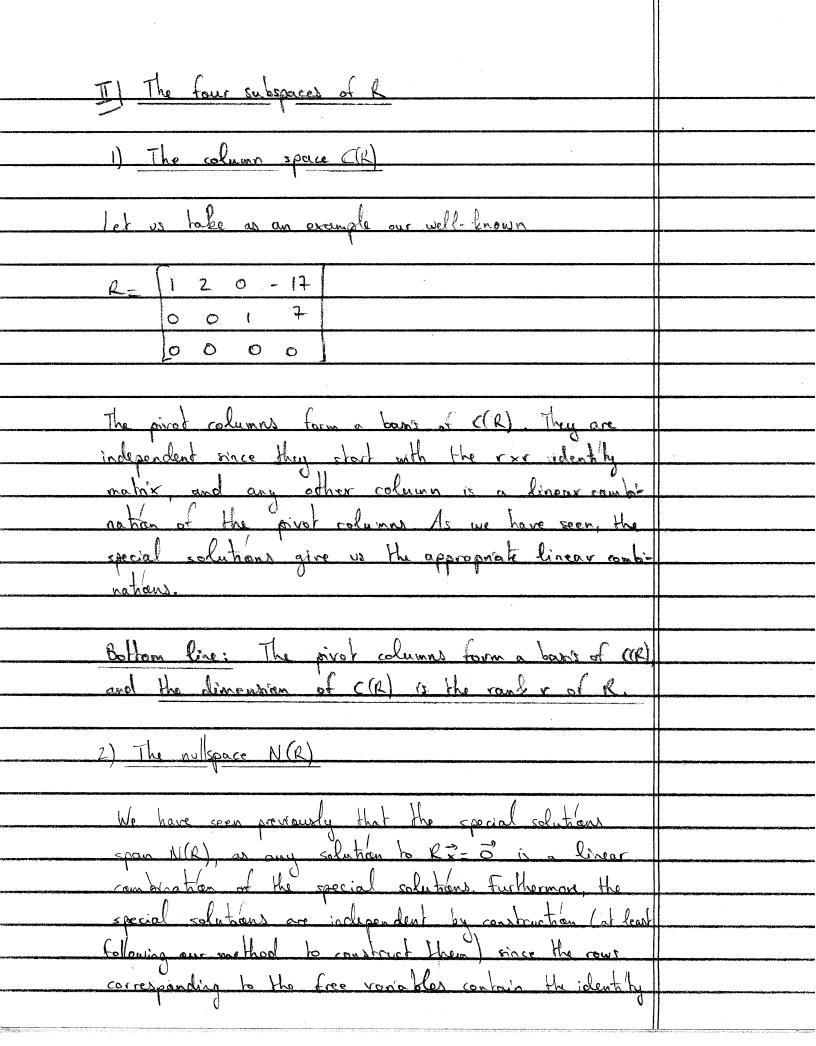
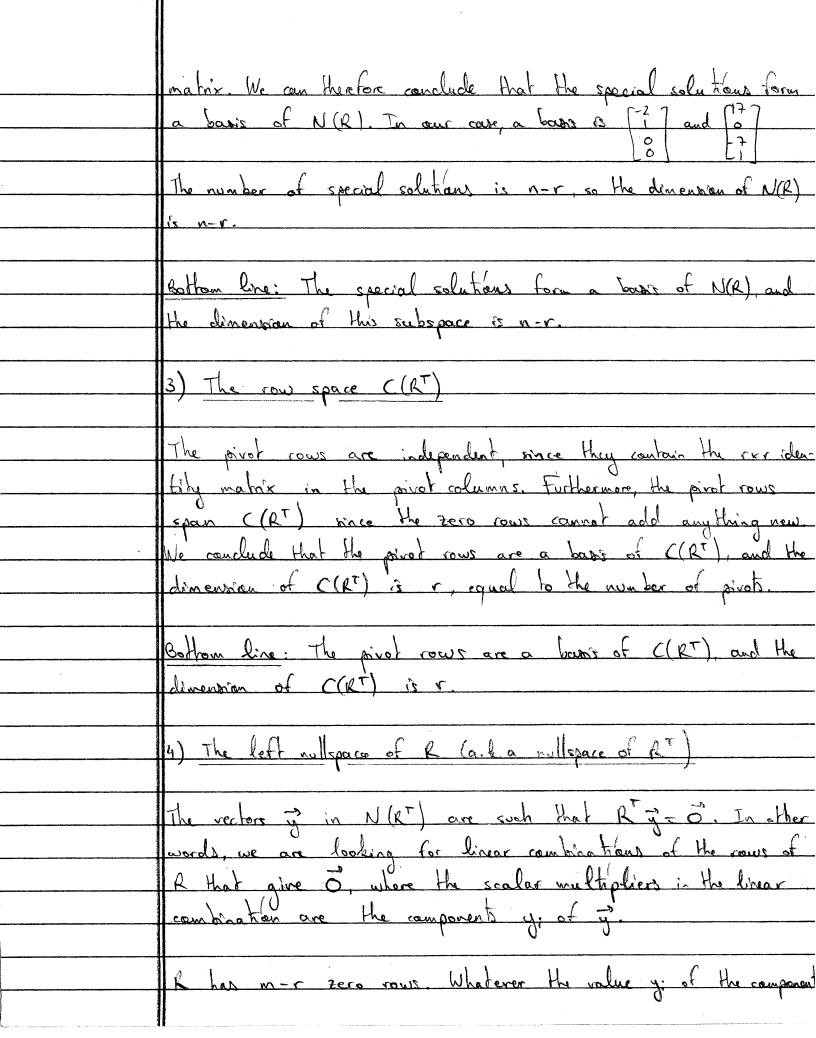
MATH-UA 140 - Linear Algebra
Lecture 13: Dimonnians of the Four Subspaces
I The four fundamental subspaces
Thus far we have encountered three fundamental enterpares
Thus far we have encountered three fundamental embagaces associated with an man makin A:
1. The column space C(A), which is a subspace of R
2. The nollegace NAI which it a subspace of R"
2. The noll space NA), which is a subspace of R"  3. The row space C(AT), which is a subspace of R"
We will now loose about the fourth fredomental or hereise
We will now learn about the fourth fundamental subspace: 4. The left nullspace N(AT) which is a subspace of R
Let us stop for a second on the name left notispace.
N(AT) is the nollspace of AT i.e the subspace of vectors is
such that AT = 0. Taking the transpore of this equality
we have (AT = ) = = = = = = = = = = = = = = = = =
such that $A^{T}y^{2} = 0$ . Taking the transpose of this equality we have $A^{T}y^{2} = 0$ (=) $y^{T}(A^{T})^{T} = 0$ (=) $y^{T}($
In the remainder of this feebers, we will see how the four subspaces are connected, which will fear us to the Fundamental
subspaces are connected, which will lead us to the Fundamental
Theorem of Linear Algebra.
V
We start by looking at the four rules paces of R- rref(A), which
We start by looking at the four rubspaces of R- rref(A), which are a bit more bransparent. No will then more to the four subspaces of A
· · · · · · · · · · · · · · · · · · ·



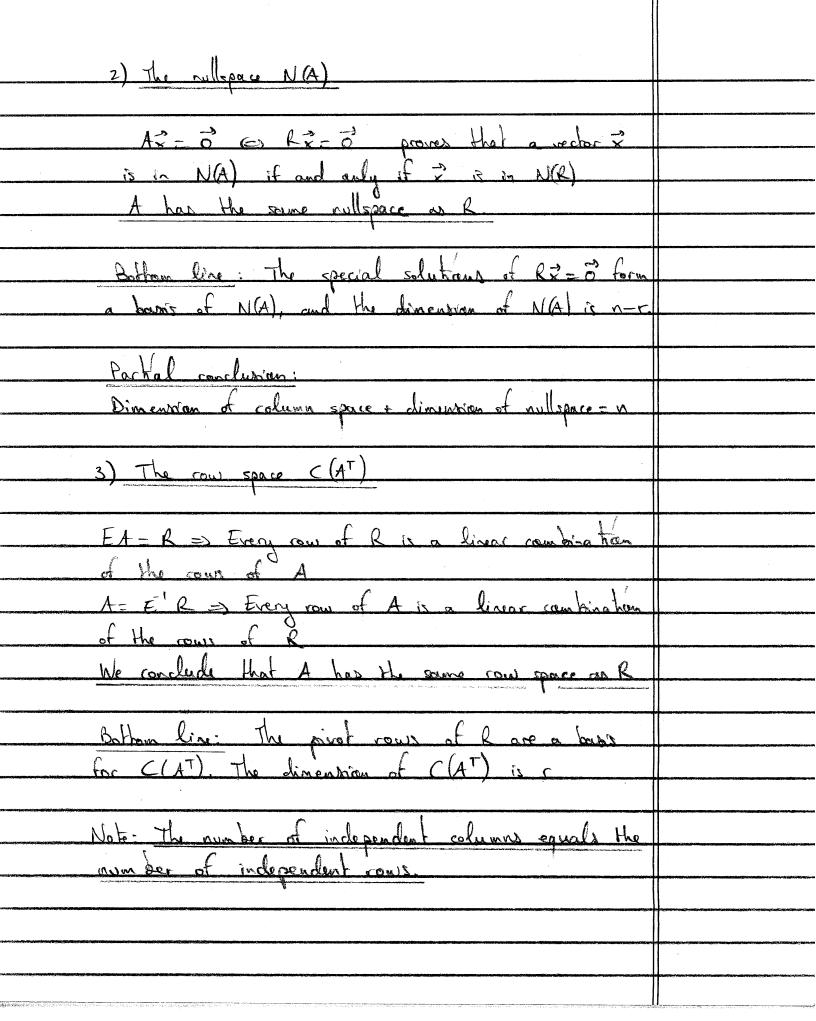


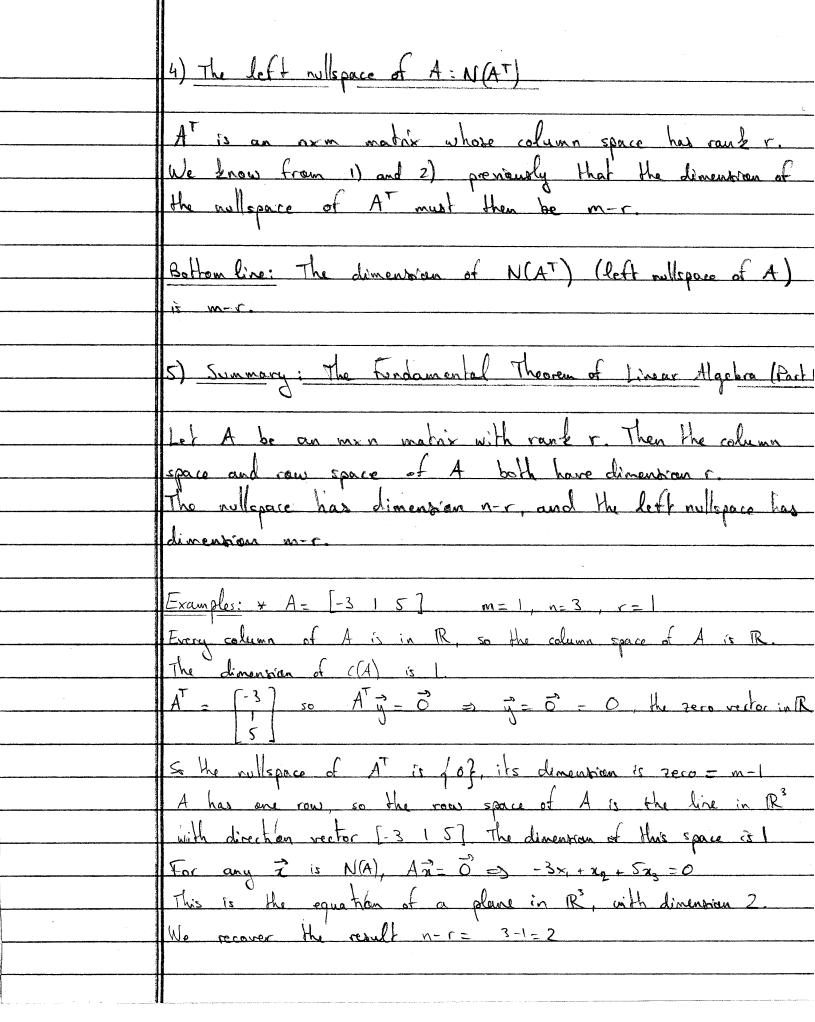
of y at these rows the linear combination of these rows gives 0.	
The repirat rows on the other hand, are not zero.	
and are independent as we have seen. Thus the only	
combination of these rows that can be o' is the hiral	
linear cambination with zero as scalar multipliers: the	
His course bungling to binot come must be sero.	
The general solution of to RT - of has record and m-r free range bles. Therefore the dimension of NRT	
and my tree rand bles. Theretore the dimention of NIR	
Bottom line: The dimension of N(RT) (left nullspace	
of R) is m-r.	
S) Summary	
The row space and nullspace of R are subspaces of	
of the dimensions is $r + n - r = n$ , the dimension of $\mathbb{R}^n$	
. The column space and left nullspace of R are subspaces	
of RM with dimensions r and on r respectively the	
sum of the dimensions is r+m-r=m, the dimension of Th.	
Decree will be a second of the	<u> </u>

M.E. Conning

II) The four subspaces of A In general, not all the subspaces of R are the same as the subspaces of A. However, there respective dimentions are equal Three is what we show now The control idea is that one relied on elimination to obtain R from A. Thur, there is an invertible elimination matrix E (in general E is the product of converal climination matricer) such that EA = R Ex A = E'R 1) The column space C(A) The columns which are linearly independent in A are also the columns which are linearly independent in R (although the vectors they contain may be different in A and in R)

Libernie, the columns which are dependent in A are also the column which are dependent in R (although the vectors they contour fray be different in A and in R). The reason for the statement above is that  $A\vec{x} = \vec{0} \iff R\vec{x} = \vec{0}$ The same linear combinations, i.e same component of x' as the scalar multiplier in the combination of the columns of A and of the column of R load to O on the right-hand ride. We conclude that the r pivot columns of A are a boxis for its column space (which may be different from C(R)). Bothom line: The pavot columns of A are a basis of CA).
The dimension of CA) is r.





6) Rank one matrices When r=1 every row is a multiple of the same of every column is a multiple of the same column. The row space is a line in R, the column space a line in R. wither as  $\vec{n} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$  times  $\begin{bmatrix} 1 - 1 & 2 & 3 \\ 1 & 1 \end{bmatrix} = \vec{v}^T$ (A 3x1 matrix times a 1x4 matrix gives Every rand one matrix has the special form A = in vt the columns are multiples of it, and the cows are multiples of it. The nullspace of Acontains any vector such that  $A\overset{\sim}{x}=0$   $(\overrightarrow{\nabla}\overset{\sim}{z}) = 0 \Rightarrow \overrightarrow{\nabla}\overset{\sim}{z} = 0$ In other words, the subspace is the plane perpendicular to v. This is the first intight into a general orthogona-lity property we will explore in the next lacher.