18.06 Exam III   Professor Strang   May 4, 2015

Your PRINTED Name is: ____________________

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R02	T10	26-322	Francesco Lin
R03	T11	26-302	Dmitry Vaintrob
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R12	T2pm	36-144	Tanya Khovanova
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Grading 1:

2:

3:

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1. (33 points)

(a) Suppose A has the eigenvalues $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -1$ with eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ in the columns of this $S = \left[\, \mathbf{x}_1 \,\middle|\, \mathbf{x}_2 \,\middle|\, \mathbf{x}_3 \,\right]$:

$$S = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

What are the eigenvalues and eigenvectors of the matrix $B = A^9 + I$?

(b) How could you find that matrix $B = A^9 + I$ using the eigenvectors in S and the eigenvalues $1, 0, -1$?

(c) Give a reason why the matrix B does have or doesn't have each of these properties:

i. B is invertible

ii. B is symmetric

iii. trace $= B_{11} + B_{22} + B_{33} = 3$.

2. (33 points)

(a) Show that $\lambda_1 = 0$ is an eigenvalue of A and find an eigenvector $\mathbf{x}_1$ with that zero eigenvalue:

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

(b) Find the other eigenvalues λ_2 and λ_3 of this symmetric matrix. Does A have two more independent eigenvectors $\mathbf{x}_2$ and $\mathbf{x}_3$? Give a reason why or why not. (Not required to find $\mathbf{x}_2$ and $\mathbf{x}_3$.)

(c) Suppose $\dfrac{d\mathbf{u}}{dt} = A\mathbf{u}$ starts from $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Explain why this $\mathbf{u}(t)$ approaches a steady state $\mathbf{u}(\infty)$ as $t \to \infty$. You can use the general formula $\mathbf{u}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + c_3 e^{\lambda_3 t}\mathbf{x}_3$ or $e^{At} = Se^{\Lambda t}S^{-1}$ without putting in all eigenvectors. **Find** that steady state $\mathbf{u}(\infty)$.

3. (34 points)

(a) If C is any symmetric matrix, show that e^C is a positive definite matrix. We can see that e^C is symmetric — which test will you use to show that e^C is positive definite?

(b) A is a 3 by 3 matrix. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are orthonormal eigenvectors (with eigenvalues $1, 2, 3$) of the symmetric matrix $A^T A$. Show that $A\mathbf{v}_1, A\mathbf{v}_2$, $A\mathbf{v}_3$ are orthogonal by rewriting and simplifying $(A\mathbf{v}_i)^T(A\mathbf{v}_j)$.

(c) For the 3 by 3 matrix A in part (b), find three matrices U, Σ, V that go into the Singular Value Decomposition $A = U\Sigma V^T$.

(d) TRUE or FALSE: If A is any symmetric 4 by 4 matrix and M is any invertible 4 by 4 matrix, then $B = M^{-1}AM$ is also symmetric. Give a reason for true or false.

Scrap Paper

Exam 3 Solutions

Question 1

(a) $A = S\text{diag}(1, 0, -1)S^{-1}$. Thus $B = A^9 + I = S(\text{diag}(1, 0, -1)^9 + I)S^{-1} = S\text{diag}(2, 1, 0)S^{-1}$. So B has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 0$ and the same eigenvectors as A.

(b) Write $A = S\text{diag}(1, 0, -1)S^{-1}$ and multiply to give $B = S\text{diag}(2, 1, 0)S^{-1}$.

(c) (i) B has 0 as an eigenvalue and so cannot be invertible.

(ii) B has distinct eigenvalues, with eigenvectors which are not orthogonal, and so it cannot be symmetric. (The point about distinct eigenvalues is not needed for full credit.)

(iii) True: the trace of B is the sum of the eigenvalues, $2 + 1 + 0 = 3$.

Question 2

(a) $A(1, 1, 1)^T = 0$ and so $x_1 = (1, 1, 1)^T$ is an eigenvector with eigenvalue $\lambda_1 = 0$.

(b) Each of the columns of $A + 3I$ is $(1, 1, 1)^T$ and so it is rank 1. In particular, the null space of $A + 3I$ has dimension 2 and so the other eigenvalues are $\lambda_2 = -3 = \lambda_3$.

(c) $u(t) = c_1e^{0t}x_1 + c_2e^{-3t}x_2 + c_3e^{-3t}x_3 \longrightarrow u(\infty) = c_1x_1$ as $t \longrightarrow \infty$. We just need to find c_1. But $u(0) = c_1x_1 + c_2x_2 + c_3x_3 = (1, 2, 3)^T$. Since x_1 is orthogonal to x_2 and x_3 we see, by dotting with x_1, that $c_1x_1 \cdot x_1 = (1, 2, 3)^T \cdot x_1$. Remembering that $x_1 = (1, 1, 1)^T$ we obtain $3c_1 = 6$ so that $c_1 = 2$ and $u(\infty) = (2, 2, 2)^T$.

Question 3

(a) We can write $C = Q\Lambda Q^T$ for some orthogonal matrix Q and some diagonal matrix Λ. Then $e^C = Qe^\Lambda Q^T$, which immediately shows that e^C is symmetric. If $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, then $e^\Lambda = \text{diag}(e^{\lambda_1}, \dots, e^{\lambda_n})$ so that each eigenvalue of e^C is of the form e^λ. In particular, it is positive, so that e^C is positive definite.

(b) $(Av_i)\cdot(Av_j) = (Av_i)^T(Av_j) = v_i^T(A^TA)v_j = jv_i^Tv_j = jv_i\cdot v_j$. Since v_1, v_2 and v_3 are orthogonal, we see that $(Av_i) \cdot (Av_j) = 0$ when $i \neq j$, i.e. Av_1, Av_2 and Av_3 are orthogonal.

(c) $V = (v_1|v_2|v_3)$, $\Sigma = \text{diag}(1, \sqrt{2}, \sqrt{3})$, and $U = (Av_1|\frac{Av_2}{\sqrt{2}}|\frac{Av_3}{\sqrt{3}})$.

(d) False. Take $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then $M^{-1}AM = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

In fact, any diagonal A with distinct eigenvalues, together with any M with nonorthogonal columns, will provide a counterexample.

Almost full credit for correctly saying false, e.g. just a rewording that says less about M. An unsymmetric B can be similar to a symmetric (diagonal) Λ as in question 1.