MATH-UA 140 - Linear Algebra

Lecture 12: Independence, Basis and Dimension

In the last few lectures, we have encountered a few important subspaces, including the column space of a matrix and the nullspace of a matrix. We have also learned how to characterize and represent them. The purpose of this lecture is to learn how one determines the dimension of any subspace, and what is the minimum amount of information required to represent the subspace. We will see that the key is to find a basis of vectors to represent the subspace, so the goal is to work our way towards defining what a basis is.

I] Linear independence

1) Definition

A sequence of vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ is linearly independent if

$$a_1\vec{u}_1 + a_2\vec{u}_2 + \ldots + a_n\vec{u}_n = \vec{0} \Rightarrow (a_1, a_2, \ldots, a_n) = (0, 0, \ldots, 0)$$

In other words, the only linear combination which gives the zero vector is the trivial linear combination with all scalars being 0

Examples: • $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\3\end{bmatrix}$ are linearly independent

• $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\begin{bmatrix}0\\0\\1\end{bmatrix}$ are linearly independent

- $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\begin{bmatrix}2\\3\\0\end{bmatrix}$ are not linearly independent.

- In $\mathbb{R}^2$, any three vectors $\begin{bmatrix}a\\b\end{bmatrix}$, $\begin{bmatrix}c\\d\end{bmatrix}$, and $\begin{bmatrix}e\\f\end{bmatrix}$ are dependent

- If among the n vectors $\vec{u_i}$, one vector $\vec{u_j} = \vec{0}$, then $\vec{u_1}, \vec{u_2}, \dots, \vec{u_n}$ are not linearly independent.

2) Alternative definition of linear independence using matrices

The columns of a matrix A are linearly independent when the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$

In other words, the nullspace of A is the zero vector.

Illustration: Let us imagine we do not immediately see that $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\begin{bmatrix}2\\3\\0\end{bmatrix}$ are linearly independent.

Let us use the matrix definition to find out the answer:

$$A = \begin{bmatrix}1 & 0 & 2\\0 & 1 & 3\\0 & 0 & 0\end{bmatrix}$$

This matrix is already in row reduced echelon form, and we can see the third column is a free column, corresponding to the special solution $\vec{s_1} = \begin{bmatrix}-2\\-3\\1\end{bmatrix}$

$-2\times$ Column 1 -3 Column 2 $+$ Column 3 $= 0$, which means that the columns are linearly dependent.

The matrix based definition also allows us to derive a general result which makes intuitive sense. Here it is.
Consider a short wide $m\times n$ matrix, with $n > m$. The rank of that matrix is at most m, the number of rows. Then there are at least $m-n$ free variables, meaning that $A\vec{x} = \vec{0}$ has nonzero solutions. We conclude that the columns of A must be linearly dependent.

In other words, any set of n vectors in $\mathbb{R}^m$ must be linearly dependent if $n > m$.

II] Vectors that span a subspace, and row space of a matrix

1) Definition

A set of vectors spans a space if all their possible linear combinations fill the space.

Examples: • $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ span the full two-dimensional space $\mathbb{R}^2$

• $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\begin{bmatrix}2\\3\\0\end{bmatrix}$ span the two-dimensional xy plane in $\mathbb{R}^3$.

So do $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$ alone

• The columns of a matrix A span $C(A)$, the column space of A (by definition)

2) The row space of a matrix

The row space of an $m \times n$ matrix is the subspace of $\mathbb{R}^n$ spanned by the rows.

In other words, the row space of A is the column space of A^T: $C(A^T)$

Example: $A = \begin{bmatrix} 2 & 7 \\ 5 & 7 \\ 3 & 1 \end{bmatrix}$

The two columns of A are linearly independent, so $C(A)$ is the plane in $\mathbb{R}^3$ spanned by the two vectors.

$\begin{bmatrix} 2 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$ are not colinear, so the row space of A is all of $\mathbb{R}^2$.

III) Basis of a vector space

1) Definition

A basis for a vector space is a sequence of vectors with two properties:

1) The basis vectors are linearly independent
2) The basis vectors span the space

Intuitively, a basis is a set of vectors which are independent and just enough in number to cover the whole vector space of interest.

Examples: • $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$

• $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$

Property: There is one and only one way to write a vector $\vec{v}$ as a combination of the basis vectors.

Proof: Suppose the basis vectors are $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ and suppose there are two ways to write $\vec{v}$:

$\vec{v} = a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_n\vec{u}_n = b_1\vec{u}_1 + b_2\vec{u}_2 + \cdots + b_n\vec{u}_n$

Then, $(a_1 - b_1)\vec{u}_1 + (a_2 - b_2)\vec{u}_2 + \cdots + (a_n - b_n)\vec{u}_n = \vec{0}$

Since the vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ are independent, this implies that $a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$.

Note that on the other hand there are infinitely many bases for a given vector space. We will see that next.

2) Bases and matrices

* Consider an invertible $n \times n$ matrix A

We know that the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$

This means that the column vectors of A are independent

Furthermore, for any $\vec{b}$ in $\mathbb{R}^n$, $A\vec{x} = \vec{b}$ has the solution $\vec{x} = A^{-1}\vec{b}$. So the column vectors of A span $\mathbb{R}^n$. We just proved that:

The vectors $\vec{u}_1, \dots, \vec{u}_n$ are a basis of $\mathbb{R}^n$ if and only if they are the columns of an $n \times n$ invertible matrix.

* What can we say about matrices that are not invertible?

In that case, the columns are not linearly independent. However, we have seen in the previous lecture that the pivot columns are linearly independent and span the column space.

The pivot columns of a matrix A are a basis for its column space

Note that even though the pivot columns of A are at the same place as the pivot columns of $R = \text{rref}(A)$, the corresponding column vectors are not the same, so that in general, the column space of A is not the same as the column space of R.

Example: $A = \begin{bmatrix} 4 & 12 \\ 1 & 3 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

The second column of A is colinear with the first column of A, so the column space of A is a line in $\mathbb{R}^2$ with $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ as a basis. $\begin{bmatrix} 12 \\ 3 \end{bmatrix}$ is also a possible basis, as is any scalar multiple of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

The column space of R is also a line, but with direction vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This is a different line in $\mathbb{R}^2$

* Thinking in terms of A^T, it is clear that <u>the pivot rows of a matrix A are a basis for its row space.</u>

What is more, <u>the pivot rows of $R = \text{rref}(A)$ are also a basis for the row space of A.</u>

Going back to the example above, we indeed see that $\begin{bmatrix} 4 \\ 12 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are two possible bases for the same line in $\mathbb{R}^2$

QUESTION: Find bases for the column and row space of

$$R = \begin{bmatrix} 1 & 2 & 0 & -17 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Summary of this section

Given a set of vectors $\vec{x}_1, \ldots, \vec{x}_n$ in $\mathbb{R}^m$, how does one find a basis for the subspace the vectors span? There are two equivalent methods:

Method 1 Construct a matrix A whose rows are the vectors $\vec{x}_1, \ldots, \vec{x}_n$. Compute $R = \text{rref}(A)$; the nonzero rows of R are a basis for the desired subspace

Method 2 Construct a matrix A whose columns are the vectors $\vec{x}_1, \ldots, \vec{x}_n$. Use elimination to find the pivot columns of A. The vectors in these columns are a basis for the desired subspace.

IV Dimension of a Vector Space

1) Theorem - Definition

All bases for a given vector space contain the same number of vectors.

The dimension of a vector space is the number of vectors in every basis.

Note: An instructive proof of the theorem is given in the textbook

2) Examples

- The dimension of $\mathbb{R}^n$ is n

- The dimension of the column space $C(A)$ of a matrix A with rank r is r (A has r pivots).

We will show in the next session that the dimension of the nullspace $N(A)$ of A is $n-r$, where n is the number of columns.

- We found in the previous lecture that the vectors $\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$ spanned the nullspace of $R = [-1 \ 4 \ 5]$. They are the basis of the plane $-x + 4y + 5z = 0$, a subspace of $\mathbb{R}^3$ with dimension 2.

The column space of R is $\mathbb{R}$, with -1 as a basis. The dimension is 1.