MATH-UA 140 - Linear Algebra

Lecture 13: Dimensions of the Four Subspaces

I) The four fundamental subspaces

Thus far, we have encountered three fundamental subspaces associated with an $m \times n$ matrix A:

1. The column space $C(A)$, which is a subspace of $\mathbb{R}^m$
2. The nullspace $N(A)$, which is a subspace of $\mathbb{R}^n$
3. The row space $C(A^T)$, which is a subspace of $\mathbb{R}^n$

We will now learn about the fourth fundamental subspace:

4. The left nullspace $N(A^T)$, which is a subspace of $\mathbb{R}^m$.

Let us stop for a second on the name left nullspace. $N(A^T)$ is the nullspace of A^T, i.e the subspace of vectors $\vec{y}$ such that $A^T\vec{y} = \vec{0}$. Taking the transpose of this equality, we have $(A^T\vec{y})^T = \vec{0}^T \iff \vec{y}^T (A^T)^T = \vec{0}^T \iff \vec{y}^T A = \vec{0}$

We see that the vector $\vec{y}^T$ multiplies A on the left.

In the remainder of this lecture, we will see how the four subspaces are connected, which will lead us to the Fundamental Theorem of Linear Algebra.

We start by looking at the four subspaces of $R = \text{rref}(A)$, which are a bit more transparent. We will then move to the four subspaces of A.

II) The four subspaces of R

1) The column space C(R)

Let us take as an example our well-known

$$R = \begin{bmatrix} 1 & 2 & 0 & -17 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns form a basis of $C(R)$. They are independent since they start with the $r \times r$ identity matrix, and any other column is a linear combination of the pivot columns. As we have seen, the special solutions give us the appropriate linear combinations.

Bottom line: The pivot columns form a basis of $C(R)$, and the dimension of $C(R)$ is the rank r of R.

2) The nullspace N(R)

We have seen previously that the special solutions span $N(R)$, as any solution to $R\vec{x} = \vec{0}$ is a linear combination of the special solutions. Furthermore, the special solutions are independent by construction (at least following our method to construct them) since the rows corresponding to the free variables contain the identity matrix. We can therefore conclude that the special solutions form a basis of $N(R)$. In our case, a basis is $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 17 \\ 0 \\ -7 \\ 1 \end{bmatrix}$

The number of special solutions is $n-r$, so the dimension of $N(R)$ is $n-r$.

Bottom line: The special solutions form a basis of $N(R)$, and the dimension of this subspace is $n-r$.

3) The row space $C(R^T)$

The pivot rows are independent, since they contain the $r \times r$ identity matrix in the pivot columns. Furthermore, the pivot rows span $C(R^T)$ since the zero rows cannot add anything new. We conclude that the pivot rows are a basis of $C(R^T)$, and the dimension of $C(R^T)$ is r, equal to the number of pivots.

Bottom line: The pivot rows are a basis of $C(R^T)$, and the dimension of $C(R^T)$ is r.

4) The left nullspace of R (a.k.a nullspace of R^T)

The vectors $\vec{y}$ in $N(R^T)$ are such that $R^T\vec{y} = \vec{0}$. In other words, we are looking for linear combinations of the rows of R that give $\vec{0}$, where the scalar multipliers in the linear combination are the components y_i of $\vec{y}$.

R has $m-r$ zero rows. Whatever the value y_i of the component of $\vec{y}$ at these rows, the linear combination of these rows gives $\vec{0}$.

The r pivot rows, on the other hand, are not zero, and are independent as we have seen. Thus, the only combination of these rows that can be $\vec{0}$ is the trivial linear combination with zero as scalar multipliers: the y_i's corresponding to pivot rows must be zero.

The general solution $\vec{y}$ to $R^T\vec{y} = \vec{0}$ has r zeros and $m-r$ free variables. Therefore, the dimension of $N(R^T)$ is $m-r$.

Bottom line: The dimension of $N(R^T)$ (left nullspace of R) is $m-r$.

5) Summary

- The row space and nullspace of R are subspaces of $\mathbb{R}^n$, with dimensions r and $n-r$ respectively. The sum of the dimensions is $r+n-r=n$, the dimension of $\mathbb{R}^n$.

- The column space and left nullspace of R are subspaces of $\mathbb{R}^m$ with dimensions r and $m-r$ respectively. The sum of the dimensions is $r+m-r=m$, the dimension of $\mathbb{R}^m$.

III) The four subspaces of A

In general, not all the subspaces of R are the same as the subspaces of A. However, there respective dimensions are equal. This is what we show now. The central idea is that one relied on elimination to obtain R from A. Thus, there is an invertible elimination matrix E (in general E is the product of several elimination matrices) such that $EA = R \Leftrightarrow A = E^{-1}R$

1) The column space C(A)

The columns which are linearly independent in A are also the columns which are linearly independent in R (although the vectors they contain may be different in A and in R)
Likewise, the columns which are dependent in A are also the columns which are dependent in R (although the vectors they contain may be different in A and in R).

The reason for the statement above is that $A\vec{x} = \vec{0} \Leftrightarrow R\vec{x} = \vec{0}$
The same linear combinations, i.e. same components of $\vec{x}$ as the scalar multipliers in the combination, of the columns of A and of the column of R lead to 0 on the right-hand side.
We conclude that the r pivot columns of A are a basis for its column space (which may be different from C(R)).

Bottom line: The pivot columns of A are a basis of C(A).
The dimension of C(A) is r.

2) The nullspace $N(A)$

$A\vec{x}=\vec{0} \Leftrightarrow R\vec{x}=\vec{0}$ proves that a vector $\vec{x}$ is in $N(A)$ if and only if $\vec{x}$ is in $N(R)$

A has the same nullspace as R.

Bottom line: The special solutions of $R\vec{x}=\vec{0}$ form a basis of $N(A)$, and the dimension of $N(A)$ is $n-r$.

Partial conclusion:

Dimension of column space + dimension of nullspace = n

3) The row space $C(A^T)$

$EA=R \Rightarrow$ Every row of R is a linear combination of the rows of A

$A=E^{-1}R \Rightarrow$ Every row of A is a linear combination of the rows of R

We conclude that A has the same row space as R

Bottom line: The pivot rows of R are a basis for $C(A^T)$. The dimension of $C(A^T)$ is r

Note: The number of independent columns equals the number of independent rows.

4) The left nullspace of A: $N(A^T)$

A^T is an $n \times m$ matrix whose column space has rank r. We know from 1) and 2) previously that the dimension of the nullspace of A^T must then be $m-r$.

Bottom line: The dimension of $N(A^T)$ (left nullspace of A) is $m-r$.

5) Summary: The Fundamental Theorem of Linear Algebra (Part I)

Let A be an $m \times n$ matrix with rank r. Then the column space and row space of A both have dimension r.
The nullspace has dimension $n-r$, and the left nullspace has dimension $m-r$.

Examples: * $A = [-3 \ \ 1 \ \ 5]$ $\quad m=1, \ n=3, \ r=1$

Every column of A is in $\mathbb{R}$, so the column space of A is $\mathbb{R}$.
The dimension of $C(A)$ is 1.

$A^T = \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}$ so $A^T \vec{y} = \vec{0} \Rightarrow \vec{y} = \vec{0} = 0$, the zero vector in $\mathbb{R}$

So the nullspace of A^T is $\{0\}$, its dimension is zero $= m-1$

A has one row, so the row space of A is the line in $\mathbb{R}^3$ with direction vector $[-3 \ \ 1 \ \ 5]$. The dimension of this space is 1

For any $\vec{x}$ is $N(A)$, $A\vec{x} = \vec{0} \Rightarrow -3x_1 + x_2 + 5x_3 = 0$

This is the equation of a plane in $\mathbb{R}^3$, with dimension 2.

We recover the result $n-r = 3-1 = 2$

6) Rank one matrices

When $r=1$, every row is a multiple of the same row, every column is a multiple of the same column.
The row space is a line in $\mathbb{R}^n$, the column space a line in $\mathbb{R}^n$.

Illustration. $A = \begin{bmatrix} 3 & -3 & 6 & 9 \\ 1 & -1 & 2 & 3 \\ -4 & 4 & -8 & -12 \end{bmatrix}$

can be written as $\vec{u} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$ times $[1 \ -1 \ 2 \ 3] = \vec{v}^T$

(A 3×1 matrix times a 1×4 matrix gives a 3×4 matrix)

$A = \vec{u}\,\vec{v}^T$

Every rank one matrix has the special form $A = \vec{u}\,\vec{v}^T$ (column times row)
the columns are multiples of $\vec{u}$, and the rows are multiples of $\vec{v}^T$.

The nullspace of A contains any vector such that $A\vec{x} = 0$
$\Leftrightarrow \vec{u}\left(\vec{v}^T\vec{x}\right) = 0 \Rightarrow \vec{v}^T\vec{x} = 0$

In other words, the nullspace is the plane perpendicular to $\vec{v}$. This is the first insight into a general orthogonality property we will explore in the next lecture.