The Derivative of $|x|$

The slope of the graph of $f(x) = |x|$ changes abruptly when $x = 0$. Does this function have a derivative? If so, what is it? If not, why not?

Solution

At first glance, this seems like a simple question. To the right of y-axis the graph of $f(x)$ has slope $+1$. To the left of the y-axis it has slope -1. It's reasonable to conclude that:

$$f'(x) = \begin{cases} 1 & x > 0, \\ -1 & x < 0. \end{cases}$$

However, this description of the derivative leaves out the value of $f'(0)$.

Our formula for the derivative tells us that:

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\Delta x)}{\Delta x}.$$

The slope of the secant line joining the points $(0, 0)$ and $(\Delta x, f(\Delta x))$ is:

$$\frac{f(0 + \Delta x) - f(0)}{\Delta x} = \frac{f(\Delta x)}{\Delta x}.$$

What is the value of this expression when Δx gets close to (but not equal to) zero?

If $\Delta x > 0$ then $f(\Delta x) = \Delta x$ and

$$\frac{f(\Delta x)}{\Delta x} = 1.$$

If $\Delta x < 0$ then $f(\Delta x) = -1 \cdot \Delta x$ and

$$\frac{f(\Delta x)}{\Delta x} = -1.$$

The value of $f(\Delta x)$ doesn't depend on the size of Δx and doesn't necessarily converge to a single value as Δx shrinks. The "limit as Δx approaches 0" isn't well defined, so $f(x)$ is not differentiable at $x = 0$.

If we try to find $f'(0)$ by finding the slope of the tangent line to the graph of $f(x)$ at $x = 0$, we have problems finding that tangent line. Our intuition about the tangent line tells us that any line tangent to the graph at $(0, 0)$ must go through $(0, 0)$ and then "follow the direction of the graph" near $(0, 0)$. The line $y = x$ goes through $(0, 0)$ and follows the positive side of the graph; the line $y = -x$ does the same in the negative direction. Neither of these two lines follow the graph away from $(0, 0)$ in both directions. The line $y = 0$ looks promising but doesn't follow the graph in *either* direction, nor is it the limit of any sequence of secant lines through $(0, 0)$. There is no tangent line to the graph of $f(x) = |x|$ at the point $(0, 0)$, so the slope $f'(x)$ is not defined for $x = 0$.

Either way, we conclude that if $f(x) = |x|$, $f'(0)$ is undefined. We say that $f(x)$ is not differentiable at $x = 0$. If a function $f(x)$ is not differentiable at even one point in its domain, $f(x)$ is not a differentiable function.