Honors Linear Algebra – Problem Set 1 Solutions

January 31, 2018

Problem 1

1. Linear combinations of $\mathbf{u}$ and $\mathbf{v}$ do not fill the plane if $\mathbf{u}$ and $\mathbf{v}$ are colinear (parallel): there exists a scalar k such that

$$\mathbf{v} = k\mathbf{u} \Leftrightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = k \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

   If u_2 and v_2 are nonzero, the condition above is equivalent to the condition

$$\frac{v_1}{v_2} = \frac{u_1}{u_2}$$

   If u_2 and v_2 are both zero, $\mathbf{u}$ and $\mathbf{v}$ are aligned whatever the values of u_1 and v_1.

2. A simple strategy here is to select vectors $\mathbf{t}$, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{v}$ so that each vector only contributes to one of the components of $\mathbf{a}$. The simplest choice of this type is

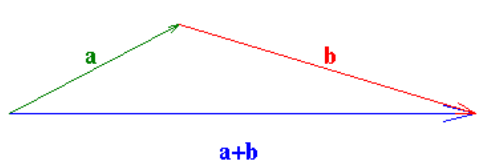
$$\mathbf{t} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

   so that any vector $\mathbf{a} = (a_1, a_2, a_3, a_4)$ is given by the linear combination

$$\mathbf{a} = a_1\mathbf{t} + a_2\mathbf{u} + a_3\mathbf{v} + a_4\mathbf{w}$$

Problem 2

1. The Triangle Inequality tells us that in a triangle, the length of the longest side of the triangle is less than or equal to the sum of the lengths of the two other sides.



2. We start by computing

$$\begin{aligned}||\mathbf{u}+\mathbf{v}||^2 =&(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})\\ =&||\mathbf{u}||^2+||\mathbf{v}||^2+2\mathbf{u}\cdot\mathbf{v}\end{aligned}$$

Now, for any scalar k, we have the property $k \leq |k|$. Using this for the scalar $\mathbf{u}\cdot\mathbf{v}$ in the previous equation, we have

$$||\mathbf{u}+\mathbf{v}||^2 \leq ||\mathbf{u}||^2+||\mathbf{v}||^2+2|\mathbf{u}\cdot\mathbf{v}|$$

At this point, the Cauchy-Schwarz inequality tells us that $|\mathbf{u}\cdot\mathbf{v}| \leq ||\mathbf{u}||||\mathbf{v}||$ so that going back to the previous equation,

$$||\mathbf{u}+\mathbf{v}||^2 \leq ||\mathbf{u}||^2+||\mathbf{v}||^2+2||\mathbf{u}||||\mathbf{v}|| = (||\mathbf{u}||+||\mathbf{v}||)^2$$

By taking the square root of both sides of the equation,

$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}||+||\mathbf{v}||$$

Problem 3

$$||\mathbf{u}-\mathbf{v}||^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = ||\mathbf{u}||^2+||\mathbf{v}||^2-2\mathbf{u}\cdot\mathbf{v} = ||\mathbf{u}||^2+||\mathbf{v}||^2-2||\mathbf{u}||||\mathbf{v}||\cos\theta$$

where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. We now replace the values $||\mathbf{u}|| = 7$ and $||\mathbf{v}|| = 4$ in the expressions above:

$$||\mathbf{u}-\mathbf{v}||^2 = 49+16-56\cos\theta = 65-56\cos\theta$$

Now, $-1 \leq \cos\theta \leq 1$, so

$$9 \leq ||\mathbf{u}-\mathbf{v}||^2 \leq 121 \;\Leftrightarrow\; 3 \leq ||\mathbf{u}-\mathbf{v}|| \leq 11$$

$$\mathbf{u}\cdot\mathbf{v} = ||\mathbf{u}||||\mathbf{v}||\cos\theta = 28\cos\theta$$

so using $-1 \leq \cos\theta \leq 1$ once more,

$$-28 \leq \mathbf{u}\cdot\mathbf{v} \leq 28$$

Problem 4

- $(1,1,1)$, $(3,-2,1)$, and $(a,2,5)$ are not independent if there exist scalars $(e,f,g) \neq (0,0,0)$ such that

$$e\begin{bmatrix}1\\1\\1\end{bmatrix}+f\begin{bmatrix}3\\-2\\1\end{bmatrix}+g\begin{bmatrix}a\\2\\5\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

This is equivalent to the following system:

$$\begin{cases} e+3f+ag=0\\ e-2f+2g=0\\ e+f+5g=0\end{cases}$$

Let us solve the system for e, f, and g, and see what the condition on a is for the system to accept a nontrivial solution. The second equation gives us $e = 2f-2g$. Plugging this into the third equation, we readily find $f = -g$. Plugging this and $e = 2f-2g = -4g$ in the first equation, we obtain

$$-4g-3g+ag=0 \;\Leftrightarrow\; (a-7)g=0$$

This solution admits two solutions: $g = 0$ or $a = 7$. If $g = 0$, then according to our intermediate steps $f = 0$, and $e = 0$. This corresponds to the trivial solution. So the condition for the three vectors to be dependent is $a = 7$.

- $(1,0,a)$, $(1,1,0)$, and $(0,1,1)$ are not independent if there exist scalars $(e,f,g) \neq (0,0,0)$ such that

$$e\begin{bmatrix}1\\0\\a\end{bmatrix}+f\begin{bmatrix}1\\1\\0\end{bmatrix}+g\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

This is equivalent to the following system:

$$\begin{cases} e+f=0\\ f+g=0\\ ea+g=0\end{cases}$$

Let us solve the system for e, f, and g, and see what the condition on a is for the system to accept a nontrivial solution. From the first two equations, we quickly get $e=-f$ and $f=-g$. Plugging the first result in the last equation, we obtain

$$ag+g=0 \;\Leftrightarrow\; (a+1)g=0$$

This solution admits two solutions: $g=0$ or $a=-1$. If $g=0$, then according to our intermediate steps $f=0$, and $e=0$. This corresponds to the trivial solution. So the condition for the three vectors to be dependent is $a=-1$.

- $(1,1,a)$, $(2,4,a)$, and $(3,5,a)$ are not independent if there exist scalars $(e,f,g) \neq (0,0,0)$ such that

$$e\begin{bmatrix}1\\1\\a\end{bmatrix}+f\begin{bmatrix}2\\4\\a\end{bmatrix}+g\begin{bmatrix}3\\5\\a\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

This is equivalent to the following system:

$$\begin{cases} e+2f+3g=0\\ e+4f+5g=0\\ a(e+f+g)=0\end{cases}$$

Let us solve the system for e, f, and g, and see what the condition on a is for the system to accept a nontrivial solution. From the first equation, $e=-2f-3g$. Using this in the second equation, we find $f=-g$. Hence, the last equation becomes

$$ga=0$$

Either $g=0$ or $a=0$. If $g=0$, then $f=0$ and $e=0$. This is the trivial solution. The condition for the dependence of the three vectors is $a=0$.

Problem 5

1. Let us assume that the system $A\mathbf{X}=\mathbf{b}$ has two solutions $\mathbf{x}=(x_1,x_2,x_3)$ and $\mathbf{y}=(y_1,y_2,y_3)$:

$$A\mathbf{x}=\mathbf{b} \qquad , \qquad A\mathbf{y}=\mathbf{b}$$

Let d be any real number. We calculate

$$A\Big(d\mathbf{x}+(1-d)\mathbf{y}\Big)=dA\mathbf{x}+(1-d)A\mathbf{y}=d\mathbf{b}+(1-d)\mathbf{b}=\mathbf{b}$$

We just proved that for any d, $d\mathbf{x}+(1-d)\mathbf{y}$ is also a solution of the system $A\mathbf{X}=\mathbf{b}$. In other words, the entire line with direction $\mathbf{y}-\mathbf{x}$ is a solution to the system. In particular, $1/2\mathbf{x}+1/2\mathbf{y}$ is a solution of the system.

2. If the 25 planes meet at two points, their intersection must be the whole line connecting these two points. Indeed, if two planes have two points in common, their intersection is the line connecting these two points. Looking at the 25 planes two by two using that argument, they must have the whole line connecting the two points in common.