MATH-UA 140 - Linear Algebra

Lecture 9: Vector Spaces

Thus far, we have learned about vectors and matrices with a "bottom up" approach: we have defined these objects in isolation, and then looked at operations that could be applied to them. For the next few lectures, we will shift gears and take a "top down" approach. Specifically, we will learn about the general spaces vectors and matrices live in, called vector spaces, and present the axioms defining rules and operations in these spaces. This may seem a little abstract at first, but will be very helpful very soon to better understand when systems have solutions and when they do not.

I] Vector spaces

1) Definition

A vector space is a set of objects called vectors, which may be added together and multiplied by a scalar. The operation of vector addition and scalar multiplication must satisfy the following axioms, written for three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in the vector space, and two scalars a and b:

1. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ Associativity of addition
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ Commutativity of addition
3. There exists an element $\vec{0}$ in the vector space, called the zero vector, such that $\vec{u} + \vec{0} = \vec{u}$ for any vector $\vec{u}$ in the vector space

4. For every element $\vec{u}$ in the vector space, there exists an element $-\vec{u}$ in the vector space, called the additive inverse of $\vec{u}$, such that $\vec{u} + (-\vec{u}) = \vec{0}$

5. $a(b\vec{u}) = (ab)\vec{u}$

6. $1\vec{u} = \vec{u}$.

7. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ Distributivity of scalar multiplication with respect to vector addition

8. $(a+b)\vec{u} = a\vec{u} + b\vec{u}$

Note 1: You probably recognize many of these axioms in the form of properties for vectors and matrices we have seen previously. This is in agreement with what I said at the beginning of this lecture: vectors and matrices live in well-known vector spaces.

Note 2: In principle, there are many possibilities for the space the multiplicative scalars live in: rational numbers, real numbers, complex numbers, etc.
In this course, the scalars will always be real numbers

Note 3: The word "vector" in the definition has to be understood in its generic sense, associated with vector spaces. As we will soon see, the vectors we are used to are vectors of particular vector spaces, but so are matrices, functions of one variable, etc.

Important note: for any $\vec{u}$, $\vec{v}$, and $\vec{w}$ in the vector space, and any scalars a and b, the operations given in the axioms yield a vector which is in the vector space itself.
If through the linear operations given in the axioms you are able to produce an object which is not in the vector space, then the set you are considering is actually not a vector space.

2) Examples

Let us look at important examples of vector spaces you have already encountered:

A) The set of real numbers, written $\mathbb{R}$, when combined with the usual addition and multiplication, is a vector space. In this particular case, the scalars and the vectors are the same: real numbers.
The zero vector is the number 0. You can easily verify that axioms 1-8 are always satisfied.

B) The set of column vectors with three real valued components when combined with vector addition and multiplication by a scalar, is a vector space. The vectors are the column vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and the zero vector is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
This vector space is three-dimensional and usually called $\mathbb{R}^3$.
Based on what we already know, you can easily check that

axioms 1 to 8 are satisfied.

c) More generally, for any positive integer n, the set of column vectors with n real valued components is a vector space when equipped with vector addition and scalar multiplication. The zero vector is $\left.\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \vdots \\ 0 \end{bmatrix}\right\} n \text{ zeros}$, the vector space is usually

called $\mathbb{R}^n$, and it has n dimensions.

D) The set of real 3 by 3 matrices is a vector space when equipped with matrix addition and multiplication by a scalar. Here, the "vectors" are 3-by-3 matrices $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and the zero vector is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Once more, you can easily convince yourself that the axioms are satisfied. The dimension of the space is $3^2 = 9$

E) The set of all real functions $f(x)$ of one variable is a vector space when equipped with the usual addition and multiplication between real numbers. The zero vector is the function which

for any input x outputs 0 (a.k.a the zero function).
Once more, it is not hard to see that the axioms are satisfied. More unusual: the vector space in this case is infinite-dimensional.

II Subspaces

Within vector spaces, there are certain sets which are such that for any two vectors $\vec{u}$ and $\vec{v}$ in the set, any linear combination of $\vec{u}$ and $\vec{v}$ is also in this set. Such sets are called subspaces, and play an important role in linear algebra

1) Definition

A subspace of a vector space is a set of vectors including the zero vector that satisfies two requirements: If $\vec{u}$ and $\vec{v}$ are vectors in the subspace and c is any scalar, then

(i) $\vec{u}+\vec{v}$ is in the subspace

(ii) $c\vec{u}$ is in the subspace

Note 1: The definition implies that all linear combinations stay in the subspace, as we said above. This is the key point

Note 2: Any subspace MUST contain the zero vector, as stated in the definition

2) Examples

• Inside the vector space of all 3 by 3 matrices, the set of upper triangular matrices $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ is a subspace: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the zero vector, is upper triangular, and the sum of two upper triangular matrices is upper triangular, as is the multiplication of an upper triangular matrix by a scalar.

• Inside the vector space $\mathbb{R}$ of the real numbers, the positive real numbers do NOT form a vector space. The zero vector 0 belongs to the set, but for any vector in the set, say $\pi + e$, the multiplication of this vector by the scalar $c = -1$, $-\pi - e$, is not in the set. Condition (ii) fails

• Any plane through the origin $(0,0,0)$ is a subspace of $\mathbb{R}^3$: it is clear that the zero vector $(0,0,0)$ is in the set, and for any two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ in the plane, the sum $\vec{u} + \vec{v}$ is in the plane, as is $c\vec{u}$ for any scalar c.

III The Column Space of A

We are now ready to make the link between solutions to a system and subspaces. We start with another important, somewhat abstract concept, namely the span of a set of vectors, and make the connection with systems subsequently, when we consider the columns of a matrix A.

1) Subspace spanned by a set S of vectors

Definition: Let S be a set of vectors in a vector space which is not necessarily a subspace. The subspace SS of all the linear combinations of the vectors in S is called the span of S.

QUESTION: Can you prove that SS is indeed a subspace?

Property: SS is the smallest subspace containing S.

2) Column space of a matrix

Definition: The column space of a matrix A is the subspace spanned by the columns of A. It is often denoted by C(A)

In other words, C(A) is the subspace of all linear combinations of the columns of A

Note that the columns of an m-by-n matrix have m components. So if A is an m-by-n matrix, $C(A)$ is a subspace of $\mathbb{R}^m$.

Examples: • The column space of $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is $\mathbb{R}^3$ itself since any vector in $\mathbb{R}^3$ can be written as the linear combination of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

• The column space of $\begin{bmatrix} 1 & 2 & -\sqrt{2} \\ 2 & 4 & -2\sqrt{2} \end{bmatrix}$ is a line in $\mathbb{R}^2$, since the last two column vectors are colinear with $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Now, we have seen several times that the matrix-vector multiplication $A\vec{x}$ could be seen as the linear combination of the columns of the matrix A. We are therefore ready to make the following important connection with systems:

$$\boxed{\text{The system } A\vec{x} = \vec{b} \text{ has at least one solution if and only if } \vec{b} \text{ is in the column space of } A.}$$