Solving systems of linear differential equations

I) Brief review of standard differential equations

1) First order, linear ordinary differential equations

Find the solution $y(t)$ to $\begin{cases} \frac{dy}{dt}(t) = \lambda y(t) \\ y(0) = y_0 \end{cases}$

Any function $y(t)$ of the form $y(t) = Ce^{\lambda t}$ solves the first equation: $\frac{dy}{dt} = \lambda C e^{\lambda t} = \lambda y(t)$

Uniqueness is obtained by imposing the initial condition:

$$y(0) = C = y_0 \Rightarrow y(t) = y_0 e^{\lambda t}$$

2) Second order linear ordinary differential equation

- Find the solution $y(t)$ to $\begin{cases} \frac{d^2y}{dt^2} + \omega^2 y = 0 \\ y(0) = y_0 \\ \left.\frac{dy}{dt}\right|_{t=0} = y_0' \end{cases}$ $\omega^2 > 0$

General solution to the equation: $y(t) = A\cos\omega t + B\sin\omega t$

Uniqueness is obtained by imposing the initial conditions:

$$y(0) = A = y_0$$
$$y'(0) = \omega B = y_0'$$

so the solution is:

$$y(t) = y_0 \cos\omega t + \frac{y_0'}{\omega} \sin\omega t$$

- Find the solution $y(t)$ to

$$\begin{cases} \frac{d^2y}{dt^2} - \lambda^2 y = 0 \\ y(0) = y_0 \qquad \lambda^2 > 0 \\ \left.\frac{dy}{dt}\right|_{t=0} = y_0' \end{cases}$$

The general solution to the equation is

$$y(t) = Ae^{\lambda t} + Be^{-\lambda t}$$

The initial conditions imply:

$$\begin{cases} y_0 = A + B \\ y_0' = \lambda(A - B) \end{cases} \iff \begin{cases} A = \frac{1}{2}\left(y_0 + \frac{y_0'}{\lambda}\right) \\ B = \frac{1}{2}\left(y_0 - \frac{y_0'}{\lambda}\right) \end{cases}$$

so the solution is

$$y(t) = \frac{1}{2}\left(y_0 + \frac{y_0'}{\lambda}\right)e^{\lambda t} + \frac{1}{2}\left(y_0 - \frac{y_0'}{\lambda}\right)e^{-\lambda t}$$

The two cases $\frac{d^2y}{dt^2} + \omega^2 y = 0, \omega^2 > 0$ and $\frac{d^2y}{dt^2} \lambda^2 y = 0, \lambda^2 > 0$

can be written as one:

$$\frac{d^2y}{dt^2} - \lambda^2 y = 0, \quad \lambda^2 \neq 0$$

Indeed, if $\lambda^2 > 0$, let $\omega^2 = -\lambda^2$ to obtain

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

and $y(t) = Ae^{\lambda t} + Be^{-\lambda t} = Ae^{i\omega t} + Be^{-i\omega t}$

$$= \frac{1}{2}\left(y_0 + \frac{y_0'}{i\omega}\right)e^{i\omega t} + \frac{1}{2}\left(y_0 - \frac{y_0'}{i\omega}\right)e^{-i\omega t}$$

$$= y_0 \cos(\omega t) + \frac{y_0'}{\omega}\sin(\omega t)$$ as we had found.

3) Coupled linear differential equations

Find the solution to the system of differential equations

$$\begin{cases} \frac{dy}{dt} = y + 2z \\ \frac{dz}{dt} = 2y + z \\ y(0) = y_0, \quad z(0) = z_0 \end{cases}$$

Adding the first two equations, we find

$$\frac{d}{dt}(y+z) = 3(y+z)$$

whose general solution is $(y+z)(t) = Ce^{3t}$

Subtracting the second equation from the first one, we find

$$\frac{d}{dt}(y-z) = -(y-z)$$

whose general solution is $(y-z)(t) = De^{-t}$

We thus have $y(t) = \frac{1}{2}\left(Ce^{3t} + De^{-t}\right)$

$$z(t) = \frac{1}{2}\left(Ce^{3t} - De^{-t}\right)$$

The initial conditions give the conditions on C and D:

$$\begin{cases} y_0 = \frac{1}{2}(C+D) \\ z_0 = \frac{1}{2}(C-D) \end{cases} \Rightarrow \begin{cases} C = y_0 + z_0 \\ D = y_0 - z_0 \end{cases}$$

so the functions y and z are given by

$$y(t) = \frac{1}{2}\left[(y_0+z_0)e^{3t} + (y_0-z_0)e^{-t}\right]$$

$$z(t) = \frac{1}{2}\left[(y_0+z_0)e^{3t} - (y_0-z_0)e^{-t}\right]$$

Connection with linear algebra and eigenvalues

In the case above, we constructed the proper combinations $y+z$ and $y-z$ by inspection. In this lecture, we will learn the robust algorithm to construct these combinations, for coupled linear systems which can be much larger and complicated.

Here is the link with linear algebra.

Let $\vec{u}(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$

Our system can be written as

$$\frac{d\vec{u}}{dt} = A\vec{u}$$

with $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

The eigenvalues of A are:
$\lambda_1 = 3$, with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda_2 = -1$, with eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

We recognize here the combination $y(t) + z(t)$ for e^{3t}, and $y(t) - z(t)$ for e^{-t}.

II) Solving coupled, linear systems of differential equations

1) First order equations

Consider a system of n linear differential equations which can be written in the form

$$\frac{d\vec{u}}{dt} = A\vec{u}, \quad \vec{u}(0) = \vec{u}_0$$

where $\vec{u} = \begin{bmatrix} y(t) \\ z(t) \\ \vdots \\ h(t) \end{bmatrix}$, and A an $n \times n$ matrix.

Let $\vec{x}$ be an eigenvector of A, with eigenvalue λ.
Consider $\vec{u} = \vec{x} e^{\lambda t}$
We have $\frac{d\vec{u}}{dt} = \lambda \vec{x} e^{\lambda t} = A\vec{u}$: $\vec{u}$ is a solution to the system!

It does however not solve the problem we are trying to solve, because $\vec{u}(0) = \vec{x} \underset{\text{in general}}{\neq} \vec{u}_0$

This issue is easy to address when A is diagonalizable, i.e when A has n linearly independent eigenvectors, which is the only situation we will cover in this course.

In that case, we can indeed expand $\vec{u}(0) = \vec{u}_0$ in the basis of eigenvectors:

$$\underline{\vec{u}_0 = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \dots + c_n \vec{x}_n}$$

The general solution to $\frac{d\vec{u}}{dt} = A\vec{u}$ is then the linear combination

$$\underline{\vec{u} = c_1 e^{\lambda_1 t} \vec{x}_1 + c_2 e^{\lambda_2 t} \vec{x}_2 + \dots + c_n e^{\lambda_n t} \vec{x}_n}$$

Indeed, $\frac{d\vec{u}}{dt} = \lambda_1 c_1 e^{\lambda_1 t} \vec{x}_1 + \lambda_2 c_2 e^{\lambda_2 t} \vec{x}_2 + \dots + \lambda_n c_n e^{\lambda_n t} \vec{x}_n$
$= A\vec{u}$

and by construction, $\vec{u}(0) = \vec{u}_0$.

Example: Solve the system of differential equations

$$\begin{cases} \frac{dw}{dt} = -2w + y - 2z \\ \frac{dy}{dt} = w - 2y + 2z \\ \frac{dz}{dt} = 3w - 3y + 5z \\ w(0) = -2, \quad y(0) = 2, \quad z(0) = 4 \end{cases}$$

This system can be rewritten as

$$\frac{d\vec{u}}{dt} = A\vec{u}$$

with $\vec{u}(t) = \begin{bmatrix} w(t) \\ y(t) \\ z(t) \end{bmatrix}$ and $A = \begin{bmatrix} -2 & 1 & -2 \\ 1 & -2 & 2 \\ 3 & -3 & 5 \end{bmatrix}$

The eigenvalues of A are 3, with eigenvector $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$, and -1, with eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

We have $\vec{u}(0) = \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} = 1 \cdot \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

Hence, the solution is:

$$\vec{u}(t) = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{-t}$$

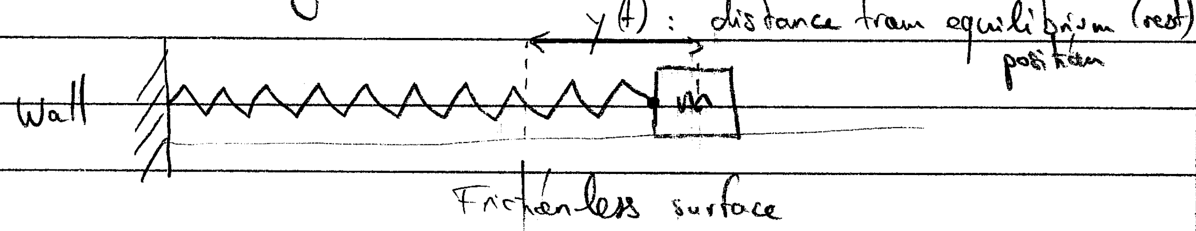
i.e. $w(t) = -e^{3t} - e^{-t}$

$y(t) = e^{3t} + e^{-t}$

$z(t) = 3e^{3t} + e^{-t}$

2) Second order equations

The motion of a mass m attached to a spring and sliding on a frictionless surface :



is given by:

$$\frac{d^2y}{dt^2} + \frac{k}{m} y = 0$$

where k is called the spring constant, and determines the strength of the spring. k large means that the spring is strong.

In physics, any equation of the type $\frac{d^2y}{dt^2} + \omega^2 y = 0$ is said to be the equation of an harmonic oscillator.

Suppose now that the surface is not frictionless, so the mass is subject to friction. The equation for an harmonic oscillator is then modified to:

$$\frac{d^2y}{dt^2} + \nu \frac{dy}{dt} + \frac{k}{m} y = 0$$

where ν characterizes the strength of friction.

This equation is often called the equation of the "Damped Harmonic Oscillator". It turns out that it appears in many areas of science. Let us now solve it with the tools we just learned:

$$\frac{d^2y}{dt^2} + \nu \frac{dy}{dt} + \frac{k}{m} y = 0 \iff \begin{cases} \frac{dy}{dt} = y'(t) \\ \frac{dy'}{dt} = -\nu y'(t) - \frac{k}{m} y(t) \end{cases}$$

Introducing $\vec{u}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$, the system can be rewritten as

$$\frac{d\vec{u}}{dt} = A\vec{u} \quad , \text{ with } A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\nu \end{bmatrix}$$

The eigenvalues λ_1 and λ_2 of A satisfy

$$\lambda^2 + \nu\lambda + \frac{k}{m} = 0$$

with eigenvectors $\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$

Hence, the general solution to $\frac{d^2y}{dt^2} + \nu\frac{dy}{dt} + \frac{k}{m} y = 0$ is

$$y(t) = Ce^{\lambda_1 t} + De^{\lambda_2 t}$$

with λ_1 and λ_2 the roots of $\lambda^2 + \nu\lambda + \frac{k}{m} = 0$ and C and D to be determined from initial conditions on y and y'.

We see that for the equation $\frac{d^2y}{dt^2} - \lambda^2 y = 0$ on page 2, the quadratic equation is $x^2 - \lambda^2 = 0$, with roots $\pm\lambda$, so the general solution is

$$y(t) = Ce^{\lambda t} + De^{-\lambda t}$$

as mentioned on page 2.

III) The exponential of a matrix

It is really tempting to write the solution of $\frac{d\vec{u}}{dt} = A\vec{u}$, $\vec{u}(0) = \vec{u}_0$ as $\vec{u}(t) = \vec{u}_0 e^{At}$

Let us define e^{At} so that this makes sense.

1) Definition

Let A be a square matrix. e^{At} is the infinite series:

$$e^{At} = I + \sum_{n=1}^{\infty} \frac{(At)^n}{n!}$$

2) Properties

- The time derivative of e^{At} is $\frac{d}{dt}\left[e^{At}\right] = A + A^2 t + \frac{1}{2}A^3 t^2 + \dots$

$$= \underline{Ae^{At}}$$

- Let $\vec{x}$ be an eigenvector of A

$$e^{At}\vec{x} = \left(I + At + \frac{1}{2}A^2t^2 + \dots\right)\vec{x} = \left(1 + \lambda t + \frac{1}{2}(\lambda t)^2 + \dots\right)\vec{x}$$
$$= e^{\lambda t}\vec{x}$$

The eigenvectors of e^{At} are the same as the eigenvectors of A. <u>The corresponding eigenvalues are $e^{\lambda t}$</u>

- If A is diagonalizable, with eigenvectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$, then so is e^{At}:

$$e^{At} = \begin{bmatrix} \vec{x}_1 & \dots & \vec{x}_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} \vec{x}_1 & \dots & \vec{x}_n \end{bmatrix}^{-1}$$

i.e. $$\underline{e^{At} = X e^{\Lambda t} X^{-1}}$$

- If two matrices A and B commute, $AB = BA$, then

$$e^A e^B = e^{A+B}$$

As a result, $e^{At}e^{-At} = e^0 = I$

e^{At} always has the inverse e^{-At}, even when A is not diagonalizable.

• If A is antisymmetric, $e^{-At} = \left(e^{At}\right)^{-1}$ and $e^{-At} = e^{A^T t} = e^{(At)^T}$

Hence $\left(e^{At}\right)^T = \left(e^{At}\right)^{-1}$: e^{At} is orthogonal.