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1. (40 points) Suppose  $u$  is a unit vector in  $R^n$ , so  $u^T u = 1$ . This problem is about the  $n$  by  $n$  symmetric matrix  $H = I - 2u u^T$ .

(a) Show directly that  $H^2 = I$ . Since  $H = H^T$ , we now know that  $H$  is not only symmetric but also \_\_\_\_\_.

**Solution** Explicitly, we find  $H^2 = (I - 2u u^T)^2 = I^2 - 4u u^T + 4u u^T u u^T$  (2 points): since  $u^T u = 1$ ,  $H^2 = I$  (3 points). Since  $H = H^T$ , we also have  $H^T H = I$ , implying that  $H$  is an orthogonal (or unitary) matrix.

(b) One eigenvector of  $H$  is  $u$  itself. Find the corresponding eigenvalue.

**Solution** Since  $Hu = (I - 2u u^T)u = u - 2u u^T u = u - 2u = -u$ ,  $\lambda = -1$ .

(c) If  $v$  is any vector perpendicular to  $u$ , show that  $v$  is an eigenvector of  $H$  and **find the eigenvalue**. With all these eigenvectors  $v$ , that eigenvalue must be repeated how many times? Is  $H$  **diagonalizable**? Why or why not?

**Solution** For any vector  $v$  orthogonal to  $u$  (i.e.  $u^T v = 0$ ), we have  $Hv = (I - 2u u^T)v = v - 2u u^T v = v$ , so the associated  $\lambda$  is 1. The orthogonal complement to the space spanned by  $u$  has dimension  $n - 1$ , so there is a basis of  $(n - 1)$  orthonormal eigenvectors with this eigenvalue. Adding in the eigenvector  $u$ , we find that  $H$  is diagonalizable.

(d) Find the diagonal entries  $H_{11}$  and  $H_{ii}$  in terms of  $u_1, \dots, u_n$ . Add up  $H_{11} + \dots + H_{nn}$  and separately add up the eigenvalues of  $H$ .

**Solution** Since  $i$ th diagonal entry of  $u u^T$  is  $u_i^2$ , the  $i$  diagonal entry of  $H$  is  $H_{ii} = 1 - 2u_i^2$  (3 points). Summing these together gives  $\sum_{i=1}^n H_{ii} = n - 2 \sum_{i=1}^n u_i^2 = n - 2$  (3 points).

2. (30 points) Suppose  $A$  is a positive definite symmetric  $n$  by  $n$  matrix.

- (a) How do you know that  $A^{-1}$  is also positive definite? (We know  $A^{-1}$  is symmetric. I just had an e-mail from the International Monetary Fund with this question.)

**Solution** Since a matrix is positive-definite if and only if all its eigenvalues are positive (5 points), and since the eigenvalues of  $A^{-1}$  are simply the inverses of the eigenvalues of  $A$ ,  $A^{-1}$  is also positive definite (the inverse of a positive number is positive) (5 points).

- (b) Suppose  $Q$  is any **orthogonal**  $n$  by  $n$  matrix. How do you know that  $Q A Q^T = Q A Q^{-1}$  is positive definite? Write down which test you are using.

**Solution** Using the energy test ( $x^T A x > 0$  for nonzero  $x$ ), we find that  $x^T Q A Q^T x = (Q^T x)^T A (Q^T x) > 0$  for all nonzero  $x$  as well (since  $Q$  is invertible). Using the positive eigenvalue test, since  $A$  is similar to  $Q A Q^{-1}$  and similar matrices have the same eigenvalues,  $Q A Q^{-1}$  also has all positive eigenvalues. (5 points for test, 5 points for application)

- (c) Show that the block matrix

$$B = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$$

is positive **semidefinite**. How do you know  $B$  is not positive definite?

**Solution** First, since  $B$  is singular, it cannot be positive definite (it has eigenvalues of 0). However, the pivots of  $B$  are the pivots of  $A$  in the first  $n$  rows followed by 0s in the remaining rows, so by the pivot test,  $B$  is still semi-definite. Similarly, the first  $n$  upper-left determinants of  $B$  are the same as those of  $A$ , while the remaining ones are 0s, giving another proof. Finally, given a nonzero vector

$$u = \begin{bmatrix} x \\ y \end{bmatrix}$$

where  $x$  and  $y$  are vectors in  $\mathbf{R}^n$ , one has  $u^T B u = (x+y)^T A (x+y)$  which is nonnegative (and zero when  $x + y = 0$ ).

3. (30 points) This question is about the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}$$

(a) Find its eigenvalues and eigenvectors.

Write the vector  $u(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$  as a combination of those eigenvectors.

**Solution** Since  $\det(A - \lambda I) = \lambda^2 + 4$ , the eigenvalues are  $2i, -2i$  (4 points). Two associated eigenvectors are  $[1 \quad -2i]^T, [1 \quad 2i]^T$ , though there are many other choices (4 points).  $u(0)$  is just the sum of these two vectors (2 points).

(b) Solve the equation  $\frac{du}{dt} = Au$  starting with the same vector  $u(0)$  at time  $t = 0$ .

In other words: the solution  $u(t)$  is what combination of the eigenvectors of  $A$ ?

**Solution** One simply adds in factors of  $e^{\lambda t}$  to each term, giving

$$u(t) = e^{2it} \begin{bmatrix} 1 \\ -2i \end{bmatrix} + e^{-2it} \begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

(c) Find the 3 matrices in the Singular Value Decomposition  $A = U \Sigma V^T$  in two steps.

–First, compute  $V$  and  $\Sigma$  using the matrix  $A^T A$ .

–Second, find the (orthonormal) columns of  $U$ .

**Solution** Note that  $A^T A = V \Sigma^T U^T U \Sigma V^T = V \Sigma^2 V^T$ , so the diagonal entries of  $\Sigma$  are simply the positive roots of the eigenvalues of

$$A^T A = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e.  $\sigma_1 = 4, \sigma_2 = 1$ . Since  $A^T A$  is already diagonal,  $V$  is the identity matrix. The columns of  $U$  should satisfy  $Au_1 = \sigma_1 v_1, Au_2 = \sigma_2 v_2$ : by inspection, one obtains

$$u_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$