18.06 Exam II Professor Strang April 7, 2014

Solutions

1. (24 points total)

(a) (6 points) What matrix P projects every vector in $\mathbf{R}^3$ onto the line that passes through origin and $a = (3, 4, 5)$?

(b) (6 points) What is the nullspace of that matrix P?

(c) (6 points) What is the row space of P^2?

(d) (6 points) What is the determinant of P?

Solution.

(a) The projection of the vector $(1, 0, 0)$ onto the line $a = (3, 4, 5)$ is $(9/50, 12/50, 15/50)$. Similarly, the projections of vectors $(0, 1, 0)$ and $(0, 0, 1)$ are $(12/50, 16/50, 20/50)$ and $(15/50, 20/50, 25/50)$ correspondingly. These are the columns of the projection matrix:

$$P = \begin{bmatrix} 9/50 & 12/50 & 15/50 \\ 12/50 & 16/50 & 20/50 \\ 15/50 & 20/50 & 25/50 \end{bmatrix} = \begin{bmatrix} 9/50 & 6/25 & 3/10 \\ 6/25 & 8/25 & 2/5 \\ 3/10 & 2/5 & 1/2 \end{bmatrix}.$$

(b) The nullspace of P is 2-dimensional. It can be generated by the following two vectors orthogonal to $a = (3, 4, 5)$: $(-5/3, 0, 1)$ and $(-4/3, 1, 0)$.

(c) Row space of P^2 is the same as row space of P, since $P^2 = P$. Row space of P is generated by $a = (3, 4, 5)$.

(d) The projection is onto 1-dimensional space, therefore, the rank of matrix P must equal to 1. Therefore, the determinant of P is 0.

2. (25 points total)

(a) (11 points) Suppose $\widehat{x}$ is the best least squares solution to $Ax = b$ and $\widehat{y}$ is the best least squares solution to $Ay = c$.

Does this tell you the best least squares solution $\widehat{z}$ to $Az = b + c$? If so, what is the best $\widehat{z}$ and *why*?

(b) (7 points) If Q is an m by n matrix with orthonormal columns, find the best least squares solution $\widehat{x}$ to $Qx = b$.

(c) (7 points) If $A = QR$, where R is square invertible and Q is the same as in (b), find the least squares solution to $Ax = b$.

Solution.

(a) Denote by P the projection onto the column space of A. We have $A\widehat{x} = Pb$ and $A\widehat{y} = Pc$. That means $A\widehat{x} + A\widehat{y} = Pb + Pc = P(b + c)$. It follows that $\widehat{x} + \widehat{y}$ is the least squares solution for $A\widehat{z} = b + c$.

(b) The least squares solution can be written as $\widehat{x} = (Q^T Q)^{-1} Q^T b$. As Q is orthonormal, $Q^T Q = I$. Therefore, $\widehat{x} = Q^T b$. Alternatively, solving least squares means finding a solution to $Q^T Q\widehat{x} = Q^T b$. As $Q^T Q = I$, we see that $\widehat{x} = Q^T b$.

(c) The least squares solution can be written as $\widehat{x} = (A^T A)^{-1} A^T b = (R^T Q^T Q R)^{-1} R^T Q^T b$. As Q is orthonormal, $Q^T Q = I$. Therefore, $\widehat{x} = (R^T R)^{-1} R^T Q^T b$. As R is invertible, we get $\widehat{x} = (R^T R)^{-1} R^T Q^T b = R^{-1} (R^T)^{-1} R^T Q^T b = R^{-1} Q^T b$.

3. (25 points total)

(a) (17 points) Find the determinant of this matrix A (with an unknown x in 4 entries).

$$A = \begin{bmatrix} x & 1 & 0 & 0 \\ 2 & x & 2 & 0 \\ 0 & 3 & x & 3 \\ 0 & 0 & 4 & x \end{bmatrix} \qquad B = \begin{bmatrix} x & 1 & 0 & 1 \\ 2 & x & 2 & 0 \\ 0 & 3 & x & 3 \\ 0 & 0 & 4 & x \end{bmatrix}$$

You could use the big formula or the cofactor formula or possibly the pivot formula.

(b) (5 points) Find the determinant for matrix B which has an additional 1 in the corner. What new contribution to the determinant does this 1 make?

(c) (3 points) If M is any 3 by 3 matrix, let $f(x) = \det(xM)$. Find the derivative of f at $x = 1$.

Solution.

(a) Using the cofactor method we can expand the determinant of A as:

$$= x \det\left(\begin{bmatrix} x & 2 & 0 \\ 3 & x & 3 \\ 0 & 4 & x \end{bmatrix}\right) - 1 \det\left(\begin{bmatrix} 2 & 2 & 0 \\ 0 & x & 3 \\ 0 & 4 & x \end{bmatrix}\right).$$

We can calculate the 3 by 3 determinants by using any formula. The first one has determinant $x^3 - 18x$, and the second one $2x^2 - 24$. The determinant of A is $x^4 - 20x^2 + 24$.

(b)

By the cofactor formula one more term is added, which is equal

$$-1 \det\left(\begin{bmatrix} 2 & x & 2 \\ 0 & 3 & x \\ 0 & 0 & 4 \end{bmatrix}\right).$$

The 3 by 3 matrix is triangular, so its determinant is the product of the diagonal elements and is equal to 24. So $\det(B) = \det(A) - 24 = x^4 - 20x^2$.

(c)

For a 3 by 3 matrix $f(x) = \det(xM) = x^3 \det(M)$. The derivative $f'(x) = 3x^2 \det(M)$.

4. (26 points total)

(a) (6 points) Find the projection p of the vector b onto the column space of A.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 2 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

(b) (7 points) Use Gram-Schmidt to find an orthogonal basis q_1, q_2 for the column space of A.

(c) (6 points) Find the projection p of the same vector b onto the column space of the new matrix Q with columns q_1 and q_2.

(d) (7 points) True or False: The best least squares solution $\widehat{x}$ to $Ax = b$ is the same as the best least squares solution $\widehat{y}$ to $Qy = b$. Explain why.

Solution.

(a) By the formula, the projection is $A(A^TA)^{-1}A^Tb$:

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 2 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 2 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 9 & 9 \\ 9 & 14 \end{bmatrix}^{-1} \begin{bmatrix} 11 \\ 12 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 14/45 & -0.2 \\ -0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -13/45 & 0.4 \\ 2/9 & 0 \\ 19/45 & -0.2 \end{bmatrix} \begin{bmatrix} 11 \\ 12 \end{bmatrix} = \begin{bmatrix} 73/45 \\ 22/9 \\ 101/45 \end{bmatrix}.$$

(b) $q_1 = (1, 2, 2)$—the first column of A. The projection of $(3, 2, 1)$ onto $(1, 2, 2)$ is $(1, 2, 2)$, with an error vector $e = (2, 0, -1)$. Thus $q_2 = (2, 0, -1)$.

(c) Columns q_1 and q_2 span the same space as columns of A. Thus the projection must be the same as before.

(d) Matrices A and Q span the same column space. Dentoe the projection of b onto that space as p. The solution $\widehat{x}$ satisfies the equation: $A\widehat{x} = p$, the solution $\widehat{y}$ satisfies the equation $Q\widehat{y} = p$. Now $A = QR$, which means $\widehat{y} = R\widehat{x}$.