MATH-UA 140 - Linear Algebra

Lecture 8: Transposes and Permutations

I) Matrix transpose

1) Definition

Let A be an m-by-n matrix, with entries a_{ij}. The transpose of A, written A^T, is the n-by-m matrix with entries $a^T_{ij} = a_{ji}$

In other words, the entry in row i, column j of A^T comes from row j, column i of the original A.

Example: $A = \begin{bmatrix} 3 & 1 & 4 \\ -1 & 8 & 10 \end{bmatrix}$, $A^T = \begin{bmatrix} 3 & -1 \\ 1 & 8 \\ 4 & 10 \end{bmatrix}$

* Note 1: $(A^T)^T = A$

* Note 2: The transpose of a lower triangular matrix is upper triangular

2) Properties

Let A and B be two matrices with the appropriate dimensions for the operations shown below. We have:

1. $(A+B)^T = A^T + B^T$
2. $(AB)^T = B^T A^T$
3. $(A^{-1})^T = (A^T)^{-1}$

* Property 3 follows from property 2. Indeed, let A be an invertible matrix. Then

$$A A^{-1} = I$$

Taking the transpose on both sides, we have

$$(A A^{-1})^T = I^T = I$$

Using property 2, we can rewrite the left-hand side,

$$(A^{-1})^T A^T = I$$

which proves that the inverse of A^T is $(A^{-1})^T$.
Note that property 3 also implies that A^T is invertible whenever A is.

* To understand Property 2, first consider the situation of a matrix vector multiplication: $A\vec{x}$
We have seen that $A\vec{x}$ can be seen as the linear combination (using the component of $\vec{x}$) of

the columns of A. $\vec{x}^T A^T$, on the other hand, is the linear combination (using the components of $\vec{x}$) of the rows of A^T. We conclude that it is the same linear combination of the same vectors, once written as columns, the other time written as rows. Hence $(A\vec{x})^T = \vec{x}^T A^T$.

We can now generalize to the product of two matrices A and B by considering the columns $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ of B:

$$AB = A\begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} = \begin{bmatrix} A\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n \end{bmatrix}$$

$$\Rightarrow (AB)^T = \begin{bmatrix} \vec{x}_1^T A^T \\ \vdots \\ \vec{x}_n^T A^T \end{bmatrix} = B^T A^T$$

Illustration: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 5 & 7 \\ -1 & -2 \end{bmatrix}$

$$AB = \begin{bmatrix} 3 & 3 \\ 11 & 13 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B^T = \begin{bmatrix} 5 & -1 \\ 7 & -2 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 3 & 11 \\ 3 & 13 \end{bmatrix} = (AB)^T$$

* As for inverses, the reverse order rule applies to a product of any number of matrices. For example, $(ABC)^T = C^T B^T A^T$

In particular, in the context of the LU factorization seen in Lecture 7,

$$A = LDU \Rightarrow A^T = U^T D^T L^T = U^T D L^T$$

since D is diagonal ($D^T = D$).

3) The dot product in terms of transpose

For any two vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, we

defined the dot product as $\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$

We are now ready to give a more linear algebra "feel" to the dot product, by writing

$$\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$$

In this context, the dot product is sometimes also called the inner product.

We can now give an alternate, slightly more mathematical definition of the transpose of a matrix:

Given a matrix A, A^T is the matrix that makes the following two inner products equal for any vectors $\vec{x}$ and $\vec{y}$:

$$(A\vec{x})^T \vec{y} = x^T (A^T y)$$

- Note that $\vec{x}^T \vec{y}$ (with the T inside) is a scalar quantity, as it should since it represents a dot product

There exists another product, with the T outside, called the rank one product or outer product:

$$\vec{x}\,\vec{y}^{\,T}$$

It is the product of an n-by-1 matrix with a 1-by-n matrix; it thus results in a square n-by-n matrix.

II] Symmetric matrices

1) Definition

A symmetric matrix A is such that $A^T = A$.
In other words, its entries a_{ij} are such that $a_{ij} = a_{ji}$

Example: $\begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 0 & 10 \\ 10 & 0 \end{bmatrix}$ are symmetric matrices

2) Immediate property

The inverse of a symmetric matrix is also symmetric.

Proof: $(A^{-1})^T \overset{\text{Property 3 seen earlier}}{=} (A^T)^{-1} = A^{-1}$

Illustration: $\begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{4}{11} & \frac{1}{11} \\ \frac{1}{11} & \frac{3}{11} \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

3) Symmetric products

Let A be any m-by-n matrix. Then $A^T A$ is an n-by-n matrix which is symmetric.

Indeed, $(A^T A)^T = A^T (A^T)^T = A^T A$

Likewise, AA^T is an m-by-m matrix which is symmetric.

Indeed, $(AA^T)^T = (A^T)^T A^T = AA^T$

Example: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 4 & 3 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 2 & 3 \end{bmatrix}$

$AA^T = \begin{bmatrix} 5 & 4 & 11 \\ 4 & 5 & 10 \\ 11 & 10 & 25 \end{bmatrix}$ $\quad A^T A = \begin{bmatrix} 21 & 16 \\ 16 & 14 \end{bmatrix}$

5) Symmetric matrices and LU factorization

In Lecture 7, we introduced the $A = LDU$ factorization of a square matrix A for which no row exchange was required during Gaussian elimination.

For symmetric matrices, this factorization takes a particularly simple form because in that case $U = L^T$

In other words, if $A = A^T$ is factored into LDU with no row exchanges, then U is exactly L^T

The symmetric LU factorization of a symmetric matrix A is $A = LDL^T$

QUESTION: Find the symmetric factorization of $A = \begin{bmatrix} 1 & 4 \\ 4 & 10 \end{bmatrix}$

III) Permutation matrices

As we have seen previously, elimination matrices are sometimes not sufficient to solve a system by Gaussian elimination. This is because the order of the rows sometimes needs to be exchanged. Row exchanges are taken care of by permutation matrices.

1) Definition

A permutation matrix P has the same rows as the identity matrix, but in any order.

Example: There are 2 2-by-2 permutation matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

There are 6 3-by-3 permutation matrices:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{21}P_{32} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_{32}P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

In general, there are $n!$ permutation matrices of dimension n-by-n. Here ! means "factoral". $n!$ is the number $n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$

$2! = 2 \times 1 = 2$, $3! = 3 \times 2 \times 1 = 6$

Permutation matrices are invertible, and their inverses P^{-1} are also permutation matrices. In fact, we can say even more:

If P is a permutation matrix, $P^{-1} = P^T$

In other words, $PP^T = I$

2) LU factorization in the most general case

In Lecture 6, we discarded the possibility of needing row exchanges to complete the Gaussian elimination process. We are now ready to tackle the most general case, for which row exchanges might be needed.
The idea is quite simple: say A is such that row exchanges are needed. The row exchange can be done by applying a permutation matrix P: $A \longrightarrow PA$
Once this is done, the rows are in the proper order, and one can proceed with LU factorization as we have seen in Lecture 6, so we can write:

$$\underline{PA = LU}$$

If there is a full set of pivots after exchanging rows, then A is invertible.

Example:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix} \xrightarrow[\text{Apply}]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 7 & 9 \end{bmatrix} \xrightarrow[\text{elimination with } l_{31}=2]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 7 \end{bmatrix} \xrightarrow[\text{elimination with } l_{32}=3]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$