Honors Linear Algebra – Problem Set 6 Solutions

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Problem 1

1. Any vector in the plane is in the nullspace of the matrix

$$M = \begin{bmatrix} 2 & 1 & -2 \end{bmatrix}$$

The reduced row echelon form of M is $[1\ 1/2\ -1]$ so a basis for the plane is $\mathbf{v}_1 = (-1/2, 1, 0)$ and $\mathbf{v}_2 = (1, 0, 1)$. Consider the matrix A whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$. $P = A(A^TA)^{-1}A^T$.

$$A^TA = \begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \frac{4}{9} \begin{bmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{4} \end{bmatrix}$$

Therefore,

$$P = \frac{4}{9} \begin{bmatrix} -\frac{1}{2} & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{4} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \frac{4}{9} \begin{bmatrix} -\frac{1}{2} & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 2 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{5}{4} \end{bmatrix} = \frac{4}{9} \begin{bmatrix} \frac{5}{4} & -\frac{1}{2} & 1 \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{5}{4} \end{bmatrix}$$

2. $\mathbf{e} = (2, 1, -2)$ is orthogonal to the plane. The projection matrix Q is given by

$$\frac{1}{||\mathbf{e}||^2}\mathbf{e}\mathbf{e}^T = \frac{1}{9} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix} = \frac{4}{9} \begin{bmatrix} 1 & \frac{1}{2} & -1 \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ -1 & -\frac{1}{2} & 1 \end{bmatrix}$$

Hence,

$$I - Q = \frac{4}{9} \begin{bmatrix} \frac{5}{4} & -\frac{1}{2} & 1 \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{5}{4} \end{bmatrix} = P$$

This is to be expected: we saw in class that any vector $\mathbf{b}$ in $\mathbf{R}^m$ can be written as $\mathbf{b} = \mathbf{p} + \mathbf{e}$, where $\mathbf{p}$ is in the column space of A, and $\mathbf{e}$ in the left nullspace of that matrix. We have $P\mathbf{b} = \mathbf{p}$ and $Q\mathbf{b} = \mathbf{e}$. Thus, $(P + Q)\mathbf{b} = P\mathbf{b} + Q\mathbf{b} = \mathbf{p} + \mathbf{e} = \mathbf{b}$. The matrix $P + Q$ is the identity.

Problem 2

The system of equations to be solved can be written as

$$\begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix} C = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

The second and third rows are incompatible, so we look for a solution in the least squares sense. Let us call $\mathbf{a}$ the vector multiplying C, and $\mathbf{b}$ the vector on the right. The best fit C is given by

$$C = \frac{\mathbf{a}^T\mathbf{b}}{||\mathbf{a}||^2} = \frac{1}{26}\begin{bmatrix} 0 & 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} = \frac{56}{13}$$

So the line going through the origin which is the best fit to the data points is given by $b = \frac{56}{13}t$. The answer is plotted in Figure 1, along with the data points.

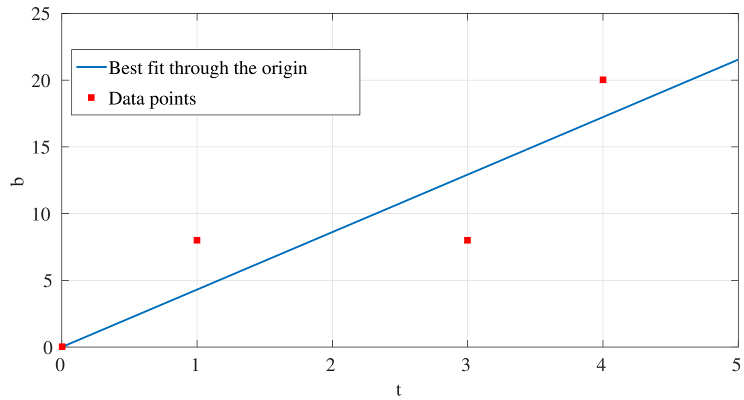

Figure 1: Line corresponding to the best fit found in this problem. The line is in blue, the data points are in red.

Problem 3

$$AP = (I - 2P)P = P - 2P^2 = P - 2P = -P$$

Now, we observe that

$$A^2 = A(I - 2P) = A - 2AP = A + 2P = I$$

Hence

$$A^{17} = A^{16}A = (A^2)^8 A = IA = A = I - 2P$$

Problem 4

Let $A = (a_{ij})1 \leq i, j \leq n$ be an orthogonal matrix, and let $\mathbf{u} = (1, 1, \ldots, 1)$.

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\right| = \left|\mathbf{u}^T \begin{bmatrix} \sum_{j=1}^{n} a_{1j} \\ \sum_{j=1}^{n} a_{2j} \\ \vdots \\ \sum_{j=1}^{n} a_{nj} \end{bmatrix}\right| = \left|\mathbf{u}^T A\mathbf{u}\right| = |\mathbf{u} \cdot (A\mathbf{u})|$$

Using the Cauchy-Schwarz inequality, we then have

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\right| \leq |A\mathbf{u}||\mathbf{u}| = |\mathbf{u}||\mathbf{u}| = |\mathbf{u}|^2$$

where we have used the fact that A is orthogonal to write $|A\mathbf{u}| = |\mathbf{u}|$. Now, $|\mathbf{u}|^2 = n$, so we have the desired result:

$$\left| \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \right| \leq n$$

The inequality in the result above came from our use of the Cauchy-Schwarz inequality. The Cauchy-Schwarz inequality becomes an equality when the two vectors are colinear:

$$\exists \lambda \in \mathbb{R} \text{ such that } A\mathbf{u} = \lambda \mathbf{u}$$

Now, taking the magnitude of the vectors on each side of the equality, we have

$$|A\mathbf{u}| = |\mathbf{u}| = |\lambda||\mathbf{u}|$$

Hence $\lambda = \pm 1$, which implies that

$$\forall i \in [\![1, n]\!] \ , \ \left| \sum_{j=1}^{n} a_{ij} \right| = 1$$

Problem 5

A turns into an upper triangular matrix if we exchange the second row and the third row. This corresponds to multiplying A with the permutation matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

We can thus write

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 \\ 0 & 0 & 5 \\ 0 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 5 \end{bmatrix}$$

Applying $P^{-1} = P$ on both sides, this can be rewritten as

$$\begin{bmatrix} 2 & 4 & 3 \\ 0 & 0 & 5 \\ 0 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 5 \end{bmatrix}$$

P is a permutation matrix, so we know that P is orthogonal, and this is the desired QR factorization of A.