MATH-UA 140 - Linear Algebra

Lecture 1 : Vectors and Linear Combinations

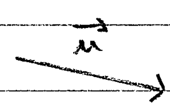
Vectors are a fundamental building block of linear algebra as we will study it in this course. We therefore start with a brief review of vectors in this lecture, as well as two operations on vectors which are at the heart of linear algebra: vector addition and multiplication by a scalar. In the following lecture, we will consider another operation between vectors, namely the dot product.

I) Vectors

1) Geometric representation

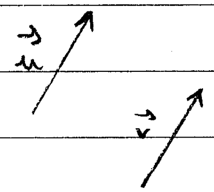
A vector is a mathematical object which has a magnitude (i.e a length) and a direction. They are therefore usually represented as arrows, the length of the arrow representing the magnitude.

EXAMPLE:



The vector $\vec{u}$ is 2cm long: its magnitude is 2cm

QUESTION: Are the vectors $\vec{u}$ and $\vec{v}$ below equal?



Answer: Yes, $\vec{u}$ and $\vec{v}$ are equal because they have the same direction and the same magnitude. Understanding this will help you to geometrically make sense of vector addition, which we will soon learn.

Vectors with the same direction and same magnitude are equal, wherever they are located in space.

Notational remark: Vectors are usually written with an arrow over their name, as we wrote $\vec{u}$. Often in textbooks you will see them written without an arrow but with a bold font.

2) Components

If the space the vectors live in (the plane, 3-D space, n-dimensional space, depending on the problem) is equipped with a Cartesian grid, then vectors can be represented algebraically, with real numbers.

In 2D (i.e in the plane), a vector is represented by two numbers: $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

In 3D (i.e in space), a vector is represented by three numbers:

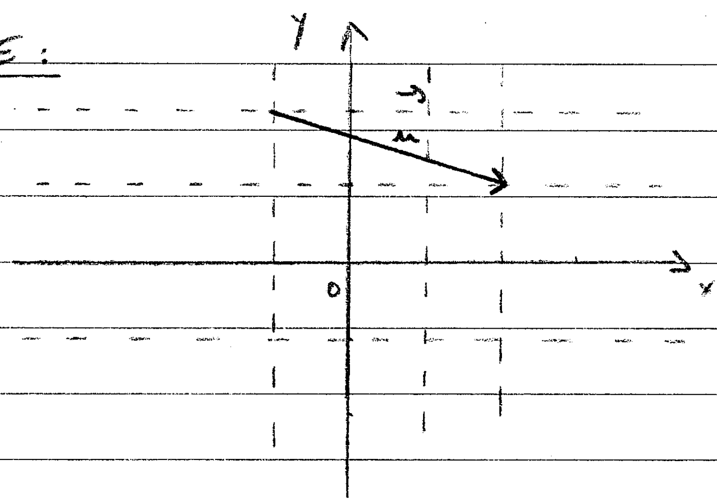
$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Notational remark: we write $\vec{u}$ as a column (not a row) for reasons that will be apparent later in this course. We may also often write $\vec{u}$ as follows: $\vec{u} = (u_1, u_2, u_3)$

The numbers u_1, u_2, u_3 are called the components of $\vec{u}$

They tell us by how much the x, y, and z-coordinates increase or decrease if one follows the vector from the beginning to the end.

EXAMPLE:



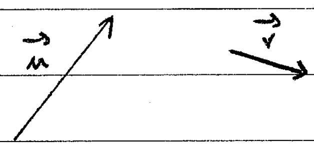
$$\vec{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

II] Vector operations

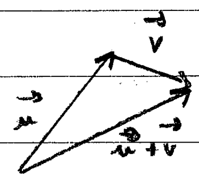
1) Definition of vector addition

If $\vec{u}$ and $\vec{v}$ are vectors positioned such that the initial point of $\vec{v}$ is at the terminal point of $\vec{u}$, then the sum $\vec{u}+\vec{v}$ is the vector from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$

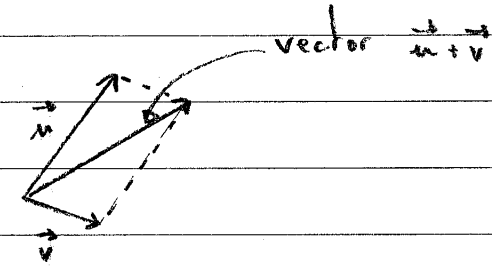
<u>Example:</u> Add the vectors $\vec{u}$ and $\vec{v}$ shown below



TRIANGLE LAW



PARALLELOGRAM LAW



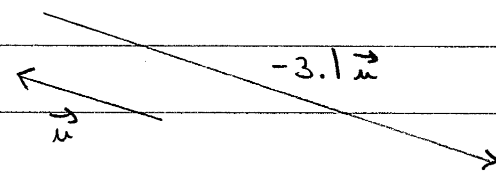
2) Definition of scalar multiplication

If c is a scalar (i.e a real number) and $\vec{u}$ is a vector, then the scalar multiple $c\vec{u}$ is the vector whose length is $|c|$ times the length of $\vec{u}$, and whose direction is the same as $\vec{u}$ if $c>0$, and is opposite to $\vec{u}$ if $c<0$.

$|\ |$ is the symbol for absolute value

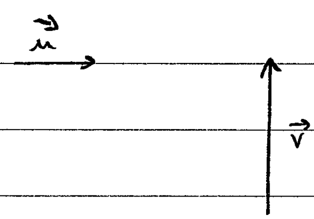
If $c=0$, or $\vec{u}=\vec{0}$, then $c\vec{u}=\vec{0}$

Example:



Subtracting vectors: Now that we know what the vector $-\vec{u}$ represents, it is easy to understand what subtracting a vector means: $u-\vec{v}=\vec{u}+(-\vec{v})$
In other words, to subtract $\vec{v}$ from $\vec{u}$, we just have to add $-\vec{v}$ to $\vec{u}$

QUESTION: What is the vector $\vec{u}-\frac{1}{2}\vec{v}$ for the vectors below?



Given the geometric interpretations we just gave, it is easy to prove the following properties:

3) Properties of vector addition and multiplication by a scalar

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are three vectors, and c and d two scalars, then

1. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
2. $\vec{u}+\vec{0}=\vec{u}$
3. $\vec{u}+(-\vec{u})=\vec{u}-\vec{u}=\vec{0}$
4. $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$
5. $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$
6. $(c+d)\vec{u}=c\vec{u}+d\vec{u}$
7. $(cd)\vec{u}=c(d\vec{u})$
8. $1\vec{u}=\vec{u}$

NOTE: It is a good exercise to visualize graphically what all these equalities means. For example, Property 1 should be fairly intuitive if one looks at the Parallelogram Law.

4) Vector operations in components

All the operations we just saw have an algebraic equivalent in terms of vector components:

* Vector addition: Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

Then: $\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$

and equivalent formulae for vectors in 2D, or general n-D.

* Scalar multiplication

Let c be a scalar, then

$$c\vec{u} = \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}$$

and equivalent formulae for vectors in 2D, or general n-D

QUESTION: Let $\vec{u} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$

What is $\vec{u} - \frac{1}{2}\vec{v}$?

III Linear combinations

Linear algebra is built on the two operations we just saw: adding vectors and multiplying by scalars. When we combine the two together, as we have done in examples above, we form what we call linear combinations.

Definition: Let $\vec{u}$ and $\vec{v}$ be two vectors, and c and d two scalars. The sum $c\vec{u} + d\vec{v}$ is a linear combination of $\vec{u}$ and $\vec{v}$.

Some linear combinations we have already encountered:

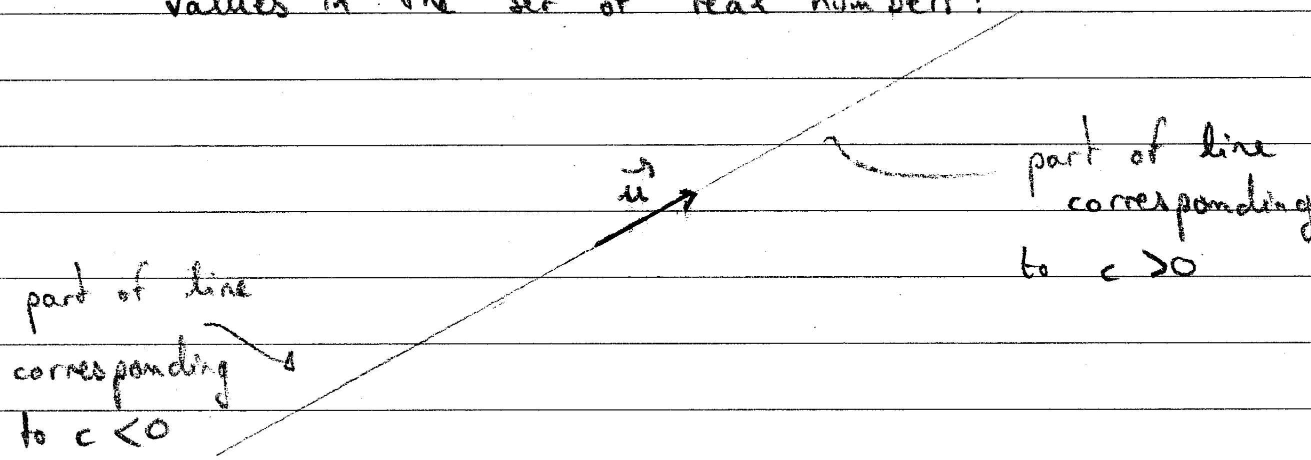
* $1\vec{u} + 1\vec{v} = \vec{u} + \vec{v}$ * $1\vec{u} + (-1)\vec{v} = \vec{u} - \vec{v}$
* $0\vec{u} + 0\vec{v} = \vec{0}$ $\left(= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ for a 3D vector} \right)$

If one has one vector $\vec{u}$, the only possible linear combinations have the form $c\vec{u}$, with c a scalar.
If one has two vectors $\vec{u}$ and $\vec{v}$, the possible linear combinations are $c\vec{u} + d\vec{v}$, c and d scalars.
If one has three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, the possible linear combinations are $c\vec{u} + d\vec{v} + e\vec{w}$, c, d, and e scalars.

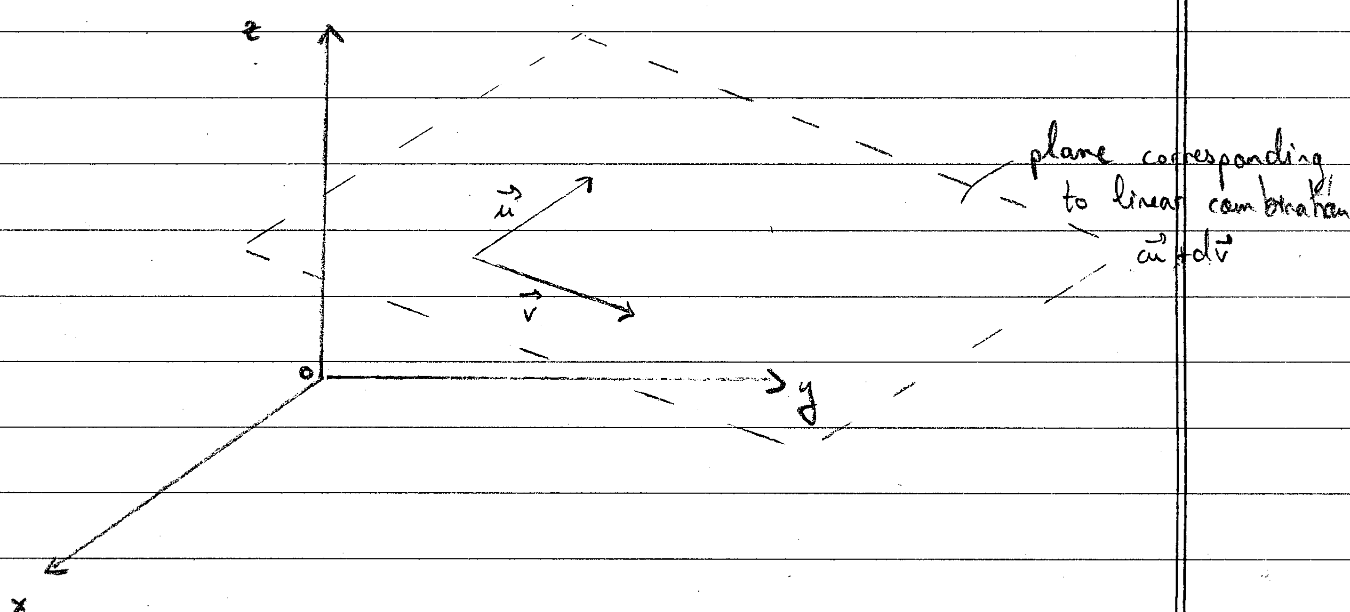
Finally, it is interesting to visualize the mathematical object which results from taking arbitrary linear combinations of vectors.

Let us for example consider three vectors $\vec{u}, \vec{v}$, and $\vec{w}$ in three-dimensional space.

* If $\vec{u}$ is not the zero vector $\vec{0}$, then the linear combinations $c\vec{u}$ describe a line as c takes all values in the set of real numbers:



* If both $\vec{u}$ and $\vec{v}$ are nonzero vectors, then the linear combinations $c\vec{u} + d\vec{v}$ describe a plane as c and d take all values in the set of real numbers



This is true unless $\vec{v}$ is aligned with $\vec{u}$, in which case the linear combinations $c\vec{u} + d\vec{v}$ just represent a line.

* If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are nonzero vectors, then the linear combination $c\vec{u} + d\vec{v} + e\vec{w}$ fill three-dimensional space as c, d, and e take values in the set of real numbers.

This is true unless $\vec{w}$ is in the plane of $\vec{u}$ and $\vec{v}$, otherwise the linear combinations $c\vec{u} + d\vec{v} + e\vec{w}$ only fill a plane.
Furthermore, if $\vec{u}$, $\vec{v}$, and $\vec{w}$ are aligned, $c\vec{u} + d\vec{v} + e\vec{w}$ only fill a line.

QUESTION: Describe geometrically all linear combinations of

a) $\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -6 \\ -2 \end{bmatrix}$

b) $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

QUESTION: Let $\vec{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Describe all points $c\vec{u} + d\vec{v}$ with c and d integers