MATH-UA 140 - Linear Algebra

Lecture 14: Orthogonality of the Four Subspaces

You may have noticed in the previous lecture that the fundamental theorem of linear algebra was not complete: we wrote (Part 1). The reason for this is that there is more one can say about the four subspaces, and this has to do with orthogonality properties. This is the purpose of this lecture.

I] Orthogonal subspaces

1) Brief review

Two vectors $\vec{u}$ and $\vec{v}$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$

In linear algebra language, we would write $\vec{u}^T \vec{v} = 0$

2) Definition

Two subspaces U and V of a vector space are orthogonal if every vector $\vec{u}$ in U is orthogonal to every vector $\vec{v}$ in V

Note: If a vector $\vec{w}$ is in two orthogonal subspaces, it is orthogonal to itself. Clearly, one then has $\vec{w} = \vec{0}$

Example of orthogonal subspaces

- The x-y plane, spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a subspace of $\mathbb{R}^3$.

The z-axis, spanned by $\begin{bmatrix}0\\0\\1\end{bmatrix}$, is a subspace of $\mathbb{R}^3$.

The two subspaces are orthogonal.

- The x-z plane, spanned by $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$ is <u>not</u> orthogonal to the x-y plane, spanned by $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$

  This is because any vector colinear with $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is part of both subspaces.

3) <u>Orthogonality for the four subspaces</u>

- Every vector $\vec{x}$ in $N(A)$ satisfies $A\vec{x} = \vec{0}$ so every row of A is orthogonal to $\vec{x}$

  <u>The nullspace $N(A)$ and the row space $C(A^T)$ are orthogonal subspaces of $\mathbb{R}^n$</u>

  <u>Example:</u> We found that $\vec{s}_1 = \begin{bmatrix}-2\\1\\0\\0\end{bmatrix}$ and $\vec{s}_2 = \begin{bmatrix}17\\0\\-7\\1\end{bmatrix}$ formed a basis for the nullspace of $A = \begin{bmatrix}1 & 2 & 3 & 4\\2 & 4 & 5 & 1\\5 & 10 & 11 & -8\end{bmatrix}$

$2\vec{s}_1 + \vec{s}_2 = \begin{bmatrix}13\\2\\-7\\1\end{bmatrix}$ is in $N(A)$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 13 \\ 2 \\ -7 \\ 1 \end{bmatrix} = 0; \quad \begin{bmatrix} 2 & 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} 13 \\ 2 \\ -7 \\ 1 \end{bmatrix} = 0;$$

$$\begin{bmatrix} 5 & 10 & 11 & -8 \end{bmatrix} \begin{bmatrix} 13 \\ 2 \\ -7 \\ 1 \end{bmatrix} = 0$$

Applying the same reasoning to A^T, we can say that <u>the left nullspace $N(A^T)$ and the column space $C(A)$ are orthogonal in $\mathbb{R}^m$.</u>

The orthogonality of the four subspaces and the first part of the fundamental theorem of linear algebra (previous lecture) are nicely summarized in the textbook with the figures 4.2 and 4.3

II) The fundamental theorem of linear algebra, Part 2

1) Orthogonal complements

<u>Definition:</u> The <u>orthogonal complement</u> of a subspace V contains <u>every</u> vector that is perpendicular to V. This subspace is often written $V^\perp$. ($\perp$ is the "perp" symbol).

Consider $C(A^T)$, the row space of A. Any vector in $C(A^T)^\perp$ satisfies $A\vec{x} = \vec{0}$. It is therefore in $N(A)$. Conversely, any vector $\vec{x}$ in $N(A)$ is orthogonal to each row of A. It is therefore orthogonal to any vector in $C(A^T)$, so $\vec{x}$ is in $C(A^T)^\perp$

We conclude that $N(A) = C(A^T)^\perp$. This is the second part of the fundamental theorem of linear algebra, when applied to A and A^T.

2) Fundamental Theorem of Linear Algebra - Part 2

$N(A)$ is the orthogonal complement of the row space $C(A^T)$ (in $\mathbb{R}^n$).

$N(A^T)$ is the orthogonal complement of the column space $C(A)$ (in $\mathbb{R}^m$)

Note: We just showed and learned that $N(A) = C(A^T)^\perp$. As you may expect, it is also true that $C(A^T) = N(A)^\perp$. Here is why: Consider $\vec{x}$ in $N(A)^\perp$. It is orthogonal to any vector in $N(A)$. Now, this $\vec{x}$ must be in $C(A^T)$. Indeed, if it were not, we could add $\vec{x}$ as an extra, linearly independent row to A without changing $N(A)$. The rank of the modified A would be $r+1$, and the dimensions of $C(A^T)$ and $N(A)$ would add up to $r+1+n-r = n+1 \neq n$. That is not possible.

We conclude: $N(A)^\perp = C(A^T)$

$N(A^T)^\perp = C(A)$

3) Nullspace component and row space component

A take home message of the Fundamental Theorem of Linear Algebra is that if A is an $m \times n$ matrix, then $N(A)$ and $C(A^T)$ combine exactly to describe all of $\mathbb{R}^n$: together they span $\mathbb{R}^n$, and there is no "overlap", i.e. no vector in both spaces.

A basis of $\mathbb{R}^n$ can thus be constructed by taking the r vectors of a basis of $C(A^T)$ and the $n-r$ vectors of a basis of $N(A)$.

Another way to say this is that any vector $\vec{x}$ in $\mathbb{R}^n$ can be written as the sum $\vec{x} = \vec{x}_r + \vec{x}_n$ of a row space vector $\vec{x}_r$ and a nullspace vector $\vec{x}_n$.

Example: Let $A = \begin{bmatrix} 6 & -2 \\ 3 & -1 \end{bmatrix}$

$$\vec{x} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ -1 \end{bmatrix}}_{\substack{\vec{x}_r \\ \text{in the row} \\ \text{space of } A}} + \underbrace{\begin{bmatrix} 2 \\ 6 \end{bmatrix}}_{\substack{\vec{x}_n \\ \text{in the nullspace} \\ \text{of } A}}$$

In the next lecture, we will learn how to decompose any vector of $\mathbb{R}^n$ in this way. We will use projections.