MATH - UA 140 - Linear Algebra

Lecture 10: The nullspace of a matrix and solving $A\vec{x} = \vec{0}$

In the previous lecture, we learned about $C(A)$, the column space of a matrix A. In this lecture, we learn about another subspace associated with A, called the nullspace of A. We first introduce it, and then learn how to identify it through computation.

I] The nullspace of A

1) Definition

Let A be an $m \times n$ matrix. The nullspace of A consists of all solutions $\vec{x}$ to $A\vec{x} = \vec{0}$. These solutions are vectors in $\mathbb{R}^n$.

Note 1: The nullspace of A is denoted $N(A)$

Note 2: Let us verify it is indeed a subspace:

- Consider two vectors $\vec{x}$ and $\vec{y}$ in $N(A)$: $A\vec{x} = \vec{0}$ and $A\vec{y} = \vec{0}$

Then $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0}$, so $\vec{x} + \vec{y}$ is also in $N(A)$

- Let c be a scalar. $A(c\vec{x}) = c(A\vec{x}) = 0$, so $c\vec{x}$ is in $N(A)$ too

This completes our proof

Note 3: If A is invertible, then $N(A)$ only contains $\vec{x} = \vec{0}$

2) Examples

- The plane $8x - 6y + 3z = 0$ can be seen as $A\vec{x} = \vec{0}$ with $A = [8 \;\; -6 \;\; 3]$
  Since the plane contains the origin $(0,0,0)$, it is a subspace of $\mathbb{R}^3$. It is the nullspace of A.

- What is the null space of $A = \begin{bmatrix} 1 & -2 \\ 3 & -6 \\ -2 & 4 \end{bmatrix}$

$N(A)$ is given by the set of solutions $\vec{x} = (x, y)$ to $A\vec{x} = \vec{0}$

$$\Leftrightarrow \begin{cases} x - 2y = 0 \\ 3x - 6y = 0 \\ -2x + 4y = 0 \end{cases}$$

The three equations are all equivalent to $x = 2y$
So $N(A)$ is the line $2y = x$.
A particular point on that line is obtained by setting $y = 1 \Rightarrow x = 2$

The nullspace of A contains all multiples of $\vec{s} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

We will now learn about a streamlined method to characterize $N(A)$ for any matrix A. Guess what? It relies on Gaussian elimination!

II] Solving $A\vec{x} = \vec{0}$ by elimination

In this section, we will see that elimination can be applied to solve problems of the form $A\vec{x} = \vec{0}$ even if A is an $m \times n$ matrix with $m \neq n$ (i.e. not square), and even if A has fewer than n pivots.

The steps are very similar to what we have seen previously:

A) Forward elimination to turn A into a matrix U which is the $m \times n$ equivalent of a triangular matrix, called an echelon matrix

B) Turn U into a matrix R which has 1's as pivots, called a reduced row echelon matrix (This step was not necessary when we were solving systems previously. It still is not something that must be done from a mathematical point of view, but it significantly simplifies the analysis.

C) Solve $R\vec{x} = \vec{0}$ by back substitution to find $\vec{x}$.

1) Echelon matrices

Here is how it works in practice. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 1 \\ 5 & 10 & 11 & -8 \end{bmatrix}$$

Row 2 - 2 Row 1 leads to

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -7 \\ 5 & 10 & 11 & -8 \end{bmatrix}$$

Row 3 − 5 Row 1 leads to:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -7 \\ 0 & 0 & -4 & -28 \end{bmatrix}$$

We see that the second column has a zero in the pivot position. Looking below, we also find a zero: row exchanges will not change the situation.
The idea here is to forge ahead and eliminate in the third column. Row 3 − 4 Row 2 leads to

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

U is the equivalent of an upper triangular matrix for a rectangular matrix. Note its "staircase" structure. U is called an <u>echelon matrix.</u>

Note that the fourth row also has a zero in the pivot position. The last row gives the equation $0=0$, which is automatically satisfied when the first two are satisfied. This was far from obvious in the original form $A\vec{x}=\vec{0}$, but becomes clear after elimination.

How do we construct any solution $A\vec{x}=\vec{0}$ from the form $U\vec{x}=\vec{0}$? The idea is back substitution, with a little twist. First, the twist: identify the columns which have pivots, and those which do not.

In our case, columns 1 and 3 have pivots, columns 2 and 4 do not. If $\vec{x}=(x_1,x_2,x_3,x_4)$ is the unknown vector, we then say that x_1 and x_3 are *pivot variables* and x_2 and x_4 are *free variables*

We get the full solution to $A\vec{x}=\vec{0}$ by choosing arbitrarily two pairs of numbers for (x_2,x_4). The simplest choice is $(x_2,x_4)=(1,0)$ and $(x_2,x_4)=(0,1)$. By back substitution, we then obtain the pivot variables x_1 and x_3:

For $(x_2,x_4)=(1,0)$, the second row $-x_3-7x_4=0$ gives $x_3=$ and the first row, $x_1+2x_2+3x_3+4x_4=0$ gives $x_1=-2$

For $(x_2,x_4)=(0,1)$, the second row gives $x_3=-7$ and the first row gives $x_1=17$

So $\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}17\\0\\-7\\1\end{bmatrix}$ are two special solutions.

What is important is that *every* solution to $A\vec{x}=\vec{0}$ is a *linear combination of the special solutions*

The complete solution in our case therefore is:

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 17 \\ 0 \\ -7 \\ 1 \end{bmatrix} = \begin{bmatrix} 17x_4 - 2x_2 \\ x_2 \\ -7x_4 \\ x_4 \end{bmatrix}$$

special solution $\vec{s}_1$ — special solution $\vec{s}_2$ — complete solution

This describes the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 17 \\ 0 \\ -7 \\ 1 \end{bmatrix}$

Let us emphasize the main point once more: every column with a zero in the pivot position is called a free column. Every free column leads to a special solution, which we can obtain by setting the free variable corresponding to that column to 1, and the other free variables to 0. The remaining variables are obtained by back substitution.

We are now ready to learn the final trick, which makes back substitution even easier.

2) The reduced row echelon matrix R

The idea is to use linear operations to further simplify the echelon matrix U. Specifically, one constructs the reduced row echelon matrix R so that the pivots equal 1, and so that the matrix has zeros above the pivots.

We had $U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Dividing the second row by -1, this becomes $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The pivots are now all 1's. To put a zero above the second pivot, we do Row 1 − 3 Row 2, to find

$$R = \underbrace{\mathrm{rref}(A)}_{\text{Row reduced echelon form}} = \begin{bmatrix} 1 & 2 & 0 & -17 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: If A is invertible, $\mathrm{rref}(A) = I$ (identity matrix)

Now, $A\vec{x} = \vec{0}$ is equivalent to $R\vec{x} = \vec{0}$ since linear operations on the rows do not change $\vec{0}$, the right hand side.

R is very convenient for the construction of the special solutions:

- $\vec{s}_1$: set $x_2 = 1$, $x_4 = 0$. The second row give $x_3 = 0$, the first row gives $x_1 = -2$, as before

- $\vec{s}_2$: set $x_2 = 0$, $x_4 = 1$. The second row gives $x_3 = -7$, and the first row gives $x_1 = -17$

Observe that we can read the special solutions directly from R, by reversing signs. This is a general result, which is very convenient.

3) Existence of nonzero solutions

Let A be an $m \times n$ matrix, and let r be the number of pivots (we will soon call r the rank of A).

- The number of free variables is $n-r$ for the system $A\vec{x} = \vec{0}$

- Suppose $n > m$. Then A has at most m pivots, so there is at least $n-m$ free variables. In other words, there are nonzero solutions to $A\vec{x} = \vec{0}$

A linear system $A\vec{x} = \vec{0}$ with more unknowns than equations has nonzero solutions

QUESTION: Find the nullspace of $U = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 0 & 9 \end{bmatrix}$