Linear Algebra – Problem Set 3 Solutions

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Problem 1

We compute the LU factorization as we have learned in class, namely by....Gaussian elimination! Row 2 - Row 1 leads to

$$\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Row 3 - Row 1 leads to

$$\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ a & b & c & d \end{bmatrix}$$

Row 4 - Row 1 leads to

$$\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

Row 3 - Row 2 leads to

$$\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

Row 4 - Row 2 leads to

$$\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$

Row 4 - Row 3 leads to

$$\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

Throughout the elimination process, the multipliers were 1s, so we can readily write

$$\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

A is invertible if one gets a full set of pivots. This is the case if $a \neq 0$,$b \neq a$,$c \neq b$, and $d \neq c$.

Problem 2

For the first matrix, Row 2 - a Row 1 leads to

$$\begin{bmatrix} 1 & a \\ 0 & b-a^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & b-a^2 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

where we have divided each row by the pivot to obtain the second form DU. The multiplier in the elimination step was a, so we can readily write

$$\begin{bmatrix} 1 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & b-a^2 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

Once more, for this 3-by-3 matrix we proceed by Gaussian elimination. Row 2 + 1/2 Row 1 leads to

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Row 3 + 2/3 Row 2 leads to

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

where once again we divided each row by the pivot to obtain the form on the right hand side. The multipliers were $l_{21} = -1/2$ and $l_{32} = -2/3$ so we can readily write

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 3

1. We start with $LDU = L_1D_1U_1$, with all matrices invertible. We multiply this equality by L_1^{-1} on the left, and by U^{-1} on the right, to obtain:

$$L_1^{-1}LD = D_1U_1U^{-1}$$

   as desired.

2. Thinking of the Gaussian elimination algorithm, the inverse L_1^{-1} of the lower triangular matrix L_1 is lower triangular. Then, $L_1^{-1}LD$ is the product of three lower triangular matrices, which must be lower triangular.

   With the same reasoning, we find that $D_1U_1U^{-1}$ is upper triangular.

3. Since $L_1^{-1}LD = D_1U_1U^{-1}$, and the matrix on the left-hand side is lower triangular and the matrix on the right-hand side is upper triangular, it must be that both matrices are diagonal. So are then $L_1^{-1}L$ and U_1U^{-1}.

   Now, by construction, all the diagonal entries of L_1, L, U, and U_1 are 1's. So are then the diagonal entries of L_1^{-1} and U^{-1}, and of $L_1^{-1}L$ and U_1U^{-1}. We conclude that $L_1^{-1}L = I$ and that $U_1U^{-1} = I$. This means that $D = D_1$, $L = L_1$ and $U = U_1$, proving the uniqueness of the LDU factorization.

Problem 4

Let us first assume that

$$M = S + A \qquad \text{with } S^T = S \text{ and } A^T = -A$$

Then

$$M + M^T = S + A + S^T + A^T = 2S \qquad \text{and} \qquad M - M^T = S + A - S^T - A^T = 2A$$

Hence, if M can be written as $M = S + A$, then we must have

$$S = \frac{1}{2}\left(M + M^T\right) \qquad \text{and} \qquad A = \frac{1}{2}\left(M - M^T\right)$$

Conversely, for any square matrix M, we can write

$$M = \frac{1}{2}\left(M + M^T\right) + \frac{1}{2}\left(M - M^T\right)$$

We have just proved that any square matrix M can be written as

$$M = S + A$$

with $S = \frac{1}{2}\left(M + M^T\right)$ a symmetric matrix, and $A = \frac{1}{2}\left(M - M^T\right)$ an antisymmetric matrix.

Let us now illustrate this with

$$M = \begin{bmatrix} -3 & 2 & 1 \\ 4 & 7 & -5 \\ -5 & -5 & 13 \end{bmatrix}$$

$$M^T = \begin{bmatrix} -3 & 4 & -5 \\ 2 & 7 & -5 \\ 1 & -5 & 13 \end{bmatrix}$$

Hence

$$S = \frac{1}{2}\left(M + M^T\right) = \begin{bmatrix} -3 & 3 & -2 \\ 3 & 7 & -5 \\ -2 & -5 & 13 \end{bmatrix} \qquad \text{and} \qquad A = \frac{1}{2}\left(M - M^T\right) = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}$$

and $M = S + A$

Problem 5

1. As discussed in Problem 3, thinking of Gaussian elimination, it is clear that lower triangular matrices with 1's on the diagonal are invertible, and that their inverses have only 1's on the diagonal.

   Furthermore, the product of two lower triangular matrices with 1's on the diagonal is a lower triangular matrix with 1's on the diagonal.

   Lower triangular matrices with 1's on the diagonal form a group under matrix multiplication.

2. Symmetric matrices do not form a group under matrix multiplication: the product of two symmetric matrices is not necessarily symmetric, as shown in the example below

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

3. Matrices with positive entries do not form a group under matrix multiplication, because the inverse of a matrix with positive entries does not need have positive entries:

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

4. Permutation matrices form a group under matrix multiplication: it is clear and we have seen in class that the product of two permutation matrices is a permutation matrix, and so is the inverse of a permutation matrix.

5. An invertible matrix which is diagonal has only nonzero entries d_i on the diagonal. If we have a second invertible matrix which is diagonal and has nonzero entries e_i, the product of the two matrices is a diagonal matrix with entries $d_i e_i$, which are also nonzero.

   Furthermore, the inverse of the diagonal matrix with nonzero entries d_i is the diagonal matrix with nonzero entries $1/d_i$.

   Hence, diagonal matrices with nonzero diagonal entries are a group under matrix multiplication.

Consider the set of matrices M such that $M^T = M^{-1}$.

For any M in that set, $(M^{-1})^T = (M^T)^{-1} = (M^{-1})^{-1} = M$, so M^{-1} is also an element of this set.

Furthermore, for any two matrices A and B in this set, $(AB)^T = B^T A^T = B^{-1} A^{-1} = (AB)^{-1}$, so AB is also in the set.

The set of matrices M such that $M^T = M^{-1}$ thus form a group under matrix multiplication.