, -\frac{2}{5}, 1)$$

And we conclude that $\mathbf{U} = (2, 1, 0)$ and $\mathbf{V} = (\frac{1}{5}, -\frac{2}{5}, 1)$ is an orthogonal basis for the plane.

To find an orthonormal basis, we simply divide $\mathbf{U}$ and $\mathbf{V}$ by their magnitude:

$$\mathbf{q}_1 = \frac{\mathbf{U}}{||\mathbf{U}||} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) \qquad q_2 = \frac{\mathbf{V}}{||\mathbf{V}||} = \sqrt{\frac{5}{6}}\left(\frac{1}{5}, -\frac{2}{5}, 1\right) = \left(\frac{1}{\sqrt{30}}, -\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}\right)$$

Problem 2

(A)

$$(A^TA)^k = (A^TA)(A^TA)\dots(A^TA)$$

where there are k terms in the product. Let us focus on the first two terms in the multiplication for the time being. Since, according to the statement of the problem, $A^TA = AA^T$,

$$(A^TA)(A^TA) = A^T(AA^T)A = (A^T)^2A^2$$

Let us now see what would happen with the third term in the multiplication:

$$(A^TA)(A^TA)(A^TA) = (A^T)^2A^2(A^TA) = (A^T)^2A(AA^T)A = (A^T)^2(AA^T)A^2 = (A^T)^3A^3$$

Repeating the process for the $k - 3$ remaining terms, we find

$$(A^TA)^k = (A^T)^kA^k = 0$$

(B) A^TA is a symmetric matrix, which can be diagonalized:

$$A^TA = Q\Lambda Q^T$$

where Q is an orthogonal matrix and Λ is a diagonal matrix. Now,

$$(A^TA)^k = Q\Lambda^kQ^T = 0 \quad \Leftrightarrow \quad \Lambda^k = 0$$

Λ^k is a diagonal matrix whose entries are λ_i^k. So it must be that all the $\lambda_i = 0$, so $\Lambda = 0$, which implies $A^TA = 0$.

Problem 3

(A) By the fundamental theorem of algebra, $V^\perp$ is the nullspace of A^T

$$A^T = \begin{bmatrix} 2 & 5 & 0 \\ 0 & 1 & 1 \\ 4 & 9 & -1 \end{bmatrix}$$

The special solutions of $A^T\mathbf{x} = \mathbf{0}$ provide a basis for the nullspace of that matrix. Let us therefore compute the row reduced echelon form of A^T:

$$\begin{bmatrix} 2 & 5 & 0 \\ 0 & 1 & 1 \\ 4 & 9 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & 5 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & 5 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{5}{2} & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{5}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

There is one free column, and the special solution is $\mathbf{u} = (5/2, -1, 1)$. This vector is a basis for $V^\perp$.

(B) Let $\mathbf{e}$ be the projection of $\mathbf{b}$ onto $V^\perp$. We know we can write $\mathbf{b} = \mathbf{p} + \mathbf{e}$ where $\mathbf{p}$ is in V. The projection $\mathbf{e}$ of $\mathbf{b}$ onto $V^\perp$ can be calculated with $\mathbf{u}$:

$$\mathbf{e} = \frac{\mathbf{u}^T\mathbf{b}}{||\mathbf{u}||^2}\mathbf{u}$$

$$\mathbf{u}^T\mathbf{b} = \begin{bmatrix} \frac{5}{2} & -1 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ -12 \\ 1 \end{bmatrix} = 33 \qquad ||\mathbf{u}||^2 = \frac{25}{4} + 1 + 1 = \frac{33}{4} \quad \Rightarrow \frac{\mathbf{u}^T\mathbf{b}}{||\mathbf{u}||^2} = 4$$

Thus,

$$\mathbf{e} = (10, -4, 4) \text{ so that } \mathbf{p} = \mathbf{b} - \mathbf{e} = (8, -12, 1) - (10, -4, 4) = (-2, -8, -3) \in V$$

Therefore, we can finally write the desired decomposition

$$\mathbf{b} = (-2, -8, -3) + (10, -4, 4)$$

Problem 4

(A) The eigenvalues λ of A satisfy

$$-\lambda(1-\lambda) - 2 = 0 \quad \Leftrightarrow \quad (\lambda+1)(\lambda-2) = 0$$

so A has two eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = 2$

Eigenvectors corresponding to the eigenvalue λ_1 are in the nullspace of

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

Thus, an eigenvector is $\mathbf{x}_1 = (-1, 1)$.

Eigenvectors corresponding to the eigenvalue λ_2 are in the nullspace of

$$\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

Thus, an eigenvector is $\mathbf{x}_2 = (2, 1)$.

(B)

$$\mathbf{u_{n+1}} = \begin{bmatrix} v_{n+2} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_{n+1} \\ v_n \end{bmatrix} = A\mathbf{u_n}$$

Hence,

$$M = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = A$$

(C) From the previous question, we can write

$$\mathbf{u}_{13} = \begin{bmatrix} v_{14} \\ v_{13} \end{bmatrix} = A^{13}\mathbf{u}_0 = A^{13}\begin{bmatrix} v_1 \\ v_0 \end{bmatrix}$$

From the first question, we can write

$$A = S\Lambda S^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

so that

$$A^{13} = S\Lambda^{13}S^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2^{13} \end{bmatrix}\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} + \frac{2^{14}}{3} & \frac{2}{3} + \frac{2^{14}}{3} \\ \frac{1}{3} + \frac{2^{13}}{3} & -\frac{2}{3} + \frac{2^{13}}{3} \end{bmatrix}$$

We conclude that

$$\begin{bmatrix} v_{14} \\ v_{13} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} + \frac{2^{14}}{3} & \frac{2}{3} + \frac{2^{14}}{3} \\ \frac{1}{3} + \frac{2^{13}}{3} & -\frac{2}{3} + \frac{2^{13}}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} + \frac{2^{14}}{3} \\ \frac{1}{3} + \frac{2^{13}}{3} \end{bmatrix}$$

Hence

$$v_{14} = \frac{2^{14} - 1}{3} = \frac{16384 - 1}{3} = 5461$$

Problem 5

$J + A$ is not invertible if and only if there exists $\mathbf{x} \neq \mathbf{0}$ such that

$$(J + A)\mathbf{x} = \mathbf{0} \quad \Leftrightarrow \quad J\mathbf{x} = -A\mathbf{x} \quad \Leftrightarrow \quad A^{-1}J\mathbf{x} = -\mathbf{x} \quad \Leftrightarrow \quad A^T J\mathbf{x} = -\mathbf{x}$$

where we used the fact that A was orthogonal for the last step. Hence, we showed that $J + A$ is not invertible if and only if -1 is an eigenvalue of $A^T J$, i.e. if and only if -1 is an eigenvalue of JA, since J is symmetric.

-1 is an eigenvalue of JA if and only if

$$\det(JA + I) = 0 \quad \Leftrightarrow \quad \begin{vmatrix} \sum_{i=1}^n a_{i1} + 1 & \sum_{i=1}^n a_{i2} & \cdots & \cdots & \sum_{i=1}^n a_{in} \\ \sum_{i=1}^n a_{i1} & \sum_{i=1}^n a_{i2} + 1 & \sum_{i=1}^n a_{i3} & \cdots & \sum_{i=1}^n a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \sum_{i=1}^n a_{in} \\ \sum_{i=1}^n a_{i1} & \sum_{i=1}^n a_{i2} & \cdots & \sum_{i=1}^n a_{i(n-1)} & \sum_{i=1}^n a_{in} + 1 \end{vmatrix} = 0$$

We proceed with this determinant exactly as we did in Problem 3 of Homework 7, to find

$$\det(JA + I) = \sum_{i,j} a_{ij} + 1$$

Hence, $(J + A)$ is invertible if and only if $\sum_{i,j} a_{ij} \neq -1$.