Continuous but not Smooth

Find values of the constants a and b for which the following function is continuous but *not* differentiable.

$$f(x) = \begin{cases} ax + b, & x > 0; \\ \sin 2x, & x \leq 0. \end{cases}$$

In other words, the graph of the function should have a sharp corner at the pont $(0, f(0))$.

Solution

Because $g(x) = ax+b$ and $h(x) = \sin 2x$ are continuous functions, the only place where $f(x)$ might be discontinuous is where $x = 0$. To make $f(x)$ continuous, we need to find values of a and b for which:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x).$$

We start with $\lim_{x\to 0^+} f(x)$ because we find linear functions easier to work with than trigonometric ones:

$$\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} ax + b \\ &= a \cdot 0 + b \\ &= b. \end{aligned}$$

Next we find $\lim_{x\to 0^-} f(x)$:

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} \sin 2x \\ &= \sin 0 \\ &= 0. \end{aligned}$$

In order for $f(x)$ to be continuous at 0, it must be true that $b = 0$.

Next we need to determine when $f(x)$ is *not* differentiable. To the right of $x = 0$, the graph of $f(x)$ is a straight line. To the left, the graph is a smooth curve. As long as $x \neq 0$, the graph of $f(x)$ has a well defined tangent line at the point $(x, f(x))$.

Therefore, if $f(x)$ is differentiable at $x = 0$ then it is differentiable overall. The function is differentiable if it has a well defined tangent line at $x = 0$. This will be true when the slope of the line $y = ax + b = ax + 0$ equals the slope of the curve $y = \sin 2x$ at the point $(0, 0)$. We'll learn later that the slope of the curve $y = \sin 2x$ equals $2 \cos 2x$; for now it's enough to know that the slope of $y = \sin 2x$ is positive when $x = 0$.

We conclude that $f(x)$ is continuous but not differentiable for $b = 0$ and any $a \leq 0$. In fact, for any value of a other than 2 the function will be continuous but not differentiable.

We can check our work by graphing $f(x)$ for a few different values of a (while $b = 0$). This increases our familiarity with graphs of continuous but non-differentiable functions.

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