MATH-UA 140 - Linear Algebra

Lecture 3: A brief introduction to matrices

A central application of linear algebra is to efficiently solve linear systems of equations. In the next lecture, we will learn how to write such systems in matrix form, and later we will learn methods to determine whether a given system has solutions, and when it does, how to compute them.
As a preparation for this key part of the course, we will introduce here all the ideas we need to be comfortable with for future lectures.

I] Matrices

1) Definition

A matrix is a rectangular array of numbers arranged in m rows and n columns, where m and n are positive integers. The matrix entries are usually written in the following format: a_{ij}, where i is a positive integer representing the row number of the entry, and j representing the column number.

Example: The generic 3-by-3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Note that in this class, we will only consider entries a_{ij} which are real numbers

Just like for vectors, we need to learn the basic operations on matrices that we will regularly use throughout the course: matrix addition, multiplication by a scalar, and matrix-matrix (or vector) multiplication.

2) Matrix addition

Matrices which have the same dimensions can be added. The entry c_{ij} of the matrix $C = A + B$ equal to the sum of the matrices A and B is given by:

$$c_{ij} = a_{ij} + b_{ij}$$

where the a_{ij} are the entries of A, and the b_{ij} the entries of B.

In other words, just like vectors, the idea is to add apples to apples.

<u>Example</u>:

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 4 & 5 & 4 \end{bmatrix}$$

3) Multiplication by a scalar

Let A be a matrix with entries a_{ij}, and c be a scalar. Then cA is a matrix with the same dimensions as A, and whose entries are $c\,a_{ij}$.

Example:

$$3\begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 0 \\ 9 & 9 & 3 \end{bmatrix}$$

4) Matrix-vector multiplication

Let A be an m-by-n matrix, and $\vec{u}$ be an n-by-1 column vector. Then $A\vec{u}$ is a m-by-1 column vector whose entries are obtained by calculating the dot product of each row of A with $\vec{u}$.

Example:

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} (2,1,0)\cdot(-1,2,1) \\ (3,3,1)\cdot(-1,2,1) \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

II] From linear combinations to linear systems

1) Linear combinations in matrix form

Consider the vectors $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

The linear combinations of these three vectors are

$c\vec{u} + d\vec{v} + e\vec{w}$, with c, d, and e scalars:

$$c\begin{bmatrix}2\\1\\1\end{bmatrix} + d\begin{bmatrix}1\\2\\1\end{bmatrix} + e\begin{bmatrix}3\\-1\\2\end{bmatrix} = \begin{bmatrix}2c+d+3e\\c+2d-e\\c+d+2e\end{bmatrix}$$

We may rewrite this in matrix form as follows:

$$\begin{bmatrix}2 & 1 & 3\\1 & 2 & -1\\1 & 1 & 2\end{bmatrix}\begin{bmatrix}c\\d\\e\end{bmatrix} = \begin{bmatrix}2c+d+3e\\c+2d-e\\c+d+2e\end{bmatrix}$$

or in concise form, $\begin{bmatrix}\vec{u} & \vec{v} & \vec{w}\\ & & \end{bmatrix}\begin{bmatrix}c\\d\\e\end{bmatrix} = \begin{bmatrix}2c+d+3e\\c+2d-e\\c+d+2e\end{bmatrix}$

If we call A the matrix $\begin{bmatrix}2 & 1 & 3\\1 & 2 & -1\\1 & 1 & 2\end{bmatrix} = \begin{bmatrix}\vec{u} & \vec{v} & \vec{w}\\ & & \end{bmatrix}$,

we see that the multiplication of A with the vector $\begin{bmatrix}c\\d\\e\end{bmatrix}$ is the same as the linear combination $c\vec{u} + d\vec{v} + e\vec{w}$ of the three columns of A.

This is another perspective for understanding matrix vector product: in section I/4), we viewed it in terms of the <u>rows of A</u>; here, we view it in terms of <u>the columns of A.</u>

2) A first look at linear systems

We have just seen how to express linear combinations in matrix form: to compute $\vec{b}$ which is the linear combination $c\vec{u}+d\vec{v}+e\vec{w}$ one evaluates the matrix-vector product $\begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Let us now change the viewpoint entirely and ask ourselves the inverse question: given a vector $\vec{b}$ and three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, find the scalars x_1, x_2, and x_3 such that $x_1\vec{u} + x_2\vec{v} + x_3\vec{w} = \vec{b}$. Calling $\vec{x}$ the unknown vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, this question can be written in matrix form:

$$\begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \text{i.e} \quad \underline{A\vec{x} = \vec{b}}$$

As an illustration, let us return to the matrix A we had in the previous section, and take $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Find x_1, x_2, and x_3 such that

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This is the matrix form of the following system of equations:

$$\begin{cases} 2x_1 + x_2 + 3x_3 = 1 \\ x_1 + 2x_2 - x_3 = 0 \\ x_1 + x_2 + 2x_3 = 0 \end{cases}$$

This system of equations is said to be linear, because the unknown x_1, x_2, and x_3 appear linearly, i.e. there is no x_1^2, or $\sqrt{x_1}$, or x_1x_3, or any such term that is non linear.

Any linear system of equations can be written in the matrix form $A\vec{x}=\vec{b}$

In this particular example, fairly straight forward algebra leads to the solution: $x_1 = \frac{5}{4}$, $x_2 = -\frac{3}{4}$, $x_3 = -\frac{1}{4}$

In general, systems cannot be solved in such a straight forward manner. We will soon learn a robust method to solve linear systems that makes use of the matrix representation.

3) Linear dependence and independence

Given a matrix $A = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$, one may wonder if for any vector $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ the linear system $A\vec{x}=\vec{b}$ has a solution $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. We will answer this question in detail in future lectures. Still, one can obtain very good intuition regarding the answer by rephrasing the question in the following terms: given 3 vectors $\vec{u}$, $\vec{v}$, and

$\vec{w}$ in 3-D space, can one always find, for any vector $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, a linear combination such that $x_1\vec{u} + x_2\vec{v} + x_3\vec{w} = \vec{b}$

Remembering Lecture 1, if $\vec{w}$ is in the plane of $\vec{u}$ and $\vec{v}$, the answer is NO, since all linear combinations $x_1\vec{u} + x_2\vec{v} + x_3\vec{w}$ only fill that plane in that case, and cannot be equal to vectors $\vec{b}$ that are not in that plane.

On the other hand, if $\vec{w}$ is not in the plane of $\vec{u}$ and $\vec{v}$, then the answer is YES, since linear combinations $x_1\vec{u} + x_2\vec{v} + x_3\vec{w}$ fill the whole space.

- When $\vec{w}$ is not in the plane of $\vec{u}$ and $\vec{v}$, we say that $\vec{u}, \vec{v}$, and $\vec{w}$ are linearly independent.

No linear combination except $0\vec{u} + 0\vec{v} + 0\vec{w} = \vec{0}$ gives $\vec{b} = 0$

- When $\vec{w}$ is in the plane of $\vec{u}$ and $\vec{v}$, we say that $\vec{u}, \vec{v}$, and $\vec{w}$ are linearly dependent.

There are nontrivial linear combinations such that $c\vec{u} + d\vec{v} + e\vec{w} = \vec{0}$