# Honors Linear Algebra – Problem Set 9 Solutions

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## Problem 1

If $a = 2$, $A$ is a symmetric matrix, which can be written as $A = Q\Lambda Q^T$.

For a general $a$, the eigenvalues $\lambda$ of $A$ satisfy $-\lambda(3-\lambda) - 2a = 0 \Leftrightarrow \lambda^2 - 3\lambda - 2a = 0$. If the discriminant of this quadratic polynomial in $\lambda$ vanishes, the polynomial has a double root. The discriminant is $9 + 8a$, so the condition on $a$ is $a = -\frac{9}{8}$. When $a = -\frac{9}{8}$, $A$ has only one eigenvalue: $\lambda = \frac{3}{2}$. The eigenvectors associated with that eigenvalue are in the nullspace of

$$\begin{bmatrix} \frac{3}{2} & -\frac{9}{8} \\ 2 & -\frac{3}{2} \end{bmatrix}$$

We see that the second column is equal to $-3/4$ times the first column, so the rank of the matrix is 1, and its nullspace also has dimension 1: there is only one eigenvector for the eigenvalue $\lambda = \frac{3}{2}$. We conclude that the matrix is not diagonalizable, so that the factorization $A = S\Lambda S^{-1}$ is not possible. The answer is $a = -\frac{9}{8}$.

## Problem 2

We showed in Lecture 15 that the matrices $A$ and $A^TA$ have the same nullspace. They therefore have the same rank. Now, $A^TA$ is a symmetric matrix with real coefficient, so it can be diagonalized as

$$A^TA = Q\Lambda Q^T$$

where $\Lambda$ is a diagonal matrix with the eigenvalues of $A^TA$ on its diagonal (as many times as the algebraic multiplicity of each eigenvalue), and $Q$ is an orthogonal matrix. Since $Q$ is invertible, the rank of $A^TA$ is equal to the rank of $\Lambda$, which is equal to the number of nonzero eigenvalues counted with their order of multiplicity. So the rank of $A$ is equal to the rank of $\Lambda$, which is equal to the number of nonzero eigenvalues counted with their order of multiplicity.

## Problem 3

The $2 \times 2$ matrix associated with this ellipse is

$$\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

The eigenvalues $\lambda$ of this matrix satisfy $(2-\lambda)(5-\lambda) - 4 = 0 \Leftrightarrow \lambda^2 - 7\lambda + 6 = 0 \Leftrightarrow (\lambda - 6)(\lambda - 1) = 0$. The eigenvalues of the matrix are 1 and 6, so the half-lengths of the ellipse are 1 and $1/\sqrt{6}$.

The eigenvector corresponding to $\lambda = 1$ is in the nullspace of

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

We may for example take $\mathbf{u}_1 = (2, -1)$.

The eigenvector corresponding to $\lambda = 6$ is in the nullspace of

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$$

We may for example take $\mathbf{u}_2 = (1, 2)$.

So the axes of the ellipse have directions $(2, -1)$ and $(1, 2)$.

# Problem 4

$J$ is the matrix

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore,

$$J^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $J^2 - \lambda I$ is upper triangular, the eigenvalues of $J^2$ satisfy $-\lambda^5 = 0$. So the only eigenvalue of $J^2$ is $\lambda = 0$. The rank of $J^2$ is 3, so $J^2$ has two linearly independent eigenvectors corresponding to $\lambda = 0$. That means that the Jordan form $H$ of $J^2$ has two Jordan blocks.

There are two possibilities (to within a reordering of the blocks) for rank 3 matrices, as $J^2$ is: 1) a $3 \times 3$ block and a $2 \times 2$ block, as shown below

$$H_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2) A Jordan form made of a $4 \times 4$ block and a $1 \times 1$ block

$$H_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We will now show that the second matrix is not possible. Assume there is a matrix $M$ such that

$$J^2 = MH_2M^{-1}$$

Then $(J^2)^3 = MH_2M^{-1}MH_2M^{-1}MH_2M^{-1} = M(H_2)^3M^{-1} \Leftrightarrow M^{-1}(J^2)^3M = (H_2)^3$.

Now,

$$(J^2)^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

while

$$H_2^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The equality $M^{-1}(J^2)^3M = (H_2)^3$ cannot be satisfied, so we conclude that the Jordan form of $J^2$ is

$$H_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$