MATH - UA 140 - Linear Algebra

Lecture 25: Similar Matrices

In Lecture 22, we saw the factorization $\Lambda = S^{-1} A S$ for matrices A which are diagonalizable. In this lecture, we generalize this type of factorization, even if the matrix A does not have a set of n linearly independent eigenvectors. In that case, the equivalent of the matrix Λ is of course not diagonal. But it can come fairly close, as we will see.

I) Similar matrices

1) Definition

Let M be <u>any</u> invertible matrix. The matrix B such that $B = M^{-1} A M$ is said to be <u>similar to A</u>.

* Note that if B is similar to A, then A is similar to B. Indeed, $B = M^{-1} A M \iff M B M^{-1} = A$

$$\iff (M^{-1})^{-1} B M^{-1} = A$$

and M^{-1} is as acceptable a matrix as M.

* The zero matrix is similar only to itself: $M^{-1} O M = O$
* For diagonalizable matrices, $\Lambda = S^{-1} A S$, so Λ and A are similar.

The word "similar" was chosen on purpose: we will see that similar matrices have much in common. We start with eigenvalues.

2) Eigenvalues of similar matrices

Similar matrices A and $M^{-1}AM$ have the same eigenvalues.

Their eigenvectors, however, are different: if $\vec{x}$ is an eigenvector of A, then $M^{-1}\vec{x}$ is an eigenvector of $M^{-1}AM$.

Here is the proof: Assume $\vec{x}$ is an eigenvector of A, with corresponding eigenvalue λ: $A\vec{x} = \lambda\vec{x}$

Consider B such that $B = M^{-1}AM \Leftrightarrow A = MBM^{-1}$

$MBM^{-1}\vec{x} = \lambda\vec{x}$

$\Leftrightarrow B(M^{-1}\vec{x}) = \lambda(M^{-1}\vec{x})$

λ is an eigenvalue of B. The corresponding eigenvector is $M^{-1}\vec{x}$

Example: Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ Its eigenvalues are 1 and 3, so it is diagonalizable: $A = S\Lambda S^{-1}$, and A is similar to $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

Now consider $M = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ $M^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$$B = M^{-1}AM = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -3 & 0 \end{bmatrix}$$

B is similar to A, and has eigenvalues 1 and 3 like A.

3) Other quantities that are common to similar matrices

We have just seen that similar matrices have the same eigenvalues.

Similar matrices thus have the same determinant, which is the product of the eigenvalues.
They also have the same trace, which is the sum of the eigenvalues.

Since the eigenvectors of a similar matrix are $M^{-1}\vec{x}$, with M invertible (so full rank), similar matrices have the same number of linearly independent eigenvectors.

Similar matrices also have the same rank. There are many ways to see this. One way is as follows, for $B = M^{-1}AM$.

$$\text{rank}(B) = \text{rank}(M^{-1}AM) \leq \text{rank}(A) \quad \text{by Problem 2, HW 5}$$
$$\text{rank}(A) = \text{rank}(MBM^{-1}) \leq \text{rank}(B) \quad \text{by Problem 2, HW 5}$$

so $\text{rank}(A) = \text{rank}\, B$.

II Jordan form of a matrix

When an $n \times n$ matrix has n linearly independent eigenvectors, it can be diagonalized: $A = S \Lambda S^{-1}$, with Λ diagonal.
A is similar to Λ, with $M = S$.

If an $n \times n$ matrix does not have n linearly independent eigenvectors, it cannot be diagonalized. However, a theorem called Jordan's Theorem (named after French mathematician Camille Jordan) tells us that this matrix is similar to a matrix J which is as close to diagonal as possible. Here is the exact statement of the theorem, followed by an example.

Jordan's theorem: If an $n \times n$ matrix A has d independent eigenvectors, then it is similar to a matrix J that has d blocks $J_1 \dots J_d$ on its diagonal, where each block J_i has the eigenvalue λ_i on the diagonal, and 1's just above the diagonal:

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & & \lambda_i \end{bmatrix}$$

Each J_i has one eigenvector.
The Jordan form of A is

$$M^{-1} A M = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & J_d \end{bmatrix} = J$$

A matrix A is similar to a matrix B if and only if they share the same Jordan form.

The Jordan form J has an off-diagonal 1 for each missing eigenvector. So there are $n-d$ off-diagonal 1's.

In this class, we will not learn to compute Jordan forms, but it is important to understand that every matrix is similar to a Jordan matrix. That means that we can always write

$$A = MJM^{-1}$$

Then, $A^2 = MJM^{-1}MJM^{-1} = MJ^2M^{-1}$, and more generally,

$$A^k = MJ^kM^{-1}$$

Computing J^k is not as simple as computing Λ^k, but still more convenient than computing A^k

Example: $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$

We saw in Lecture 22 that A had only $\lambda = 2$ as its eigenvalue, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as an eigenvector.

A is similar to $J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$$\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$