Linear Algebra – Problem Set 4 Solutions

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Problem 1

(a) The zero vector $(0, 0, 0)$ belongs to this set.

Let us take two vectors $\mathbf{u}$ and $\mathbf{v}$ in the set: $\mathbf{u} = (u_1, u_1, u_3)$, and $\mathbf{v} = (v_1, v_1, v_3)$. Then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_1 + v_1, u_3 + v_3)$$

The first two components of $\mathbf{u} + \mathbf{v}$ are equal, so $\mathbf{u} + \mathbf{v}$ is in the set.

Let c be a scalar

$$c\mathbf{u} = (cu_1, cu_1, cu_3)$$

The first two components of $c\mathbf{u}$ are equal, so $c\mathbf{u}$ is in the set.

The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$ is indeed a subspace.

(b) $(0, 0, 0)$ does not belong to the set since its first component is not 1. The plane of vectors (b_1, b_2, b_3) with $b_1 = 1$ is not a subspace.

(c) $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (0, 1, 1)$ are two vectors in this set. However, $\mathbf{u} + \mathbf{v} = (1, 1, 2)$ does not belong to the set. The vectors (b_1, b_2, b_3) with $b_1 b_2 b_3 = 0$ do not form a subspace.

(d) By definition, this set is the subspace spanned by the vectors $\mathbf{u}$ and $\mathbf{v}$.

(e) The zero vector $(0, 0, 0)$ belongs to this set.

Let us take two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ in the set: $u_1 + u_2 + u_3 = 0$ and $v_1 + v_2 + v_3 = 0$.

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

The sum of the components is

$$u_1 + v_1 + u_2 + v_2 + u_3 + v_3 = (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3) = 0$$

We conclude that $\mathbf{u} + \mathbf{v}$ is in the set.

Let c be a scalar

$$c\mathbf{u} = (cu_1, cu_2, cu_3)$$

The sum of the components is

$$cu_1 + cu_2 + cu_3 = c(u_1 + u_2 + u_3) = 0$$

We conclude that $c\mathbf{u}$ is in the set.

The vectors (b_1, b_2, b_3) with $b_1 + b_2 + b_3 = 0$ do form a subspace.

(f) $\mathbf{u} = (1, 2, 3)$ is a vector in this set. However, $(-1)\mathbf{u} = (-1, -2, -3)$ is not in this set. So the set of all vectors (b_1, b_2, b_3) with $b_1 \leq b_2 \leq b_3$ is not a subspace.

Problem 2

(a) We see that all the column vectors are colinear with $(1, -2, \frac{1}{2})$. For $\mathbf{b} = (b_1, b_2, b_3)$ to be in the column space of the matrix, $\mathbf{b}$ must also be colinear with $(1, -2, \frac{1}{2})$. Hence the condition is $b_2 = -2b_1$ and $2b_3 = b_1$.

(b) We solve the system by elimination on the augmented matrix:

$$\begin{bmatrix} -2 & 3 & b_1 \\ 6 & -9 & b_2 \\ 5 & 7 & b_3 \end{bmatrix} \to \begin{bmatrix} -2 & 3 & b_1 \\ 0 & 0 & b_2 + 3b_1 \\ 5 & 7 & b_3 \end{bmatrix} \to \begin{bmatrix} -2 & 3 & b_1 \\ 0 & 0 & b_2 + 3b_1 \\ 0 & \frac{29}{2} & b_3 + \frac{5}{2}b_1 \end{bmatrix}$$

Looking at the second row, we see that the system only has a solution if $b_2 + 3b_1 = 0 \Leftrightarrow b_2 = -3b_1$.

Problem 3

1. $3x + 2y - z = 0$ may be viewed as the equation $A\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = (x, y, z)$ and $A = [3\ 2\ -1]$.

   The first column is a pivot column, so the free variables are y and z.

   The row reduced echelon form of A is $R = [1\ \frac{2}{3}\ -\frac{1}{3}]$. To get the first special solution, set $y = 1$, $z = 0$, and read x in the second column of R (by flipping the sign): $\mathbf{s}_1 = (-\frac{2}{3}, 1, 0)$. For the second special solution, set $y = 0$, $z = 1$, and read x in the third column of R (by flipping the sign): $\mathbf{s}_2 = (\frac{1}{3}, 0, 1)$

2. Any point in the plane can be written as the sum of the particular solution (which is in the plane) and the linear combination of the special solutions $\mathbf{s}_1$ and $\mathbf{s}_2$ which can be viewed as two non-parallel "direction" vectors for the plane.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

Problem 4

(a) Impossible: When going from A to U, the elimination process does not change the first row of A. So if U has zeroes in the first row, A must also have the same zeroes.

(b) We have seen in class that the row reduced echelon form of invertible matrices is the identity. So all we have to do here is to pick an invertible matrix which does not have zero entries. One way to do this is start with an invertible upper triangular matrix U, say

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and to eliminate the zeroes with linear combinations of the rows. Replacing Row 3 with Row 3 + Row 2 gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Replacing Row 3 with Row 3 + Row 1 gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Finally, replacing Row 2 with Row 2 + Row 1 gives A:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

(c) Since A does not have zeroes in the first row, $U = R$ cannot have zeroes in the first row. This means that U and R have only one pivot: 1. A matrix A which satisfies these conditions is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

(d) For $U = 2R$ to hold, all the pivots in U must have the same value: 2. Furthermore, U cannot have zeroes above the pivots. We take for example

$$A = U = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 5

1. Let B be an $m \times n$ matrix, and $\mathbf{b}_j$ the column vector corresponding to its jth column. According to the problem statement, there exists scalars c_1, c_2,…, c_{j-1} which are not all zero such that

$$\mathbf{b}_j = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots + c_{j-1}\mathbf{b}_{j-1}$$

where the vectors $\mathbf{b}_1$,…,$\mathbf{b}_{j-1}$ correspond to the columns 1 to $j-1$ of B.

Now, let A be a $p \times m$ matrix, with rows $\mathbf{a}_1^T$,…,$\mathbf{a}_p^T$, where the $\mathbf{a}_1$, …, $\mathbf{a}_p$ are vectors in $\mathbb{R}^m$.

We can write

$$AB = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_p^T \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 & \ldots & \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T\mathbf{b}_1 & \ldots & \mathbf{a}_1^T\mathbf{b}_n \\ \vdots & \vdots & \vdots \\ \mathbf{a}_p^T\mathbf{b}_1 & \ldots & \mathbf{a}_p^T\mathbf{b}_n \end{bmatrix}$$

We conclude that the jth column of AB is

$$\begin{bmatrix} \mathbf{a}_1^T\mathbf{b}_j \\ \vdots \\ \mathbf{a}_p^T\mathbf{b}_j \end{bmatrix}$$

We can now use the fact that $\mathbf{b}_j$ is a linear combination of the previous columns:

$$\begin{bmatrix} \mathbf{a}_1^T\mathbf{b}_j \\ \vdots \\ \mathbf{a}_p^T\mathbf{b}_j \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T(c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots + c_{j-1}\mathbf{b}_{j-1}) \\ \vdots \\ \mathbf{a}_p^T(c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots + c_{j-1}\mathbf{b}_{j-1}) \end{bmatrix} = \begin{bmatrix} c_1\mathbf{a}_1^T\mathbf{b}_1 + c_2\mathbf{a}_1^T\mathbf{b}_2 + \ldots + c_{j-1}\mathbf{a}_1^T\mathbf{b}_{j-1} \\ \vdots \\ c_1\mathbf{a}_p^T\mathbf{b}_1 + c_2\mathbf{a}_p^T\mathbf{b}_2 + \ldots + c_{j-1}\mathbf{a}_p^T\mathbf{b}_{j-1} \end{bmatrix}$$
$$= c_1\begin{bmatrix} \mathbf{a}_1^T\mathbf{b}_1 \\ \vdots \\ \mathbf{a}_p^T\mathbf{b}_1 \end{bmatrix} + c_2\begin{bmatrix} \mathbf{a}_1^T\mathbf{b}_2 \\ \vdots \\ \mathbf{a}_p^T\mathbf{b}_2 \end{bmatrix} + \ldots + c_{j-1}\begin{bmatrix} \mathbf{a}_1^T\mathbf{b}_{j-1} \\ \vdots \\ \mathbf{a}_p^T\mathbf{b}_{j-1} \end{bmatrix}$$

We see that c_1 is multiplying the first column of AB, c_2 the second column of AB, etc., until c_{j-1} multiplying the $(j-1)$th column of AB. In other words, the jth column of AB is the same linear combination of the previous columns of AB as the jth column of B is of the previous columns of B.

AB cannot have new pivot columns as compared to B, so $rank(AB) \leq rank(B)$.

2. Let

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{, and} \qquad A_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A_1 B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

so $rank(A_1 B) = rank(B) = 1$.

$$A_2 B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so $rank(A_2 B) = 0 < rank(B)$.