MATH - UA 140 - Linear Algebra

Lecture 20: Cramer's rule, inverses, and volumes

Determinants are useful to determine if a matrix is singular or not. They also play a major role in the calculation of the eigenvalues of a matrix, as we will find out in Lecture 21. Before we do so, we will see in this lecture that determinants can be used to compute the inverse A^{-1} of a matrix A, and to solve for $\vec{x}$ in $A\vec{x} = \vec{b}$. While not always very efficient from a computational point of view, the formulas we obtain are the first explicit formulas for A^{-1} and $\vec{x}$, as opposed to their being the result of elimination steps.
At the end of the lecture, we will also show how determinants can be used to compute areas and volumes.

I] Explicit formulas in terms of determinants

1) Cramer's rule

Cramer's rule is a method to obtain an explicit expression for the solution $\vec{x}$ to $A\vec{x} = \vec{b}$ in terms of A and $\vec{b}$. This idea is quite elegant, explained here for a 3×3 matrix, but general in nature.

Say we want to solve $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

We observe that $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}}_{\text{call this matrix } B_1}$

Evaluating the determinants on both sides of the equality, we obtain a formula for x_1:

$$\det A \; x_1 = \det B_1 \underset{\substack{\text{(assuming} \\ A \text{ invertible)}}}{\iff} x_1 = \frac{\det B_1}{\det A}$$

A very similar idea can be applied for a formula for x_2:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}}_{= B_2}$$

so $\det A \; x_2 = \det B_2 \iff x_2 = \dfrac{\det B_2}{\det A}$

Now that we get the general idea, we can state Cramer's rule in general:

If $\det A$ is not zero, $A\vec{x} = \vec{b}$ can be solved by determinants as follows:

$$x_1 = \frac{\det B_1}{\det A}, \quad x_2 = \frac{\det B_2}{\det A}, \quad \ldots, \quad x_n = \frac{\det B_n}{\det A}$$

where the matrix B_j has the j^{th} column of A replaced by the vector $\vec{b}$

Note that Cramer's rule requires the evaluation of $n+1$ determinants (the determinant of A and the determinants of the B_j). If one uses the general formula for determinants, each determinant has $n!$ terms. This makes Cramer's rule highly inefficient for computing determinants. Still, let us show a 2×2 example.

Example:

$$\begin{cases} x+y=-1 \\ 2x-y=-5 \end{cases} \iff A\vec{x}=\vec{b} \text{ with } A=\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \vec{x}=\begin{bmatrix} x \\ y \end{bmatrix}, \vec{b}=\begin{bmatrix} -1 \\ -5 \end{bmatrix}$$

$$B_1=\begin{bmatrix} -1 & 1 \\ -5 & -1 \end{bmatrix}, \quad B_2=\begin{bmatrix} 1 & -1 \\ 2 & -5 \end{bmatrix}$$

$$\det A = -3; \quad \det B_1 = 6; \quad \det B_2 = -3$$

$$x = \frac{6}{-3} = -2; \quad y = \frac{-3}{-3} = 1$$

While Cramer's rule is inefficient in general to get answers which are numbers, it can be convenient to derive general formulas since it is explicit. Here is an example for the general formula of the inverse A^{-1} of a 2×2 matrix A:

Inverting $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ amounts to solving $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

for x_1, x_2, y_1 and y_2, i.e

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Using Cramer's rule, we have $x_1 = \dfrac{\begin{vmatrix} 1 & b \\ 0 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \dfrac{d}{ad-bc}$

$$x_2 = \frac{\begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-c}{ad-bc}, \quad y_1 = \frac{\begin{vmatrix} 0 & b \\ 1 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-b}{ad-bc}, \quad y_2 = \frac{\begin{vmatrix} a & 0 \\ c & 1 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{a}{ad-bc}$$

We conclude that $A^{-1} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

This result motivates the next section:

2) Inverses using determinants

The formula for A^{-1} above may be written as

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \frac{C^T}{\det A}$$

where the C_{ij} are the cofactors of the matrix A, and C the matrix whose entries are the C_{ij}. It turns out that this formula holds for any invertible $n \times n$ matrix. Let us see why with a 3×3 matrix.

Let A be a 3×3 matrix, and C the matrix with the cofactors of A as entries. We have

$$AC^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix}$$

Let us see why this is true. The diagonal entries of AC^T are:

$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ cofactor formula for 1st row
$a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$ cofactor formula for 2nd row
$a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$ cofactor formula for 3rd row

We recognize the cofactor expansions for each row, so they all are equal to det A.

Why are the off-diagonal terms 0? Let us for instance look at the first row, second column, given by

$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}$. Now, we have
$C_{21} = -(a_{12}a_{33} - a_{32}a_{13})$; $C_{22} = (a_{11}a_{33} - a_{31}a_{13})$; $C_{23} = -(a_{11}a_{32} - a_{31}a_{12})$
so $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}$ is the cofactor formula for

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \quad \text{since the first two rows are equal}$$

Hence $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$, and we could show in a similar way that all the off-diagonal terms are zero.

We conclude that $AC^T = \det A \; I$

$$\iff A^{-1} = \frac{C^T}{\det A}$$

This result holds for the determinant of any invertible $n \times n$ matrix:

The i,j entry of A^{-1} is the cofactor C_{ji} divided by $\det A$:

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det A} \qquad \text{so that} \qquad A^{-1} = \frac{C^T}{\det A}$$

Example: $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

The cofactor matrix is $C = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

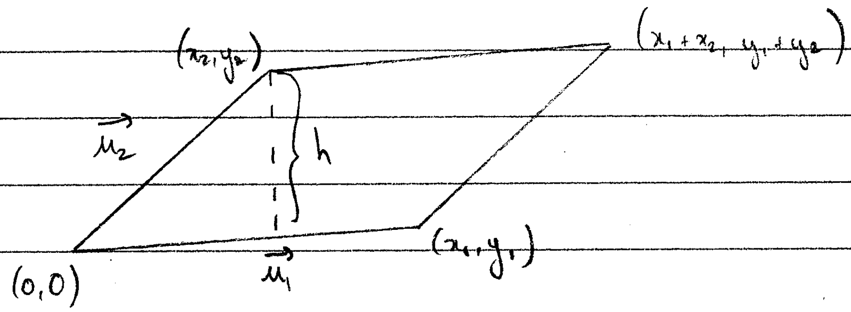
We saw in the last lecture that $\det A = 1$, so

$$A^{-1} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

II] Computing volumes using determinants

1) Area of a parallelogram starting from (0,0)

Consider the parallelogram with vertices $(0,0)$, (x_1,y_1), (x_2,y_2), and, necessarily $(x_1+x_2,\ y_1+y_2)$:



The area of the parallelogram is the base $b = \|\vec{u}_1\|$ times the height h: $A = bh$

To compute h, we view h as $\|\vec{e}\|$, where $\vec{e} = \vec{u}_2 - \vec{p}$ and $\vec{p}$ is the projection of $\vec{u}_2$ onto $\vec{u}_1$:

$$\vec{e} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \frac{1}{x_1^2 + y_1^2} \begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{x_2 x_1^2 + x_2 y_1^2 - x_1^2 x_2 - x_1 y_1 y_2}{x_1^2 + y_1^2} \\[2ex] \dfrac{x_1^2 y_2 + y_1^2 y_2 - x_1 x_2 y_1 - y_1^2 y_2}{x_1^2 + y_1^2} \end{bmatrix} = \begin{bmatrix} \dfrac{y_1 (x_2 y_1 - x_1 y_2)}{x_1^2 + y_1^2} \\[2ex] \dfrac{x_1 (x_1 y_2 - x_2 y_1)}{x_1^2 + y_1^2} \end{bmatrix}$$

$$\|\vec{e}\| = \frac{|x_2 y_1 - x_1 y_2|}{\|\vec{u}_1\|}$$

so $A = \|\vec{e}\| \|\vec{u}_1\| = |x_2 y_1 - x_1 y_2|$

We conclude that the area of the parallelogram starting from $(0,0)$ is the absolute value of the determinant $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$!

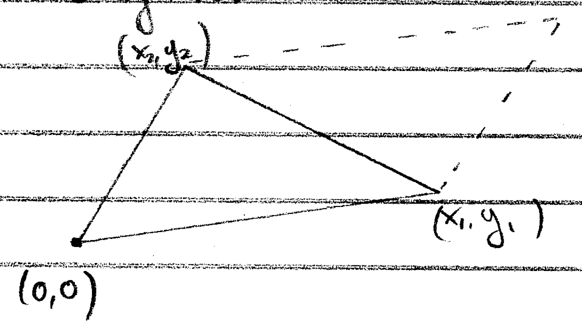
<u>Example</u>: What is the area of the parallelogram with vertices $(0,0)$, $(3,1)$, $(1,2)$, and $(4,3)$?

$$\text{Area} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5$$

2) Area of a triangle

Case 1: The triangle has $(0,0)$ as one of its vertices

Then the area of the triangle is obtained by dividing our formula for the area of the parallelogram by 2. Indeed, the triangle can be seen as $\frac{1}{2}$ of a parallelogram:



Case 2: The triangle does not have $(0,0)$ as one of its vertices

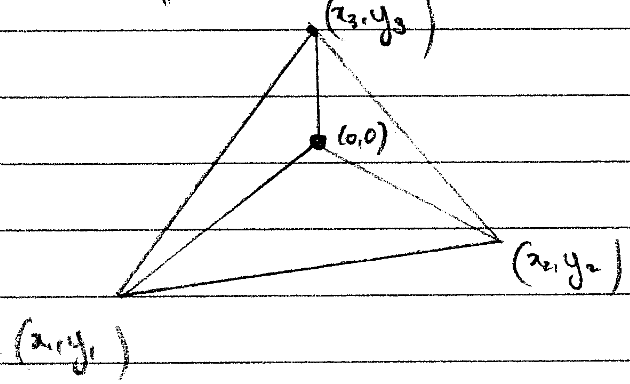
Then the formula for the area of the triangle is:

$$\text{Area} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$, $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ vertices of the triangle

which is a generalization of the formula $\text{Area} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$

which holds when $(x_3, y_3) = 0$.

To see why the general formula holds, one splits the triangle into three triangle which all have the origin as one corner:



Using a cofactor expansion, we have

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}(x_2y_3 - x_3y_2) + \frac{1}{2}(x_3y_1 - x_1y_3) + \frac{1}{2}(x_1y_2 - x_2y_1)$$

which is the sum of the areas of the three triangles.

3) Volume of an n-dimensional box

The formula can be generalized further: the volume of an n-dimensional box is $|\det A|$, where A is the matrix whose rows are the vectors corresponding to the n edges of the box coming out from the origin.

Example: The volume of a 3-dimensional box with edges $\vec{u}$, $\vec{v}$, and $\vec{w}$ is

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$