Linear Algebra – Problem Set 2 Solutions

February 7, 2018

Problem 1

Let us start by solving the system by Gaussian elimination. The first step of the elimination process is to replace row 2 with (row 2)-5(row 1). The resulting system is

$$\begin{cases} x + 3y + 2z = 0 \\ (a-15)y - 9z = 3 \\ y - 2z = -2 \end{cases}$$

We see that if $a = 15$, the second pivot is 0, requiring a row exchange between row 2 and row 3. The resulting upper triangular system is

$$\begin{cases} x + 3y + 2z = 0 \\ y - 2z = -2 \\ -9z = 3 \end{cases}$$

Let us now assume that $a \neq 15$, and return to the system

$$\begin{cases} x + 3y + 2z = 0 \\ (a-15)y - 9z = 3 \\ y - 2z = -2 \end{cases}$$

The elimination operation (row 3)-1/$(a-15)$(row 2) leads to the following upper triangular system:

$$\begin{cases} x + 3y + 2z = 0 \\ (a-15)y - 9z = 3 \\ (\frac{9}{a-15} - 2)z = -2 \end{cases}$$

This system is singular if

$$\frac{9}{a-15} = 2 \quad \Leftrightarrow \quad a = \frac{39}{2}$$

Problem 2

We notice that we get the desired matrix by subtracting, in Pascal's matrix, row 3 from row 4, then by subtracting row 2 from row 3, and finally row 1 from row 2. The matrices corresponding to each operation are the elimination matrices

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad , \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, $E = E_{21}E_{32}E_{43}$, i.e.

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Now, the resulting matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Subtracting row 3 from row 4 (using E_{43}), and then row 2 from row 3 (using E_{32}), we obtain the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Lastly, subtracting row 3 from row 4 once more, using E_{43}, we obtain the identity matrix, as desired. Hence

$$M = E_{43}E_{32}E_{43}E$$

i.e.

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Problem 3

For all a, b, c, and d,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} a+2b & 3a+4b \\ c+2d & 3c+4d \end{bmatrix} = \begin{bmatrix} a+3c & b+3d \\ 2a+4c & 2b+4d \end{bmatrix}$$

The equality on the right hand side holds if and only if

$$\begin{cases} a+2b = a+3c \\ 3a+4b = b+3d \\ c+2d = 2a+4c \\ 3c+4d = 2b+4d \end{cases} \quad \Leftrightarrow \quad \begin{cases} b = \frac{3}{2}c \\ a = d-b = d-\frac{3}{2}c \\ c+2d = 2a+4c \\ 3c+4d = 2b+4d \end{cases}$$

Using the first equation to replace b in the last one, we obtain

$$3c+4d = 3c+4d$$

This equation always holds, no matter what c and d are.

Using the second equation to replace a in the third equation, we find

$$c+2d = 2d+c$$

This equation always holds, no matter what c and d are. We conclude that if $b = 3/2c$ and $a = d - 3/2c$, any pair (c, d) of real numbers satisfy the system. Therefore, the matrices which commute with

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

have the general form

$$M = \begin{bmatrix} d - \frac{3}{2}c & \frac{3}{2}c \\ c & d \end{bmatrix}$$

with (c, d) any pair of real numbers.

Problem 4

To find A^{-1}, we proceed as we saw in class, and construct the augmented matrix

$$M = \begin{bmatrix} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We see that A is already in its "Gaussian" form. All one needs to do is use elimination to turn the entries $-c$, $-b$ and $-a$ above the diagonal of A into 0s. The operation Row 3 + c Row 4 leads to

$$\begin{bmatrix} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The operation Row 2 + b Row 3 then leads to

$$\begin{bmatrix} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & b & bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The operation Row 1+ a Row 2 then leads to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & a & ab & abc \\ 0 & 1 & 0 & 0 & 0 & 1 & b & bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This is in the desired form $[I_4 \ A^{-1}]$. We conclude that the inverse of A is

$$A^{-1} = \begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 5

1. In analogy with standard elimination matrices, let us consider the block matrix

$$E = \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}$$

where the letter I stands for identity matrices. Block multiplication indeed leads to

$$EM = \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

which is the desired form for M, with S the matrix $S = D - CA^{-1}B$.

2. We can compute

$$F = NG = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix}$$

F is block diagonal, with A and S as the diagonal blocks.

3. There are two ways to split the matrix

$$M = \begin{bmatrix} 4 & 2 & 8 \\ 6 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

into blocks such that A and D are square matrices. In the first way, let

$$A = 4 \ , \ \ B = [2 \ \ 8] \ , \ \ C = \begin{bmatrix} 6 \\ 3 \end{bmatrix} \ , \ \ D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A^{-1} = \frac{1}{4}$, so

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 6 \\ 3 \end{bmatrix} [2 \ \ 8] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 12 \\ \frac{3}{2} & 6 \end{bmatrix} = \begin{bmatrix} -2 & -12 \\ -\frac{3}{2} & -5 \end{bmatrix}$$

In the second way of splitting the matrix, we let

$$A = \begin{bmatrix} 4 & 2 \\ 6 & 1 \end{bmatrix} \ , \ \ B = \begin{bmatrix} 8 \\ 0 \end{bmatrix} \ , \ \ C = [3 \ \ 0] \ , \ \ D = 1$$

We then have

$$A^{-1} = -\frac{1}{8} \begin{bmatrix} 1 & -2 \\ -6 & 4 \end{bmatrix}$$

so that

$$S = 1 + \frac{1}{8} [3 \ \ 0] \begin{bmatrix} 1 & -2 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \end{bmatrix} = 1 + 3 = 4$$