

**Example 2.**  $f(x) = x^n$  where  $n = 1, 2, 3, \dots$

In this example we answer the question “What is  $\frac{d}{dx}x^n$ ?” Once we know the answer we can use it to, for example, find the derivative of  $f(x) = x^4$  by replacing  $n$  by 4.

At this point in our studies, we only know one tool for finding derivatives – the difference quotient. So we plug  $y = f(x)$  into the definition of the difference quotient:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{(x_0 + \Delta x)^n - x_0^n}{\Delta x}$$

Because writing little zeros under all our  $x$ ’s is a nuisance and a waste of chalk (or of photons?), and because there’s no other variable named  $x$  to get confused with, from here on we’ll replace  $x_0$  with  $x$ .

$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

Remember that when we use the difference quotient, we’re thinking of  $x$  as fixed and of  $\Delta x$  as getting closer to zero. We want to simplify this fraction so that we can plug in 0 for  $\Delta x$  without any danger of dividing by zero. To do this we must expand the expression  $(x + \Delta x)^n$ .

A famous formula called the **binomial theorem** tells us that:

$$(x + \Delta x)^n = (x + \Delta x)(x + \Delta x)\dots(x + \Delta x) \quad n \text{ times}$$

We can rewrite this as

$$x^n + n(\Delta x)x^{n-1} + O((\Delta x)^2)$$

where  $O(\Delta x)^2$  is shorthand for “all of the terms with  $(\Delta x)^2$ ,  $(\Delta x)^3$ , and so on up to  $(\Delta x)^n$ .”

One way to begin to understand this is to think about multiplying all the  $x$ ’s together from

$$(x + \Delta x)^n = (x + \Delta x)(x + \Delta x)\dots(x + \Delta x) \quad n \text{ times.}$$

There are  $n$  of these  $x$ ’s, so multiplying them together gives you one term of  $x^n$ . What if you only multiply together  $n - 1$  of the  $x$ ’s? Then you have one  $(x + \Delta x)$  left that you haven’t taken an  $x$  from, and you can multiply your  $x^{n-1}$  by  $\Delta x$ . (If you multiplied by  $x$ , you’d just have the  $x^n$  that you already got.) There were  $n$  different  $\Delta x$ ’s that you could have chosen to use, so you can get this result  $n$  different ways. That’s where the  $n(\Delta x)x^{n-1}$  comes from.

We could keep going, and figure out how many different ways there are to multiply  $n - 2$   $x$ ’s by two  $\Delta x$ ’s, and so on, but it turns out we don’t need to. Every other way of multiplying together one thing from each  $(x + \Delta x)$  gives you at least two  $\Delta x$ ’s, and  $\Delta x \cdot \Delta x$  is going to be too small to matter to us as  $\Delta x \rightarrow 0$ .

Now that we have some idea of what  $(x + \Delta x)^n$  is, let's go back to our difference quotient.

$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x} = \frac{(x^n + n(\Delta x)(x^{n-1}) + O(\Delta x)^2) - x^n}{\Delta x} = nx^{n-1} + O(\Delta x)$$

As it turns out, we *can* simplify the quotient by canceling a  $\Delta x$  in all of the terms in the numerator. When we divide a term that contains  $\Delta x^2$  by  $\Delta x$ , the  $\Delta x^2$  becomes  $\Delta x$  and so our  $O(\Delta x^2)$  becomes  $O(\Delta x)$ .

When we take the limit as  $x$  approaches 0 we get:

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = nx^{n-1}$$

and therefore,

$$\frac{d}{dx} x^n = nx^{n-1}$$

This result is sometimes called the “power rule”. We will use it often to find derivatives of polynomials; for example,

$$\frac{d}{dx} (x^2 + 3x^{10}) = 2x + 30x^9$$

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18.01SC Single Variable Calculus  
Fall 2010

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