MATH - UA 140 - Linear Algebra

Lecture 21: Introduction to eigenvectors and eigenvalues

I] Eigenvectors and eigenvalues

1) Definition

Consider a square matrix A, and a vector $\vec{x}$. We know that $A\vec{x}$ can be viewed as a linear combination of the columns of A, with coefficients given by the components of $\vec{x}$.
Now, there exist some special vectors for this matrix A such that $A\vec{x}$ is aligned with $\vec{x}$. This means that there exists a scalar λ such that

$$A\vec{x} = \lambda\vec{x}$$

The special vectors with this property are called eigenvectors of the matrix A.
The number λ is called an eigenvalue of A.

2) Examples

* For any vector $\vec{x}$, $I\vec{x} = \vec{x}$
Every vector $\vec{x}$ therefore is an eigenvector of the identity matrix. The identity matrix has only one eigenvalue: $\lambda = 1$

This is a very unusual situation, as we will see with other examples.

* Let A be a singular matrix. Then there are vectors $\vec{x}$ in the nullspace of A such that

$$A\vec{x} = \vec{0} = 0\vec{x}$$

The vectors in the nullspace of a singular matrix A are eigenvectors of A, with corresponding eigenvalue 0.

Note: By convention, the zero vector is NOT an eigenvector. This is because for the zero vector, $A\vec{x} = \lambda\vec{x}$ is satisfied for any λ.

On the other hand, having $\lambda = 0$ is perfectly fine. A matrix has $\lambda = 0$ as its eigenvalue if and only if it is singular.

* Let us now turn to a more usual example

$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ has $\vec{x} = \begin{bmatrix} -4/5 \\ \frac{3}{5} \end{bmatrix}$ as an eigenvector, with corresponding eigenvalue $\lambda = -2$. Indeed,

$$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{8}{5} \\ -\frac{6}{5} \end{bmatrix} = -2 \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$$

II Computing eigenvalues and eigenvectors

We will now learn how to compute the eigenvectors and eigenvalues of a square matrix. The first step of the calculation consists in identifying all the eigenvalues of the matrix. Once all the eigenvalues are known, we look for each of them, one by one, for their corresponding eigenvectors.

1) Calculating eigenvalues

Let A be a square matrix, $\vec{x}$ an eigenvector of A, and λ the corresponding eigenvalue. We can write

$$A\vec{x} = \lambda \vec{x} \iff A\vec{x} - \lambda\vec{x} = \vec{0}$$
$$\iff (A - \lambda I)\vec{x} = \vec{0}$$

In other words, λ is an eigenvalue of A if and only if the matrix $A - \lambda I$ is singular.
This is the case if and only if $\det(A - \lambda I) = 0$

This criterion gives us the method to compute all the eigenvalues:

1) Compute $\det(A - \lambda I)$
2) Solve the polynomial equation $\det(A - \lambda I) = 0$ to find all the eigenvalues λ

If A is an $n \times n$ matrix, $\det(A - \lambda I)$ is a polynomial in λ of degree n, so $\det(A - \lambda I) = 0$ has n roots, which may or may not be distinct, and may or may not be real numbers, as we will soon see.

<u>Example</u> $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 12$$
$$= \lambda^2 - 3\lambda - 10$$
$$= (\lambda+2)(\lambda-5)$$

The two eigenvalues of A are $\lambda = -2$ (we knew that from the example before) and $\lambda = 5$.

2) <u>Calculating eigenvectors</u>

The eigenvectors corresponding to a given eigenvalue λ are in the nullspace of $A - \lambda I$. Thus, we compute eigenvectors by solving, for each distinct eigenvalue, the equation

$$(A - \lambda I)\vec{x} = \vec{0}$$

with the usual method seen in Lecture 10.

<u>Example</u>: Let us compute the eigenvectors of $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

- <u>For $\lambda = -2$</u>, we solve $\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \vec{x} = \vec{0}$

The row reduced echelon form of $\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}$ is $\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 0 \end{bmatrix}$

so an eigenvector is $\vec{x}_1 = \left(-\frac{4}{3}, 1\right)$, in the nullspace of that matrix.

Note that in our early example, we had given $\begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$ as an eigenvector corresponding to $\lambda = -2$. We have $\begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} = \frac{3}{5}\vec{x}_1$, as expected since the nullspace of $\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 0 \end{bmatrix}$ has dimension 1.

- For $\lambda = 5$, we look for the nullspace of $\begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}$, ie the nullspace of $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. An eigenvector is $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

III| A few important properties

1) Eigenvalues of a triangular matrix

Consider a triangular matrix A (upper or lower triangular) with diagonal entries $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$.
then $A - \lambda I$ is also triangular, with entries $a_{11}-\lambda, a_{22}-\lambda, \dots, a_{nn}-\lambda$. $\det(A-\lambda I) = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$

We conclude that the eigenvalues of a triangular matrix lie on its diagonal.

2) Product of eigenvalues; sum of eigenvalues

We will state here without proving two important properties:

- The product of the n eigenvalues of a matrix equals its determinant : $\lambda_1 \lambda_2 \dots \lambda_n = \det A$

- The sum of the n eigenvalues of a matrix equals the sum of the diagonal entries:

$$\lambda_1 + \lambda_2 + \dots + \lambda_n = a_{11} + a_{22} + \dots + a_{nn}$$

The sum of the diagonal entries is called the trace of the matrix:

$a_{11} + a_{22} + \dots + a_{nn} = \mathrm{Tr}(A)$ (trace of A)

You can easily verify that these 2 properties hold for $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

3) Some annoying facts

- Gaussian elimination does NOT preserve the eigenvalues. So if we write $A = LU$, the eigenvalues of U, which we can read on its diagonal, are NOT the eigenvalues of A in general.

- Eigenvalues may not be real numbers. Here is an example of this:

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \det(B - \lambda I) = \begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1$$

$$\det(B - \lambda I) = 0 \iff \lambda^2 = -1$$
$$\iff \lambda = \pm i$$

B has two eigenvalues: the complex numbers i and $-i$. One can easily show that its eigenvectors have complex components. We will encounter complex numbers every so often in the next few lectures.

4) Eigenvalues and matrix powers

Let A be a square matrix with eigenvector $\vec{x}$ and eigenvalue λ: $\quad A\vec{x} = \lambda \vec{x}$

Applying A on the left on both sides, we get
$$A^2 \vec{x} = \lambda A\vec{x} = \lambda^2 \vec{x}$$
So $\vec{x}$ is an eigenvector of A^2 with corresponding eigenvalue λ^2.

Applying A on the left once more, we get
$$A^3 \vec{x} = \lambda^2 A\vec{x} = \lambda^3 \vec{x}$$
$\vec{x}$ is an eigenvector of A^3 with corresponding eigenvalue λ^3.

Generally, $\vec{x}$ is an eigenvector of A^k with corresponding eigenvalue λ^k.

This gives a first interpretación of the value of the concept of eigenvalue: it gives a dynamical sense of the acción of a matrix. It tells us what happens to a vector if we apply a matrix to it several times. If $\vec{x}$ is an eigenvector, and the eigenvalue λ is such that $|\lambda| > 1$, the vector will be amplified. If $|\lambda| < 1$, the vector will be contracted.

Let us return to our example $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$. The two eigenvectors $\vec{x}_1$ and $\vec{x}_2$ we found are not colinear, so they are a basis for $\mathbb{R}^2$. Any vector $\vec{x}$ in $\mathbb{R}^2$ can be written as the linear combinación:

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$$

Applying A once, we get $A\vec{x} = -2c_1\vec{x}_1 + 5c_2\vec{x}_2$
Applying A once more, we get $A^2\vec{x} = 4c_1\vec{x}_1 + 25c_2\vec{x}_2$
Applying A again, we get $A^3\vec{x} = 8c_1\vec{x}_1 + 125c_2\vec{x}_2$

Because $|\lambda_1| = 2 < |\lambda_2| = 5$, we see that if we keep applying A to $\vec{x}$, the result will align with the eigenvector $\vec{x}_2$ more and more.