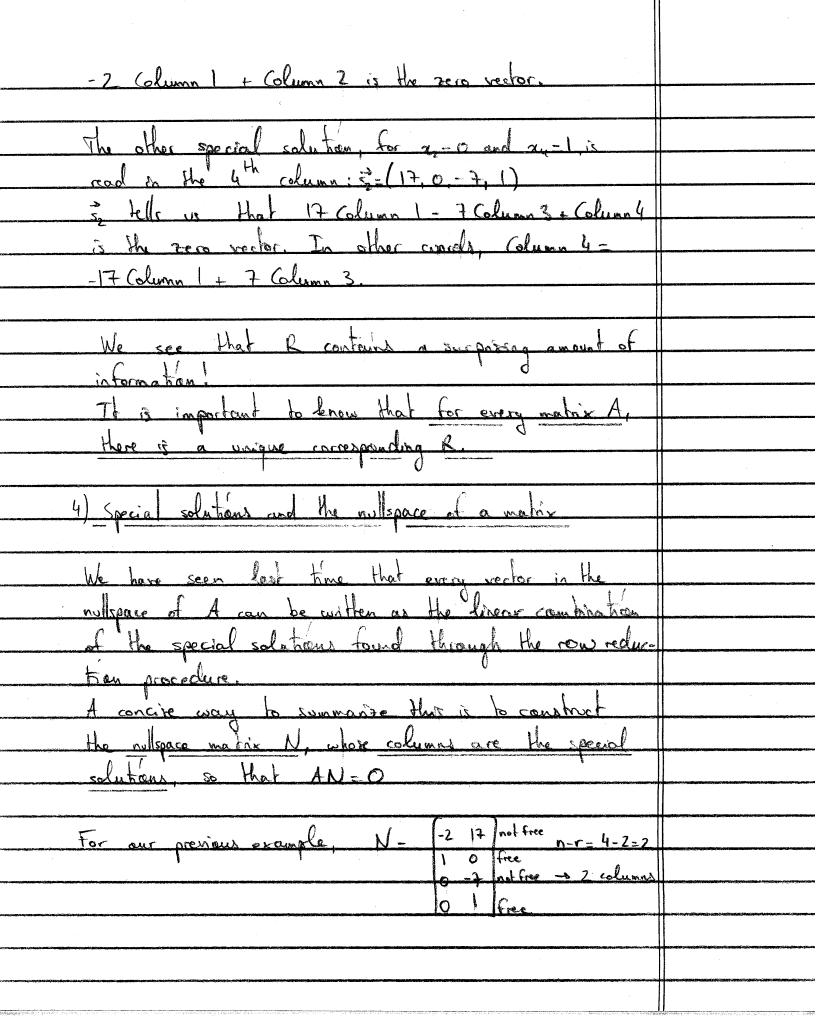
MATH-UA 140 - Linear Algebra
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Lecture 11: Solving Az = b in general
This lecture starts with a review of a few concepts we already
This lecture stort with a review of a few concepts we already concered in the last lecture, and then connects all we have
learned in the last few lectures to provide a general technique for solving the equation $A\overset{\sim}{\times} - \overset{\circ}{\text{L}}$ , whether $A$ is a square matrix or not.
que for solving the equation A== B whether A is a
square matrix of not
I Rank of a matrix
1) Computational definition
The rank of a matrix A is the number of pivots of A. This
number is often withen r.
In the following, we will see the Endamental importance of
the rank beyond its computational definition. It tells us
the true size of a linear system, i.e the number of
equations which are not redundant.
Example: A= 1131
-2312
LO 5 7 4 J
$\begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 1 & 3 & 1 \end{bmatrix}$
Elimination leads to 0 5 7 4 -0 0 5 7 4
05747 0000

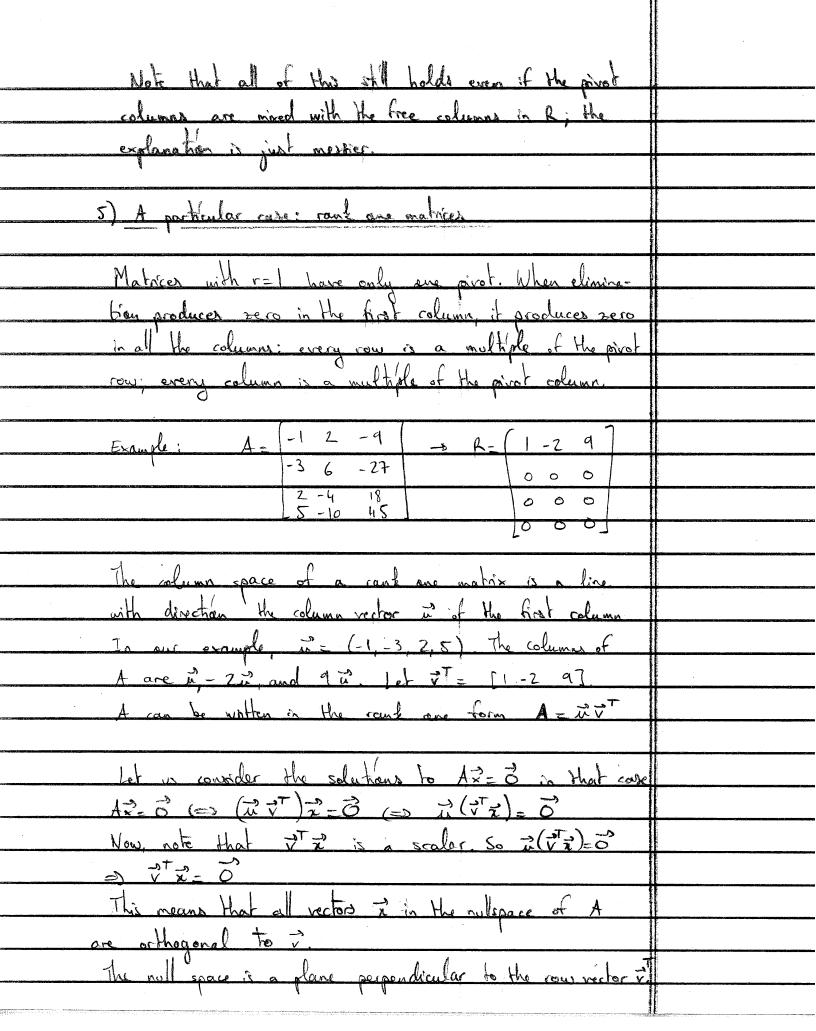
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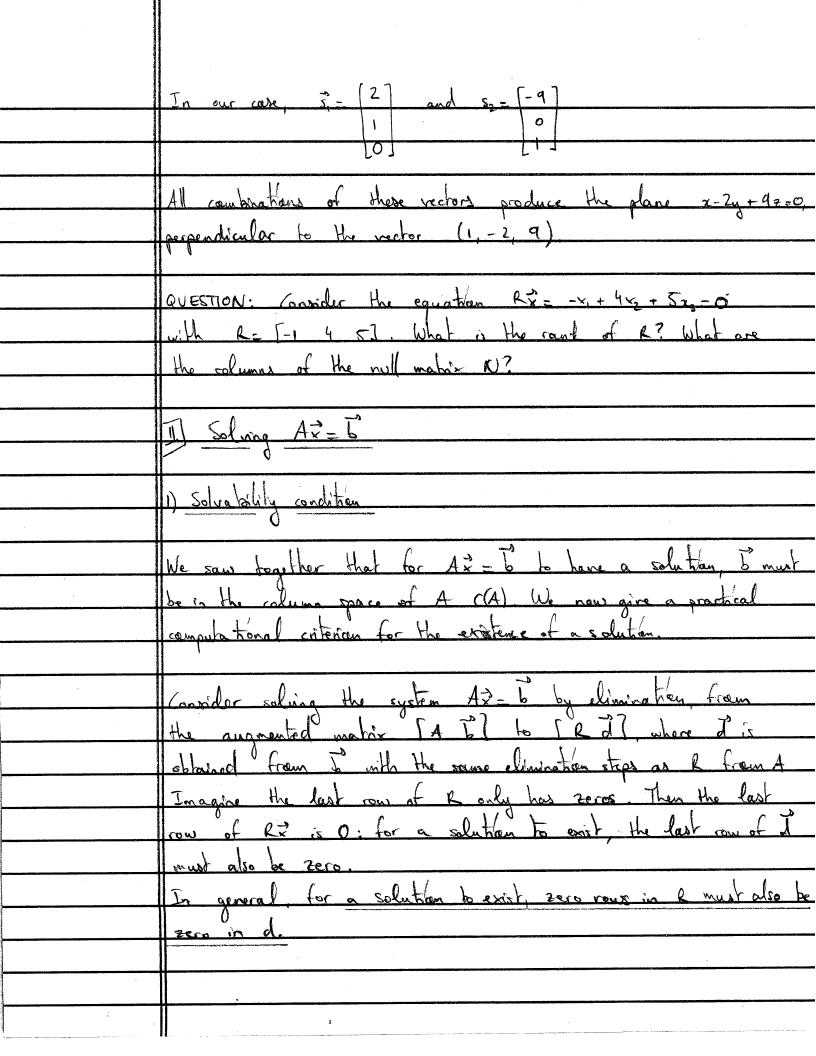
The rank of A is 2. In the system AZ=0, the last equation really is redundant, equivalent to 0=0	
2) Rant and column dependence	
The rank is the number of pivots of a matrix, so also the number of pivot columns. The columns which are not pivot columns are called free columns.	
The pivot columns are not linear combinations of earlier columns. The free columns are combinations of earlier columns	
there, the word early applies to the order with which one applies elimination column by column.	
As we will soon see me say that the reprivate columns are independent columns and the ner free	
columns are dependent columns.	
2 4 0 3 7	
We have $\begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$	
The rank of A is 2	

	3) Echelon form R, special solutions, and linear counterations
	In the last lecture we saw how to construct the special
	solutions to A2 - 3 from the reduced row echelon making
	Rassociated with A. There are no special solutions. For
	each special solution are free variable is equal to I and
	the other free variables are set to O. The pivot variables can then
	be read in the column in R associated with the nonzero
	free variable, by revering the signs
	Romander aux example last time:
	1234] (120-17)
	A= 2 4 5 1 - R= 0 0 1 7
	5 10 11 -8 ] 0 0 0 0
	P P P
	privat free pivat free column column
	We will now show that the pivot columns of A are the same
	as the pivot columns of R. This is a general result.
	It is clear that column 2 is just 2 times column 1.
	Column 4 is a linear combination of columns 1 and 3, and
	it is a general result that the special solutions tell us
	what the linear combination is
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	We can read the special solutions off R: for x=1, x=0,
	the special solution is obtained from the second column of
	R. 52- (-2, 1, 0, 0). This solution tells us that



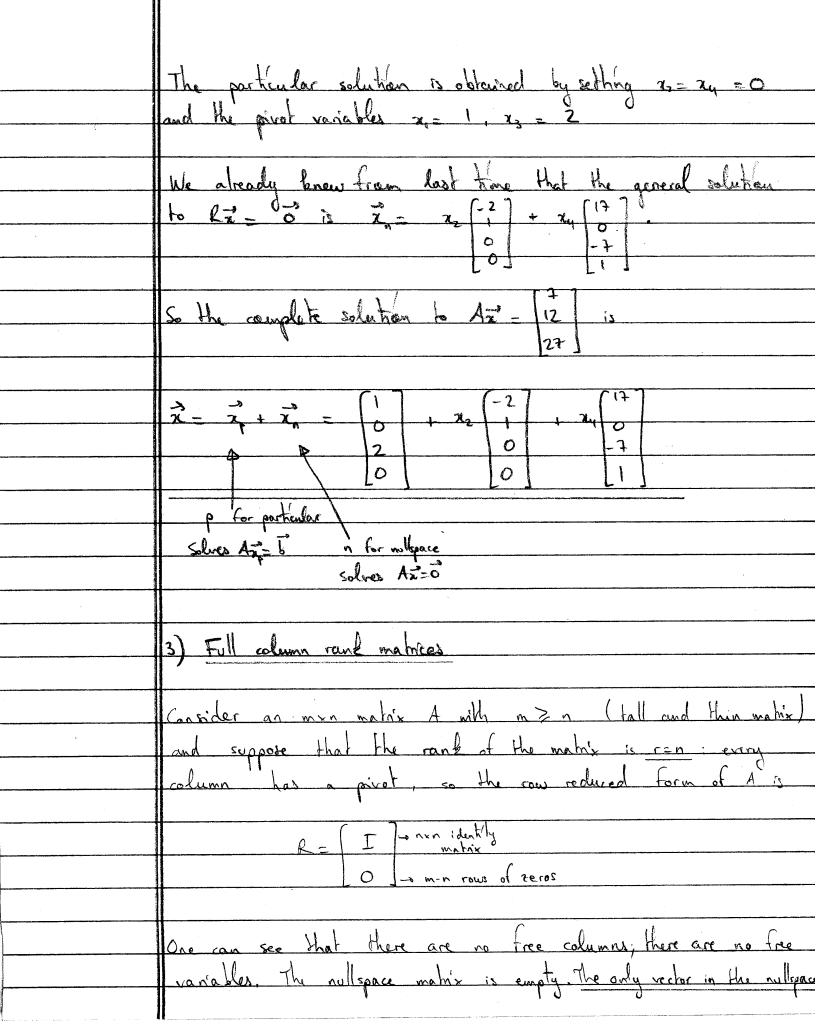
	Observe that R had the identity matrix (2x2 in our case) in
	Observe that R had the identity matrix (2x2 in our case) in the private columns, and N has the identity matrix (also 2x2)
	in its free rows.
	a distribution de recommendant de la materia
	General Summary
	- (-
	Consider an man matrix A with road i. This means that $4x=0$
<b>Q</b> :	has r pivots and n r free variables.
	The nullspace matrix N contains the n-r special solutions.
	The special solutions are easy to obtain from the row reduced
	form R= 5. Suppose for simplicity that the pivot columns
	lace the trait of continues became the continue of the same
	following blocks:
	R = [I F   r pivot rows
	O O m-r zero rows
	r pivot n-r free
	columns columns
	Then the nullspace makix N is given by
	N-(-F) r pivet variables  I n-v free variables
	I n-v free variables
	Indeed, multiplying the matrices by blocks, we have RN=-FI+FI=0
٦	Indeed, multiplying the matrices by blocks, we have RNFI+FI=0 as desired. This illustrates why the special solutions, which are in N, can always be read in R.
	can always be read in R.





2) Complete solution In the previous bechires, we bearned how to compute the general solution to  $R\ddot{x}=\ddot{0}$ . All we need to obtain the general solution to  $R\ddot{x}=\ddot{d}$  is to construct a particular solution to that equation, which we will add to the general solution to  $R\ddot{x}=\ddot{0}$ .

Here is the simple trick to do so: set the free variable to zero, and read the pivot variables right off  $\ddot{d}$ . This works because the pivot rows and columns of R 00-4-28  $\begin{vmatrix} 7 & 1 & 2 & 0 & -17 & | \times 1 \\ 2 & -1 & 0 & 0 & 1 & 7 & | \times 3 \end{vmatrix}$ 0000



Ax = To has a solution if I is in the column space of and in that case it is the only solution [13] Find the condition on Tifor  $A\vec{x} = \vec{b}$  to have a solution  $\vec{x}$  and give the solution when the condition is satisfied 4) Full row rank matrices We now consider the other extreme corre: m & n (short and orde) and of A is r= m Every row has a pivot The matrix R has no zero rows: R= [I F] Ar-b has a solution for every b: (A)- RM
There are n-m=n-r special solutions in the nullspace of A Example: Consider the system of equations 1 3x + y + 22 = 4 The augmented matrix is  $\begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & -2 & -1 & -11 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & \frac{1}{2} & \frac{11}{2} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{11}{2} \end{bmatrix}$ 

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	A partiular sole	ation is obtained 6	y setting 2-0	and		
	A particular solution is obtained by setting 2=0 and rading $\vec{d}$ : $(x,y,z)=(-\frac{1}{2},\frac{11}{2},0)$					
	The special soluti	fan is obtained by	setting z = 1, a	ed reversing		
	The special solution is obtained by setting z=1, and reversing the signs in the third column: (x,y,z)= (-\frac{1}{2},-\frac{1}{2},1)					
	The complete solution $\vec{x} = \vec{x} + \vec{x} - \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + z \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$					
	Summary  In this class, we have now covered the four different possibility for the system of equations $A\vec{x} = \vec{b}$ with $A$ an man matrix with  rank $r$ :					
	M	(2) r=m and r <n< th=""><th>(3) r=n and c/m</th><th>(9) Fly and -1</th></n<>	(3) r=n and c/m	(9) Fly and -1		
	1	A is short and mide		· A		
	invertible	R- FT F7	R- [I]	I have full		
	R- [I]	$A\overrightarrow{z}=\overrightarrow{b}$ has $\infty$	[0]	rank		
	A= Thas I solution	solutans	43-1 has 0	R=[IF		
		1	or I solute	[00]		
		1		A== 6 has 0		
				or solution		
			·			