18.06 Professor Edelman Final Exam December 22, 2011

Grading

1

2

3

Your PRINTED name is: ____________________ **4**

5

6

7

Please circle your recitation:

1	T 9	2-132	Kestutis Cesnavicius	2-089	2-1195	`kestutis`
2	T 10	2-132	Niels Moeller	2-588	3-4110	`moller`
3	T 10	2-146	Kestutis Cesnavicius	2-089	2-1195	`kestutis`
4	T 11	2-132	Niels Moeller	2-588	3-4110	`moller`
5	T 12	2-132	Yan Zhang	2-487	3-4083	`yanzhang`
6	T 1	2-132	Taedong Yun	2-342	3-7578	`tedyun`

1 (13 pts.)

Suppose the matrix A is the product

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) (3 pts.) What is the rank of A?

A has rank 2. (Since the first matrix is non-singular, it does not affect the rank.)

(b) (5 pts.) Give a basis for the nullspace of A.

$\begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ Columns 1 and 2 are pivot columns. The other two are free. We assign 1,0 and 0,1 to the free variables.

(c) (5 pts.) For what values of t (if any) are there solutions to $Ax = \begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix}$?

$t = 2$. Elimination on $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & t \end{pmatrix}$ yields $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & t-2 \end{pmatrix}$.

2 (12 pts.)

Let $A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$.

(a) (3 pts.) Find a basis for the column space of A.

> $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. This matrix is familar from class. The first two columns are pivot columns, the third is free.

(b) (3 pts.) Find a basis for the column space of Σ where $A = U\Sigma V^T$ is the svd of A.

> Σ is diagonal with first two diagonal elements positive. Hence a basis for the column space is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(c) (3 pts.) Find a basis for the column space of the matrix exponential e^A

> The matrix exponential has full rank, so the three columns of the identity or any linearly independent set of three vectors will do.

(d) (3 pts.) Find a non-zero constant solution (meaning no dependence on t) to $\frac{d}{dt}u(t) = Au(t)$.

> $\frac{d}{dt}u(t) = 0 = Au \implies u(t) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, the eigenvector corresonding to 0.

3 (12 pts.)

(a) (3 pts.) Give an example of a nondiagonalizable matrix A which satisfies $\det(tI - A) = (4-t)^4$

$$\begin{pmatrix} 4 & 1 & & \\ & 4 & 1 & \\ & & 4 & 1 \\ & & & 4 \end{pmatrix}$$

is a Jordan block hence is non-diagonalizable.

(b) (3 pts.) Give an example of two different matrices that are similar and both satisfy $\det(tI - A) = (1-t)(2-t)(3-t)(4-t)$.

$$\begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 4 & & & \\ & 3 & & \\ & & 2 & \\ & & & 1 \end{pmatrix}$$

(c) (3 pts.) Give an example if possible of two matrices that are not similar and both satisfy $\det(tI - A) = (1-t)(2-t)(3-t)(4-t)$.

All matrices with distinct eigenvalues 1,2,3,4 are similar, so this is impossible.

(d) (3 pts) Give an example of two different 4x4 matrices that have singular values 1,2,3,4.

$$\begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & & & \\ & -2 & & \\ & & -3 & \\ & & & -4 \end{pmatrix}$$

4 (16 pts.)

The matrix $G = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$.

(a) (3 pts.) This matrix has two eigenvalues $\lambda = 2$, and one eigenvalue $\lambda = -2$. Given that, find the fourth eigenvalue.

The trace is $2 - 2i = 2 + 2 - 2 + ?$ so the fourth eigenvalue is $-2i$.

(b) (3 pts.) Find a real eigenvector and show that it is indeed an eigenvector.

$G \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$. One can write down $G - 2I = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -2-i & -1 & i \\ 1 & -1 & -1 & -1 \\ 1 & i & -1 & -2-i \end{pmatrix}$ and notice that columns 1 and columns 3 add to 0.

(Problem 4 continued.) The matrix $G = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$.

(c) (4 pts.) Is G a Hermitian matrix? Why or why not. (Remember Hermitian means that $H_{jk} = \bar{H}_{kj}$ where the bar indicates complex conjugate.)

No, the diagonals are not real.

(d) (4 pts.) Give an example of a real non-diagonal matrix X for which $G^H X G$ is Hermitian.

Any real symmetrix non-diagonal X will do, for example $X == \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

5 (16 pts.)

The following operators apply to differentiable functions $f(x)$ transforming them to another function $g(x)$. For each one state clearly whether it is linear or not, (expalnations not needed). (2 pts each problem)

(a) $g(x) = \frac{d}{dx} f(x)$ linear (for all linear cases check $cf(x)$ goes to $cg(x)$ and $f_1(x) + f_2(x)$ goes to $g_1(x)+g_2(x)$

(b) $g(x) = \frac{d}{dx} f(x) + 2$ not linear (zero does not go to 0)

(c) $g(x) = \frac{d}{dx} f(2x)$ linear

(d) $g(x) = f(x + 2)$ linear

(e) $g(x) = f(x)^2$ not linear (the function $cf(x)$ should go to $cg(x)$ but it goes to $c^2 g(x)$.)

(f) $g(x) = f(x^2)$ linear

(g) $g(x) = 0$ linear

(h) $g(x) = f(x)+f(2)$ linear (don't be fooled, this one is indeed linear)

6 (20 pts.)

Let $A = I_3 - cE_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - c \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(a) (4 pts.) There are two values of c that make A a projection matrix. Find them by guessing, calculating, or understanding projection matrices. Check that A is a projection matrix for these two c.

$A = A^2 = I - 2cE + 3c^2E \implies 3c^2 = c$ so $c = 0$ or $c = 1/3$. Thus $A = I$ or $A = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ which upon squaring is itself.

(b) (4 pts.) There are two values of c that make A an orthogonal matrix. Find them and check that A is orthogonal for these two c.

$I = A^TA = A^2 = I - 2c + 3c^2E \implies 3c^2 = 2c$ so $c = 0$ or $c = 2/3$. Thus $A = I$ or $A = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$ which upon squaring is the identity.

(c) (4 pts.) For which values of c is A diagonalizable?

The matrix is symmetric, so all values of c make A diagonalizable.

(Problem 6 Continued) Let $A = I_3 - cE_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - c \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(d) (4 pts.) Find the eigenvalues of A^{-1} (if it exists) in terms of c. (Hint: find the eigenvalues of E_3 first.)

$E_3$is rank 1 and trace 3 so the eigenvalues are 3,0,0. Then A has eigenvalues 1-3c,1,1. Finally A^{-1} has eigenvalues $\frac{1}{1-3c}, 1, 1$.

(e) (4 pts.) For which values of c is A positive definite?

$\frac{1}{1-3c} > 0$ so $c < 1/3$.

7 (11 pts.)

The general equation of a circle in the plane has the form $x^2 + y^2 + Cx + Dy + E = 0$. Suppose you are trying to fit $n \geq 3$ distinct points (x_i, y_i), $i = 1, ..., n$ to obtain a "best" least squares circle, it is reasonable to write a generally unsolvable equation $Ax = b$.

(a) (7 pts.) Describe A and b clearly, indicating the number of rows and columns of A and the number of elements in b.

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & 1 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} -x_1^2 - y_1^2 \\ -x_2^2 - y_2^2 \\ \vdots \\ -x_n^2 - y_{n^2} \end{pmatrix}$$

. The matrix A has n rows and 3 columns, while b has n elements.

(b) (4 pts.) When $n = 3$ it is possible to describe when the equation is and is not solvable. You can use your geometric intuition, or a determinant area formula to describe when A is singular. Give a simple geometrical description. (We are looking for a specific word – so only a short answer will be accepted.)

A circle is determined by three points as long as they are not colinear. The matrix A is the area matrix for a triangle, when n=3, so the interpretation is that we can solve the equation, when the area of the triangle is not-zero, i.e. the triangle does not collapse to a line.