Definition of double integration

In this note we will work abstractly, defining double integration as a sum, technically a limit of Riemann sums. It is best to learn this first before getting into the details of computing the value of a double integral –we will learn how to do that next.

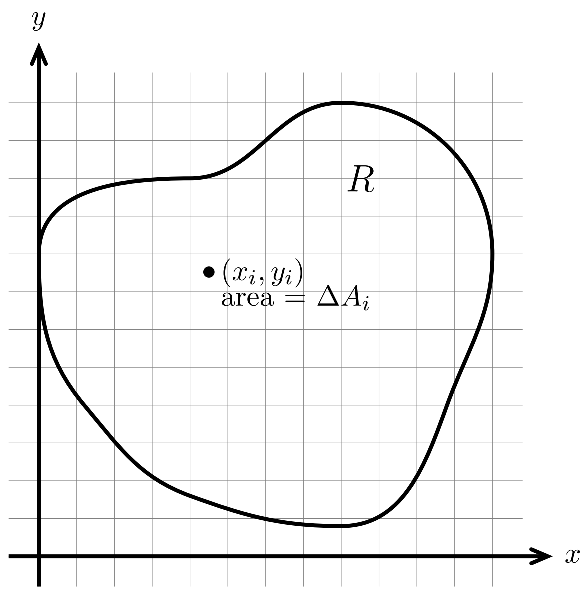
Definition of double integrals

Suppose we have a region in the plane R and a function $f(x, y)$, Then the double integral

$$\int\int_R f(x, y)\, dA$$

is defined as follows.

Divide the region R into small pieces, numbered from 1 to n. Let ΔA_i be the area of the i^{th} piece and also pick a point (x_i, y_i) in that piece. The figure shows a region R divided into small pieces and shows the i^{th} piece with its area, and choice of a point in the little region.



Now form the sum

$$\sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i,$$

and then, finally

$$\int\int_R f(x, y)\, dA = \lim_{\Delta A \to 0} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i.$$

Here, the limit is taken by letting the number of pieces go to infinity and the area of each piece go to 0. There are technical requirements that the limit exist and be independent of the specific limiting process. In 18.02 these requirements are always met. (Later you might study fractals and other strange objects which don't satisfy them.)

Interpretations of the double integral

As you saw in single variable calculus, these sums can be used to compute areas, volumes, mass, work, moment of inertia and many other quantities. Again, before focusung on some computational issues we will show you how easy it is to setup a double integral to compute certain quantities.

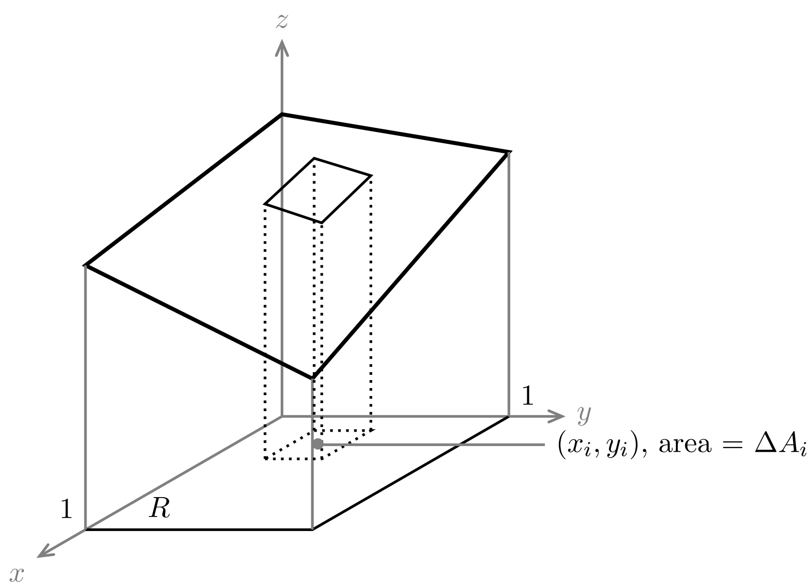
Example 1: Set up a double integral to compute the area of a region R in the plane.

Answer: Use the figure above for visualization. The area of R is just the sum of the areas of the pieces. That is,

$$\text{area} = \int\int_R dA.$$

Example 2: Set up a double integral to compute the volume of the solid below the graph of $z = f(x, y) = 2 - .5(x + y)$ and above the unit square in the xy-plane.

Answer: The figure below shows the graph of $f(x, y)$ above the unit square in the plane. The unit square is labeled R. We also show a little piece of the R and the solid region above that piece. We are imagining we've divided R into n small pieces and this is the i^{th} one. It contains the point (x_i, y_i) and has area ΔA_i.



The small solid region is almost a box and so its volume, ΔV_i, is roughly its base times its height, i.e.,

$$\Delta V_i \approx \Delta A_i \times f(x_i, y_i).$$

The total volume is the sum of the volumes of all the small pieces, i.e.,

$$\text{volume} = \sum_{i=1}^{n} \Delta V_i \approx \sum_{i=1}^{n} \Delta A_i \times f(x_i, y_i).$$

In the limit this becomes an exact integral for volume

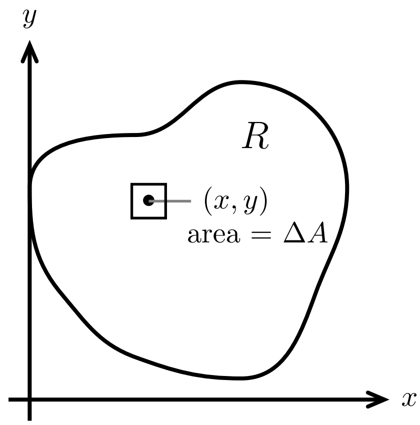
$$\text{volume} = \int\int_R f(x, y)\, dA.$$

Double integration

1. Set up a double integral for the mass of the planar region R with a variable density $\delta(x, y)$. Use a simple sketch to illustrate the setup.

Note: the problem as stated is abstract, so your solution will also be abstract. The goal here is to focus on the simple idea of slicing the region into small pieces and summing the contributions of each of the pieces that is the basis of integration.

Answer: First note: since this is a planar region the density is in mass/unit area.



Over a small piece the density (which we assume is continuous) won't vary very much. Thus, if Δm is the mass of the small piece shown in the figure, we have

$$\Delta m \approx \delta(x, y)\Delta A,$$

where (x, y) is a point in the piece and ΔA is the area of the piece.

Now, imagine we slice the region into small pieces and find the mass of each piece, then the total mass of the region is just the sum of the masses of the pieces.

If we let $\Delta A \to 0$ the approximation will become better and the sum will become an integral.

$$M = \int\int_R \delta(x, y)\, dA.$$

If we want to do this more formally, we would slice the region into n pieces and label each piece with and index from 1 to n. If Δm_i is the mass of the i$^{\text{th}}$ piece then

$$\Delta m_i \approx \delta(x_i, y_i)\Delta A_i,$$

where (x_i, y_i) is an arbitrary point in the the piece and ΔA_i is its area. Summing the masses of all the pieces we get

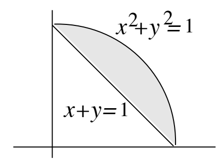
$$\text{Mass of } R \; = \sum_{i=1}^{n} \Delta m_i \approx \sum_{i=1}^{n} \delta(x_i, y_i)\Delta A_i.$$

This is called a Riemann sum. In the limit as n goes to infinity and the area of each piece goes to 0, the sum becomes the integral.

Limits in Iterated Integrals

For most students, the trickiest part of evaluating multiple integrals by iteration is to put in the limits of integration. Fortunately, a fairly uniform procedure is available which works in any coordinate system. *You must always begin by sketching the region; in what follows we'll assume you've done this.*

1. Double integrals in rectangular coordinates.



Let's illustrate this procedure on the first case that's usually taken up: double integrals in rectangular coordinates. Suppose we want to evaluate over the region R pictured the integral

$$\iint_R f(x,y)\,dy\,dx\ , \qquad\qquad R = \text{ region between } x^2+y^2=1 \quad\text{and}\quad x+y=1\ ;$$

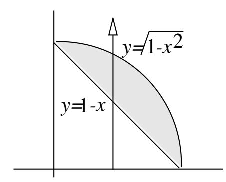
we are integrating first with respect to y. Then to put in the limits,

1. Hold x fixed, and let y increase (since we are integrating with respect to y). As the point (x,y) moves, it traces out a vertical line.

2. Integrate from the y-value where this vertical line enters the region R, to the y-value where it leaves R.

3. Then let x increase, integrating from the lowest x-value for which the vertical line intersects R, to the highest such x-value.

Carrying out this program for the region R pictured, the vertical line enters R where $y = 1-x$, and leaves where $y=\sqrt{1-x^2}$.



The vertical lines which intersect R are those between $x=0$ and $x=1$. Thus we get for the limits:

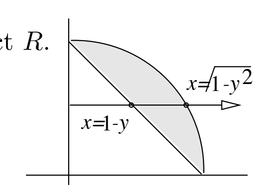
$$\iint_R f(x,y)\,dy\,dx \;=\; \int_0^1\int_{1-x}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx.$$

To calculate the double integral, integrating in the reverse order $\iint_R f(x,y)\,dx\,dy$,

1. Hold y fixed, let x increase (since we are integrating first with respect to x). This traces out a horizontal line.

2. Integrate from the x-value where the horizontal line enters R to the x-value where it leaves.

3. Choose the y-limits to include all of the horizontal lines which intersect R.



Following this prescription with our integral we get:

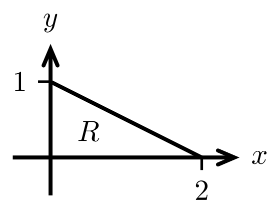
$$\iint_R f(x,y)\,dx\,dy \;=\; \int_0^1\int_{1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$

Limits for double integrals

1. Evaluate $\iint_R x\,dA$, where R is the finite region bounded by the axes and $2y + x = 2$.

Answer:

First we sketch the region.



Next, we find limits of integration. By using vertical stripes we get limits

Inner: y goes from 0 to $1 - x/2$; outer: x goes from 0 to 2.

Thus the integral is

$$\int_0^2 \int_0^{1-x/2} x\,dy\,dx$$

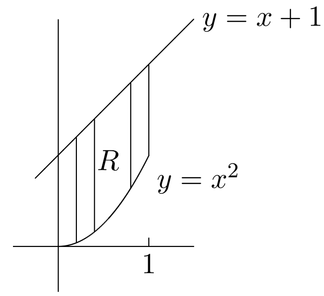
Finally, we compute the inner, then the outer integrals.

Inner: $xy|_0^{1-x/2} = x - \dfrac{x^2}{2}$.

Outer: $\dfrac{x^2}{2} - \dfrac{x^3}{6}\Big|_0^2 = \dfrac{2}{3}$.

Problems: Regions of Integration

1. Find the mass of the region R bounded by
$y = x + 1$; $y = x^2$; $x = 0$ and $x = 1$, if density $= \delta(x, y) = xy$.



Answer:
Inner limits: y from x^2 to $x + 1$. Outer limits: x from 0 to 1.

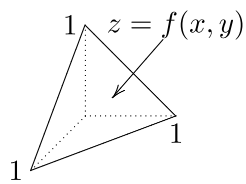
$$\Rightarrow \; M = \int\int_R \delta(x, y)\, dA = \int_{x=0}^{1} \int_{y=x^2}^{x+1} xy\, dy\, dx$$

Inner: $\displaystyle\int_{x^2}^{x+1} xy\, dy = x\frac{y^2}{2}\Big|_{x^2}^{x+1} = \frac{x(x+1)^2}{2} - \frac{x^5}{2} = \frac{x^3}{2} + x^2 + \frac{x}{2} - \frac{x^5}{2}.$

Outer: $\displaystyle\int_0^1 \frac{x^3}{2} + x^2 + \frac{x}{2} - \frac{x^5}{2}\, dx = \frac{x^4}{8} + \frac{x^3}{3} + \frac{x^2}{4} - \frac{x^6}{12}\Big|_0^1 = \frac{1}{8} + \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{5}{8}.$

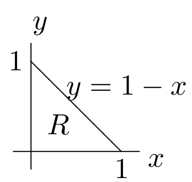
Note: The syntax $y = x^2$ in limits is redundant but useful. We know it must be y because of the dy matching the integral sign.

2. Find the volume of the tetrahedron shown below.



Tetrahedron

Answer: The surface has height: $z = 1 - x - y$.



Region R

Limits: inner: $0 < y < 1 - x$, outer: $0 < x < 1$. $\Rightarrow \; V = \displaystyle\int_{x=0}^{1} \int_{y=0}^{1-x} 1 - x - y\, dy\, dx.$

Inner: $\displaystyle\int_{y=0}^{1-x} 1 - x - y\, dy = y - xy - \frac{y^2}{2}\Big|_0^{1-x} = 1 - x - x + x^2 - \frac{1}{2} + x - \frac{x^2}{2}.$

Outer: $\displaystyle\int_0^1 \frac{1}{2} - x + \frac{x^2}{2}\, dx = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}.$

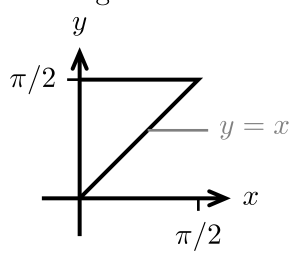
Changing the order of integration

1. Evaluate

$$I = \int_0^{\pi/2} \int_x^{\pi/2} \frac{\sin y}{y}\, dy\, dx$$

by changing the order of integration.

<u>**Answer:**</u>
The given limits are (inner) y from x to $\pi/2$; (outer) x from 0 to $\pi/2$.
We use these to sketch the region of integration.



The given limits have inner variable y. To reverse the order of integration we use horizontal stripes. The limits in this order are

(inner) x from 0 to y; (outer) y from 0 to $\pi/2$.

So the integral becomes

$$I = \int_0^{\pi/2} \int_0^{y} \frac{\sin y}{y}\, dx\, dy$$

We compute the inner, then the outer integrals.

Inner: $\left.\frac{\sin y}{y} x\right|_0^y = \sin y$. Outer: $-\cos y|_0^{\pi/2} = 1$.

Problems: Exchanging the Order of Integration

Calculate $\int_0^2 \int_x^2 e^{-y^2}\, dy\, dx$.

Answer: As you may recall, the function e^{-y^2} has no simple antiderivative. However, this double integral can be computed by reversing the order of integration.

The region R is the triangle with vertices at $(0,0)$, $(0,2)$ and $(2,2)$ (sketch it!) Thus:

$$\int_{x=0}^2 \int_{y=x}^2 e^{-y^2}\, dy\, dx = \int_{y=0}^2 \int_{x=0}^y e^{-y^2}\, dx\, dy.$$

Inner: $\displaystyle\int_{x=0}^y e^{-y^2}\, dx = ye^{-y^2}$.

Outer: $\displaystyle\int_{y=0}^2 ye{-}y^2\, dy = -\frac{1}{2}e^{-y^2}\Big|_0^2 = \frac{1}{2}(1 - e^{-4}) \approx \frac{1}{2}$.

We're finding the area under a surface with maximum height 1 and minimum height $e^{-4} \approx 0.1$ over a triangle of area 2. This answer seems plausible.

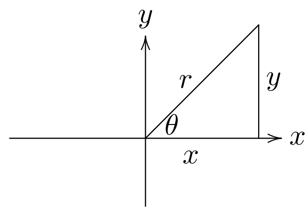
Integration in polar coordinates

Polar Coordinates

Polar coordinates are a different way of describing points in the plane. The polar coordinates (r, θ) are related to the usual *rectangular coordinates* (x, y) by by

$$x = r\cos\theta, \quad y = r\sin\theta$$

The figure below shows the standard polar triangle relating x, y, r and θ.



Because cos and sin are periodic, different (r, θ) can represent the same point in the plane. The table below shows this for a few points.

(x, y)	$(1, 0)$	$(0, 1)$	$(2, 0)$	$(1, 1)$	$(-1, 1)$	$(-1, -1)$	$(0, 0)$
(r, θ)	$(1, 0)$	$(1, \pi/2)$	$(2, 0)$	$(\sqrt{2}, \pi/4)$	$(\sqrt{2}, 3\pi/4)$	$(\sqrt{2}, 5\pi/4)$	$(0, \pi/2)$
(r, θ)	$(1, 2\pi)$			$(\sqrt{2}, 9\pi/4)$		$(-\sqrt{2}, \pi/4)$	$(0, -7.2)$
(r, θ)	$(1, 4\pi)$						

In fact, you can add any multiple of 2π to θ and the polar coordinates will still represent the same point.

Because θ is not uniquely specified it's a little trickier going from rectangular to polar coordinates. The equations are easily deduced from the standard polar triangle.

$$r = \sqrt{x^2 + y^2}, \qquad \text{"}\theta = \tan^{-1}(y/x)\text{"}.$$

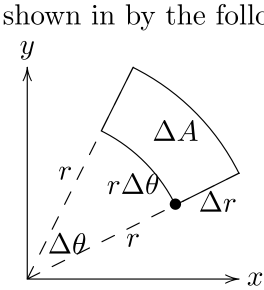
We use quotes around $\tan^{-1}$ to indicate it is not a single valued function.

The area element in polar coordinates

In polar coordinates the area element is given by

$$dA = r\,dr\,d\theta.$$

The geometric justification for this is shown in by the following figure.

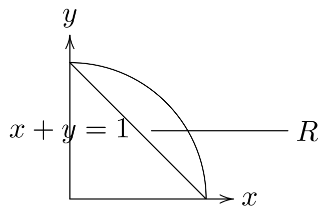


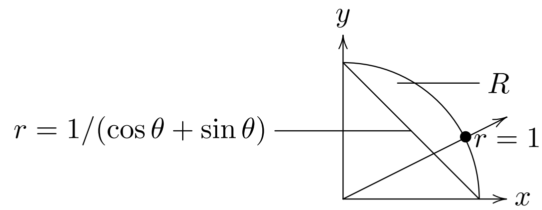
The small curvy rectangle has sides Δr and $r\Delta\theta$, thus its area satisfies $\Delta A \approx (\Delta r)(r\,\Delta\theta)$. As usual, in the limit this becomes $dA = r\,dr\,d\theta$.

Double integrals in polar coordinates

The area element is one piece of a double integral, the other piece is the limits of integration which describe the region being integrated over.

Finding procedure for finding the limits in polar coordinates is the same as for rectangular coordinates. Suppose we want to evaluate $\int\int_R dr\,d\theta$ over the region R shown.





(The integrand, including the r that usually goes with $r\,dr\,d\theta$, is irrelevant here, and therefore omitted.)
As usual, we integrate first with respect to r. Therefore, we

1. Hold θ fixed, and let r increase (since we are integrating with respect to r). As the point moves, it traces out a ray going out from the origin.

2. Integrate from the r-value where the ray enters R to the r-value where it leaves. This gives the limits on r.

3. Integrate from the lowest value of θ for which the corresponding ray intersects R to the highest value of θ.

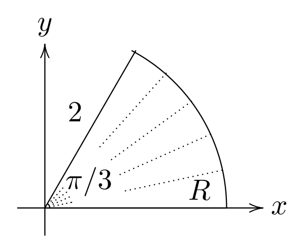
To follow this procedure, we need the equation of the line in polar coordinates. We have

$$x + y = 1 \quad \to \quad r\cos\theta + \mathbf{r}\sin\theta = 1, \quad \text{or} \quad r = \frac{1}{\cos\theta + \sin\theta}.$$

This is the r value where the ray enters the region; it leaves where $r = 1$. The rays which intersect R lie between $\theta = 0$ and $\theta = \pi/2$. Thus the double iterated integral in polar coordinates has the limits

$$\int_0^{\pi/2}\int_{1/(\cos\theta+\sin\theta)}^{1} dr\,d\theta.$$

Example: Find the mass of the region R shown if it has density $\delta(x, y) = xy$ (in units of mass/unit area)
In polar coordinates: $\delta = r^2\cos\theta\sin\theta$.
Limits of integration: (radial lines sweep out R):
inner (fix θ): $0 < r < 2$, outer: $0 < \theta < \pi/3$.



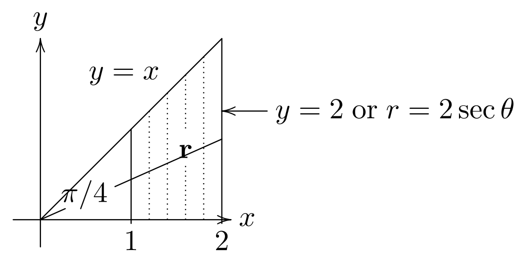
$\Rightarrow$ Mass $M = \int\int_R \delta(x, y)\,dA = \int_{\theta=0}^{\pi/3}\int_{r=0}^{2} r^2\cos\theta\sin\theta\,r\,d\theta\,dr$

Inner: $\displaystyle\int_0^2 r^3\cos\theta\sin\theta\,dr = \frac{r^4}{4}\cos\theta\sin\theta\Big|_0^2 = 4\cos\theta\sin\theta$

Outer: $\displaystyle M = \int_0^{\pi/3} 4\cos\theta\sin\theta\,d\theta = 2\sin^2\theta\Big|_0^{\pi/3} = \frac{3}{2}.$

Example: Let $I = \int_1^2 \int_0^x \frac{1}{(x^2+y^2)^{3/2}}\, dy\, dx$. Compute I using polar coordinates.

<u>Answer:</u> Here are the steps we take.
Draw the region.



Find limits in polar coordinates:
Inner (fix θ): $\sec\theta < r < 2\sec\theta$, outer: $0 < \theta < \pi/4$.

$$\Rightarrow \; I = \int_{\theta=0}^{\pi/4} \int_{r=\sec\theta}^{2\sec\theta} \frac{1}{r^3} r\, dr\, d\theta.$$

Compute the integral:

Inner: $\displaystyle\int_{\sec\theta}^{2\sec\theta} \frac{1}{r^2}\, dr = -\frac{1}{r}\Big|_{\sec\theta}^{2\sec\theta} = \frac{1}{2}\cos\theta.$

Outer: $\displaystyle I = \int_0^{\pi/4} \frac{1}{2}\cos\theta\, d\theta = \frac{1}{2}\sin\theta\Big|_0^{\pi/4} = \frac{\sqrt{2}}{4}.$

Example: Find the volume of the region above the xy-plane and below the graph of $z = 1 - x^2 - y^2$.

You should draw a picture of this.

In polar coordinates we have $z = 1 - r^2$ and we want the volume under the graph and above the inside of the unit disk.

$$\Rightarrow \text{ volume } V = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r dr\, d\theta.$$

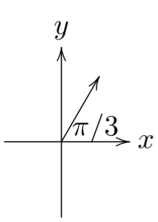
Inner integral: $\displaystyle\int_0^1 (1 - r^2)\, r dr = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$

Outer integral: $\displaystyle V = \int_0^{2\pi} \frac{1}{4}\, d\theta = \frac{\pi}{2}.$

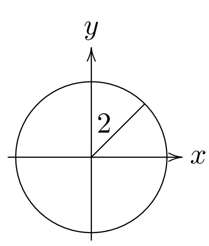
Gallery of polar graphs ($r = f(\theta)$)

A point P is on the graph **if any** representation of P satisfies the equation.

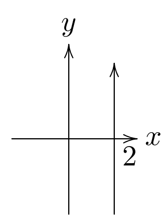
Examples:



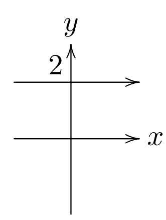
Ray: $\theta = \pi/3$



Circle centered on 0: $r = 2$

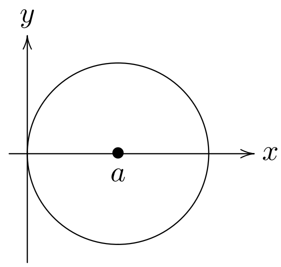


Vertical line $x = 2 \Leftrightarrow r = 2\sec\theta$.

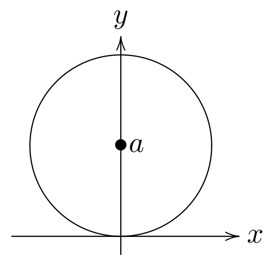


Horizontal line $y = 2 \Leftrightarrow r = 2/\sin\theta$.

Example: Show the graph of $r = 2a\cos\theta$ is a circle of radius a centered at $(a, 0)$.
Some simple algebra gives $r^2 = 2ar\cos\theta = 2ax \quad \Rightarrow\ x^2+y^2 = 2ax \quad \Rightarrow\ (x-a)^2+y^2 = a^2$.
This is a circle or radius a centered at $(a, 0)$.
Note: we can determine from the graph that the range of theta is $-\pi/2 \le \theta \le \pi/2$.



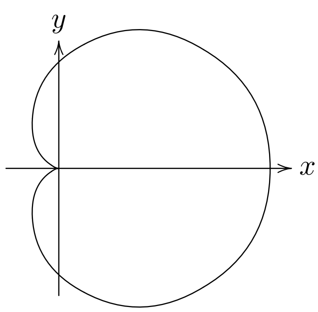
$r = 2a\cos\theta$
$-\pi/2 \le \theta \le \pi/2.$



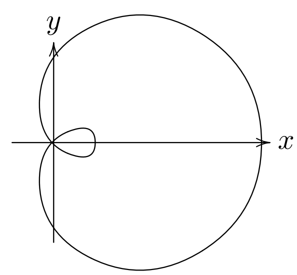
$r = 2a\sin\theta$
$0 \le \theta \le \pi.$

Warning: We can use negative values of r for plotting. You should never use it in integration. In integration it is better to make use of symmetry and only integrate over regions where r is positive.

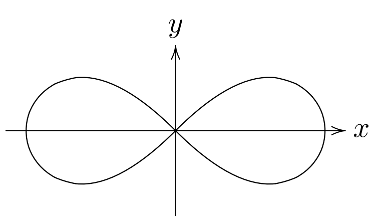
Here are a few more curves.



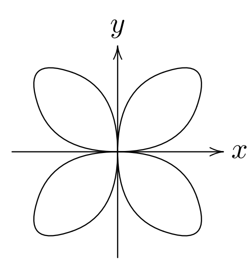
Cardiod: $r = a(1 + \cos\theta)$



Limaçon: $r = a(1 + b\cos\theta)$ $(b > 1)$



Lemniscate: $r^2 = 2a^2\cos 2\theta$

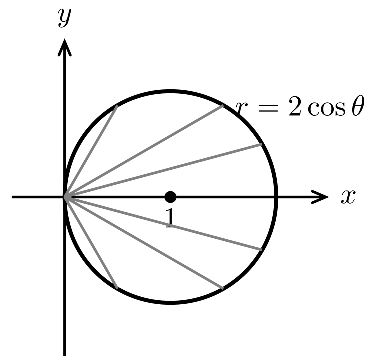


Four leaved rose: $r = a\sin 2\theta$

Double integration in polar coordinates

1. Compute $\iint_R f(x,y)\,dx\,dy$, where $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$ and R is the region inside the circle of radius 1, centered at (1,0).

<u>Answer:</u> First we sketch the region R



Both the integrand and the region support using polar coordinates. The equation of the circle in polar coordinates is $r = 2\cos\theta$, so using radial stripes the limits are

(inner) r from 0 to $2\cos\theta$; (outer) θ from $-\pi/2$ to $\pi/2$.

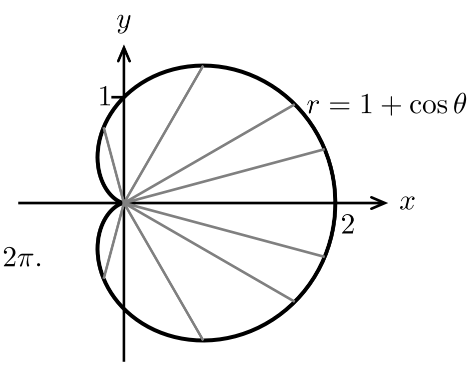
Thus,

$$\iint_R f(x,y)\,dx\,dy = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} \frac{1}{r}\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} dr\,d\theta.$$

Inner integral: $2\cos\theta$.

Outer integral: $2\sin\theta|_{-\pi/2}^{\pi/2} = 4$.

2. Find the area inside the cardioid $r = 1 + \cos\theta$.



<u>Answer:</u> The cardioid is so-named because it is heart-shaped.

Using radial stripes, the limits of integration are

(inner) r from 0 to $1+\cos\theta$; (outer) θ from 0 to 2π.

So, the area is

$$\iint_R dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta.$$

Inner integral: $\dfrac{(1+\cos\theta)^2}{2}$.

Side work:

$$\int \cos^2\theta\,d\theta = \int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C \;\Rightarrow\; \int_0^{2\pi} \cos^2\theta\,d\theta = \pi.$$

Outer integral:

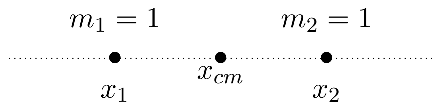
$$\int_0^{2\pi} \frac{(1+\cos\theta)^2}{2} = \int_0^{2\pi} \frac{1}{2} + \cos\theta + \frac{\cos^2\theta}{2}\,d\theta = \pi + 0 + \frac{\pi}{2} = \frac{3\pi}{2}.$$

The area of the cardioid is $\dfrac{3\pi}{2}$.

Mass and average value

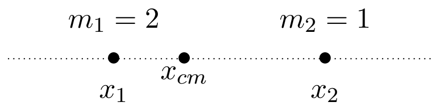
Center of Mass

Example 1: For two equal masses, the center of mass is at the midpoint between them.



$x_{cm} = \frac{x_1+x_2}{2}$.

Example 2: For unequal masses the center of mass is a weighted average of their positions.

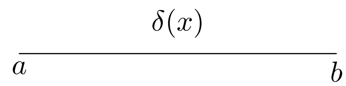


$x_{cm} = \frac{2x_1+x_2}{3}$.

In general, x_{cm} = weighted average of position $= \frac{\sum m_i x_i}{\sum m_i}$.

For a continuous density, $\delta(x)$, on the segment $[a, b]$ (units of density are mass/unit length) the sums become integrals. We will skip running through the logic of this since we are about to show it for two dimensions.

$$M = \int_a^b \delta(x)\,dx, \quad x_{cm} = \frac{1}{M}\int_a^b x\delta(x)\,dx.$$

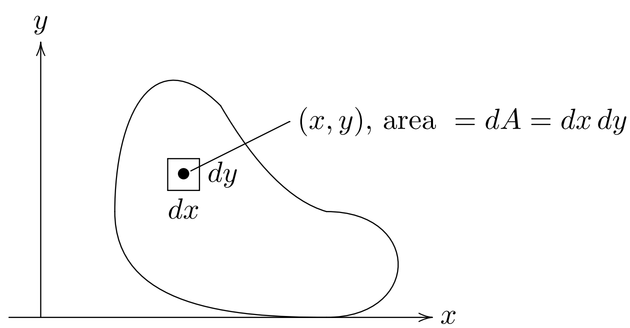


In 2 dimensions we label the center of mass as (x_{cm}, y_{cm}) and we have the following formulas

$$M = \int\int_R \delta(x, y)\,dA, \quad x_{cm} = \frac{1}{M}\int\int_R x\delta(x, y)\,dA, \quad y_{cm} = \frac{1}{M}\int\int_R y\delta(x, y)\,dA.$$

These formulas are easy to justify using our usual method for building integrals.

In this case, we divide our region into little pieces and we sum up the contributions of each piece using an integral. To keep the figure below uncluttered we only show one piece and we don't bother to label it as the i^{th}. In the end we will go directly to the integral, by thinking if it as a sum.



The little piece shown has mass $\delta(x, y)\,dA$ and the total mass is just the sum the pieces. That is, it's just the integral

$$M = \int\int_R \delta\,dA.$$

Likewise the x and y coordinates of the center of mass are just the weighted average of the x and y coordinates of each of the pieces. So, we get the formulas given above.

Example: Suppose the unit square has density $\delta = xy$; Find its mass and center of mass.

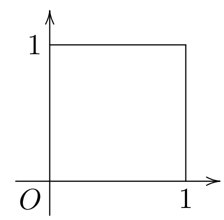
<u>**Answer:**</u>

$M = \int\int_R \delta \, dA = \int_0^1 \int_0^1 xy \, dx \, dy = \frac{1}{4}.$

$x_{cm} = \frac{1}{M} \int\int x\delta \, dA = \frac{1}{M} \int_0^1 \int_0^1 x^2 y \, dy \, dx.$

This is easily computed as $x_{cm} = \frac{2}{3}$.

By symmetry of the region and the density δ, we also have $y_{cm} = \frac{2}{3}$.



Average Value

We can think of center of mass as the average position of the mass. That is, it's the average of position with respect to mass. We can also take averages of functions with respect to other things. For instance, the *average value* of $f(x, y)$ *with respect to area* on a region R is

$$\frac{1}{\text{area } R} \int\int_R f(x, y) \, dA.$$

In general, if we simply say the average value of a function, it means average value with respect to area (or later, when we do triple integrals it can mean with respect to volume).

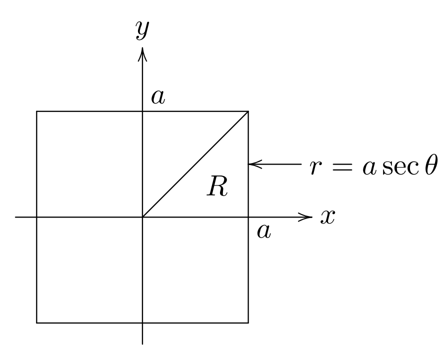
Example: What's the average distance of a point in a unit square from the center?

<u>**Answer:**</u> We center the square on the origin. Notice each side has length $2a$ and the area of the square is $4a^2$. So,

$$\text{average distance} = \frac{1}{4a^2} \int\int_{\text{square}} \sqrt{x^2 + y^2} \, dx \, dy.$$

By symmetry the integral is 8 times the integral of the triangular region R shown. We actually compute the integral in polar coordinates.

$$\begin{aligned} \text{average} &= \frac{8}{4a^2} \int\int_R r \, r \, dr \, d\theta \\ &= \frac{2}{a^2} \int_0^{\pi/4} \int_0^{a \sec\theta} r^2 \, dr \, d\theta \\ &= \frac{2}{a^2} \int_0^{\pi/4} \frac{a^3 \sec^3\theta}{3} \, d\theta \\ &= \frac{a}{3}(\sqrt{2} + \ln(\sqrt{2} + 1)). \end{aligned}$$



This last integral was computed using integration by parts as

$$\int \sec^3\theta \, d\theta = \frac{\int \sec\theta \, d\theta + \sec\theta \tan\theta}{2} = \frac{\ln(\sec\theta + \tan\theta) + \sec\theta \tan\theta}{2}.$$

Note: the center of mass is the average value of x and y with respect to mass.

The *geometric center* has coordinates given by the average value of x and y with respect to area, i.e., the center of mass when $\delta = 1$.

Problems: Mass and Average Value

Let R be the quarter of the unit circle in the first quadrant with density $\delta(x, y) = y$.

1. Find the mass of R.

Because R is a circular sector, it makes sense to use polar coordinates. The limits of integration are then $0 \le r \le 1$ and $0 \le \theta \le \pi/2$. In addition we have $\delta = r \sin\theta$. To find the mass of the region, we integrate the product of density and area.

$$\begin{aligned} M &= \iint_R \delta \, dA \\ &= \int_0^{\pi/2} \int_0^1 (r \sin\theta)\, r \, dr \, d\theta \\ &= \int_0^{\pi/2} \int_0^1 r^2 \sin\theta \, dr \, d\theta. \end{aligned}$$

Inner: $\frac{1}{3} r^3 \sin\theta \Big|_0^1 = \frac{1}{3} \sin\theta$.
Outer: $-\frac{1}{3} \cos\theta \Big|_0^{\pi/2} = \frac{1}{3}$.
The region has mass 1/3.

This seems like a reasonable conclusion – the region has area a little greater than 1/2 and average density around 1/2.

2. Find the center of mass.

The center of mass (x_{cm}, y_{cm}) is described by

$$x_{cm} = \frac{1}{M} \iint_R x\delta \, dA \quad \text{and} \quad y_{cm} = \frac{1}{M} \iint_R y\delta \, dA.$$

From (1), $M = \frac{1}{3}$.

$$\begin{aligned} x_{cm} &= \frac{1}{M} \iint_R x\delta \, dA \\ &= 3 \int_0^{\pi/2} \int_0^1 (r\cos\theta)(r\sin\theta) r \, dr \, d\theta \\ &= \int_0^{\pi/2} \int_0^1 3r^3 \cos\theta \sin\theta \, dr \, d\theta. \end{aligned}$$

Inner: $\frac{3}{4} r^4 \cos\theta \sin\theta \Big|_0^1 = \frac{3}{4} \cos\theta \sin\theta$.

Outer: $\frac{3}{4} \frac{1}{2} (\sin\theta)^2 \Big|_0^{\pi/2} = \frac{3}{8} = x_{cm}$.

$$\begin{aligned}
y_{cm} &= \frac{1}{M}\iint_R y\delta\, dA \\
&= 3\int_0^{\pi/2}\int_0^1 (r\sin\theta)(r\sin\theta)r\, dr\, d\theta \\
&= \int_0^{\pi/2}\int_0^1 3r^3\sin^2\theta\, dr\, d\theta.
\end{aligned}$$

Inner: $\left.\frac{3}{4}r^4\sin^2\theta\right|_0^1 = \frac{3}{4}\sin^2\theta.$

Outer: $\left.\frac{3}{4}\left(\frac{\theta}{2}-\frac{1}{4}\sin(2\theta)\right)\right|_0^{\pi/2} = \frac{3\pi}{16} = y_{cm}.$

The center of mass is at $\left(\frac{3}{8},\frac{3}{4}\right) \approx (.4, .6)$.

This point is within R and agrees with our intuition that $x_{cm} < 1/2$ and $y_{cm} > x_{cm}$.

3. Find the average distance from a point in R to the x axis.

To find the average of a function $f(x,y)$ over an area, we compute $\frac{1}{\text{Area}}\iint_R f(x,y)\, dA$. Here $f(x,y) = y$.

$$\begin{aligned}
\frac{1}{\text{Area}}\iint_R y\, dA &= \frac{1}{\pi/4}\int_0^{\pi/2}\int_0^1 (r\sin\theta)r\, dr\, d\theta \\
&= \frac{4}{\pi}\int_0^{\pi/2}\int_0^1 r^2\sin\theta\, dr\, d\theta.
\end{aligned}$$

This should look familiar – we computed in (1) that $\int_0^{\pi/2}\int_0^1 r^2\sin\theta\, dr\, d\theta = \frac{1}{3}$. The average distance from a point in R to the x axis is $\frac{4}{\pi}\cdot\frac{1}{3} = \frac{4}{3\pi}$.

Moment of Inertia

Moment of inertia

We will leave it to your physics class to really explain what moment of inertia means. Very briefly it measures an object's resistance (inertia) to a change in its rotational motion. It is analogous to the way mass measure the resistance to changes in the object's linear motion.

Because it has to do with rotational motion the moment of inertia is always measured about a reference line, which is thought of as the axis of rotation.

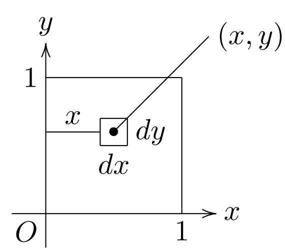
For a point mass, m, the moment of inertia about the line is

$$I = m\, d^2,$$

where d is the distance from the mass to the line. (The letter I is a standard notation for moment of inertia.)

If we have a distributed mass we compute the moment of inertia by summing the contributions of each of its parts. If the mass has a continuous distribution, this sum is, of course, an integral.

Example 1: Suppose the unit square, R, has density $\delta = xy$.
Find its moment of inertia about the y-axis.



<u>**Answer:**</u> The distance from the small piece of the square (shown in the figure) to the y-axis is x. If the piece has mass dm then its moment of inertia is

$$dI = x^2\, dm = x^2 \delta(x, y)\, dA = x^3 y\, dx\, dy.$$

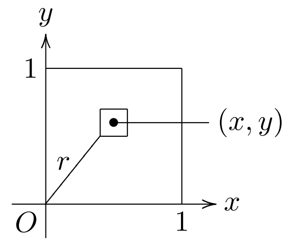
We took a shortcut here: we went straight to the notation for infinitesimal pieces, dI, dA and used equalities. rather than using more formal notation ΔI, ΔA, using approximations and then taking limits.

In the equation above, we used the notation dI to indicate it is just a small bit of moment of inertia. We also used that the mass of a piece is density times area. Now it's a simple matter to sum up all the bits of moment of inertia using an integral

$$I = \int\int_R dI = \int_0^1 \int_0^1 x^3 y\, dy\, dx = \frac{1}{8}.$$

(Note: this integral is so easy to compute that we don't give the details.)

Example 2: For the same square as in example 1, find the *polar moment of inertia*.



Answer: The polar moment of inertia of a planar region is the moment of inertia about the origin (the axis of rotation is the z-axis). Finding this is exactly the same as in example 1, except the distance to the axis is now the polar distance r. We get,

$$dI = r^2\,dm = (x^2 + y^2)\delta(x, y)\,dA = (x^3y + xy^3)\,dx\,dy.$$

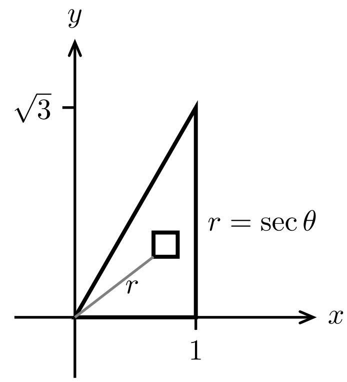
Summing, using an integral gives

$$I = \int\int_R dI = \int_0^1 \int_0^1 x^3y + xy^3\,dy\,dx = \frac{1}{4}.$$

Moment of inertia

1. Let R be the triangle with vertices $(0,0)$, $(1,0)$, $(1,\sqrt{3})$ and density $\delta = 1$. Find the polar moment of inertia.

__**Answer:**__ The region R is a 30, 60 , 90 triangle.



The polar moment of inertia is the moment of inertia around the origin (that is, the z-axis). The figure shows the triangle and a small square piece within R. If the piece has area dA then its polar moment of inertia is $dI = r^2\delta\, dA$. Summing the contributions of all such pieces and using $\delta = 1$, $dA = r\, dr\, d\theta$, we get the total moment of inertia is

$$I = \iint_R r^2 \delta\, dA = \iint_R r^2\, r\, dr\, d\theta = \iint_R r^3\, dr\, d\theta.$$

Next we find the limits of integration in polar coordinates. The line

$$x = 1 \;\Leftrightarrow\; r\cos\theta = 1 \;\Leftrightarrow\; r = \sec\theta.$$

So, using radial stripes, the limits are: (inner) r from 0 to $\sec\theta$; (outer) θ from 0 to $\pi/3$.

Thus,

$$I = \int_0^{\pi/3}\int_0^{\sec\theta} r^3\, dr\, d\theta.$$

Inner integral: $\dfrac{\sec^4\theta}{4}$.

Outer integral: Use $\sec^4\theta = \sec^2\theta\sec^2\theta = (1+\tan^2\theta)\, d(\tan\theta) \;\Rightarrow$ the outer integral is

$$\frac{1}{4}\left(\tan\theta + \frac{\tan^3\theta}{3}\right)\Bigg|_0^{\pi/3} = \frac{1}{4}\left(\sqrt{3} + \frac{(\sqrt{3})^3}{3}\right) = \frac{\sqrt{3}}{2}.$$

The polar moment of inertia is $\dfrac{\sqrt{3}}{2}$.

Changing Variables in Multiple Integrals

1. Changing variables.

Double integrals in x, y coordinates which are taken over circular regions, or have integrands involving the combination $x^2 + y^2$, are often better done in polar coordinates:

$$\iint_R f(x,y)\,dA = \iint_R g(r,\theta)\,r\,drd\theta\ . \tag{1}$$

This involves introducing the new variables r and θ, together with the equations relating them to x, y in both the forward and backward directions:

$$r = \sqrt{x^2+y^2}, \qquad \theta = \tan^{-1}(y/x); \qquad\qquad x = r\cos\theta, \qquad y = r\sin\theta\ . \tag{2}$$

Changing the integral to polar coordinates then requires three steps:

A. Changing the integrand $f(x,y)$ to $g(r,\theta)$, by using (2);

B. Supplying the area element in the r, θ system: $dA = \ r\,dr\,d\theta$;

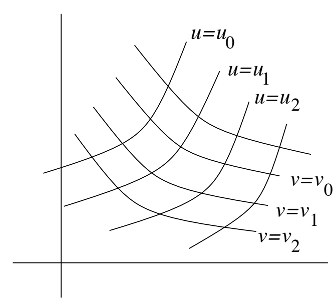
C. Using the region R to determine the limits of integration in the r, θ system.

In the same way, double integrals involving other types of regions or integrands can sometimes be simplified by changing the coordinate system from x, y to one better adapted to the region or integrand. Let's call the new coordinates u and v; then there will be equations introducing the new coordinates, going in both directions:

$$u = u(x,y), \qquad v = v(x,y); \qquad\qquad x = x(u,v), \qquad y = y(u,v) \tag{3}$$

(often one will only get or use the equations in one of these directions). To change the integral to u, v-coordinates, we then have to carry out the three steps **A**, **B**, **C** above. A first step is to picture the new coordinate system; for this we use the same idea as for polar coordinates, namely, we consider the grid formed by the level curves of the new coordinate functions:

$$u(x,y) = u_0, \qquad v(x,y) = v_0\ . \tag{4}$$



Once we have this, algebraic and geometric intuition will usually handle steps **A** and **C**, but for **B** we will need a formula: it uses a determinant called the **Jacobian**, whose notation and definition are

$$\frac{\partial(x,y)}{\partial(u,v)} \ = \ \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}\ . \tag{5}$$

Using it, the formula for the area element in the u, v-system is

$$dA \ = \ \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv\ , \tag{6}$$

so the change of variable formula is

$$\iint_R f(x,y)\,dx\,dy \;=\; \iint_R g(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv\ , \tag{7}$$

where $g(u,v)$ is obtained from $f(x,y)$ by substitution, using the equations (3).

We will derive the formula (5) for the new area element in the next section; for now let's check that it works for polar coordinates.

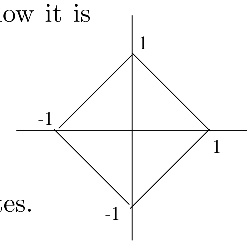
Example 1. Verify (1) using the general formulas (5) and (6).

Solution. Using (2), we calculate:

$$\frac{\partial(x,y)}{\partial(r,\theta)} \;=\; \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} \;=\; \begin{vmatrix} \cos\theta & -r\,\sin\theta \\ \sin\theta & r\,\cos\theta \end{vmatrix} \;=\; r(\cos^2\theta + \sin^2\theta) \;=\; r\ ,$$

so that $dA \;=\; r\,dr\,d\theta$, according to (5) and (6); note that we can omit the absolute value, since by convention, in integration problems we always assume $r \geq 0$, as is implied already by the equations (2).

We now work an example illustrating why the general formula is needed and how it is used; it illustrates step **C** also — putting in the new limits of integration.



Example 2. Evaluate $\displaystyle\iint_R \left(\frac{x-y}{x+y+2}\right)^2 dx\,dy$ over the region R pictured.

Solution. This would be a painful integral to work out in rectangular coordinates. But the region is bounded by the lines

$$x+y=\pm 1, \qquad x-y=\pm 1 \tag{8}$$

and the integrand also contains the combinations $x-y$ and $x+y$. These powerfully suggest that the integral will be simplified by the change of variable (we give it also in the inverse direction, by solving the first pair of equations for x and y):

$$u = x+y, \qquad v = x-y; \qquad\qquad x = \frac{u+v}{2}, \qquad y = \frac{u-v}{2}\ . \tag{9}$$

We will also need the new area element; using (5) and (9) above. we get

$$\frac{\partial(x,y)}{\partial(u,v)} \;=\; \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} \;=\; -\frac{1}{2}\ ; \tag{10}$$

note that it is the second pair of equations in (9) that were used, not the ones introducing u and v. Thus the new area element is (this time we do need the absolute value sign in (6))

$$dA \;=\; \frac{1}{2}\,du\,dv\ . \tag{11}$$

We now combine steps **A** and **B** to get the new double integral; substituting into the integrand by using the first pair of equations in (9), we get

$$\iint_R \left(\frac{x-y}{x+y+2}\right)^2 dx\,dy \;=\; \iint_R \left(\frac{v}{u+2}\right)^2 \frac{1}{2}\,du\,dv\ . \tag{12}$$

In uv-coordinates, the boundaries (8) of the region are simply $u = \pm 1$, $v = \pm 1$, so the integral (12) becomes

$$\iint_R \left(\frac{v}{u+2}\right)^2 \frac{1}{2}\, du\, dv \;=\; \int_{-1}^{1}\int_{-1}^{1} \left(\frac{v}{u+2}\right)^2 \frac{1}{2}\, du\, dv$$

We have

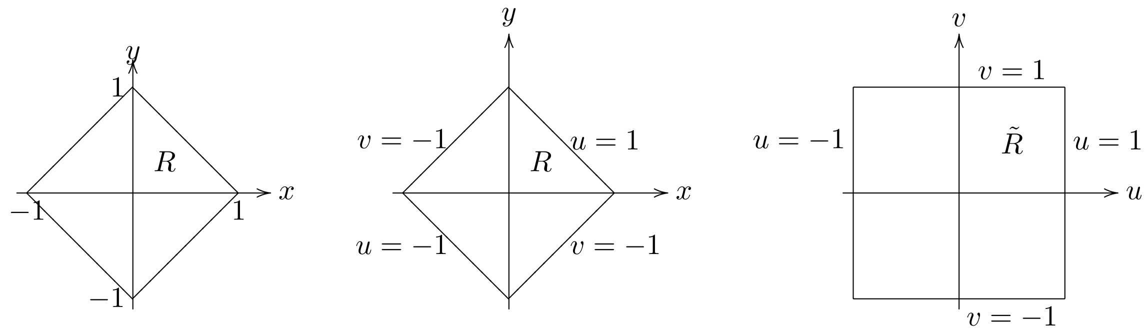
$$\text{inner integral} \;=\; -\frac{v^2}{2(u+2)}\Bigg]_{u=-1}^{u=1} \;=\; \frac{v^2}{3}\,; \qquad \text{outer integral} \;=\; \frac{v^3}{9}\Bigg]_{-1}^{1} \;=\; \frac{2}{9}\,.$$

Problems: Change of Variables

Compute $\displaystyle\iint_R \left(\frac{x+y}{2-x+y}\right)^4 dx\,dy$, where R is the square with vertices at $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$.

Answer: Since the region is bounded by the lines $x+y=\pm1$ and $x-y=\pm1$, we make a change of variables:

$$u = x+y \quad v = x-y.$$



Computing the Jacobian: $\dfrac{\partial(u,v)}{\partial(x,y)} = -2 \;\Rightarrow\; \dfrac{\partial(x,y)}{\partial(u,v)} = -1/2.$

Thus, $dx\,dy = \dfrac{1}{2}\,du\,dv.$

Using either method 1 or method 2 we see the boundaries are given by $u=\pm1$, $v=\pm1$ $\Rightarrow$ the integral is $\displaystyle\iint_R \left(\frac{x+y}{w-x+y}\right)^4 dx\,dy = \int_{-1}^{1}\int_{-1}^{1}\left(\frac{u}{2-v}\right)^4 \frac{1}{2}\,du\,dv.$

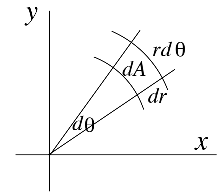
Inner integral $= \left.\dfrac{u^5}{10(2-v)^4}\right|_{u=-1}^{u=1} = \dfrac{1}{5(2-v)^4}.$

Outer integral $= \left.\dfrac{1}{15(2-v)^3}\right|_{-1}^{1} = \dfrac{26}{405} \approx .06.$

We're integrating the fourth power of a fraction whose numerator ranges between -1 and 1 and whose denominator ranges between 1 and 3. The value of this integrand will always be positive and will often be small, so this answer seems reasonable.

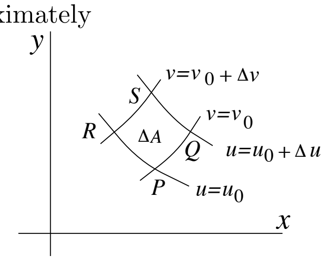
Changing Variables in Multiple Integrals

2. The area element.



In polar coordinates, we found the formula $dA = r\,dr\,d\theta$ for the area element by drawing the grid curves $r = r_0$ and $\theta = \theta_0$ for the r, θ-system, and determining (see the picture) the infinitesimal area of one of the little elements of the grid.

For general u, v-coordinates, we do the same thing. The grid curves (4) divide up the plane into small regions ΔA bounded by these contour curves. If the contour curves are close together, they will be approximately parallel, so that the grid element will be approximately a small parallelogram, and



(13) $$\Delta A \approx \text{area of parallelogram PQRS} = |PQ \times PR|$$

In the uv-system, the points P, Q, R have the coordinates

$$P : (u_0, v_0), \qquad Q : (u_0 + \Delta u, v_0), \qquad R : (u_0, v_0 + \Delta v)\ ;$$

to use the cross-product however in (13), we need PQ and PR in $\mathbf{i}\,\mathbf{j}$- coordinates. Consider PQ first; we have

(14) $$PQ = \Delta x\,\mathbf{i} + \Delta y\,\mathbf{j}\ ,$$

where Δx and Δy are the changes in x and y as you hold $v = v_0$ and change u_0 to $u_0 + \Delta u$. According to the definition of partial derivative,

$$\Delta x \approx \left(\frac{\partial x}{\partial u}\right)_0 \Delta u, \qquad \Delta y \approx \left(\frac{\partial y}{\partial u}\right)_0 \Delta u;$$

so that by (14),

(15) $$PQ \approx \left(\frac{\partial x}{\partial u}\right)_0 \Delta u\ \mathbf{i} + \left(\frac{\partial y}{\partial u}\right)_0 \Delta u\ \mathbf{j}\ .$$

In the same way, since in moving from P to R we hold u fixed and increase v_0 by Δv,

(16) $$PR \approx \left(\frac{\partial x}{\partial v}\right)_0 \Delta v\ \mathbf{i} + \left(\frac{\partial y}{\partial v}\right)_0 \Delta v\ \mathbf{j}\ .$$

We now use (13); since the vectors are in the xy-plane, $PQ \times PR$ has only a $\mathbf{k}$-component, and we calculate from (15) and (16) that

$$\mathbf{k}\text{-component of } PQ \times PR \approx \left|\begin{matrix} x_u\Delta u & y_u\Delta u \\ x_v\Delta v & y_v\Delta v \end{matrix}\right|_0$$

(17) $$= \left|\begin{matrix} x_u & x_v \\ y_u & y_v \end{matrix}\right|_0 \Delta u \Delta v\ ,$$

where we have first taken the transpose of the determinant (which doesn't change its value), and then factored the Δu and Δv out of the two columns. Finally, taking the absolute value, we get from (13) and (17), and the definition (5) of Jacobian,

$$\Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right|_0 \Delta u \Delta v\ ;$$

passing to the limit as $\Delta u, \Delta v \to 0$ and dropping the subscript 0 (so that P becomes any point in the plane), we get the desired formula for the area element,

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\ .$$

Changing Variables in Multiple Integrals

3. Examples and comments; putting in limits.

If we write the change of variable formula as

$$\iint_R f(x,y)\,dx\,dy \;=\; \iint_R g(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\;, \tag{18}$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} \;=\; \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}\;, \qquad g(u,v) = f(x(u,v), y(u,v)), \tag{19}$$

it looks as if the essential equations we need are the inverse equations:

$$x = x(u,v), \qquad y = y(u,v) \tag{20}$$

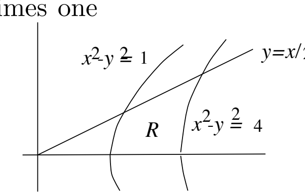
rather than the direct equations we are usually given:

$$u = u(x,y), \qquad v = v(x,y)\;. \tag{21}$$

If it is awkward to get (20) by solving (21) simultaneously for x and y in terms of u and v, sometimes one can avoid having to do this by using the following relation (whose proof is an application of the chain rule, and left for the Exercises):

$$\frac{\partial(x,y)}{\partial(u,v)}\,\frac{\partial(u,v)}{\partial(x,y)} \;=\; 1 \tag{22}$$

The right-hand Jacobian is easy to calculate if you know $u(x,y)$ and $v(x,y)$; then the left-hand one — the one needed in (19) — will be its reciprocal. Unfortunately, it will be in terms of x and y instead of u and v, so (20) still ought to be needed, but sometimes one gets lucky. The next example illustrates.



Example 3. Evaluate $\displaystyle\iint_R \frac{y}{x}\,dx\,dy$, where R is the region pictured, having as boundaries the curves $x^2 - y^2 = 1, \quad x^2 - y^2 = 4, \quad y = 0, \quad y = x/2\;.$

Solution. Since the boundaries of the region are contour curves of $x^2 - y^2$ and y/x, and the integrand is y/x, this suggests making the change of variable

$$u \;=\; x^2 - y^2, \qquad v \;=\; \frac{y}{x}\;. \tag{23}$$

We will try to get through without solving these backwards for x, y in terms of u, v. Since changing the integrand to the u, v variables will give no trouble, the question is whether we can get the Jacobian in terms of u and v easily. It all works out, using (22):

$$\frac{\partial(u,v)}{\partial(x,y)} \;=\; \begin{vmatrix} 2x & -2y \\ -y/x^2 & 1/x \end{vmatrix} \;=\; 2 - 2y^2/x^2 \;=\; 2 - 2v^2; \qquad \text{so} \quad \frac{\partial(x,y)}{\partial(u,v)} \;=\; \frac{1}{2(1-v^2)}\;,$$

according to (22). We use now (18), put in the limits, and evaluate; note that the answer is positive, as it should be, since the integrand is positive.

$$\begin{aligned}\iint_R \frac{y}{x}\,dx\,dy &= \iint_R \frac{v}{2(1-v^2)}\,du\,dv \\ &= \int_0^{1/2}\int_1^4 \frac{v}{2(1-v^2)}\,du\,dv \\ &= -\frac{3}{4}\ln(1-v^2)\Big]_0^{1/2} = -\frac{3}{4}\ \ln\ \frac{3}{4}\ .\end{aligned}$$

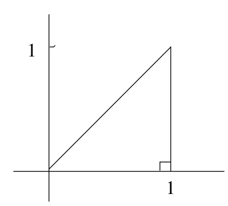
Putting in the limits

In the examples worked out so far, we had no trouble finding the limits of integration, since the region R was bounded by contour curves of u and v, which meant that the limits were constants.

If the region is not bounded by contour curves, maybe you should use a different change of variables, but if this isn't possible, you'll have to figure out the uv-equations of the boundary curves. The two examples below illustrate.

Example 4. Let $u = x + y,\ v = x - y$; change $\int_0^1 \int_0^x dy\,dx$ to an iterated integral $du\,dv$.

Solution. Using (19) and (22), we calculate $\dfrac{\partial(x,y)}{\partial(u.v)} = -1/2$, so the Jacobian factor in the area element will be $1/2$.



To put in the new limits, we sketch the region of integration, as shown at the right. The diagonal boundary is the contour curve $v = 0$; the horizontal and vertical boundaries are not contour curves — what are their uv-equations? There are two ways to answer this; the first is more widely applicable, but requires a separate calculation for each boundary curve.

Method 1 *Eliminate x and y from the three simultaneous equations $u = u(x,y)$, $v = v(x,y)$, and the xy-equation of the boundary curve.* For the x-axis and $x = 1$, this gives

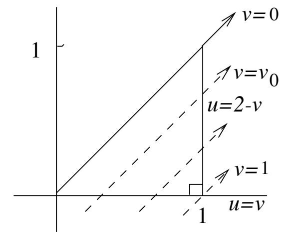
$$\begin{cases} u = x + y \\ v = x - y \\ y = 0 \end{cases} \quad\Rightarrow\quad u = v; \qquad\qquad \begin{cases} u = x + y \\ v = x - y \\ x = 1 \end{cases} \quad\Rightarrow\quad \begin{cases} u = 1 + y \\ v = 1 - y \end{cases} \quad\Rightarrow\quad u + v = 2.$$

Method 2 *Solve for x and y in terms of u, v; then substitute $x = x(u,v),\ y = y(u,v)$ into the xy-equation of the curve.*

Using this method, we get $x = \frac{1}{2}(u+v),\ \ y = \frac{1}{2}(u-v)$; substituting into the xy-equations:

$$y = 0 \ \Rightarrow\ \frac{1}{2}(u-v) = 0 \ \Rightarrow\ u = v; \qquad\qquad x = 1 \ \Rightarrow\ \frac{1}{2}(u+v) = 1 \ \Rightarrow\ u + v = 2.$$

To supply the limits for the integration order $\iint du\,dv$, we



1. first hold v fixed, let u increase; this gives us the dashed lines shown;
2. integrate with respect to u from the u-value where a dashed line enters R (namely, $u = v$), to the u-value where it leaves (namely, $u = 2 - v$).
3. integrate with respect to v from the lowest v-values for which the dashed lines intersect the region R (namely, $v = 0$), to the highest such v-value (namely, $v = 1$).

Therefore the integral is $\displaystyle\int_0^1 \int_v^{2-v} \frac{1}{2}\, du\, dv$.

(As a check, evaluate it, and confirm that its value is the area of R. Then try setting up the iterated integral in the order $dv\, du$; you'll have to break it into two parts.)

Example 5. Using the change of coordinates $u = x^2 - y^2$, $v = y/x$ of Example 3, supply limits and integrand for $\displaystyle\iint_R \frac{dxdy}{x^2}$, where R is the infinite region in the first quadrant under $y = 1/x$ and to the right of $x^2 - y^2 = 1$.

Solution. We have to change the integrand, supply the Jacobian factor, and put in the right limits.

To change the integrand, we want to express x^2 in terms of u and v; this suggests eliminating y from the u, v equations; we get

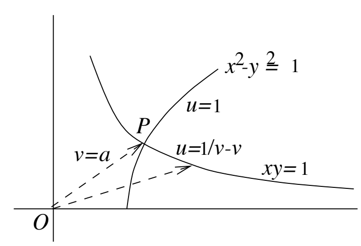
$$u = x^2 - y^2, \quad y = vx \quad \Rightarrow \quad u = x^2 - v^2x^2 \quad \Rightarrow \quad x^2 = \frac{u}{1 - v^2} \ .$$

From Example 3, we know that the Jacobian factor is $\dfrac{1}{2(1 - v^2)}$; since in the region R we have by inspection $0 \le v < 1$, the Jacobian factor is always positive and we don't need the absolute value sign. So by (18) our integral becomes

$$\iint_R \frac{dx\, dxy}{x^2} = \iint_R \frac{1 - v^2}{2u(1 - v^2)}\, du\, dv = \iint_R \frac{du\, dv}{2u}$$

Finally, we have to put in the limits. The x-axis and the left-hand boundary curve $x^2 - y^2 = 1$ are respectively the contour curves $v = 0$ and $u = 1$; our problem is the upper boundary curve $xy = 1$. To change this to $u - v$ coordinates, we follow Method 1:

$$\begin{cases} u = x^2 - y^2 \\ y = vx \\ xy = 1 \end{cases} \Rightarrow \begin{cases} u = x^2 - 1/x^2 \\ v = 1/x^2 \end{cases} \Rightarrow u = \frac{1}{v} - v \ .$$



The form of this upper limit suggests that we should integrate first with respect to u. Therefore we hold v fixed, and let u increase; this gives the dashed ray shown in the picture; we integrate from where it enters R at $u = 1$ to where it leaves, at $u = \dfrac{1}{v} - v$.

The rays we use are those intersecting R: they start from the lowest ray, corresponding to $v = 0$, and go to the ray $v = a$, where a is the slope of OP. Thus our integral is

$$\int_0^a \int_1^{1/v - v} \frac{du\, dv}{2u} \ .$$

Problems: Polar Coordinates and the Jacobian

1. Let $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$. Directly calculate the Jacobian $\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$.

<u>**Answer:**</u> Because we are familiar with the change of variables from rectangular to polar coordinates and we know that $\frac{\partial(r,\theta)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = 1$, this result should not come as a surprise.

$$
\begin{aligned}
\frac{\partial(r,\theta)}{\partial(x,y)} &= \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix} \\
&= \begin{vmatrix} \frac{2x}{2\sqrt{x^2+y^2}} & \frac{2y}{2\sqrt{x^2+y^2}} \\ \frac{-y/x^2}{1+(y/x)^2} & \frac{1/x}{1+(y/x)^2} \end{vmatrix} \\
&= \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} \\
&= \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{vmatrix} \\
&= \frac{x^2+y^2}{r^3} = \frac{1}{r}.
\end{aligned}
$$

2. For the change of variables $x = u$, $y = \sqrt{r^2 - u^2}$, write $dx\,dy$ in terms of u and r.

<u>**Answer:**</u> We know $dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,r)}\right| du\,dr$.

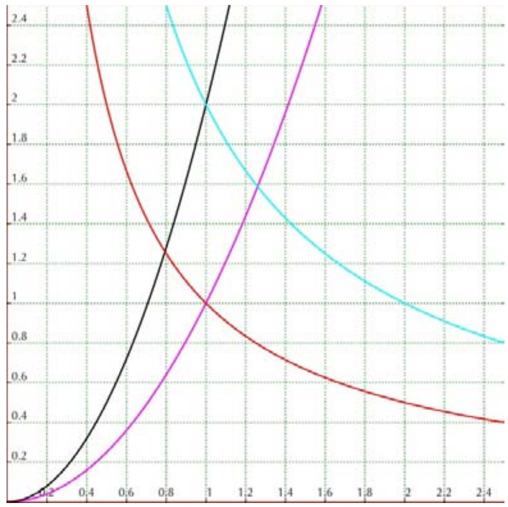
$$
\begin{aligned}
\frac{\partial(x,y)}{\partial(u,r)} &= \begin{vmatrix} x_u & x_r \\ y_u & y_r \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 \\ \frac{-u}{\sqrt{r^2-u^2}} & \frac{r}{\sqrt{r^2-u^2}} \end{vmatrix} \\
&= \frac{r}{\sqrt{r^2-u^2}}
\end{aligned}
$$

Hence $dx\,dy = \frac{r}{\sqrt{r^2-u^2}}\,du\,dr$.

Problems: Change of Variables Example

Use a change of variables to find the area of the region bounded by the curves $y = x^2$, $y = 2x^2$, $y = 1/x$, and $y = 2/x$.

Answer: Draw a picture:



The region is roughly diamond shaped. The top and bottom of the diamond have x coordinate 1, so we could split the area into left and right halves and compute the area using techniques from single variable calculus.

Instead, we rewrite the equations describing the boundary of the region as follows:

$$xy = 1, \quad xy = 2, \quad y/x^2 = 1, \quad y/x^2 = 2.$$

If we let $u = xy$ and $v = y/x^2$, the boundary curves become $u = 1$, $u = 2$, $v = 1$ and $v = 2$. The Jacobian is then:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -\frac{2y}{x^3} & \frac{1}{x^2} \end{vmatrix} = \frac{3y}{x^2} = 3v.$$

Noting that v is positive throughout the region, we get:

$$dx\,dy = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} du\,dv = \frac{1}{3v} du\,dv.$$

Finally, we compute:

$$\text{Area} = \iint_R dx\,dy = \int_1^2 \int_1^2 \frac{1}{3v} du\,dv = \frac{1}{3}\ln 2 \approx 0.23.$$

We refer to our original sketch to confirm that this is a plausible result. We could also check our work by computing the area using single variable calculus.