

## Limits

Last class we talked about a series of secant lines approaching the “limit” of a tangent line, and about how as  $\Delta x$  approaches zero,  $\frac{\Delta y}{\Delta x}$  approaches the “limit”  $y' = \frac{dy}{dx}$ . Now we want to talk about limits more carefully; this will include some of our first steps towards our goal of being able to differentiate every function you know.

Some limits are easy to compute:

$$\lim_{x \rightarrow 3} \frac{x^2 + x}{x + 1} = \frac{3^2 + 3}{3 + 1} = \frac{12}{4} = 3$$

With an easy limit, you can get a meaningful answer just by plugging in the limiting value. This is because when  $x$  is close to 3, the value of the function  $f(x) = \frac{x^2 + x}{x + 1}$  is close to  $f(3)$ .

Some limits are not easy to compute. For example, the definition of the derivative:

$$\lim_{x \rightarrow x_0} \frac{\Delta f}{\Delta x} = \lim_{x \rightarrow x_0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is never an easy limit, because the denominator  $\Delta x = 0$  is not allowed. (The limit  $x \rightarrow x_0$  is computed under the implicit assumption that  $x \neq x_0$ .) We'll always need to cancel  $\Delta x$  before we can make sense out of the limit.

Other “hard” limits would be:

$$\lim_{x \rightarrow -1} \frac{x^2 + x}{x + 1} \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{x^2 + x}{x + 1}.$$

Any limit involving infinity or division by zero is going to be harder to compute; sometimes the answer will be that there is no limit.

To complete our discussion of limits, we need just one more piece of notation — the concepts of left hand and right hand limits.

The limit

$$\lim_{x \rightarrow x_0^+} f(x)$$

is known as the *right-hand limit* and means that you should use values of  $x$  that are greater than  $x_0$  (to the right of  $x_0$  on the number line) to compute the limit. Shown below is the graph of the function:

$$f(x) = \begin{cases} x + 1 & x > 0 \\ -x & x \leq 0 \end{cases}$$

The right-hand limit  $\lim_{x \rightarrow 0^+} f(x)$  equals 1.

The *left-hand limit*

$$\lim_{x \rightarrow x_0^-} f(x)$$

is found by looking at values of  $f(x)$  when  $x$  is less than  $x_0$  (to the left of  $x_0$  on the number line). For this function,  $\lim_{x \rightarrow 0^-} f(x) = 0$ .

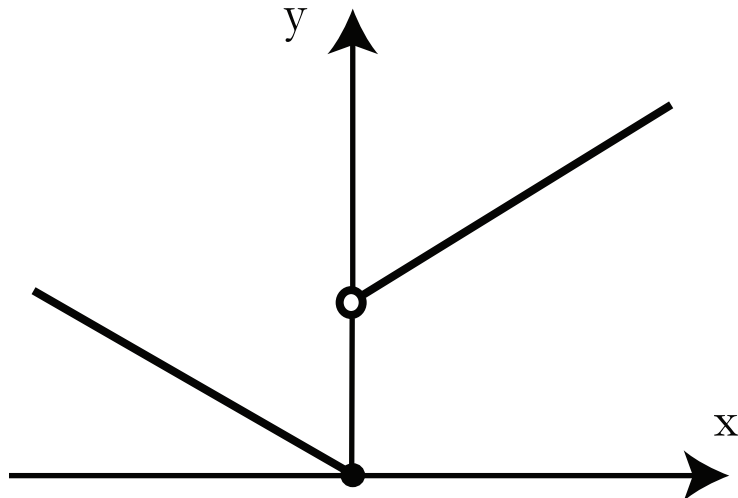


Figure 1: Graph of  $f(x)$

The notions of left- and right- hand limits will make things much easier for us as we discuss continuity, next.

Let's talk more about the example graphed above. To calculate

$$\lim_{x \rightarrow x_0^+} f(x)$$

we use only values of  $x$  that are greater than 0. When  $x > 0$ ,  $f(x)$  is defined to equal  $x + 1$ . So we plugged  $x = 0$  into the expression  $x + 1$  to calculate the right-hand limit.

When calculating

$$\lim_{x \rightarrow x_0^-} f(x),$$

we have  $x < 0$ . Here  $f(x)$  is defined to equal  $-x$ ; when we plug  $x = 0$  into this expression we get  $\lim_{x \rightarrow x_0^-} f(x) = 0$ .

Notice that it doesn't matter that  $f(0) = 0$ . Our calculations would have been exactly the same if  $f(0)$  were 1 or even if  $f(0) = 2$ .

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