Proof of the Second Fundamental Theorem of Calculus

Theorem: (The Second Fundamental Theorem of Calculus) If f is continuous and $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$.

Proof: Here we use the interpretation that $F(x)$ (formerly known as $G(x)$) equals the area under the curve between a and x. Our goal is to take the derivative of F and discover that it's equal to f.

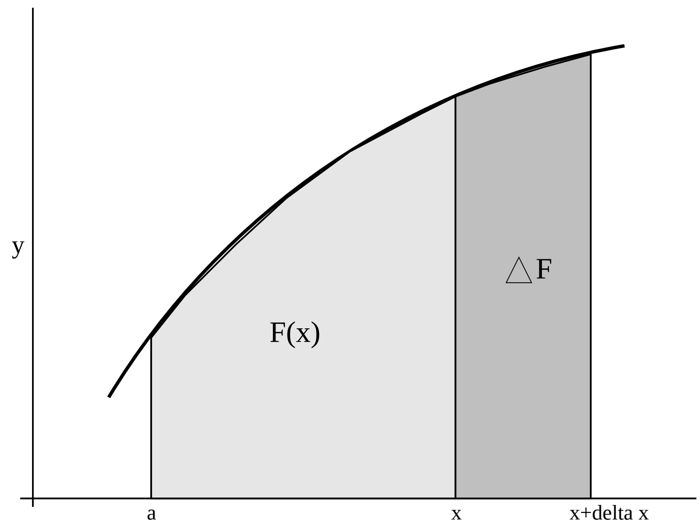

Figure 1: Graph of $f(x)$ with shaded area $F(x)$.

We graph the equation $y = f(x)$ and keep track of where a, x and $x + \Delta x$ are. This splits the area under the curve into pieces. The first piece is the area under the curve between a and x which is, by definition, $F(x)$. The second piece is a thin region; its area is ΔF, which is the change in the area under the curve as x increases by Δx.

We now approximate this thin region with area ΔF by a rectangle. Its base has width Δx and its height is close to $f(x)$ (because f is continuous). So

Thanks to Leibniz notation!!

$$\Delta F \approx \Delta x f(x).$$

Divide both sides by Δx to get $\frac{\Delta F}{\Delta x} \approx f(x)$, then take the limit as Δx goes to zero to get the derivative:

$$F'(x) = \lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = f(x).$$