MATH-UA 140 - Linear Algebra

Lecture 4: Solving Linear Systems

The purpose of this lecture is to learn a streamlined method for solving linear systems of equations, called "Gaussian elimination" or sometimes simply "elimination". Before we do so, we will review part of the material from the last lecture, on the geometric interpretation of linear systems of equations and their geometric interpretation.

I) Linear systems of equations

1) 2-D situation: 2 equations and 2 unknowns

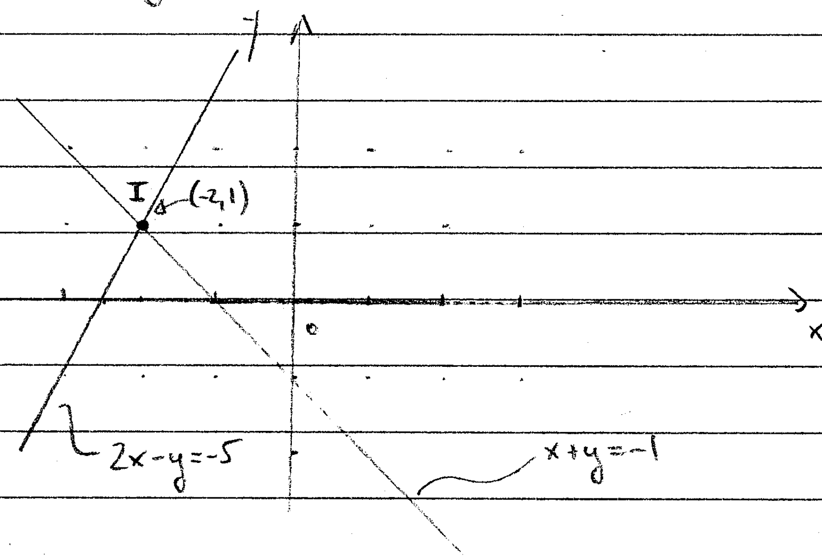
* "Row picture"

Consider the following system of equations

$$\begin{cases} x + y = -1 \\ 2x - y = -5 \end{cases}$$

for the unknowns x and y.

Geometrically, the system is equivalent to the following question: what are the points with cartesian coordinates (x, y) which satisfy both $x+y=-1$ and $2x-y=5$. Both equations are equations of lines, plotted below. Since the lines are not parallel, only one point with coordinates (x_I, y_I) is on both lines. This is of course the intersection of point I of the two lines, which is the unique solution to the system of equations. In our particular case, we can read the solution off the plot: $x_I = -2$, $y_I = 1$



As we saw last time, the system $\begin{cases} x+y = -1 \\ 2x-y = -5 \end{cases}$ can be seen as the <u>row reading</u> of the matrix representation $A\vec{x} = \vec{b}$:

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}}_{\text{Matrix } A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{Unknown vector } \vec{x}} = \underbrace{\begin{bmatrix} -1 \\ -5 \end{bmatrix}}_{\text{vector } \vec{b}}$$

The "row reading" thus views the solution of the system of 2 equations for 2 unknowns (when it exists) as the intersection of 2 lines (when they are not parallel)

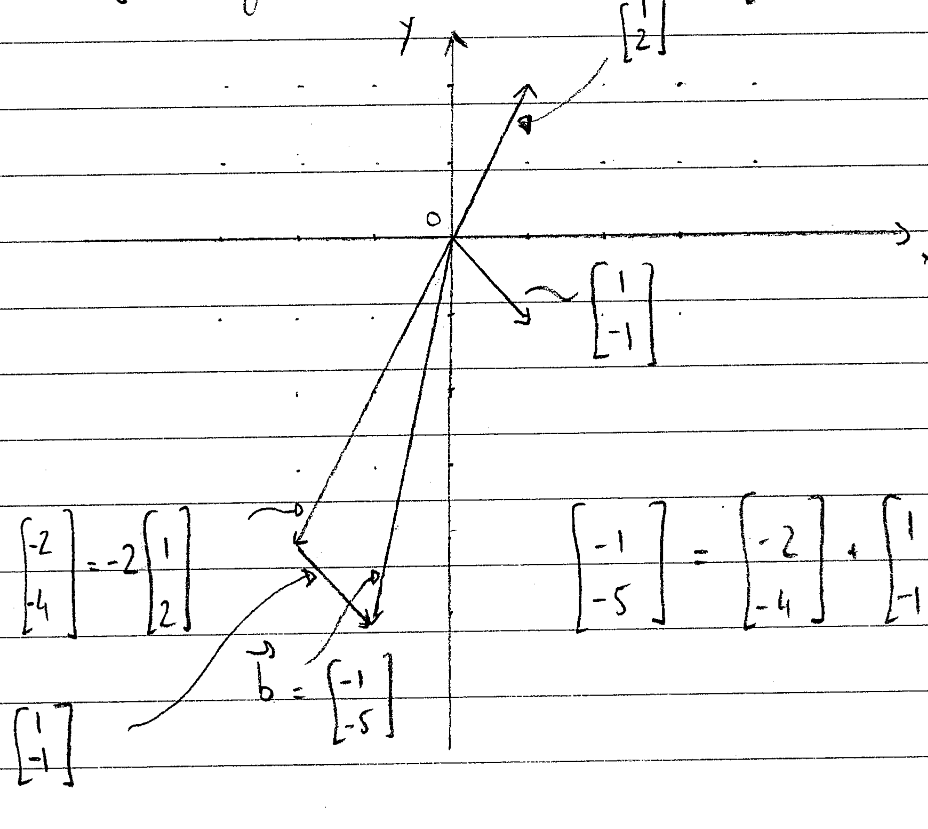
* "Column picture"

We saw in the previous lecture that the matrix representation $A\vec{x} = \vec{b}$ could also be read in terms of columns, corresponding to the linear combinations:

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$$

In that case, solving the linear system means looking for x and y such that the linear combination $x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ equals the target vector $\vec{b}$.

By inspection, one quickly sees that the combination such that $x = -2$, $y = 1$ gives $\vec{b}$. Geometrically, this looks as follows



$$\begin{bmatrix} -1 \\ -5 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

In other words, the column picture finds linear combinations of the column vectors on the left-hand side to produce the vector on the right hand side.

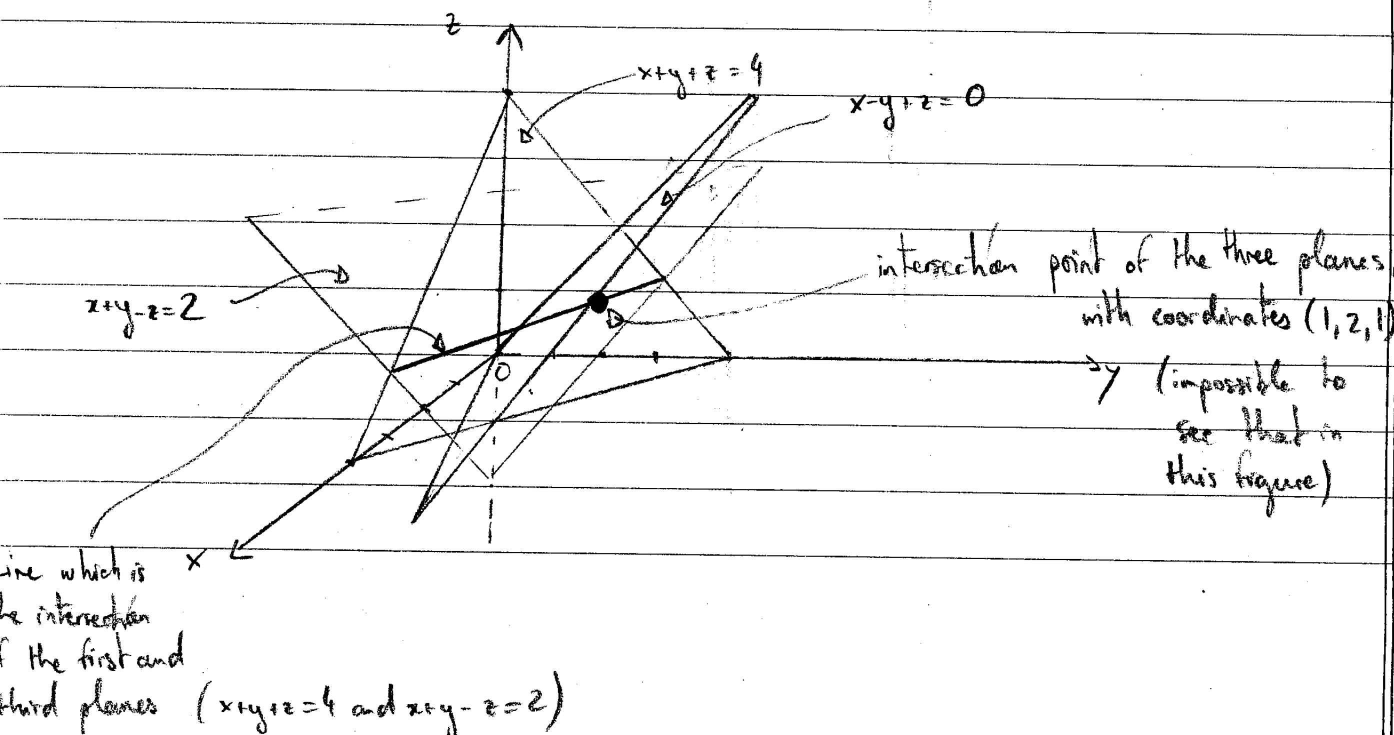
2) 3-D situation: 3 equations and 3 unknowns

* Row picture

Consider the linear system of equations

$$\begin{cases} x+y+z=4 \\ x-y+z=0 \\ x+y-z=2 \end{cases}$$

Geometrically, one can view the solution(s) as the points with coordinates (x,y,z) which satisfy the three equations at once. Each equation is the equation of a plane, so we are looking for the intersection of 3 planes:



As you can see, the row picture is quite hard to visualize here, and nearly impossible to draw properly. The take home message is that when the system admits a unique solution, the intersection of two of the planes is a line, and the third plane intersects the line in a unique point.

* Column picture

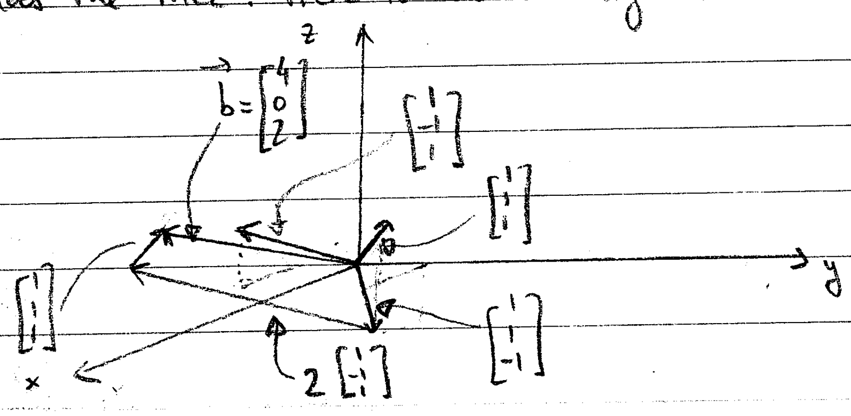
The matrix formulation of the linear system is

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}}_{\vec{b}}$$

which we can read in column form as:

$$x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

By inspection, we see that the linear combination such that $x=1, y=2, z=1$ does the trick. Here is how we may visualize this



The column picture is not so easy to draw or visualize either, but still a bit easier and more convenient than the row picture ... it is also easier to imagine for dimensions bigger than 3.

III] Solving linear systems of equations

1) Gaussian elimination

Here we present a general technique for solving a linear system of equations with as many unknowns, using the following system of 3 equations and three unknowns:

$$\begin{cases} 2x + y + 3z = 1 \\ x + 2y - z = 0 \\ x + y + 2z = 0 \end{cases}$$

The idea is to eliminate x from equations 2 and 3, and y from equation 3. This is done as follows:

① Identify the factor multiplying x in the first equation, called the first pivot. Here, it is 2.

② Identify the factor in front of x in the second equation: 1 in this case.
To eliminate x in the second equation, we multiply

the pivot equation by the number $l_{21} = \frac{1 \leftarrow \text{factor in 2nd equation}}{2 \leftarrow \text{1st pivot}}$, called the first multiplier, and subtract from the second equation, which becomes

$$\frac{3}{2}y - \frac{5}{2}z = -\frac{1}{2}$$

(3) Identify factor in front of x in the third equation, and follow the same procedure, with multiplier $l_{31} = \frac{1 \leftarrow \text{factor in 3rd equation}}{2 \leftarrow \text{1st pivot}}$. The third equation becomes:

$$\frac{1}{2}y + \frac{1}{2}z = -\frac{1}{2}$$

At this point, x is eliminated from the last two equations, which we repeat here:

$$\begin{cases} \frac{3}{2}y - \frac{5}{2}z = -\frac{1}{2} \\ \frac{1}{2}y + \frac{1}{2}z = -\frac{1}{2} \end{cases}$$

(4) Identify the second pivot, i.e the factor multiplying y in the first equation of the remaining system. Here, it is $\frac{3}{2}$

(5) Identify the factor multiplying y in second equation. Multiply the first equation by the multiplier $l_{32} = \frac{\frac{1}{2} \leftarrow \text{factor in 2nd equation}}{\frac{3}{2} \leftarrow \text{second pivot}} = \frac{1}{3}$

from 2nd equation:

$$\frac{1}{2}z + \frac{5}{6}z = -\frac{1}{2} + \frac{1}{6} \iff \frac{4}{3}z = -\frac{1}{3}$$

⑥ Now the system is reduced to

$$\begin{cases} 2x + y + 3z = 1 & \longrightarrow \quad 2x - \frac{3}{4} - \frac{3}{4} = 1 \Rightarrow x = \frac{5}{4} \\ \frac{3}{2}y - \frac{5}{2}z = -\frac{1}{2} & \longrightarrow \quad 3y + \frac{5}{4} = -1 \Rightarrow y = -\frac{3}{4} \\ \frac{4}{3}z = -\frac{1}{3} & \longrightarrow \quad z = -\frac{1}{4} \end{cases}$$

solve by direct back substitution, from top to bottom, as shown by arrows above.

2) Matrix picture of Gaussian elimination

With Gaussian elimination, we have shown that the following system

$$\underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{b}} \qquad A\vec{x} = \vec{b}$$

was equivalent to the following system.

$$\underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 0 & \frac{3}{2} & -\frac{5}{2} \\ 0 & 0 & \frac{4}{3} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{3} \end{bmatrix}}_{\vec{c}} \qquad U\vec{x} = \vec{c}$$

The matrix U only has nonzero entries above the main diagonal (diagonal included). We say that U is upper triangular

It is because U is upper triangular that back substitution is straight forward.

Note finally that the pivots of the system are the diagonal entries of U.

3) General algorithm for Gaussian elimination

Let us generalize the elimination algorithm for 4-by-4, 5-by-5, and general n-by-n matrices.

Consider an n-by-n matrix A and the linear system $A\vec{x}=\vec{b}$. Here is how to transform it to $U\vec{x}=\vec{c}$ with U upper triangular

Step 1 (Column 1): Use the first equation to create zeros below the first pivot (by multiplying with the multipliers l_{i1} and subtracting, as we have done)

Step 2 (Column 2): Use the new equation 2 to create zeros below the second pivot, with the same multiplication/elimination technique

Step 3 to n (Columns 3 to n): Keep going with the same procedure to find all n pivots and U

QUESTION: Use Gaussian elimination to solve the following system

$$\begin{cases} 2w + x - y + 2z = 2 \\ w + x + 3y + z = 1 \\ -w + 2x + 2y + 2z = 0 \\ w + x + y + 2z = 1 \end{cases}$$

4) When the method fails

As you may already know, not all linear systems of n equations for n unknowns have solutions, and for systems that have a solution, the solution may not be unique. Let us see what that means for what we just learned.

One of the keys in Gaussian elimination is the multiplier l_{ij}, constructed by dividing the multiplying factor in equation i by the j^{th} pivot.

Clearly, for l_{ij} to exist, the j^{th} pivot cannot be zero.

Zero is never allowed as a pivot for a unique solution to the system to exist.

Let us see what happens in simple 2-by-2 linear systems when the pivot is 0.

<u>Example 1</u>: Consider the system

$$\begin{cases} 2x + y = 3 \\ 4x + 2y = 2 \end{cases}$$

Elimination leads to:

$$\begin{cases} 2x + y = 3 \\ \quad 0y = -4 \end{cases}$$

We have the desired upper triangular form. However, the second pivot is 0, and the equation $0y = -4$ does not have a solution: the system does not have a solution.

Mathematically, the reason for this is clear:

$4x + 2y = 2 \iff 2x + y = 1$

The line given by the second equation is <u>parallel</u> to the line given by the first equation. Therefore, they do never intersect.

<u>Example 2</u>: Consider the system

$$\begin{cases} 2x + y = 3 \\ 4x + 2y = 6 \end{cases}$$

Elimination leads to:

$$\begin{cases} 2x + y = 3 \\ \quad 0y = 0 \end{cases}$$

Once more, the system is upper triangular, as desired. However, any y now solves the second equation. So any y can be picked, and then $x = \frac{1}{2}(3-y)$ has to be picked to satisfy the system.

The geometric interpretation here, of course, is that $2x+y=3$ and $4x+2y=6$ are the same line, and any point on this line solves the system.

<u>A cautionary tale about apparent 0 pivots</u>

The appearance of a 0 pivot can be fixed in certain situations, by exchanging the order of the equations. Consider for example the system

$$\begin{cases} 0x - 3y = 6 \\ 2x + y = 6 \end{cases}$$

This system is already in upper triangular form; all one has to do is exchange the order of the equations:

$$\begin{cases} 2x + y = 6 & \longrightarrow \quad x = 4 \\ -3y = 6 & \longrightarrow \quad y = -2 \end{cases}$$

This is called a <u>permutation</u>

The general rule is as follows:
A zero in the pivot can be repaired if there is a nonzero below it. The repair consists in flipping the order of the equations

If the problem cannot be fixed through the flipping of rows, then the upper triangular matrix U has at least one ze on its diagonal, and the system has no solutions or infinitely many.