MATH-UA 140 - Linear Algebra

Lecture 11: Solving $A\vec{x} = \vec{b}$ in general

This lecture starts with a review of a few concepts we already covered in the last lecture, and then connects all we have learned in the last few lectures to provide a general technique for solving the equation $A\vec{x} = \vec{b}$, whether A is a square matrix or not.

I] Rank of a matrix

1) Computational definition

The rank of a matrix A is the number of pivots of A. This number is often written r.

In the following, we will see the fundamental importance of the rank, beyond its computational definition. It tells us the true size of a linear system, i.e the number of equations which are not redundant.

Example: $A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ -2 & 3 & 1 & 2 \\ 0 & 5 & 7 & 4 \end{bmatrix}$,

Elimination leads to $\begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 5 & 7 & 4 \\ 0 & 5 & 7 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 5 & 7 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The rank of A is 2. In the system $A\vec{x} = \vec{0}$, the last equation really is redundant, equivalent to $0=0$

2) Rank and column dependence

The rank is the number of pivots of a matrix, so also the number of pivot columns. The columns which are not pivot columns are called free columns.

The pivot columns are not linear combinations of earlier columns. The free columns are combinations of earlier columns

Here, the word early applies to the order with which one applies elimination column by column.

As we will soon see, we say that the r pivot columns are independent columns, and the $n-r$ free columns are dependent columns.

Example: $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 7 & 1 \end{bmatrix}$

We have $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

The rank of A is 2

3) Echelon form R, special solutions, and linear combinations

In the last lecture, we saw how to construct the special solutions to $A\vec{x} = \vec{0}$ from the reduced row echelon matrix R associated with A. There are $n-r$ special solutions. For each special solution, one free variable is equal to 1 and the other free variables are set to 0. The pivot variables can then be read in the column in R associated with the nonzero free variable, by reversing the signs.

Remember our example last time:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 1 \\ 5 & 10 & 11 & -8 \end{bmatrix} \longrightarrow R = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & -17 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \text{pivot column} & \text{free column} & \text{pivot column} & \text{free column} \end{array} \end{array}$$

We will now show that the pivot columns of A are the same as the pivot columns of R. This is a general result.
It is clear that column 2 is just 2 times column 1. Column 4 is a linear combination of columns 1 and 3, and it is a general result that the special solutions tell us what the linear combination is

We can read the special solutions off R: for $x_2 = 1$, $x_4 = 0$, the special solution is obtained from the second column of R, $\vec{s}_1 = (-2, 1, 0, 0)$. This solution tells us that

-2 Column 1 + Column 2 is the zero vector.

The other special solution, for $x_2 = 0$ and $x_4 = 1$, is read in the 4^{th} column: $\vec{s}_2 = (17, 0, -7, 1)$
$\vec{s}_2$ tells us that 17 Column 1 $-$ 7 Column 3 + Column 4 is the zero vector. In other words, Column 4 = -17 Column 1 + 7 Column 3.

We see that R contains a surprising amount of information!
It is important to know that <u>for every matrix A, there is a unique corresponding R.</u>

4) Special solutions and the nullspace of a matrix

We have seen last time that every vector in the nullspace of A can be written as the linear combination of the special solutions found through the row reduction procedure.
A concise way to summarize this is to construct the <u>nullspace matrix N</u>, whose <u>columns are the special solutions</u>, so that $AN = 0$

For our previous example, $N = \begin{bmatrix} -2 & 17 \\ 1 & 0 \\ 0 & -7 \\ 0 & 1 \end{bmatrix}$ not free / free / not free / free

$n - r = 4 - 2 = 2$
$\rightarrow$ 2 columns

Observe that R had the identity matrix (2×2 in our case) in the pivot columns, and N has the identity matrix (also 2×2) in its free rows.

General summary

Consider an $m\times n$ matrix A with rank r. This means that $A\vec{x}=\vec{0}$ has r pivots and $n-r$ free variables.
The nullspace matrix N contains the $n-r$ special solutions.

The special solutions are easy to obtain from the row reduced form $R\vec{x}=\vec{0}$. Suppose for simplicity that the pivot columns are the first r columns. Then R can be written in the following blocks:

identity

$$R=\begin{array}{c}\left[\begin{array}{cc} I & F \\ 0 & 0 \end{array}\right] \\ \begin{array}{cc} \text{r pivot columns} & \text{n-r free columns} \end{array}\end{array}\begin{array}{l} \text{r pivot rows} \\ \text{m-r zero rows} \\ \end{array}$$

Then the nullspace matrix N is given by

$$N=\left[\begin{array}{c} -F \\ I \end{array}\right]\begin{array}{l} \text{r pivot variables} \\ \text{n-r free variables} \end{array}$$

Indeed, multiplying the matrices by blocks, we have $RN=-FI+FI=0$ as desired. This illustrates why the special solutions, which are in N, can always be read in R.

Note that all of this still holds even if the pivot columns are mixed with the free columns in R; the explanation is just messier.

5) A particular case: rank one matrices

Matrices with $r=1$ have only one pivot. When elimination produces zero in the first column, it produces zero in all the columns: every row is a multiple of the pivot row; every column is a multiple of the pivot column.

Example: $A = \begin{bmatrix} -1 & 2 & -9 \\ -3 & 6 & -27 \\ 2 & -4 & 18 \\ 5 & -10 & 45 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & -2 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The column space of a rank one matrix is a line with direction the column vector $\vec{u}$ of the first column. In our example, $\vec{u} = (-1, -3, 2, 5)$. The columns of A are $\vec{u}$, $-2\vec{u}$, and $9\vec{u}$. Let $\vec{v}^T = [1 \ -2 \ \ 9]$.
A can be written in the rank one form $\mathbf{A} = \vec{u}\,\vec{v}^T$

Let us consider the solutions to $A\vec{x} = \vec{0}$ in that case.
$A\vec{x} = \vec{0} \iff (\vec{u}\,\vec{v}^T)\vec{x} = \vec{0} \iff \vec{u}(\vec{v}^T\vec{x}) = \vec{0}$
Now, note that $\vec{v}^T\vec{x}$ is a scalar. So $\vec{u}(\vec{v}^T\vec{x}) = \vec{0}$
$\Rightarrow \vec{v}^T\vec{x} = \vec{0}$
This means that all vectors $\vec{x}$ in the nullspace of A are orthogonal to $\vec{v}$.
The null space is a plane perpendicular to the row vector $\vec{v}$.

In our case, $\vec{s}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $s_2 = \begin{bmatrix} -9 \\ 0 \\ 1 \end{bmatrix}$

All combinations of these vectors produce the plane $x - 2y + 9z = 0$, perpendicular to the vector $(1, -2, 9)$.

QUESTION: Consider the equation $R\vec{x} = -x_1 + 4x_2 + 5x_3 = 0$ with $R = [-1 \ \ 4 \ \ 5]$. What is the rank of R? What are the columns of the null matrix N?

$\boxed{\text{II.}}$ Solving $A\vec{x} = \vec{b}$

1) Solvability condition

We saw together that for $A\vec{x} = \vec{b}$ to have a solution, $\vec{b}$ must be in the column space of A $C(A)$. We now give a practical computational criterion for the existence of a solution.

Consider solving the system $A\vec{x} = \vec{b}$ by elimination, from the augmented matrix $[A \ \vec{b}]$ to $[R \ \vec{d}]$, where $\vec{d}$ is obtained from $\vec{b}$ with the same elimination steps as R from A.
Imagine the last row of R only has zeros. Then the last row of $R\vec{x}$ is 0: for a solution to exist, the last row of $\vec{d}$ must also be zero.
In general, for a solution to exist, zero rows in R must also be zero in d.

2) Complete solution

In the previous lectures, we learned how to compute the general solution to $R\vec{x} = \vec{0}$. All we need to obtain the general solution to $R\vec{x} = \vec{d}$ is to construct a particular solution to that equation, which we will add to the general solution to $R\vec{x} = \vec{0}$.
Here is the simple trick to do so: <u>set the free variables to zero, and read the pivot variables right off $\vec{d}$.</u>

This works because the pivot rows and columns of R are the identity matrix.

Example:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 1 \\ 5 & 10 & 11 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ 27 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -7 \\ 5 & 10 & 11 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 27 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -7 \\ 0 & 0 & -4 & -28 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ -8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 1 & 2 & 0 & -17 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}_{\vec{d}}$$

The particular solution is obtained by setting $x_2 = x_4 = 0$ and the pivot variables $x_1 = 1$, $x_3 = 2$

We already knew from last time that the general solution to $R\vec{x} = \vec{0}$ is $\vec{x}_n = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 17 \\ 0 \\ -7 \\ 1 \end{bmatrix}$.

So the complete solution to $A\vec{x} = \begin{bmatrix} 7 \\ 12 \\ 27 \end{bmatrix}$ is

$$\vec{x} = \vec{x}_p + \vec{x}_n = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 17 \\ 0 \\ -7 \\ 1 \end{bmatrix}$$

p for particular
Solves $A\vec{x}_p = \vec{b}$

n for nullspace
solves $A\vec{x} = \vec{0}$

3) Full column rank matrices

Consider an $m \times n$ matrix A with $m \geq n$ (tall and thin matrix) and suppose that the rank of the matrix is $r = n$: every column has a pivot, so the row reduced form of A is

$$R = \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{array}{l} \to n \times n \text{ identity matrix} \\ \to m-n \text{ rows of zeros} \end{array}$$

One can see that there are no free columns; there are no free variables. The nullspace matrix is empty. The only vector in the nullspace is $\vec{0}$.

$A\vec{x} = \vec{b}$ has a solution if $\vec{b}$ is in the column space of A, and in that case it is the only solution

QUESTION: Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$ Find the condition on $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ for $A\vec{x} = \vec{b}$ to have a solution $\vec{x}$, and give the solution when the condition is satisfied

4) Full row rank matrices

We now consider the other extreme case: A is an $m \times n$ matrix with $m \le n$ (short and wide), and the rank of A is $r = m$. Every row has a pivot. The matrix R has no zero rows: $R = [\underset{\substack{m\times m \\ \text{identity}}}{I} \quad \underset{\substack{n-m \\ \text{free columns}}}{F}]$

$A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$: $C(A) = \mathbb{R}^m$

There are $n - m = n - r$ special solutions in the nullspace of A.

Example: Consider the system of equations

$$\begin{cases} x + y + z = 5 \\ 3x + y + 2z = 4 \end{cases}$$

The augmented matrix is $\begin{bmatrix} 1 & 1 & 1 & 5 \\ 3 & 1 & 2 & 4 \end{bmatrix}$

$$\to \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & -2 & -1 & -11 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & \frac{1}{2} & \frac{11}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{11}{2} \end{bmatrix}$$

A particular solution is obtained by setting $z=0$ and reading $\vec{d}$: $(x,y,z) = \left(-\frac{1}{2}, \frac{11}{2}, 0\right)$

The special solution is obtained by setting $z=1$, and reversing the signs in the third column: $(x,y,z) = \left(-\frac{1}{2}, -\frac{1}{2}, 1\right)$

The complete solution is $\vec{x} = \vec{x}_p + \vec{x}_n = \begin{bmatrix} -\frac{1}{2} \\ \frac{11}{2} \\ 0 \end{bmatrix} + z \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$

Summary

In this class, we have now covered the four different possibilit for the system of equations $A\vec{x} = \vec{b}$ with A an $m \times n$ matrix with rank r:

① $r=m=n$	② $r=m$ and $r<n$	③ $r=n$ and $r<m$	④ $r<m$ and $r<$
A is square and invertible	A is short and wide	A is tall and thin	A does not have full rank
$R = [I]$	$R = [I \ F]$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
$A\vec{x} = \vec{b}$ has 1 solution	$A\vec{x} = \vec{b}$ has ∞ solutions	$A\vec{x} = \vec{b}$ has 0 or 1 solution	$A\vec{x} = \vec{b}$ has 0 or ∞ solution