MATH-UA 140 - Linear Algebra

Lecture 19: Permutations and Cofactors

We have learned how to compute the determinant of a matrix from its pivots. The purpose of this lecture is to learn about formulae for the determinant involving the entries a_{ij} of A themselves. Computers calculate determinants using the pivots, but we will often find the "direct" formulae more convenient.
First, we will review and discuss a few points regarding the pivot approach.

I) Determinants and Pivots

1) Review

Consider the LU decomposition of a matrix A, possibly requiring permutation. Let $d_1, \dots, d_n$ be the n pivots on the diagonal of U.

We have $PA = LU$

$$\Rightarrow \det P \det A = d_1 d_2 \cdots d_n$$

$$\Rightarrow \underline{\det A = \pm (d_1 d_2 \cdots d_n)}$$

If the number of row exchanges is even, the $\oplus$ sign applies. Otherwise, the $\ominus$ sign applies.
If A has fewer than n pivots, $\det A = 0$

Example:

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 0 & 5 \\ 0 & 5 & 7 \end{bmatrix} \qquad PA = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 5 \end{bmatrix}$$

where P is the permutation matrix which exchanges the last two rows. So we conclude that

$$\det A = - 2 \times 5 \times 5 = -50$$

2) Calculating pivots from determinants

Imagine that we have a matrix A and that we use elimination to make pivots appear. If row exchange is not needed, then the first k pivots only depend on the matrix entries a_{ij} in the $k \times k$ matrix A_k in the left-hand corner of A.

Consider the LU decomposition of A_k: $A_k = L_k U_k$, where L_k is a $k \times k$ lower triangular matrix with 1's on the diagonal, and U_k is a $k \times k$ upper triangular matrix with the first k pivots of A $d_1, d_2, \dots, d_k$ on the boundary.

We have
$$\det A_k = \det U_k = d_1 d_2 \cdots d_{k-1} d_k$$
$$= \det A_{k} d_k$$

This provides a formula for the pivot d_k in terms of sub-determinants of the matrix A:

The k^{th} pivot is $d_k = \dfrac{d_1 d_2 \cdots d_{k-1} d_k}{d_1 d_2 \cdots d_{k-1}} = \dfrac{\det A_k}{\det A_{k-1}}$

Example: $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}$

$\det A_1 = 2$; $\det A_2 = 2\times 2 - 1\times 1 = 3$; $\det A_3 = 2(2\times 2 - 1\times(-1)) + 1((-1)\times 1 - 1\times 2)$
$+ 3(1\times 1 - 2\times 1)$
$= 10 - 3 - 3 = 4$

$d_1 = 2$; $d_2 = \dfrac{\det A_2}{\det A_1} = \dfrac{3}{2}$; $d_3 = \dfrac{\det A_3}{\det A_2} = \dfrac{4}{3}$

If you go back to Lecture 7, this is indeed the value of the pivots we found in the LU factorization of A.

II | General formula for $\det A$ in terms of the entries of A

Calculating the determinant of a matrix in terms of its pivot values is quite powerful, but also indirect, as it relies on the calculation of the pivots in the first place, through elimination. We will now construct a formula involving the matrix entries directly.

1) General formula: construction

The formula is a direct consequence of the linearity property of the determinant. Let us see this with 2x2 and 3x3 matrices:

- $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \underbrace{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}_{=0 \text{ because matrices are not invertible}} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}$$

$$= ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= ad - bc$$

- For 3x3 matrices, the process is more tedious as expanding the determinant in the same way would yield 27 terms. What we will do instead is to expand and keep only the non zero terms:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

We have 6 non zero determinants. Determinants of the type $\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix}$ are zero because the rows are linearly dependent. This is why they do not appear in the expansion. So we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + a_{12}a_{23}a_{31}\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} + a_{13}a_{21}a_{32}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$+ a_{11}a_{23}a_{32}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} + a_{12}a_{21}a_{33}\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + a_{13}a_{22}a_{31}\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

where the + signs are used when an even number of row exchanges are required to go from the identity matrix to the desired matrix, and the - signs are used when an odd number of row exchanges is required.

Before we move, note two important facts regarding our calculation of the determinant using linearity in each row:

* In the determinants that are nonzero, the nonzero entries are in different columns and different rows
* There are six ways to order the columns (1, 2, 3) row by row: (1, 2, 3), (2, 3, 1), (3, 1, 2), (1, 3, 2), (2, 1, 3), and (3, 2, 1)
The first three correspond to even permutations of the rows, the last three to odd permutations of the rows

2) General formula for any $n \times n$ matrix

In general, for an $n \times n$ matrix, there are $n! = n(n-1)(n-2)\cdots(3)(2)(1)$ possible orderings of the columns. For $n=3$, $3! = 6$, as we have seen.

The terms appearing in the formula are all of the form $a_{1i}a_{2j}a_{3\ell}$, and we get all the terms by choosing all the possible column sequences $(i,j,\ell) = (1,2,3), (2,3,1), (3,1,2), (1,3,2), (2,1,3), (3,2,1)$

When the column sequence is in the proper order, including "wrapping around", i.e $(1,2,3)$, $(2,3,1)$ and $(3,1,2)$, there is a $+$ sign in front of the term.
When the column sequence is out of order, i.e $(1,3,2)$, $(2,1,3)$, and $(3,2,1)$, there is a $-$ sign in front of the term.

This is generalized as follows for an $n \times n$ matrix:

$$\boxed{\begin{aligned} \det A &= \text{sum over all } n! \text{ column permutations } P = (i,j,\ldots,z) \\ &= \sum (\det P)\, a_{1i}\, a_{2j} \cdots a_{nz} \end{aligned}}$$

We are sloppy on purpose with the expression above, because it is rarely used except for 2×2 and 3×3 matrices, for which we gave the exact expressions previously. The reason is that the formula leads to a very large number of terms: for a 4×4 matrix, we have 24 terms; for a 12×12 matrix, we have 479 001 600 terms!

In terms of direct calculations, the computation through cofactors, which we will see next, is much more convenient.

Example

$$A = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}$$

The only nonzero entry in the first row is a_{13}; the only nonzero entry in the last row is a_{42}. So the only nonzero term in det A has column 4 for row 2 and column 1 for row 3. The permutation is $(3, 4, 1, 2)$, which is in the right order, so $\det A = + a_{13} a_{24} a_{31} a_{42} = 1$

III) Determinant by Cofactors

We start this section by rewriting the determinant of 3×3 matrices as follows:

$$\det A = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

This expression can be interpreted as follows: a_{11}, a_{12}, and a_{13} come from the first row, and the terms in parenthesis are determinants of 2×2 matrices in the second and third row of A.
The expression is the result of the following expansion

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

So the determinant of the 3×3 matrix can be computed as

$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$, where the scalars C_{11}, C_{12}, and C_{13}, called cofactors, are 2×2 determinants.

This is an efficient way to compute determinants, but there is one little subtlety: one has to pay attention to the signs. In our 3×3 example, C_{12} is not $a_{21}a_{33} - a_{31}a_{23}$ but $-(a_{21}a_{33} - a_{31}a_{23}) = a_{31}a_{23} - a_{21}a_{33}$

Here is the general rule to use the cofactor expansion for the determinant of an $n\times n$ matrix:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$$

where $C_{1j} = (-1)^{1+j} \det M_{1j}$

and M_{1j} is the submatrix obtained by crossing out row 1 and column j in A.

Since there is nothing special about the first row in the calculation of determinants (think about Property 2), the formula can be generalized for any row:

The determinant is the dot product of any row i of A with its cofactors coming from other rows:

$$\boxed{\begin{aligned}&\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}\\ &\text{with } C_{ij} = (-1)^{i+j}\det M_{ij} \text{ the cofactors}\end{aligned}}$$

and M_{ij} the matrix obtained by crossing off row i and column j from A.

Note that the alternating sign pattern $(-1)^{i+j}$ is nicely visualized as:

$$\text{sign}(-1)^{i+j} = \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Lastly, what is true for A is also true for A^T, so the cofactor expansion can also be computed down a given column j as $\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$, with the cofactors defined as before (including alternating signs).

Example: The cofactor expansion is particularly effective when applied to a matrix with many 0's, as below

$$\begin{vmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & d & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & b & 0 \\ 0 & c & 0 \\ 0 & d & 1 \end{vmatrix} + a \underbrace{\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}}_{0}$$

$$= 1 \times \begin{vmatrix} c & 0 \\ d & 1 \end{vmatrix} - b \underbrace{\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}}_{0}$$

$$= c$$