MATH - UA 140 - Linear Algebra

Lecture 18: Introduction to Determinants

In this lecture, we will get acquainted with a very important concept in linear algebra, namely the determinant of a square matrix.

The general formula for the determinant is quite complicated. We will soon get there, but it is better to wait a little bit. Instead, we will adopt the following strategy: we will learn the formula for the determinant of 2×2 and 3×3 matrices, and then use these formulas to learn about the key properties of determinants.

I] Formulas for the determinant in terms of matrix entries

1) 2×2 matrices

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix. The determinant of A is the scalar quantity written $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example: $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 1\times4 - 2\times2 = 0$

2) 3x3 matrices

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. The determinant of A

is the scalar quantity given by:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) + a_{12}(a_{31}a_{23} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Example

$$\begin{vmatrix} 1 & 0 & 3 \\ 1 & 5 & 6 \\ 3 & -1 & 4 \end{vmatrix} = 1\times(5\times4 - (-1)\times6) + 0\times(3\times6 - 1\times4) + 3\times(1\times(-1) - 3\times5) = 26 - 48 = -22$$

II Defining properties of the determinant

1) Identity matrix

The determinant of the $n\times n$ identity matrix is 1.
You can easily see that this indeed holds for the 2×2 and 3×3 identity matrix.

2) Determinant and row exchange

Let us compare the determinants of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Likewise, let us compute

$$\begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} = a_{31}(a_{22}a_{13} - a_{12}a_{23}) + a_{32}(a_{11}a_{23} - a_{21}a_{13}) + a_{33}(a_{21}a_{12} - a_{11}a_{22})$$

$$= a_{11}(a_{32}a_{23} - a_{22}a_{33}) + a_{12}(a_{33}a_{21} - a_{31}a_{23}) + a_{13}(a_{22}a_{31} - a_{21}a_{32})$$

$$= - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The general property is as follows: <u>the determinant changes sign when two rows are exchanged.</u>

From Property 1 and 2, we are now able to compute the determinant of any permutation matrix P: $\det P = 1$ if P is obtained from I with an even number of row exchanges; $\det P = -1$ otherwise.

3) <u>Determinant and linearity</u>

Let us look at the determinant of a 2×2 matrix in which we multiplied the first row by a scalar m:

$$\begin{vmatrix} ma & mb \\ c & d \end{vmatrix} = mad - mbc = m(ad - bc) = m \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

It is clear from the formula for the determinant of a 3×3 matrix that the result also holds in this situation.

In general, if we multiply the first row of a matrix by m, the determinant is multiplied by m

What about the addition of scalars?

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = (a+a')d - (b+b')c = (ad-bc) + a'd - b'c$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

The two properties also extend to $n \times n$ matrices: if the other rows are left unchanged, the determinant is a linear function of its first row.

The fact that it is the first row is irrelevant, however, since we can always flip the order of the rows and then flip back using property 2. The bottom line is:

The determinant is a linear function of each row separately.

This property implies a result which may at first be counterintuitive to some:

$$\begin{vmatrix} ma & mb \\ mc & md \end{vmatrix} = m \begin{vmatrix} a & b \\ mc & md \end{vmatrix} = m^2 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} ma_{11} & ma_{12} & ma_{13} \\ ma_{21} & ma_{22} & ma_{23} \\ ma_{31} & ma_{32} & ma_{33} \end{vmatrix} = m \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ma_{21} & ma_{22} & ma_{23} \\ ma_{31} & ma_{32} & ma_{33} \end{vmatrix} = m^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ma_{31} & ma_{32} & ma_{33} \end{vmatrix} = m^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

For an $n \times n$ matrix, the determinant would be multiplied by m^n.

We will soon see that the determinant of an $n \times n$ matrix can be interpreted as the volume of an n-dimensional box. It then makes sense that if one multiplies all the lengths by the scalar m, the volume is increased by m^n.

So, to summarize, we have seen that

- $\det I = 1$ (Property 1)
- The determinant changes sign when two rows are exchanged (Property 2)
- The determinant is a linear function of each row separately (Property 3)

These three properties are sufficient to compute the determinant of any $n \times n$ matrix. They completely determine $\det A$ for any $n \times n$ A, even though that is not how we calculate it in practice. Before we learn practical formulas to compute $\det A$, let us highlight 7 important properties of the determinant which are consequences of the three properties above.

III) Other important properties

1) If two rows of A are equal, then $\det A = 0$ (Property 4)

This follows immediately from Property 2: exchanging the two equal rows does not change A but changes the sign of the determinant.
We have $\det A = -\det A \Rightarrow \det A = 0$

2) Subtracting a multiple of one row from another row leaves $\det A$ unchanged (Property 5)

This follows immediately from Property 3 and from property 4. Let us use a 3×3 matrix to visualize this, and you can easily convince yourself that it is true in general:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}-ma_{11} & a_{22}-ma_{12} & a_{23}-ma_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \overset{\text{Property 3}}{=} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - \ell \underbrace{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}_{0 \text{ by Property 4}}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Property 5 has an important consequence when combined with Property 2: if U is the upper triangular matrix such that $PA = LU$, then $\det A = \pm \det U$

Indeed, all we do when we compute the LU decomposition of a matrix is to subtract multiple of rows from other rows (elimination) and potentially exchange rows (permutation P). The + or − sign depends on the number of row exchanges, even or odd.

Bottom line: if we know how to compute determinant of triangular matrices U, we know how to find determinants of all matrices A.

3) A matrix with a row of zeros has $\det A = 0$ (Property 6)

This follows immediately from Properties 4 and 5: by Property 5, the determinant is unchanged if we add a nonzero row to the row of zeros. The resulting matrix has two equal rows, so $\det A = 0$ by Property 4.

4) If A is triangular, then $\det A = a_{11} a_{22} \cdots a_{nn}$, i.e the product of the diagonal entries (Property 7)

For the proof, we consider two situations:

- Suppose first that A is triangular, with all diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$ nonzero. Then by Jordan elimination, we know that we can turn A into a diagonal matrix $\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$

and by Property 5 the determinant remains unchanged. Now, let us evaluate, using property 3, the determinant of the diagonal matrix:

$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix}$$

$$\cdots = a_{11}a_{22}\cdots a_{nn} \det I = \underline{a_{11}a_{22}\cdots a_{nn}} \blacksquare$$

- What happens if a diagonal entry a_{ii} is zero? Then A does not have full rank. In that case, we have seen that elimination will eventually produce a zero row. By property 5, the determinant is unchanged, and by property 6, the determinant is zero. So $\det A = 0 = a_{11}a_{22}\cdots a_{ii}\cdots a_{nn}$. The formula also holds.

5) If A is singular then $\det A = 0$. If A is invertible, then $\det A \neq 0$ (Property 8)

Indeed, if A is singular, then the upper triangular U resulting from elimination has a zero in the pivot position, and by Property 7 this means $\det A = 0$

If A is invertible, U has n pivots, and $\det U = u_{11}\cdots u_{nn} \neq 0$

$\underline{\det A = \pm \det U = \pm u_{11}u_{22}\cdots u_{nn}}$

+ sign if the number of row exchanges is even, − sign if the number of row exchanges is odd.

Illustration: say $a \neq 0$, and consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $U = \begin{bmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{bmatrix}$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{vmatrix} = ad - bc$, in agreement with the formula we learned at the beginning of this lecture.

6) <u>The determinant of the matrix product</u> AB is $\det(AB) = \det(A)\det(B)$ (Property 9)

A direct proof of Property 9 can be tedious. Instead, we will show that the number $d = \dfrac{\det(AB)}{\det(B)}$ satisfies the same properties 1, 2, and 3

as $\det(A)$, so since properties 1, 2, and 3 fully define the determinant, we must have $d = \det(A)$

* First, of course, d only makes sense if $\det B \neq 0$. So let us treat $\det B = 0$, and forget this case afterwards.
If $\det B = 0$, B is not invertible, so its rank is less than n.
We saw in a homework problem that $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$, so AB is not invertible either, and $\det(AB) = 0$.
So $\det(AB) = \det(A)\det(B)$ is true in that case.

* Now $\det B \neq 0$, and we verify properties 1, 2, and 3
  - Property 1: If $A = I$, $\det A = I$ and $d = \dfrac{\det B}{\det B} = 1$ ✓
  - Property 2: When two rows of A are exchanged, the same rows

of AB are exchanged. So $\det(AB)$ changes sign just like $\det A$, and so does $d = \frac{\det(AB)}{\det B}$. ✓

• Property 3: When a row of A is multiplied by the scalar m, so is the same row of AB. So $\det(AB)$ is multiplied by m just like $\det A$, and so is $d = \frac{\det(AB)}{\det B}$

If one adds a row of a matrix A' to a row of A, then the same row of the matrix $A'B$ is added to that row of AB. By Property 3, we know that the determinants add, and since they are both divided by $\det B$, the ratios add.

We conclude that $\det A$ and d satisfy properties 1, 2 and 3 in the identical way, so we must have

$$d = \det A \iff \frac{\det(AB)}{\det B} = \det A$$

$$\iff \underline{\det(AB) = \det A \det B}$$

For A invertible, $AA^{-1} = I$ so property 9 means $\boxed{\det(A^{-1}) = \frac{1}{\det A}}$

7) $\det A = \det(A^T)$ (Property 10)

* If A is not invertible, then A^T is not invertible either. We then have $\det A = 0$ and $\det A^T = 0$ (Property 8) so $\det A = \det A^T$
* If A is invertible, then there exist a permutation matrix P, a lower triangular matrix L with 1's for all the diagonal entries, and an upper triangular matrix U with nonzero diagonal entries such that

$PA = LU \iff A^T P^T = U^T L^T$

Taking the determinant on both sides of the equality and using Property 9, we can write separately

$\det P \det A = \det L \det U$ and $\det A^T \det P^T = \det U^T \det L^T$

Now, from Property 7, $\det L = \det L^T = 1$ and $\det U = \det U^T$ (The transpose operation does not change the diagonal entries)
Thus,

$$\det P \det A = \det U = \det A^T \det P^T \quad (*)$$

Now, we saw in Lecture 17 that $PP^T = I$. By Property 9, this means $\det P^T = \frac{1}{\det P}$

We also know that $\det P = \pm 1$, from earlier in this lecture.
So $\underline{\det P^T = \det P}$

Returning to $(*)$, this means that $\underline{\det A = \det A^T}$, which completes our proof.

Note: The rows of A are the columns of A^T, so $\det A = \det A^T$ implies that all the properties of determinants we have learned regarding rows also apply to columns.