V1. Plane Vector Fields

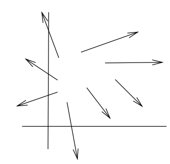
1. Vector fields in the plane; gradient fields.

We consider a function of the type

$$\mathbf{F}(x, y) \;=\; M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}\ . \tag{1}$$

where M and N are both functions of two variables. To each pair of values (x_0, y_0) for which both M and N are defined, such a function assigns a vector $\mathbf{F}(x_0, y_0)$ in the plane. $\mathbf{F}$ is therefore called a **vector function of two variables**. The set of points (x, y) for which $\mathbf{F}$ is defined is called the *domain* of $\mathbf{F}$.

To visualize the function $\mathbf{F}(x, y)$, at each point (x_0, y_0) in the domain we place the corresponding vector $\mathbf{F}(x_0, y_0)$ so that its tail is at (x_0, y_0). Thus each point of the domain is the tail end of a vector, and what we get is called a **vector field**. This vector field gives a picture of the vector function $\mathbf{F}(x, y)$.



Conversely, given a vector field in a region of the xy-plane, it determines a vector function of the type (1), by expressing each vector of the field in terms of its $\mathbf{i}$ and $\mathbf{j}$ components. Thus there is no real distinction between "vector function" and "vector field". Mindful of the applications to physics, in these notes we will mostly use "vector field". We will use the same symbol $\mathbf{F}$ to denote both the field and the function, saying "the vector field $\mathbf{F}$", rather than "the vector field corresponding to the vector function $\mathbf{F}$".

We say the vector field $\mathbf{F}$ is *continuous* in some region of the plane if both $M(x, y)$ and $N(x, y)$ are continuous functions in that region. The intuitive picture of a continuous vector field is that the vectors associated to points sufficiently near (x_0, y_0) should have direction and magnitude very close to that of $\mathbf{F}(x_0, y_0)$ — in other words, as you move around the field, the vectors should change direction and magnitude smoothly, without sudden jumps in size or direction.

In the same way, we say $\mathbf{F}$ is *differentiable* in some region if M and N are differentiable, that is, if all the partial derivatives

$$\frac{\partial M}{\partial x}\,, \qquad \frac{\partial M}{\partial y}\,, \qquad \frac{\partial N}{\partial x}\,, \qquad \frac{\partial N}{\partial y}$$

exist in the region. We say $\mathbf{F}$ is *continuously differentiable* in the region if all these partial derivatives are themselves continuous there. In general, all the commonly used vector fields are continuously differentiable, except perhaps at isolated points, or along certain curves. But as you will see, these points or curves affect the properties of the field in very important ways.

Where do vector fields arise in science and engineering?

One important way is as **gradient vector fields**. If

$$w = f(x,y) \tag{2}$$

is a differentiable function of two variables, then its *gradient*

$$\nabla w = \frac{\partial w}{\partial x}\,\mathbf{i} + \frac{\partial w}{\partial y}\,\mathbf{j} \tag{3}$$

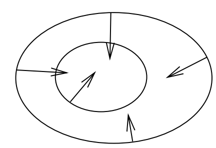
is a vector field, since both partial derivatives are functions of x and y. We recall the geometric interpretation of the gradient:

$$\begin{aligned} \text{dir } \nabla w &= \text{the direction } \mathbf{u} \text{ in which } \left.\frac{dw}{ds}\right|_{\mathbf{u}} \text{ is greatest;} \\ |\nabla w| &= \text{this greatest value of } \left.\frac{dw}{ds}\right|_{\mathbf{u}}, \end{aligned} \tag{4}$$

where $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = \nabla w \cdot \mathbf{u}$ is the directional derivative of w in the direction $\mathbf{u}$.

Another important fact about the gradient is that if one draws the contour curves of $f(x,y)$, which by definition are the curves

$$f(x,y) = c, \qquad c \text{ constant},$$

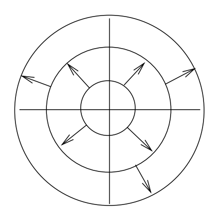


then at every point (x_0, y_0), the gradient vector ∇w at this point is perpendicular to the contour line passing through this point, i.e.,

(5) *the gradient field of f is perpendicular to the contour curves of f .*

Example 1. Let $w = \sqrt{x^2+y^2} = r$. Using the definition (3) of gradient, we find

$$\nabla w = \frac{x}{r}\,\mathbf{i} + \frac{y}{r}\,\mathbf{j} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{r}\ .$$



The domain of ∇w is the xy-plane with $(0,0)$ deleted, and it is continuously differentiable in this region. Since $|x\,i + y\,j| = r$, we see that $|\nabla w| = 1$. Thus all the vectors of the vector field ∇w are unit vectors, and they point radially outward from the origin. This makes sense by (4), since the definition of w shows that dw/ds should be greatest in the radially outward direction, and have the value 1 in that direction.

Finally, the contour curves for w are circles centered at $(0,0)$, which are perpendicular to the vectors ∇w everywhere, as (5) predicts.

2. Force and velocity fields.

Continuing our search for ways in which vector fields arise, here are two physical situations which are described mathematically by vector fields. We shall refer to them often in the sequel, using our physical intuition to suggest the sort of mathematical properties that vector fields ought to have.

Force fields.

From physics, we have the two-dimensional electrostatic force fields arising from a distribution of static (i.e., not moving) charges in the plane. At each point (x_0, y_0) of the plane, we put a vector representing the force which would act on a unit positive charge placed at that point.

In the same way, we get vector fields arising from a distribution of masses in the xy-plane, representing the gravitational force acting at each point on a unit mass. There are also the electromagnetic fields arising from moving electric charges and/or a distribution of magnets, representing the magnetic force at each point.

Any of these we shall simply refer to as a **force field**.

Example 2. Express in $\mathbf{i}-\mathbf{j}$ form the electrostatic force field $\mathbf{F}$ in the xy-plane arising from a unit positive charge placed at the origin, given that the force vector at (x, y) is directed radially away from the origin and that it has magnitude c/r^2, c constant.

Solution. Since the vector $x\,\mathbf{i} + y\,\mathbf{j}$ with tail at (x, y) is directed radially outward and has magnitude r, it has the right direction, and we need only change its magnitude to c/r^2. We do this by multiplying it by c/r^3, which gives

$$\mathbf{F} \;=\; \frac{cx}{r^3}\,\mathbf{i} + \frac{cy}{r^3}\,\mathbf{j} \;=\; c\,\frac{x\,\mathbf{i} + y\,\mathbf{j}}{(x^2+y^2)^{3/2}}\;.$$

Flow fields and velocity fields

A second way vector fields arise is as the steady-state *flow fields* and *velocity fields*.

Imagine a fluid flowing in a horizontal shallow tank of uniform depth, and assume that the flow pattern at any point is purely horizontal and not changing with time. We will call this a *two-dimensional steady-state flow* or for short, simply a *flow*. The fluid can either be compressible (like a gas), or incompressible (like water). We also allow for the possibility that at various points, fluid is beiong added to or subtracted from the flow; for instance, someone could be standing over the tank pouring in water at a certain point, or over a certain area. We also allow the density to vary from point to point, as it would for an unevenly heated gas.

With such a flow we can associate two vector fields.

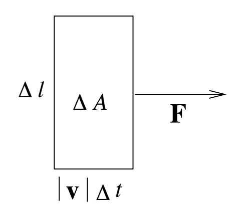
There is the **velocity field** $\mathbf{v}(x, y)$ where the vector $\mathbf{v}(x, y)$ at the point (x, y) represents the velocity vector of the flow at that point — that is, its direction gives the direction of flow, and its magnitude gives the speed of the flow.

Then there is the **flow field**, defined by

$$\mathbf{F} \;=\; \delta(x, y)\,\mathbf{v}(x, y) \tag{6}$$

where $\delta(x, y)$ gives the density of the fluid at the point (x, y), in terms of mass per unit area. Assuming it is not 0 at a point (x, y), we can interpret $\mathbf{F}(x, y)$ as follows:

$$\begin{aligned} \text{dir}\,\mathbf{F} &= \text{direction of fluid flow at } (x, y); \\ |\mathbf{F}| &= \begin{cases} \text{rate (per unit length per second) of mass transport} \\ \text{across a line perpendicular to the flow direction at } (x, y). \end{cases} \end{aligned} \tag{7}$$



Namely, we see that first by (6) and then by the picture,

$$|\mathbf{F}|\,\Delta l\,\Delta t \;=\; \delta\,|\mathbf{v}|\,\Delta t\,\Delta l \;=\; \text{mass in } \Delta A,$$

from which (7) follows by dividing by $\Delta l\,\Delta t$ and letting Δl and $\Delta t \to 0$.

If the density is a constant δ_0, as it would be for an incompressible fluid at a uniform temperature, then the flow field and velocity field are essentially the same, by (6) — the vectors of one are just a constant scalar multiple of the vectors of the other.

Example 3. Describe and interpret $\mathbf{F} \;=\; \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2+y^2}$ as a flow field and a force field.

Solution. As in Example 2, the field $\mathbf{F}$ is defined everywhere except $(0,0)$ and its direction is radially outward; now, however, its magnitude is r/r^2, i.e., $|\mathbf{F}| = 1/r$.

$\mathbf{F}$ is the *flow field* for a source of magnitude 2π at the origin. To see this, look at a circle of radius a centered at the origin. At each point P on this circle, the flow is radially outward and by (7),

$$\begin{aligned} \text{mass transport rate at P} \;&=\; \frac{1}{a}, \qquad \text{so that} \\ \text{mass transport rate across circle} \;&=\; \frac{1}{a}\cdot 2\pi a \;=\; 2\pi\;. \end{aligned}$$

This shows that in one second, 2π mass flows out through every circle centered at the origin. This is the flow field for a source of magnitude 2π at the origin — for example, one could imagine a narrow pipe placed over the tank, introducting 2π mass units per second at the point $(0,0)$.

We know that $|\mathbf{F}| \;=\; \delta\,|\mathbf{v}| \;=\; 1/r$. Two important cases are:

• if the fluid is incompressible, like water, then its density is constant, so the flow speed must decrease like $1/r$ — the flow outward gets slower the further you are from the origin;

• if it is compressible like a gas, and its flow speed stays constant, then the density must decrease like $1/r$.

We now interpret the same field as a *force field*.

Suppose we think of the z-axis in space as a long straight wire, bearing a uniform positive electrostatic charge. This gives us a vector field in space, representing the electrostatic force field.

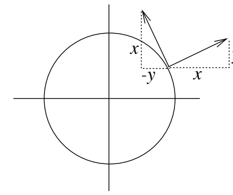
Since one part of the wire looks just like any other part, radial symmetry shows first that the vectors in the force field have 0 as their $\mathbf{k}$-component, i.e., they are pointed radially outward from the wire, and second that their magnitude depends only on their distance r from the wire. It can be shown in fact that the resulting force field is $\mathbf{F}$, up to a constant factor.

Such a field is called "two-dimensional", even though it is a vector field in space, because z and $\mathbf{k}$ don't enter into its description — once you know how it looks in the xy-plane, you know how it looks all through space.

The important thing to notice is that the magnitude of the force field in the xy-plane decreases like $1/r$, *not* like $1/r^2$, as it would if the charge were all at a point.

In the same way, the gravitational field of a uniform mass distribution along the z-axis would be $-\mathbf{F}$, up to a constant factor, and would be called a "two-dimensional gravitational field". Naturally, we don't have infinite long straight wires, but if you have a long straight wire, and stay away from its ends, or have only a short straight wire, but stay close to it, the force field will look like $\mathbf{F}$ near the wire.

Example 4. Find the velocity field of a fluid with density 1 in a shallow tank, rotating with constant angular velocity ω counterclockwise around the origin.



Solution. First we find the field direction at each point (x, y).

We know the vector $x\,\mathbf{i} + y\,\mathbf{j}$ is directed radially outward. Therefore a vector perpendicular to it in the counterclockwise direction (see picture) will be $-y\,\mathbf{i} + x\,\mathbf{j}$ (since its scalar product with $x\,\mathbf{i} + y\,\mathbf{j}$ is 0 and the signs are correct).

The preceding vector has magnitude r. If the angular velocity is ω, then the linear velocity is given by

$$|\mathbf{v}| = \omega r,$$

so to get the velocity field, we should multiply the above field by ω :

$$\mathbf{v} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j} .$$

Problems: Vector Fields

1. Sketch the following vector fields. Pay attention to their names because we will be encountering these fields frequently.

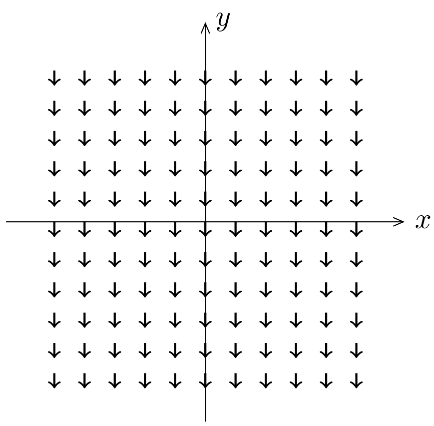
(a) Force, constant gravitational field $\mathbf{F}(x, y) = -g\mathbf{j}$.

(b) Velocity $\mathbf{V}(x, y) = \dfrac{x}{x^2 + y^2}\mathbf{i} + \dfrac{y}{x^2 + y^2}\mathbf{j} = \langle x, y\rangle / r^2$. (This is a shrinking radial field –like water pouring from a source at (0,0).)

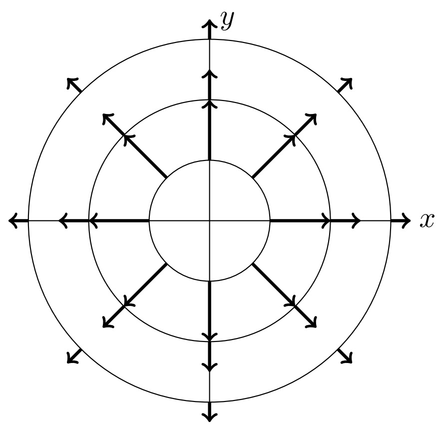
(c) Unit tangential field $\mathbf{F} = \langle -y, x\rangle / r$.

(d) Gradient $\mathbf{F} = \boldsymbol{\nabla} f$, where $f(x, y) = \dfrac{xy}{3}$ and $\boldsymbol{\nabla} f = \left\langle \dfrac{y}{3}, \dfrac{x}{3} \right\rangle$.

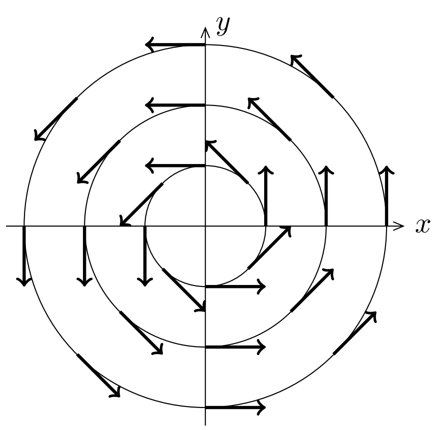
<u>Answer:</u> We visualize vector fields by drawing little arrows in the plane whose length and direction correspond to the magnitude and direction of the vector field at the base of the arrow.



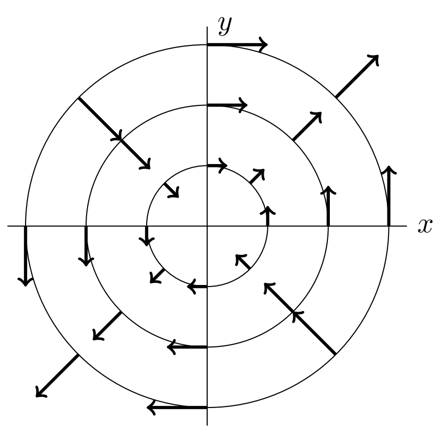
(a) Constant vector field



(b) Shrinking radial field



(c) Unit tangential field



(d) Gradient field $\boldsymbol{\nabla} f = \langle \frac{y}{3}, \frac{x}{3} \rangle$.

2. Compute the gradient field of $f(x, y) = xy^2$.

$$\begin{aligned}
\boldsymbol{\nabla} f(x,y) &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \\
&= y^2\,\mathbf{i} + 2xy\,\mathbf{j}.
\end{aligned}$$

Work and line integrals

Line integrals: (also called *path integrals*)
Ingredients:

Field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} = \langle M, N \rangle$

Curve C: $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = \langle x, y \rangle \ \Rightarrow \ d\mathbf{r} = \langle dx, dy \rangle$.

Line integral:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \langle M, N \rangle \cdot \langle dx, dy \rangle = \int_C M\,dx + N\,dy.$$

We need to discuss:

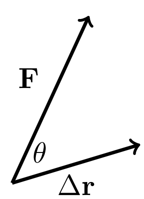
a) What this notation means and how line integrals arise.

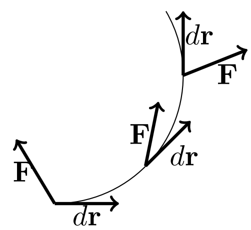
b) How to compute them.

c) Their properties and notation.

a) **How line integrals arise.**

The figure on the left shows a force $\mathbf{F}$ being applied over a displacement $\boldsymbol{\Delta}\mathbf{r}$. Work is force times distance, but only the component of the force in the direction of the displacement does any work. So, work $= |\mathbf{F}| \cos\theta\, |\Delta\mathbf{r}| = \mathbf{F} \cdot \Delta\mathbf{r}$.





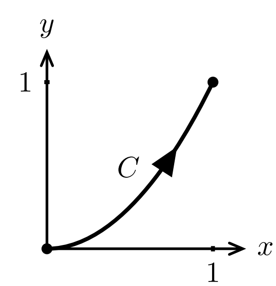
For a variable force applied over a curve the total work is found by 'summing' the infinitesimal pieces. We call this a line integral and denote it

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

It is illustrated in the above figure on the right.

b) **Computing line integrals.**

We show this by steps by example.



Example 1: Evaluate $I = \displaystyle\int_C x^2 y\,dx + (x - 2y)\,dy$

over the part of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.
First, parametrize the curve:

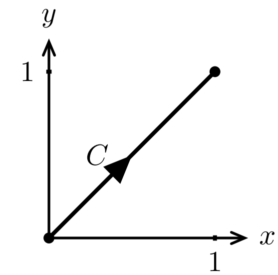
$$x = t,\ y = t^2, \quad 0 \le t \le 1.$$

Note, we specified the range of t to get exactly the part of the curve we wanted.
Next, compute the differentials of x and y:

$$dx = dt, \quad dy = 2t\,dt.$$

Finally substitute everything in the integral and compute the standard single variable integral:

$$I = \int_0^1 t^2(t^2)\,dt + (t-2t^2)2t\,dt = \int_0^1 t^4 + 2t^2 - 4t^3\,dt = -\frac{2}{15}.$$



Example 2: (Line integrals depend on the path.)
Same integral as previous example except C is the straight line from $(0,0)$ to $(1,1)$.
Parametrize curve: $x = t$, $y = t$, $0 \le t \le 1$. $\Rightarrow$ $dx = dt$, $dy = dt$

$$\Rightarrow\ I = \int_0^1 t^2 \cdot t\,dt + (t-2t)\,dt = \int_0^1 t^3 - t\,dt = -\frac{1}{4}.$$

Note: this is a different value from example 1 and illustrates the very important fact that, in general, the line integral depends on the path. Later we will learn how to spot the cases when the line integral will be independent of path.

Example 3: (Line integrals are independent of the parametrization.)
Here we do the same integral as in example 1 except use a different parametrization of C.
Parametrize C: $x = \sin t$, $y = \sin^2 t$, $0 \le t \le \pi/2$ $\Rightarrow$ $dx = \cos t\,dt$, $dy = 2\sin t\cos t\,dt$.

$$\begin{aligned}\Rightarrow\ I &= \int_0^{\pi/2} \sin^4 t\cos t\,dt + (\sin t - 2\sin^2 t)2\sin t\cos t\,dt \\ &= \int_0^{\pi/2} (\sin^4 t + 2\sin^2 t - 4\sin^3 t)\cos t\,dt = \frac{\sin^5 t}{5} + \frac{2}{3}\sin^3 t - \sin^4 t\Big|_0^{\pi/2} = -\frac{2}{15}.\end{aligned}$$

This is same value as example 1 and illustrates the very important point that the line integral is independent of how the curve is parametrized.

c) **Properties and notation of line integrals**

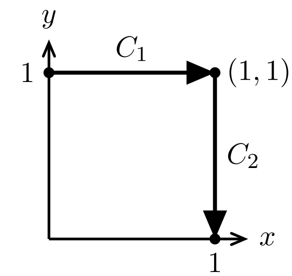
1. They are independent of parametrization.
2. If you reverse direction on curve then the line integral changes sign. That is,

$$\int_{-C} \mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\mathbf{r}.$$

($-C$ means the same curve traversed in the opposite direction.)

3. If C is closed we use the notation $\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C M\,dx + N\,dy$.

Example 4: Evaluate $I = \int_C y\,dx + (x+2y)\,dy$ where C is the curve shown.



<u>Answer:</u> The curve has two pieces so the integral will also

$$I = \int_{C_1} y\,dx + (x+2y)\,dy + \int_{C_2} y\,dx + (x+2y)\,dy.$$

Here we see that we don't always need to introduce a new variable t.

$C_1:$ $y = 1$, use x as parameter. $0 \le x \le 1 \;\Rightarrow\; dx = dx,\ dy = 0.$

Substituting in the integral $\Rightarrow \displaystyle\int_{C_1} y\,dx + (x+2y)\,dy = \int_0^1 dx = 1.$

$C_2:$ $x = 1$, use y as parameter. y goes from 1 to 0 and $dx = 0$

$\Rightarrow \displaystyle\int_{C_2} y\,dx + (x+2y)\,dy = \int_1^0 (1+2y)\,dy = -\int_0^1 1 + 2y\,dy = -2.$

We get $I = 1 - 2 = -1$.

Problems: Work and Line Integrals

1. Evaluate $I = \int_C y\,dx + (x+2y)\,dy$ where C is the curve shown.

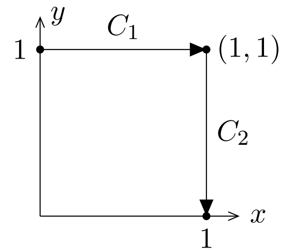


Figure 1: Curve C is C_1 followed by C_2.

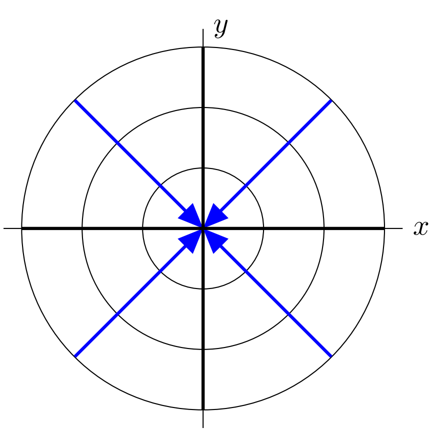
<u>**Answer:**</u> The curve C is made up of two pieces, so

$$I = \int_{C_1} y\,dx + (x+2y)\,dy + \int_{C_2} y\,dx + (x+2y)\,dy.$$

Note that we don't always need to introduce the variable t.
$C_1:\quad y = 1$, use x as parameter. $0 \le x \le 1 \;\Rightarrow\; dx = dx,\ dy = 0$.
$\Rightarrow \displaystyle\int_{C_1} y\,dx + (2+2y)\,dy = \int_0^1 dx = 1.$
$C_2:\quad x = 1$, use y as parameter. y goes from 1 to 0.
$\Rightarrow \displaystyle\int_{C_2} y\,dx + (2+2y)\,dy = \int_1^0 (1+2y)\,dy = -\int_0^1 1+2y\,dy = -2.$
So $I = 1 - 2 = -1$.

2. Let $\mathbf{F} = -x\mathbf{i} - y\mathbf{j}$. Sketch this vector field and describe it in words.

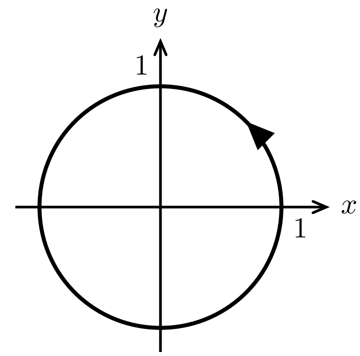
<u>**Answer:**</u>



Each arrow starts at (x,y) and ends at the origin. The further a vector in this field is from $(0,0)$, the longer it is.

Geometric Approach to Line Integrals

Line integrals are intrinsically geometric, so we should sometimes be able to use geometric reasoning to avoid the tedious calculations used in computing certain line integrals. The geometry can also give us some insight into the situation that calculation sometimes obscures.



We start with a line integral that we compute directly,

Example 1: Evaluate $I = \oint_C -y\,dx + x\,dy$ where C is the unit circle traversed in a counterclockwise (CCW) direction.
Parametrization: $x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$.
$\Rightarrow\; dx = -\sin t\,dt,\; dy = \cos t\,dt$.
$\Rightarrow\; I = \int_0^{2\pi} -\sin t(-\sin t)\,dt + \cos t(\cos t)\,dt = \int_0^{2\pi} dt = 2\pi$.

The intrinsic formula Recall that we know $\dfrac{d\mathbf{r}}{dt} = \mathbf{T}\dfrac{ds}{dt}$, where $\mathbf{T}$ = unit tangent and s = arclength. Removing the dt gives $d\mathbf{r} = \mathbf{T}\,ds$. We can use this in our formula for line integrals and get a form that we call the *intrinsic formula*

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds.$$

Example 2: Redo example 1 using the intrinsic formula: $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$.
$\mathbf{T}$ = unit tangent $\;\Rightarrow\; \mathbf{T} = -y\,\mathbf{i} + x\,\mathbf{j}$ (on the unit circle $x^2+y^2=1$).
$\Rightarrow\; \mathbf{F}\cdot\mathbf{T} = y^2 + x^2 = 1$ (on the unit circle) $\;\Rightarrow\; I = \int_C ds$ = arclength of circle $= 2\pi$.
Lesson: it can pay to think geometrically.

Problems: Geometric Approach to Line Integrals

1. Let $\mathbf{F}(x, y) = e^x y\,\mathbf{i}$ describe a force field. Show without computation that the work integral along the line segment from $(2, 0)$ to $(2, 4)$ is 0.

<u>**Answer:**</u> Since the vector $d\mathbf{r}$ points in the $\mathbf{j}$ direction we have $\mathbf{F} \cdot d\mathbf{r} = 0$. Therefore $\displaystyle\int \mathbf{F} \cdot d\mathbf{r} = 0$.

2. Let C be the curve $g(x, y) = x^3 y + xy^3 = 5$. Find $\displaystyle\int_C \boldsymbol{\nabla} g \cdot d\mathbf{r}$.

<u>**Answer:**</u> Since C is a level curve for G we know $\boldsymbol{\nabla} g \cdot d\mathbf{r} = 0$. Therefore, $\displaystyle\int_C \boldsymbol{\nabla} g \cdot d\mathbf{r} = 0$.

Work integrals

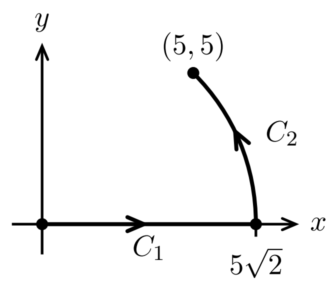
1. Let C be the path from (0,0) to (5,5) consisting of the straight line from (0,0) to $(5\sqrt{2}, 0)$ followed by the arc from $(5\sqrt{2}, 0)$ to (5,5) that is part of the circle of radius $5\sqrt{2}$ centered at the origin.

Compute $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ for the following vector fields $\mathbf{F}$

a) $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$; b) $\mathbf{F} = x\,\mathbf{j}$.

(Remember to work smart and exploit geometry where possible.)

<u>**Answer:**</u> We have $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.



a) We note that $\mathbf{F} \perp C_2$ everywhere, so $\displaystyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0 \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}$.

On C_1 we have $y = 0$, so $dy = 0$, and x runs from 0 to $5\sqrt{2}$. Taking $M = x$, $N = y$ we have

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} M\,dx + N\,dy = \int_{C_1} x\,dx + y\,dy = \int_0^{5\sqrt{2}} x\,dx = 25.$$

The work integral = 25.

b) In this case, $\mathbf{F} \perp C_1$, so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} x\,dy.$$

We parametrize C_2 by $x = 5\sqrt{2}\cos t$; $y = 5\sqrt{2}\sin t$; $0 \le t \le \pi/4$. This gives

$$\int_{C_2} x\,dy = \int_0^{\pi/4} 50\cos^2 t\,dt = \int_0^{\pi/4} 50\left(\frac{1 + \cos 2t}{2}\right)\,dt = \frac{25\pi}{4} + \frac{25}{2}.$$

The work integral is $\dfrac{25\pi}{4} + \dfrac{25}{2}$.

Fundamental Theorem for Line Integrals

Gradient fields and potential functions

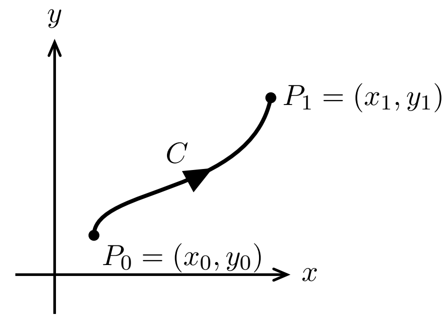
Earlier we learned about the gradient of a scalar valued function

$$\nabla f(x,y) = \langle f_x, f_y \rangle.$$

For example, $\nabla x^3y^4 = \langle 3x^2y^4, 4x^3y^3 \rangle$.

Now that we know about vector fields, we recognize this as a special case. We will call it a *gradient field*. The function f will be called a *potential function* for the field.

For gradient fields we get the following theorem, which you should recognize as being similar to the fundamental theorem of calculus.



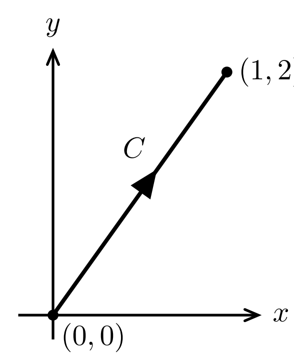
Theorem (Fundamental Theorem for line integrals)
If $\mathbf{F} = \nabla f$ is a gradient field and C is *any* curve with endpoints $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(x,y)|_{P_0}^{P_1} = f(x_1, y_1) - f(x_0, y_0).$$

That is, for *gradient fields* the line integral is independent of the path taken, i.e., it depends only on the endpoints of C.

Example 1: Let $f(x,y) = xy^3 + x^2 \;\Rightarrow\; \mathbf{F} = \nabla f = \langle y^3 + 2x, 3xy^2 \rangle$

Let C be the curve shown and compute $I = \int_C \mathbf{F} \cdot d\mathbf{r}$.



Do this both directly (as in the previous topic) and using the above formula.
Method 1: parametrize C: $x = x$, $y = 2x$, $0 \le x \le 1$.

$$\Rightarrow I = \int_C (y^3 + 2x)\,dx + 3xy^2\,dy = \int_0^1 (8x^3 + 2x)\,dx + 12x^3 2\,dx$$
$$= \int_0^1 32x^3 + 2x\,dx = 9.$$

Method 2: $\int_C \nabla f \cdot d\mathbf{r} = f(1,2) - f(0,0) = 9.$

Proof of the fundamental theorem

$$\begin{aligned}\int_C \nabla f \cdot d\mathbf{r} &= \int_C f_x\,dx + f_y\,dy = \int_{t_0}^{t_1} \left[f_x(x(t),y(t))\,\frac{dx}{dt} + f_y(x(t),y(t))\,\frac{dy}{dt} \right] dt \\ &= \int_{t_0}^{t_1} \frac{d}{dt} f(x(t),y(t))\,dt = f(x(t),y(t))|_{t_0}^{t_1} = f(P_1) - f(P_0) \quad \blacksquare\end{aligned}$$

The third equality above follows from the chain rule.

Significance of the fundamental theorem

For gradient fields $\mathbf{F}$ the work integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of the path.
We call such a line integral *path independent*.
The special case of this for closed curves C gives:

$$\mathbf{F} = \nabla f \;\Rightarrow\; \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \quad \text{(proof below)}.$$

Following physics, where a conservative force does no work around a closed loop, we say $\mathbf{F} = \nabla f$ is a *conservative* field.

Example 1: If $\mathbf{F}$ is the electric field of an electric charge it is conservative.

Example 2: The gravitational field of a mass is conservative.

Differentials: Here we can use differentials to rephrase what we've done before. First recall:

a) $\nabla f = f_x\,\mathbf{i} + f_y\,\mathbf{j} \;\Rightarrow\; \nabla f \cdot d\mathbf{r} = f_x\,dx + f_y\,dy$.

b) $\displaystyle\int_C \nabla f \cdot d\mathbf{r} = f(P_1) - f(p_0)$.

Using differentials we have $df = f_x\,dx + f_y\,dy$. (This is the same as $\nabla f \cdot d\mathbf{r}$.) We say $M\,dx + N\,dy$ is an *exact differential* if $M\,dx + N\,dy = df$ for some function f.

As in (b) above we have $\displaystyle\int_C M\,dx + N\,dy = \int_C df = f(P_1) - f(P_0)$.

Proof that path independence is equivalent to conservative

We show that

$$\int_C \mathbf{F} \cdot d\mathbf{r} \quad \text{is path independent for any curve } C$$

is equivalent to

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \quad \text{for any closed path.}$$

This is not hard, it is really an exercise to demonstrate the logical structure of a proof showing equivalence. We have to show:

i) Path independence $\Rightarrow$ the line integral around any closed path is 0.

ii) The line integral around all closed paths is 0 $\Rightarrow$ path independence.

i) Assume path independence and consider the closed path C shown in figure (i) below. Since the starting point P_0 is the same as the endpoint P_1 we get $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = f(P_1) - f(P_0) = 0$ (this proves (i)).

ii) Assume $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve. If C_1 and C_2 are both paths between P_0 and P_1 (see fig. 2) then $C_1 - C_2$ is a closed path. So by hypothesis

$$\oint_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0 \;\Rightarrow\; \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

That is the line integral is path independent, which proves (ii).

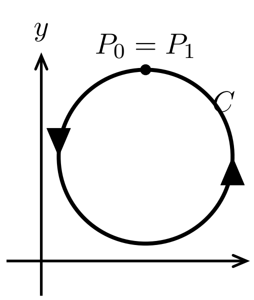


Figure (i)

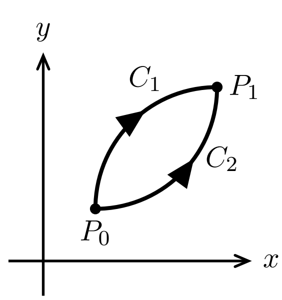


Figure (ii)

Fundamental Theorem for Line Integrals

1. Let $f = xy + e^x$.

a) Compute $\mathbf{F} = \nabla f$.

b) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for each of the following paths from (0,0) to (2,1).

i) The path consisting of a horizontal segment followed by a vertical segment.

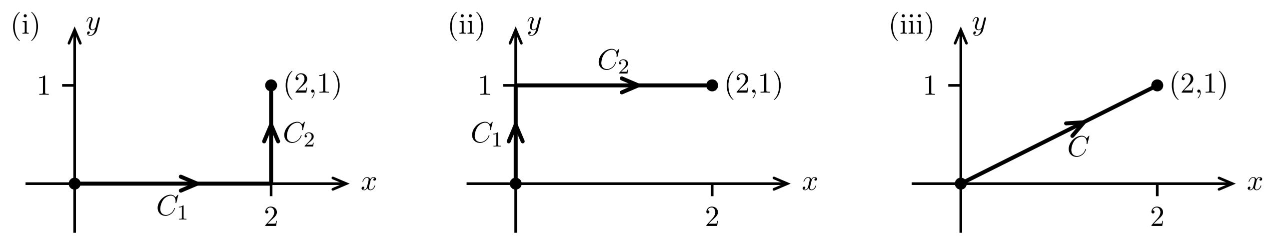
ii) The path consisting of a vertical segment followed by a horizontal segment.

iii) The straight line from (0,0) to (2,1).

c) All of the answers to part (b) should be the same. Show they agree with the answer given by the fundamental theorem for line integrals.

Answer: a) $\mathbf{F} = \nabla f = \langle f_x, f_y \rangle = \langle y + e^x, x \rangle$.

b) We have $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (y + e^x)\, dx + x\, dy$.



i) The curve C has two pieces C_1 and C_2. We compute the integral over each piece separately.

C_1: x runs from 0 to 2; $y = 0$, $dy = 0$. So,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 e^x\, dx = e^2 - 1.$$

C_2: $x = 2$, $dx = 0$; y runs from 0 to 1. So,

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 2\, dy = 2.$$

Summing the two pieces: $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1 + C_2} \mathbf{F} \cdot d\mathbf{r} = e^2 + 1.$

ii) This is similar to part (i).

C_1: $x = 0$, $dx = 0$; y runs from 0 to 1. So,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 0\, dy = 0.$$

C_2: x runs from 0 to 2; $y = 1$, $dy = 0$. So,

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 1 + e^x\, dx = 2 + e^2 - 1 = 1 + e^2.$$

Summing the two pieces: $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1 + C_2} \mathbf{F} \cdot d\mathbf{r} = e^2 + 1.$

iii) We parametrize C by $x = 2t;\quad y = t;\quad t$ runs from 0 to 1 $\Rightarrow\ dx = 2\,dt,\ dy = dt$. Thus,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (t + \mathrm{e}^{2t})2\,dt + 2t\,dt = \int_0^1 (4t + 2\mathrm{e}^{2t})\,dt = 2t^2 + \mathrm{e}^{2t}\Big|_0^1 = 2 + \mathrm{e}^2 - 1 = 1 + \mathrm{e}^2.$$

c) The fundamental theorem for line integrals says (for any of the paths in part (b))

$$\int_C \boldsymbol{\nabla} f\cdot d\mathbf{r} = f(2,1) - f(0,0) = 2 + \mathrm{e}^2 - 1 = 1 + \mathrm{e}^2.$$

All the answers agree.

Line Integrals of Vector Fields

In lecture, Professor Auroux discussed the non-conservative vector field

$$\mathbf{F} = \langle -y, x \rangle.$$

For this field:

1. Compute the line integral along the path that goes from $(0,0)$ to $(1,1)$ by first going along the x-axis to $(1,0)$ and then going up one unit to $(1,1)$.

Answer: To compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ we break the curve into two pieces, then add the line integrals along each piece.

First, fix $y = 0$ (so $dy = 0$) and let x range from 0 to 1.

$$\int_{x=0}^{x=1} \mathbf{F} \cdot d\mathbf{r} = \int_{x=0}^{x=1} -y\,dx + x\,dy = \int_0^1 0\,dx = 0.$$

Next, fix $x = 1$ (so $dx = 0$) and let y range from 0 to 1:

$$\int_{y=0}^{y=1} \mathbf{F} \cdot d\mathbf{r} = \int_{y=0}^{y=1} -y\,dx + 1\,dy = 1.$$

We conclude that $\int_C \mathbf{F} \cdot d\mathbf{r} = 1$.

2. Compute the line integral along the path from $(0,0)$ to $(1,1)$ that first goes up the y-axis to $(0,1)$.

Answer: Again we split the curve into two parts. We start by fixing $x = 0$ (so $dx = 0$) and letting y range from 0 to 1:

$$\int_{y=0}^{y=1} -y\,dx + 0\,dy = 0.$$

Next fix $y = 1$ and let x range from 0 to 1:

$$\int_{x=0}^{x=1} -1\,dx + x\,dy = -x\Big|_0^1 = -1.$$

Here, $\int_C \mathbf{F} \cdot d\mathbf{r} = -1$.

3. Should you expect your answers to the preceding problems to be the same? Why or why not?

Answer: If $\mathbf{F}$ were conservative, the value of a line integral starting at $(0,0)$ and ending at $(1,1)$ would be independent of the path taken. We know from lecture that $\mathbf{F}$ is non-conservative, so we don't expect line integrals along different paths to have the same values.

4. Compute the line integral of $\mathbf{F}$ along a path that runs counterclockwise around the unit circle.

Answer: We parametrize C by $x = \cos\theta$, $y = \sin\theta$ with $0 \le \theta < 2\pi$. Then $dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$, and:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\theta=0}^{\theta=2\pi} -\sin\theta\, dx + \cos\theta\, dy = \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\, d\theta = 2\pi.$$

5. Should your answer to the previous problem be 0? Why or why not?

Answer: The vector field is not conservative, so its line integral around a closed curve need not be zero.

Answer the following questions for the field

$$\mathbf{F} = \langle 0, x \rangle.$$

6. Compute the line integral along the path that goes from $(0, 0)$ to $(1, 1)$ by first going along the x-axis to $(1, 0)$ and then going up one unit to $(1, 1)$.

Answer: We split the curve into two pieces as in problem (1).

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left(\int_{x=0}^{x=1} 0\, dx + x\, dy \right) + \left(\int_{y=0}^{y=1} 0\, dx + 1\, dy \right) = 1.$$

7. Compute the line integral along the path from $(0, 0)$ to $(1, 1)$ which first goes up the y-axis to $(0, 1)$.

Answer: Proceed as in problem (2):

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left(\int_{y=0}^{y=1} 0\, dx + 0\, dy \right) + \left(\int_{x=0}^{x=1} 0\, dx + x\, dy \right) = 0.$$

8. Compute the line integral of $\mathbf{F}$ along the line segment from $(0, 0)$ to $(1, 1)$.

Answer: Parametrize C by $x = y = t$ where $0 \le t \le 1$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=0}^{t=1} 0\, dt + t\, dt = \left. \frac{t^2}{2} \right|_0^1 = 1/2.$$

9. Is the vector field $\mathbf{F} = \langle 0, x \rangle$ conservative? How do you know?

Answer: The field $\mathbf{F}$ is not conservative. If it were, the line integrals in problems 6, 7 and 8 would depend only on the endpoints of C and so would have the same values.

V2.1 Gradient Fields and Exact Differentials

1. Criterion for gradient fields.

Let $\mathbf{F} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$ be a two-dimensional vector field, where M and N are continuous functions. There are three equivalent ways of saying that $\mathbf{F}$ is conservative, i.e., a gradient field:

$$(1) \qquad \mathbf{F} = \nabla f \;\Leftrightarrow\; \int_P^Q \mathbf{F}\cdot d\mathbf{r} \;\text{ is path-independent } \;\Leftrightarrow\; \oint_C \mathbf{F}\cdot d\mathbf{r} = 0 \;\text{ for any closed } C$$

Unfortunately, these equivalent formulations don't give us any effective way of deciding if a given field $\mathbf{F}$ is a conservative field or not. However, if we assume that $\mathbf{F}$ is not just continuous but is even continuously differentiable (meaning: M_x, M_y, N_x, N_y all exist and are continuous), then there is a simple and elegant criterion for deciding whether or not $\mathbf{F}$ is a gradient field in some region.

Criterion. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ be continuously differentiable in a region D. Then, in D,

$$(2) \qquad \mathbf{F} = \nabla f \;\text{ for some } f(x,y) \quad \Rightarrow \quad M_y = N_x\ .$$

Proof. Since $\mathbf{F} = \nabla f$, this means

$$\begin{array}{ccccc} M = f_x & \text{and} & N = f_y\ . & & \text{Therefore,} \\ M_y = f_{xy} & \text{and} & N_x = f_{yx}\ . & & \end{array}$$

But since these two mixed partial derivatives are continuous (since M_y and N_x are, by hypothesis), a standard 18.02 theorem says they are equal. Thus $M_y = N_x$. $\square$

This theorem may be expressed in a slightly different form, if we define the scalar function called the **two-dimensional curl** of $\mathbf{F}$ by

$$(3) \qquad \text{curl}\,\mathbf{F} \;=\; N_x - M_y\ .$$

Then (2) becomes

$$(2') \qquad \mathbf{F} = \nabla f \quad \Rightarrow \quad \text{curl}\,\mathbf{F} = 0\ .$$

This criterion allows us to test $\mathbf{F}$ to see if it is a gradient field. Naturally, we would also like to know that the converse is true: if curl $\mathbf{F} = 0$, then $\mathbf{F}$ is a gradient field. As we shall see, however, this requires some additional hypotheses on the region D. For now, we will assume D is the whole plane. Then we have

Converse to Criterion. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ be continuously differentiable for all x, y.

$$(4) \qquad M_y = N_x \text{ for all } x,y \quad \Rightarrow \quad \mathbf{F} = \nabla f \;\text{ for some differentiable } f \text{ and all } x,y.$$

The proof of (4) will be postponed until we have more technique. For now we will illustrate the use of the criterion and its converse.

Example 1. For which value(s), if any of the constants a, b will $axy\,\mathbf{i} + (x^2 + by)\,\mathbf{j}$ be a gradient field?

Solution. The partial derivatives are continuous for all x, y and $M_y = ax,\ N_x = 2x$. Thus by (2) and (4), $\mathbf{F} = \nabla f \ \Leftrightarrow\ a = 2;\ b$ is arbitrary.

Example 2. Are the fields $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2 + y^2}$, $\mathbf{G} = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$ conservative?

Solution. We have (the second line follows from the first by interchanging x and y):

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{-x^2+y^2}{(x^2+y^2)^2}; \qquad \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = \frac{2xy}{(x^2+y^2)^2};$$
$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{x^2-y^2}{(x^2+y^2)^2}; \qquad \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{2yx}{(x^2+y^2)^2};$$

from this, we see immediately that

$$\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right); \qquad \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right);$$

the two equations in the last line show respectively that $\mathbf{F}$ and $\mathbf{G}$ satisfy the criterion (2). However, neither field is defined at $(0,0)$, so that the converse (4) is not applicable. So the question cannot be decided just on the basis of (2) and (4). In fact, it turns out that $\mathbf{F}$ is a gradient field, since one can check that

$$\mathbf{F} = \nabla \ln \sqrt{x^2+y^2} = \nabla \ln r\ .$$

On the other hand, $\mathbf{G}$ is not conservative, since if C is the unit circle $x = \cos t,\ y = \sin t$, we have

$$\oint_C \mathbf{G}\cdot d\mathbf{r} = \oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^{2\pi} \frac{\sin^2 t\,dt + \cos^2 t\,dt}{\sin^2 t + \cos^2 t} = 2\pi \neq 0.$$

We will return to this example later on.

Identifying Gradient Fields and Exact Differentials

1. Determine whether each of the vector fields below is conservative.

a) $\mathbf{F} = \langle xe^x + y, x\rangle$

b) $\mathbf{F} = \langle xe^x + y, x + 2\rangle$

c) $\mathbf{F} = \langle xe^x + y + x, x\rangle$

<u>Answer:</u> We know from lecture that if $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is continuously differentiable for all x and y, then

$$M_y = N_x \text{ for all } x \text{ and } y \implies \mathbf{F} \text{ is conservative.}$$

Each of the fields in question is continuously differentiable for all x and y.

a) $M = xe^x + y$, $N = x$. $M_y = 1, N_x = 1$. The field is conservative.

b) $M = xe^x + y$, $N = x + 2$. $M_y = 1, N_x = 1$. The field is conservative.

c) $M = xe^x + y + x$, $N = x$. $M_y = 1, N_x = 1$. The field is conservative.

In fact, we can add any function of x to M and any function of y to N without affecting M_y and N_x.

2. Show $(xe^x + y)\,dx + x\,dy$ is exact.

<u>Answer:</u> We know from lecture that $M\,dx + N\,dy$ is an exact differential if and only if $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is a gradient field. To show $\mathbf{F}$ is a gradient field, we must show that $\mathbf{F}$ is continuously differentiable and $M_y = N_x$ for all x, y.

Indeed, $\mathbf{F}$ is continuously differentiable for all x, y by inspection. Here $M = xe^x + y$ and $N = x$, so $M_y = N_x = 1$. We conclude that $(xe^x + y)\,dx + x\,dy$ is exact.

3. Compute the two dimensional curl of $\mathbf{F}$ for each of the vector fields below.

a) $\mathbf{F} = \langle x, xe^x + y\rangle$

b) $\mathbf{F} = \mathbf{i} + \mathbf{j}$

c) $\mathbf{F} = \langle xy^2, x^2y\rangle$

<u>Answer:</u> We know that if $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ then $\text{curl}\mathbf{F} = N_x - M_y$.

a) $\text{curl}\mathbf{F} = (e^x + xe^x) - 0 = e^x(1 + x)$.

(This looks similar to the conservative vector fields from previous problems, but its components have been swapped.)

b) $M = N = 1$ so $\text{curl}\mathbf{F} = 0 - 0 = 0$.

c) $N_x = 2xy$ and $M_y = 2yx$, so $\text{curl}\mathbf{F} = 0$.

V2.2-3 Gradient Fields and Exact Differentials

2. Finding the potential function.

In example 2 in the previous reading we saw that

$$\mathbf{F} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2+y^2} = \nabla \ln \sqrt{x^2+y^2} \;=\; \nabla \ln r.$$

This raises the question of how we found the function $\frac{1}{2}\ln(x^2+y^2)$. More generally, if we know that $\mathbf{F} = \nabla f$ — for example if curl $\mathbf{F} = 0$ in the whole xy-plane — how do we find the function $f(x,y)$? There are two methods; some students prefer one, some the other.

Method 1. Suppose $\mathbf{F} = \nabla f$. By the Fundamental Theorem for Line Integrals,

(5) $$\int_{(x_0,y_0)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r} \;=\; f(x,y) - f(x_0,y_0)\ .$$

Read from left to right, (5) gives us an easy way of finding the line integral in terms of $f(x,y)$. But read right to left, it gives us a way of finding $f(x,y)$ by using the line integral:

(5′) $$f(x,y) \;=\; \int_{(x_0,y_0)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r} \;+\; c\ .$$

(Here c is an arbitrary constant of integration; as (5′) shows, $c = f(x_0,y_0)$.)

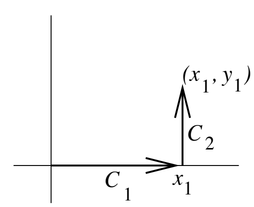
Remark. It is common to refer to $f(x,y)$ as the (mathematical) **potential function**. The potential function used in physics is $-f(x,y)$. The negative sign is used by physicists so that the potential difference will represent work done *against* the field $\mathbf{F}$, rather than work done *by* the field, as the convention is in mathematics.

Example 3. Let $\mathbf{F} = (x+y^2)\,\mathbf{i} + (2xy+3y^2)\,\mathbf{j}$. Verify that $\mathbf{F}$ satisfies the Criterion (2), and use method 1 above to find the potential function $f(x,y)$.

Solution. We verify (2) immediately: $\dfrac{\partial(y^2)}{\partial y} \;=\; 2y \;=\; \dfrac{\partial(2xy)}{\partial x}$.

We use (5′). The point (x_0,y_0) can be any convenient starting point; $(0,0)$ is the usual choice, if the integrand is defined there. (We will subscript the variables to avoid confusion with the variables of integration, but you don't have to.) By (5′),

(6) $$f(x_1,y_1) \;=\; \int_{(0,0)}^{(x_1,y_1)} (x+y^2)\,dx + (2xy+3y^2)\,dy\ .$$



Since the integral is path-independent, we can choose any path we like. The usual choice is the one on the right, as it simplifies the computations. (Most of what follows you can do mentally, with a little practice.)

On C_1, we have $y=0,\ dy=0$, so the integral on C_1 becomes $\displaystyle\int_0^{x_1} x\,dx \;=\; \frac{1}{2}x_1^2.$

On C_2, we have $x = x_1,\ dx = 0$, so the integral is $\displaystyle\int_0^{y_1} (2x_1y+3y^2)\,dy \;=\; x_1y_1^2 + y_1^3\ .$

Adding the integrals on C_1 and C_2 to get the integral along the entire path, and dropping the subscripts, we get by (6) and (5′)

$$f(x,y) = \frac{1}{2}x^2 + xy^2 + y^3 + c\ .$$

(The constant of integration is added by (5′), since the choice of starting point was arbitrary. You should always confirm the answer by checking that $\nabla f = \mathbf{F}$.) □

Method 2. Once again suppose $\mathbf{F} = \nabla f$, that is $M\,\mathbf{i} + N\,\mathbf{j} = f_x\,\mathbf{i} + f_y\,\mathbf{j}$. It follows that

$$f_x = M \qquad \text{and} \qquad f_y = N\ . \tag{7}$$

These are two equations involving partial derivatives, which we can solve simultaneously by integration. We illustrate using the previous example: $\mathbf{F} = (x + y^2,\ 2xy + 3y^2)$.

Solution by Method 2. Using the first equation in (7),

$$\frac{\partial f}{\partial x} = x + y^2. \qquad \text{Hold } y \text{ fixed, integrate with respect to } x\text{:}$$

$$f = \frac{1}{2}\,x^2 + y^2x + g(y). \qquad \text{where } g(y) \text{ is an arbitrary function of } y. \tag{8}$$

To find $g(y)$, we calculate $\dfrac{\partial f}{\partial y}$ two ways:

$$\frac{\partial f}{\partial y} = 2yx + g'(y) \qquad \text{by (8), while}$$

$$\frac{\partial f}{\partial y} = 2xy + 3y^2 \qquad \text{from (7), second equation.}$$

Comparing these two expressions, we see that $g'(y) = 3y^2$, so $g(y) = y^3 + c$. Putting it all together, using (8), we get $f(x,y) = \frac{1}{2}\,x^2 + y^2x + y^3 + c$, as before. □

In the first method, the answer is written down immediately as a line integral; the rest of the work is in evaluating the integral, which goes quickly, since on a horizontal or vertical path either $dx = 0$ or $dy = 0$.

In the second method, the answer is obtained by an algorithm involving several steps which should be carried out in the right order.

The first method has the advantage of reminding you each time how $f(x,y)$ is defined and what it means, facts of theoretical and practical importance. The second method has the advantage of requiring no knowledge of line integrals, which makes it popular with students; on the other hand, when done in three dimensions, the bookkeeping gets more complicated, whereas in the first method it does not; overall, the first method is faster, provided you are confident enough to do some of the work mentally.

3. Exact differentials.

The formal expressions $M(x,y)\,dx + N(x,y)\,dy$ which have appeared as the integrands in our line integrals are called **differentials**. In some applications, most notably thermodynamics, one usually works directly with the differential $M\,dx + N\,dy$ and its line integral $\int M\,dx + N\,dy$, without considering or using the associated vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$. Therefore it is important to have the results about gradient fields in this section translated into the language of differentials. We do this now.

If $f(x,y)$ is a differentiable function, its **total differential** df (or simply *differential*) is by definition the expression

$$df = f_x dx + f_y dy \ . \tag{9}$$

For example, if $f(x,y) = x^2y^3$, then $d(x^2y^3) = 2xy^3dx + 3x^2y^2dy$.

We call the differential $M\,dx + N\,dy$ **exact**, in a region D where M and N are defined, if it is the total differential of some function $f(x,y)$ in this region, i.e., if in D,

$$M = f_x \qquad \text{and} \qquad N = f_y, \qquad \text{for some } \ f(x,y). \tag{10}$$

From this we see that the relation between differentials and vector fields is

$$\begin{aligned} M\,dx + N\,dy \quad \text{is exact} \quad &\Leftrightarrow \quad M\,\mathbf{i} + N\,\mathbf{j} \quad \text{is a gradient field} \\ M\,dx + N\,dy \ = \ df \quad &\Leftrightarrow \quad M\,\mathbf{i} + N\,\mathbf{j} \ = \ \nabla f \ . \end{aligned}$$

In this language, the criterion (we use the same equation number as in the section where they were first presented)

$$\mathbf{F} = \nabla f \ \text{ for some } f(x,y) \quad \Rightarrow \quad M_y = N_x \ . \tag{2}$$

and its partial converse:

Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ be continuously differentiable for all x, y.

$$M_y = N_x \text{ for all } x,y \quad \Rightarrow \quad \mathbf{F} = \nabla f \ \text{ for some differentiable } f \text{ and all } x,y. \tag{4}$$

become the

Exactness Criterion. *Assume M and N are continuously differentiable in a region D of the plane. Then in this region,*

$$M\,dx + N\,dy \ \ \textit{exact} \quad \Rightarrow \quad M_y = N_x \ ; \tag{11}$$

$$\textit{if } D \textit{ is the whole } xy\textit{-plane}, \quad M_y = N_x \quad \Rightarrow \quad M\,dx + N\,dy \ \ \textit{exact}. \tag{12}$$

If the exactness criterion shows that $M\,dx + N\,dy$ is exact, then the function $f(x,y)$ may be found by either of the two methods previously described.

Identifying Potential Functions

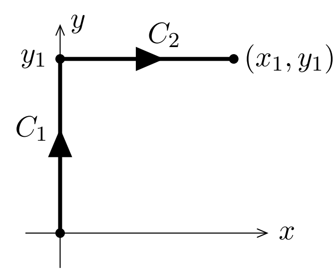
1. Show $\mathbf{F} = \langle 3x^2 + 6xy, 3x^2 + 6y \rangle$ is conservative and find the potential function f such that $\mathbf{F} = \nabla f$.

Answer: First, $M_y = 6x = N_x$. Since $\mathbf{F}$ is defined for all (x, y), $\mathbf{F}$ is conservative.

Method 1 (for finding f): Use

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(P_1) - f(P_0) \;\Rightarrow\; f(P_1) = f(P_0) + \int_C \mathbf{F} \cdot d\mathbf{r}.$$

$P_1 = (x_1, y_1)$ must be arbitrary. We can fix P_0 and C any way we want.
For this problem take $P_0 = (0, 0)$ and C as the path shown.



Path for problem 1

$C_1 : x = 0, y = y, \;\Rightarrow\; dx = 0, dy = dy$
$C_2 : x = x, y = y_1, \;\Rightarrow\; dx = dx, dy = 0$

$$\Rightarrow \quad f(x_1, y_1, z_1) - f(0, 0, 0) = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\,dx + N\,dy$$
$$= \int_0^{x_1} M(x, 0)\,dx + \int_0^{y_1} N(x_1, y)\,dy$$
$$= \int_0^{y_1} 6y\,dy + \int_0^{x_1} 3x^2 + 6y_1\,dx = 3y_1^2 + x_1^3 + 3x_1^2 y_1$$

$\Rightarrow\; f(x_1, y_1) - f(0, 0) = 3y_1^2 + x_1^3 + 3x_1^2 y_1 = 3y_1^2 + x_1^3 + 3x_1^2 y_1.$

$\Rightarrow\; \boxed{f(x, y) = 3y^2 + x^3 + 3x^2 y + C.}$

Method 2: $f_x = 3x^2 + 6xy \;\Rightarrow\; f = x^3 + 3x^2 y + g(y).$

$\Rightarrow\; f_y = 3x^2 + g'(y) = 3x^2 + 6y \;\Rightarrow\; g'(y) = 6y \;\Rightarrow\; g(y) = 3y^2 + C.$

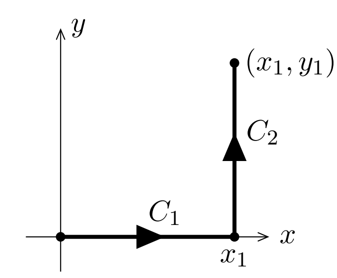
$\Rightarrow\; \boxed{f(x, y) = x^3 + 3x^2 y + 3y^2 + C.}$

In general method 1 is preferred because in 3 dimensions it will be easier.

2. Let $\mathbf{F} = (x + xy^2)\,\mathbf{i} + (x^2 y + 3y^2)\,\mathbf{j}$. Show $\mathbf{F}$ is a gradient field and find the potential function using both methods.

Answer: We have $M(x, y) = x + xy^2$ and $N(x, y) = x^2 y + 3y^2$, so $M_y = 2xy = N_x$ and $\mathbf{F}$ is defined on all (x, y). Thusy, by Theorem 1, $\mathbf{F}$ is conservative.

Method 1: Use the path shown.



Path for problem 2

$f(P_1) - f(0, 0) = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\,dx + N\,dy.$

$C_1 : x = x,\; y = 0, \;\Rightarrow\; dx = dx,\; dy = 0 \;\Rightarrow\; M(x, 0) = x.$

$C_2 : x = x_1,\; y = y, \;\Rightarrow\; dx = 0,\; dy = dy \;\Rightarrow\; N(x_1, y) = x_1^2 y + 3y^2.$

$$\Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} \;=\; \int_0^{x_1} x\,dx + \int_0^{y_1} x_1^2 y + 3y^2\,dy \;=\; x_1^2/2 + x_1^2 y_1^2/2 + y_1^3.$$

$\Rightarrow\; f(x_1, y_1) - f(0, 0) = x_1^2/2 + x_1^2 y_1^2/2 + y_1^3$

$\Leftrightarrow\; \boxed{f(x, y) = x^2/2 + x^2 y^2/2 + y^3 + C.}$

Method 2: $f_x = x + y^2 \;\Rightarrow\; f = x^2/2 + x^2 y^2/2 + g(y)$

$\Rightarrow\ f_y = x^2y + g'(y) = x^2y + 3y^2 \ \Rightarrow\ g'(y) = 3y^2 \ \Rightarrow\ g(y) = y^3 + C.$

$\Rightarrow\ \boxed{f(x,y) = x^2/2 + x^2y^2/2 + y^3 + C.}$

Two Dimensional Curl

We have learned about the curl for two dimensional vector fields.

By definition, if $\mathbf{F} = \langle M, N \rangle$ then the two dimensional curl of $\mathbf{F}$ is $\operatorname{curl} \mathbf{F} = N_x - M_y$

Example: If $\mathbf{F} = x^3 y^2 \, \mathbf{i} + x \, \mathbf{j}$ then $M = x^3 y^2$ and $N = x$, so $\operatorname{curl} \mathbf{F} = 1 - 2x^3 y$.

Notice that $\mathbf{F}(x, y)$ is a vector valued function and its curl is a scalar valued function. It is important that we label this as the two dimensional curl because it is only for vector fields in the plane. Later we will see that the two dimensional curl is really just the $\mathbf{k}$ component of the (vector valued) three dimensional curl.

Problems: Two Dimensional Curl

Imagine a flat arrangement of particles covering the plane. Suppose all the particles are moving in counterclockwise circles about the origin with constant angular speed ω.

Let $\mathbf{F}(x, y)$ be the velocity field described by the velocity of the particles at point (x, y). Find $\mathbf{F}$ and show $\text{curl}(\mathbf{F}) = 2\omega$.

Answer: Because the particles have a constant angular speed ω and no radial velocity, the motion of the particles can be parametrized by $r = r_0$, $\theta = \theta_0 + \omega t$. In polar coordinates we have $(x(t), y(t)) = (r_0 \cos(\theta_0 + \omega t), r_0 \sin(\theta_0 + \omega t))$.

Taking derivatives with respect to t we find

$$\begin{aligned}
\mathbf{F} &= -\omega\, r_0 \sin(\theta_0 + \omega t)\mathbf{i} + \omega\, r_0 \cos(\theta_0 + \omega t)\mathbf{j} = \langle -\omega y, \omega x \rangle, \\
\text{curl}\mathbf{F} &= N_x - M_y \\
&= \omega - (-\omega) \\
&= 2\omega.
\end{aligned}$$