Slope of a line tangent to a circle – implicit version

We just finished calculating the slope of the line tangent to a point (x, y) on the top half of the unit circle. In this calculation we started by solving the equation $x^2 + y^2 = 1$ for y, chose one "branch" of the solution to work with, then used the chain rule, the power rule and some algebra of exponents to compute the derivative $\frac{dy}{dx} = -\frac{x}{y}$.

We'll now see how we could have used implicit differentiation to do the same calculation much more easily. In fact, we'll find the slope of a line tangent to *any* point on the unit circle.

We don't need to solve for y — we can just apply the operator $\frac{d}{dx}$ to both sides of the original equation:

$$\begin{aligned} x^2 + y^2 &= 1 \\ \frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}(1) \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0 \end{aligned}$$

We can easily take the derivative of the first term. For the second term, applying the chain rule with the inside function y and outside function u^2 gives us:

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

This is the same answer, but we didn't have to restrict ourselves to just the top half of the circle or use any square roots. Implicit differentiation made this calculation much easier.