MATH-UA 140 - Linear Algebra

Lecture 5: Elimination in terms of matrix operations

The purpose of this lecture is to reexpress the two ideas covered in the previous lecture — Gaussian elimination and permutation — in terms of simple operations on the matrix A for the system. To do so, we start by learning a new key operation: multiplication of a matrix by another matrix

I] Matrix multiplication

1) Definition

Let A be an m-by-n matrix, and B be an n-by-p matrix, with m, n and p positive integers. Then $C = AB$ is an m-by-p matrix given by:

$$C = AB = A\left[\vec{b}_1 \ \vec{b}_2 \cdots \vec{b}_p\right] = \left[A\vec{b}_1 \ A\vec{b}_2 \cdots A\vec{b}_p\right]$$

where the vectors $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p$ are the columns of the matrix B, and the vector matrix products $A\vec{b}_1, \ldots, A\vec{b}_p$ have been defined in Lecture 3.

Example

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & -2 \\ 3 & 1 \end{bmatrix} = \left[\begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 1 & -6 \\ 3 & -11 \end{bmatrix}$$

2) The identity matrix

Just like the number 1 is the identity element (or neutral element) for multiplication between scalar, the matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

whose only nonzero entries are 1s on the diagonal is the identity element for matrix multiplication. Provided the size of the matrices are compatible, for any matrix A, we can write

$$AI = A \qquad IA = A$$

Illustration:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

3) Properties of matrix multiplication

Let A, B, and C be three matrices whose dimensions are compatible with the operations below, c be a scalar, and I the identity matrix (also with compatible dimensions). We have:

1. $A(BC) = (AB)C$ 2. $c(AB) = (cA)B = A(cB)$
3. $IA = A$, $AI = A$

Note that in general, $AB \neq BA$

Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$ while $BA = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$

II | Gaussian elimination in terms of matrix operations

1) Elimination matrices

Let us return to the example

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

seen in Lecture 4

The first step of the Gaussian elimination process was to subtract $\frac{1}{2}$ of the first equation from the second equation.

The vector $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ became $\vec{b}_{New} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix}$

This operation can be written in matrix form, $\vec{b}_{New} = E\vec{b}$ where E_{21} is the elimination matrix

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Indeed, for any vector $\vec{b} = (b_1, b_2, b_3)$,

$$E_{21}\vec{b} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ -\frac{b_1}{2} + b_2 \\ b_3 \end{bmatrix}$$

In our case, applying E to $\vec{b}$, you would indeed find $\vec{b}_{New}$ (I recommend you check!)

Our next step was to subtract $\frac{1}{2}$ of equation 1 from equation 3. This corresponds to the elimination matrix $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}$

You start to see the general pattern: the identity matrix I such that $I\vec{b} = \vec{b}$ for any vector $\vec{b}$ has 1's on the diagonal and zeros otherwise; the elimination matrix E_{ij} that subtracts a multiple l of row j from row i has the extra nonzero entry $-l$ in the i,j position, in addition to the diagonal of 1's.

We did not only apply the elimination algorithm to $\vec{b}$ of course; we also applied it to the left-hand side, i.e the matrix A. Let us see how that looks in matrix form:

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & \frac{3}{2} & -\frac{5}{2} \\ 1 & 1 & 2 \end{bmatrix}$$

The matrix on the right-hand side indeed corresponds to the intermediate system we had after the first elimination operation.

2) Permutation matrices

We saw that exchanging rows of a system was sometimes necessary for the elimination algorithm to function properly. Let us see what row exchanges look like in matrix form. Specifically, what matrix P_{12} exchanges row 1 and row 2? The answer is obtained by exchanging row 1 and 2 of the identity matrix

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Indeed, for any vector $\vec{b} = (b_1, b_2, b_3)$,

$$P_{12}\vec{b} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_1 \\ b_3 \end{bmatrix}$$

P_{12} exchanges the rows 1 and 2 of the column vector $\vec{b}$. Consequently, it also exchanges rows 1 and 2 of any matrix:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 5 \\ 2 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 6 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

In the example above, P_{21} does exactly what it was made for: it acts on $\begin{bmatrix} 0 & 6 & 5 \\ 2 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ to turn it into an upper triangular matrix.

General rule for P_{ij}: The permutation matrix P_{ij} which exchanges rows i and j of a matrix A when multiplied to A is obtained by exchanging rows i and j in the identity matrix

3) The augmented matrix

A slick way of visualizing operations necessary to solve $A\vec{x} = \vec{b}$ is to consider the augmented matrix which is made of the columns of A and of $\vec{b}$:

$$\text{Augmented matrix} = \begin{bmatrix} A & \vec{b} \end{bmatrix}$$

Elimination steps and permutation steps can then be visualized in one compact form by applying the appropriate matrix to the augmented matrix. For example, the first elimination step seen in II 1) results in the augmented matrix

$$\begin{bmatrix} E_{21} A & E_{21} \vec{b} \end{bmatrix}$$

More explicitly, the original augmented matrix is

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & -1 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

and the first elimination step results in

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & -1 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & \frac{3}{2} & -\frac{5}{2} & -\frac{1}{2} \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

which compactly summarizes the first step.

QUESTION: Write the augmented matrix $[A \ \vec{b}]$ for the linear system

$$\begin{cases} x - 2y - 3z = 0 \\ -x + y + 2z = 3 \\ 2y + z = -8 \end{cases}$$

Apply E_{21} and E_{32} to reach a triangular system, and solve by back substitution