Linear Algebra – Problem Set 5 Solutions

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Problem 1

The vectors (x, y, z, t) satisfying $x + 2y - 3z - t = 0$ are in the nullspace of the matrix

$$A = [1 \;\; 2 \;\; -3 \;\; -1]$$

This matrix has rank 1, so it has three free columns. Consequently, it has three linearly independent special solutions (the three columns contain the identity at the position of the free variables):

$$\mathbf{s}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \mathbf{s}_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad , \quad \mathbf{s}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The three special solutions span the nullspace, so they are a basis for the nullspace, i.e. a basis of the plane $x + 2y - 3z - t = 0$. Any vector in this plane can then be written as a linear combination of the three special solutions. There cannot be a fourth independent vector.

Problem 2

1. We write

$$\begin{aligned} A &= [\mathbf{u}\;\mathbf{w}] \begin{bmatrix} \mathbf{v}^T \\ \mathbf{z}^T \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

A has its column space spanned by $\mathbf{u} = (3, 2, 5)$ and $\mathbf{w} = (1, 1, 3)$, and its row space spanned by $\mathbf{v} = (-1, 1)$ and $\mathbf{z} = (0, 1)$. We can calculate

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -2 & 3 \\ -5 & 8 \end{bmatrix}$$

2. Following the previous question, we see that in this case $A = \mathbf{u}\mathbf{v}^T + \mathbf{w}\mathbf{z}^T$, with $\mathbf{u} = (7, -2, 5)$ and $\mathbf{w} = (1, 3, 1)$ a basis for the column space of A, and $\mathbf{v} = (9, 9, 0)$ and $\mathbf{z} = (2, 1, 2)$ a basis for its row space.

   We know from Problem Set 4 that the rank of a product of matrices cannot be larger than the rank of either matrix in the product. Each matrix in the product that gives A has rank 2, so the rank of A is less than or equal to 2. Since A is a 3×3 matrix, this means that A is not invertible.

Problem 3

1. All vectors in $\mathbf{R}^3$ are orthogonal to the zero vector, so $S^\perp = \mathbb{R}^3$

2. $S^\perp$ is the nullspace of the matrix $[-1\ 1\ -1]$. The row reduced echelon form of this matrix is $[1\ -1\ 1]$. The special solutions are $(1,1,0)$ and $(-1,0,1)$, and are a basis for the nullspace. We conclude that $S^\perp$ is the subspace spanned by $(1,1,0)$ and $(-1,0,1)$. In other words, it is the plane in $\mathbb{R}^3$ containing the vectors $(1,1,0)$ and $(-1,0,1)$.

3. $S^\perp$ is the nullspace of the matrix
$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$
The row reduced echelon form of this matrix is
$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
The special solution is $(1,1,0)$, which is a basis of $S^\perp$. So $S^\perp$ is the line in $\mathbb{R}^3$ aligned with $(1,1,0)$.

Problem 4

By the Fundamental Theorem of Linear Algebra, the row space of each matrix is the orthogonal complement of the nullspace of each matrix in $\mathbb{R}^n$. So all we have to show is that A and BA have the same nullspace.

- Let $\mathbf{x}$ be a vector in the nullspace of A: $A\mathbf{x} = \mathbf{0}$. Then multiplying the equality by B on the left, we find $B(A\mathbf{x}) = \mathbf{0} \;\;\Leftrightarrow\;\; (BA)\mathbf{x} = \mathbf{0}$. So $\mathbf{x}$ is also in the nullspace of BA.

- Let $\mathbf{x}$ be a vector in the nullspace of BA: $BA\mathbf{x} = \mathbf{0}$. Then since B is invertible we can multiply both sides of the equality by B^{-1} on the left, to obtain
$$A\mathbf{x} = B^{-1}\mathbf{0} \;\;\Leftrightarrow\;\; A\mathbf{x} = \mathbf{0}$$
This means that $\mathbf{x}$ is also in the nullspace of A.

  We proved that a vector $\mathbf{x}$ is in the nullspace of A if and only if it is in the nullspace of BA, so A and BA have the same nullspace, and by the Fundamental Theorem of Linear Algebra, they also have the same row space.

Problem 5

1. (a)
$$A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$
$$\Rightarrow\;\; (A^TA)^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \;\;\Rightarrow\;\; A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
Therefore,
$$\mathbf{p} = A(A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

   (b)
$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\Rightarrow \ (A^TA)^{-1} = \begin{bmatrix} \frac{3}{2} & -1 \\ -1 & 1 \end{bmatrix} \ \Rightarrow \ A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Therefore,

$$\mathbf{p} = A(A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$$

2. Along the way in the previous question, we computed

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad , \qquad P_2 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$