MATH-UA 140 - Linear Algebra

Lecture 2: Lengths and the Dot product

I| The dot product

1) Definition

If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are two vectors, then the dot product of $\vec{u}$ and $\vec{v}$ is the number written $\vec{u} \cdot \vec{v}$ defined by

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

NOTE: The result of the dot product operation is a scalar (i.e a number). In some textbooks, you might therefore find that the dot product is also called the scalar product. Be careful not to get confused between scalar product (i.e dot product, the topic of this lecture) and scalar multiplication (covered in the last lecture)

QUESTION: Can you guess the formula defining the dot product for two vectors in 2D? For two vectors in n-D?

Example: Let $\vec{u} = \begin{bmatrix} \sqrt{2} \\ 1 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} \sqrt{2} \\ 3 \\ 2 \end{bmatrix}$

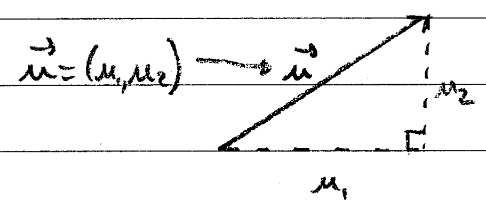
$\vec{u} \cdot \vec{v} = \sqrt{2} \times \sqrt{2} + 1 \times 3 + (-1) \times 2 = 3$

2) Length of a vector

Definition: The length (or magnitude) $\|\vec{u}\|$ of a vector $\vec{u}$ is the square root of $\vec{u}\cdot\vec{u}$:

$$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}}$$

- In 2D, the formula makes sense, considering the Pythagorean theorem:



According to the Pythagorean theorem, $\|\vec{u}\|^2 = u_1^2 + u_2^2$

$$\Rightarrow \|\vec{u}\| = \sqrt{u_1^2+u_2^2} = \sqrt{\vec{u}\cdot\vec{u}}$$

- It is fairly easy to convince yourself that the formula also makes sense in 3D, by using the Pythagorean theorem twice. I recommend the exercise!

- Consider the 4-D vector $(1,1,1,1)$ corresponding to the diagonal of a unit cube in four-dimensional space. It's length is $\sqrt{1^2+1^2+1^2+1^2} = 2$.
In general, the diagonal of a unit cube in n-dimensions has length $\sqrt{n}$

• <u>Definition</u>: A unit vector $\vec{u}$ is a vector whose length equals one: $\|\vec{u}\|=1$. In other words, $\vec{u}\cdot\vec{u}=1$.

For any nonzero vector $\vec{u}$, $\frac{\vec{u}}{\|\vec{u}\|}$ is the unit vector in the same direction as $\vec{u}$

<u>QUESTION</u>: Let $\vec{u}=\begin{bmatrix}2\\1\\1\end{bmatrix}$ and $\vec{v}=\begin{bmatrix}1\\4\\-1\end{bmatrix}$

Compute $\vec{w}=\vec{u}-\vec{v}$ and find the unit vector $\vec{a}$ in the same direction as $\vec{w}$

3) <u>Properties of the dot product</u>

Using the definition of the dot product, one can readily prove algebraically the following properties:

1. $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$ 2. $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$
3. $(c\vec{u})\cdot\vec{v}=c(\vec{u}\cdot\vec{v})=\vec{u}\cdot(c\vec{v})$ 4. $\vec{0}\cdot\vec{u}=0$

for any vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, and any scalar c.

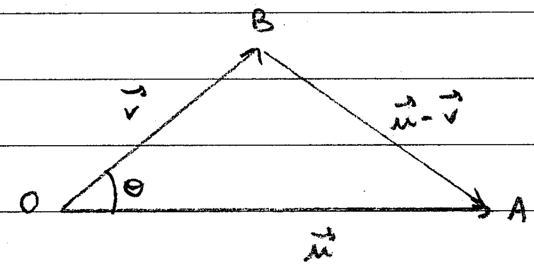
Note that Property 2 is the <u>commutative property</u> of the dot product: it does not matter in which order one takes the vectors for the dot product, one always obtains the same result.

It is a good exercise to verify that properties 2 and 3 are indeed true algebraically.

II] Geometric interpretation of the dot product

1) Geometric interpretation

Let us consider the vectors $\vec{u}$ and $\vec{v}$ below:



θ is the angle between the vectors $\vec{u}$ and $\vec{v}$. This angle is most easily defined and visualized when $\vec{u}$ and $\vec{v}$ have the same starting point, such as the point O in the figure.
We have the following theorem linking the angle θ to the vectors $\vec{u}$ and $\vec{v}$ and the dot product $\vec{u}\cdot\vec{v}$:

Theorem: If θ is the angle between the vectors $\vec{u}$ and $\vec{v}$, then $\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$

Corollary: If θ is the angle between the non zero vectors $\vec{u}$ and $\vec{v}$, then $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$

Proof: The proof relies on the law of cosines in a triangle.
In the triangle OAB above, the law says the following:

$$|AB|^2 = |OA|^2 + |OB|^2 - 2\,|OA|\,|OB|\cos\theta$$
$$\Leftrightarrow \|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$
$$\Leftrightarrow (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

use definition of the length of a vector

(Expand using property 2) $\iff \|\vec{u}\|^2 - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \|\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$

(Property 1) $\iff -2\vec{u}\cdot\vec{v} = -2\|\vec{u}\|\|\vec{v}\|\cos\theta$

$\iff \vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$ ■

In practice, we often use the theorem to calculate the dot product of two vectors given the length of the vectors and the angle between them.

We use the corollary to calculate the angle between two vectors that are given to us.

It turns out that in this class we will not do much of either because we will not often look at / discuss angles. However, the dot product will play an important role in our course, and it would have been unfortunate not to discuss these interesting geometric properties.

<u>Examples</u>: • Find the angle between the vector $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and the vector $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\|\vec{u}\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$ $\qquad \|\vec{v}\| = \sqrt{0^2+1^2+0^2} = 1$

$\vec{u}\cdot\vec{v} = 1\times 0 + 1\times 1 + 0\times 0 = 1$

$\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \dfrac{1}{\sqrt{2}} \Rightarrow \theta = \dfrac{\pi}{4}$ radians or $45°$

• If two vectors are aligned and pointing in the same direction, the angle between them is $\theta = 0$

- If two vectors are aligned and pointing in opposite directions, the angle between them is $\theta = \pi$ $(180°)$

2) <u>Orthogonal vectors</u>

<u>Definition</u>: Two nonzero vectors are said to be orthogonal (or perpendicular) if the angle between them is $\theta = \frac{\pi}{2}$ $(90°)$

It is easy to see that orthogonal vectors have a very particular dot product: If $\vec{u}$ and $\vec{v}$ are perpendicular, then $\theta = \pi/2$ so $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\left(\frac{\pi}{2}\right) = 0$

Conversely, if $\vec{u} \cdot \vec{v} = 0$, we find by applying the corollary that $\cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

We have just proved the following property:

Two vectors $\vec{u}$ and $\vec{v}$ are orthogonal <u>if and only if</u> $\underline{\vec{u} \cdot \vec{v} = 0}$

This is a convenient test to find out if 2 vectors are orthogonal.

<u>QUESTION</u>: Let $\vec{u} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 3 \\ -3 \end{bmatrix}$

Are $\vec{u}$ and $\vec{v}$ orthogonal?

3) Important inequalities

* For any two vectors $\vec{u}$ and $\vec{v}$, we have
$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$
$$\Rightarrow \quad |\vec{u}\cdot\vec{v}| = \|\vec{u}\|\,\|\vec{v}\|\,|\cos\theta|$$

Since for any θ, $|\cos\theta| \leq 1$, we conclude that for any two vectors $\vec{u}$ and $\vec{v}$,

$$|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\,\|\vec{v}\|$$

This is an extremely important inequality, that plays a major role in almost all fields of mathematics. It is called the Schwarz inequality, or sometimes the Cauchy-Schwarz-Buniakowsky

* In HW #1, you will prove another important inequality, called the triangle inequality, which says, for any two vectors $\vec{u}$ and $\vec{v}$:
$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$