18.06 Quiz 3 Professor Strang May 6, 2011

Your PRINTED name is ____________________ **1.**

Your Recitation Instructor (and time) is ____________________ **2.**

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Please show enough work so we can see your method and give due credit.

1. (a) Find two eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}.$$

$p(\lambda) = \det(A - \lambda I) = \lambda^2 - 7\lambda + 10 = (\lambda-2)(\lambda-5) = 0 \Rightarrow \lambda_1 = 2$, $\lambda_2 = 5$

$N(A - 2I) = N\left(\begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix}\right) = \text{span}\left\{\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x_1}\right\}.$

$N(A - 5I) = N\left(\begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\left\{\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{x_2}\right\}.$

(b) Express any vector $u_0 = \begin{bmatrix} a \\ b \end{bmatrix}$ as a combination of the eigenvectors.

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S^{-1} u_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a-b \\ b \end{bmatrix}$

So $u_0 = c_1 x_1 + c_2 x_2 = (a-b)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c) What is the solution $u(t)$ to $\frac{du}{dt} = Au$ starting from $u(0) = u_0$?

$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = (a-b) e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

(d) Find a formula $u_k =$ ____________ for the solution to $u_{k+1} = Au_k$ which starts from that vector u_0. **Set $k = -1$ to find $A^{-1} u_0$.**

$u_k = A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 = (a-b) 2^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\, 5^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

$u_{-1} = (a-b) 2^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\, 5^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{a-b}{2} + \frac{b}{5} \\ \frac{b}{5} \end{bmatrix}$

2. This problem is about the matrix

$$A = \begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix}.$$

(a) Find all eigenvectors of A. Exactly why is it impossible to diagonalize A in the form $A = S\Lambda S^{-1}$?

$p(\lambda) = (\lambda - \sqrt{2})(\lambda - \sqrt{2}) = 0 \Rightarrow \lambda_1 = \lambda_2 = \sqrt{2}$ repeated eigenvalues

$N(A - \sqrt{2} I) = N\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

There are not enough independent eigenvectors to form an invertible matrix S with eigenvectors as its columns.

(b) Find the matrices U, Σ, V^T in the Singular Value Decomposition $A = U\,\Sigma\,V^T$.

Tell me *two orthogonal vectors* v_1, v_2 in the plane so that Av_1 and Av_2 are also orthogonal. $B = A^T A = \begin{bmatrix} \sqrt{2} & 0 \\ 1 & \sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$

eigenvalues of B

$\Rightarrow P_B(\lambda) = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4) = 0 \Rightarrow \lambda_1 = 4,\ \lambda_2 = 1$ for B.

$\Rightarrow \sigma_1 = \sqrt{\lambda_1} = 2,\ \sigma_2 = \sqrt{\lambda_2} = 1. \Rightarrow \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$

eigenvectors of B

$\begin{cases} N(B - 4I) = N\begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}\right\} \xRightarrow{\text{normalize}} v_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \\ N(B - I) = N\begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}\right\} \Rightarrow v_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}. \end{cases}$

$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}.$

$u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}.$

$A = [u_1 | u_2]\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}[v_1 | v_2]^T.$

$\boxed{v_1 \perp v_2 \text{ and } Av_1 \perp Av_2 \text{ because } u_1 \perp u_2.}$

(c) Find a matrix B that is similar to A (but different from A).

Show that A and B meet the requirement to be similar *(what is it?)*.

we say $B \underset{\text{similar}}{\sim} A$ if $B = MAM^{-1}$ for some invertible M.

choose for example $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. (You can choose any M you like!)

Then $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \cdots = \begin{bmatrix} \sqrt{2} & 0 \\ 1 & \sqrt{2} \end{bmatrix}$

and $B \neq A$ but $B \sim A$!

3. Suppose A is a real m by n matrix.

(a) Prove that the symmetric matrix A^TA has the property $x^T(A^TA)x \geq 0$ for every vector x in R^n. Explain each step in your reason.

$$x^T(A^TA)x = (x^TA^T)Ax = (Ax)^TAx = (Ax)\cdot(Ax) \geq 0.$$

(b) According to part (a), the matrix A^TA is positive semidefinite at least — and possibly positive definite. Under what condition on A is A^TA positive definite?

we want to see under what condition

$x^T(A^TA)x = 0$ implies $x = 0$.

So let $x^TA^TAx = 0$. By (a) we get $(Ax)\cdot(Ax) = 0$.

So $Ax = 0$. Hence to get $x = 0$ from $Ax = 0$,

we need $N(A) = \{0\}$ or A must have independent columns.

(c) If $m < n$ prove that A^TA is *not* positive definite.

we use (b) and show that if $m < n$ then $N(A) \neq \{0\}$.

OK! we know that $\dim N(A) = n - r$ where $r = \text{rank}(A)$.

But $r \leq m$. So

$\dim N(A) = n - r \geq n - m$. Since $m < n$, $n - m > 0$

therefore: $N(A) \neq \{0\}$.