Honors Linear Algebra – Problem Set 8 Solutions

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Problem 1

1. Consider the big permutation formula for determinants. If we choose an entry from the B block in that formula, then we must also choose an entry from the bottom left block, which is zero. So any term in the permutation formula involving entries from B is zero. The determinant of the block matrix thus is

$$a_{11}a_{22}d_{11}d_{22} - a_{12}a_{21}d_{11}d_{22} - a_{11}a_{22}d_{21}d_{12} + a_{12}a_{21}d_{21}d_{12} = \det A \det D$$

2. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

We have $|A| = |B| = |C| = |D| = 0$, and yet

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = 1$$

Hence

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |B||C|$$

3. Let us take the same matrices A, B, C, and D as above

$$AD = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad CB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad AD - CB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore, $\det(AD - CB) = 0 \neq 1 = \det M$

Problem 2

As we saw in Lecture 20, if a square matrix A has an inverse A^{-1}, then we can write

$$A^{-1} = \frac{C^T}{\det A}$$

where C is the cofactor matrix.

Consider a matrix A whose cofactors are all zero, and assume A has an inverse. Then the formula says that A^{-1} is the zero matrix. This is clearly impossible: the zero matrix is not invertible. We conclude that A does not have an inverse.

Regarding the second question, let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

All the cofactors of A are equal to 1. However, A has rank 1, so A is clearly not invertible. Having a cofactor matrix without zero entries does not imply invertibility.

Problem 3

1. The eigenvalues of P are give by solving

$$\begin{vmatrix} \frac{2}{10} - \lambda & \frac{4}{10} & 0 \\ \frac{4}{10} & \frac{8}{10} - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{vmatrix} = 0$$

   for λ. Using a cofactor expansion, we can rewrite this as

$$\frac{(1-\lambda)}{100}\left[(2-10\lambda)(8-10\lambda) - 16\right] = 0 \quad \Leftrightarrow \lambda(1-\lambda)^2 = 0$$

   The eigenvalues of P are $\lambda_1 = 0$ and $\lambda_2 = 1$.

   In general, the eigenvalues of any matrix such that $P^2 = P$ can only be 0 and 1. Indeed, let λ be an eigenvalue of such P, with corresponding eigenvector $\mathbf{x}$. We have $P\mathbf{x} = \lambda\mathbf{x}$. Now, multiplying this equality by P on the left, we have $P^2\mathbf{x} = \lambda P\mathbf{x} = \lambda^2\mathbf{x}$. Since $\mathbf{x}$ is not the zero vector, it must be that $\lambda^2 = \lambda$. Only $\lambda = 0$ and $\lambda = 1$ satisfy this equality.

2. The eigenvector corresponding to $\lambda_1 = 0$ is in the nullspace of

$$\begin{bmatrix} 2 & 4 & 0 \\ 4 & 8 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

   After exchanging the second and third rows, the row reduced echelon form of this matrix is

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   So an eigenvector for $\lambda_1 = 0$ is $(-2, 1, 0)$.

   The eigenvectors corresponding to $\lambda_2 = 1$ are in the nullspace of

$$\begin{bmatrix} -8 & 4 & 0 \\ 4 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   The corresponding row reduced echelon form is

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   So we conclude that two linearly independent eigenvectors for $\lambda_2 = 1$ are $(1/2, 1, 0)$ and $(0, 0, 1)$.

3. $(1/2, 1, 0)$ and $(0, 0, 1)$ are both eigenvectors for the eigenvalue $\lambda_2 = 1$, and so is there sum $(1/2, 1, 1)$, which is an eigenvector with no zero components.

Problem 4

1. A has three distinct eigenvalues, so three linearly independent eigenvectors, which are a basis of $\mathbb{R}^3$. The dimension of the subspace spanned by the eigenvector for the eigenvalue $\lambda = 0$ is 1. This is the dimension of its nullspace. So A has rank 2.

2. $\det(A^T A) = \det(A^T) \det A = (\det A)^2 = 0$ since A is singular (rank 2).

3. We cannot say anything about the eigenvalues of $A^T A$.

4. We know that for any matrix A, the eigenvalues of A^2 are the squares of the eigenvalues of A. In our case, the eigenvalues of A^2 are therefore 0 , 9, and 36.

   Now, for any matrix B, if the eigenvalues of B are λ_i, the eigenvalues of $B+I$ are λ_i+1. In our case, we can then say that the eigenvalues of A^2+I are 1, 10, and 37, and A^2+I is invertible.

   Finally, for any invertible matrix C, the eigenvalues of C^{-1} are the inverses of the eigenvalues of C. In our case, we can thus conclude that the eigenvalues of $(A^2+I)^{-1}$ are 1, 1/10, and 1/37.

Problem 5

The eigenvalues of A satisfy $(2-\lambda)^2-1=0 \Leftrightarrow (3-\lambda)(1-\lambda)=0$. So the eigenvalues of A are $\lambda_1=3$ and $\lambda_2=1$.

The eigenvector corresponding to λ_1 is in the nullspace of

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

So we may take $\mathbf{x}_1=(1,-1)$ as an eigenvector.

The eigenvector corresponding to λ_2 is in the nullspace of

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

So we may take $\mathbf{x}_2=(1,1)$ as an eigenvector.

The eigenvector matrix S thus is

$$S=\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

We have

$$S^{-1}=\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Hence, A may be diagonalized as

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Therefore,

$$A^k=\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3^k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}=\begin{bmatrix} (3^k+1)/2 & (1-3^k)/2 \\ (1-3^k)/2 & (3^k+1)/2 \end{bmatrix}$$