Honors Linear Algebra – Problem Set 7 Solutions

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Problem 1

We start by decomposing $\mathbf{a}$ as $\mathbf{a} = \mathbf{p} + \mathbf{e}$, where $\mathbf{p}$ is the projection of $\mathbf{a}$ on the subspace spanned by the orthonormal basis $\mathbf{q}_1$, ..., $\mathbf{q}_n$: $\mathbf{p} = QQ^T\mathbf{a}$. By construction, $\mathbf{e} = \mathbf{a} - \mathbf{p} = (I - QQ^T)\mathbf{a}$ is orthogonal to the subset spanned by $\mathbf{q}_1$, ..., $\mathbf{q}_n$, and thus orthogonal to each of them. The desired orthonormal basis $(\mathbf{q}_1, \ldots, \mathbf{q}_n, \mathbf{q}_{n+1})$ is then obtained by setting

$$\mathbf{q}_{n+1} = \frac{\mathbf{e}}{||\mathbf{e}||} = \frac{\mathbf{a} - QQ^T\mathbf{a}}{||\mathbf{a} - QQ^T\mathbf{a}||}$$

Problem 2

1. Both matrices are upper triangular, so their determinants are simply the product of the diagonal entries:

$$\det A = ad \qquad \det B = 72$$

   We saw that the determinant of the inverse of a matrix is the inverse of the determinant of the matrix, so

$$\det(A^{-1}) = \frac{1}{ad} \qquad \det(B^{-1}) = \frac{1}{72}$$

   We also saw that $\det(AB) = \det A \det B$, so

$$\det(A^2) = (\det A)^2 = (ad)^2 \qquad \det(B^2) = (\det B)^2 = 72^2 = 5184$$

2. If $a = b$, then the determinant is zero because the first two rows are equal. The formula also gives zero in that case.

   Now, let us assume $a \neq b$. Using three steps of Gaussian elimination, we can write

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2 - a^2 \\ 0 & 0 & c^2 - a^2 - \frac{(b^2-a^2)(c-a)}{b-a} \end{vmatrix} = (b-a)(c^2-a^2) - (b^2-a^2)(c-a)$$

   The determinant therefore is

$$(b-a)(c^2-a^2) - (b^2-a^2)(c-a) = (b-a)(c-a)\left[(c+a) - (b+a)\right] = (b-a)(c-a)(c-b)$$

Problem 3

Let us adopt the notation

$$\det H = \det(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$$

where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are the columns of H. Now, for each column $\mathbf{x}_j$, we write

$$\mathbf{x}_j = \mathbf{E}_j + a\mathbf{C}$$

where $\mathbf{E}_j$ is the vector in $\mathbf{R}^n$ with entry λ_j in the j^{th} row, and entry 0 in all other rows, and $\mathbf{C}$ is the vector in $\mathbf{R}^n$ whose entries are all 1.

We can thus write

$$\det H = \det(\mathbf{E}_1 + a\mathbf{C}, \mathbf{E}_2 + a\mathbf{C}, \ldots, \mathbf{E}_n + a\mathbf{C})$$

Now, we know that determinants are linear with respect to each column, so we write

$$\begin{aligned}
&\det(\mathbf{E}_1 + a\mathbf{C}, \mathbf{E}_2 + a\mathbf{C}, \ldots, \mathbf{E}_n + a\mathbf{C}) = \det(\mathbf{E}_1, \mathbf{E}_2 + a\mathbf{C}, \ldots, \mathbf{E}_n + a\mathbf{C}) + a\det(\mathbf{C}, \mathbf{E}_2 + a\mathbf{C}, \ldots, \mathbf{E}_n + \mathbf{C})\\
&= \det(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3 + a\mathbf{C} \ldots, \mathbf{E}_n + a\mathbf{C}) + a\det(\mathbf{E}_1, \mathbf{C}, \mathbf{E}_3 + a\mathbf{C}, \ldots, \mathbf{E}_n + a\mathbf{C}) + a\det(\mathbf{C}, \mathbf{E}_2, \mathbf{E}_3 + a\mathbf{C}, \ldots, \mathbf{E}_n + a\mathbf{C})
\end{aligned}$$

where we used the fact that $\det(\mathbf{C}, \mathbf{C}, \mathbf{E}_3 + a\mathbf{C}, \ldots, \mathbf{E}_n + a\mathbf{C}) = 0$ because two columns are identical.

Continuing the expansion using the linearity with respect to the remaining columns, we find

$$\det H = \det(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \ldots, \mathbf{E}_n) + a\det(\mathbf{C}, \mathbf{E}_2, \mathbf{E}_3, \ldots, \mathbf{E}_n) + a\det(\mathbf{E}_1, \mathbf{C}, \mathbf{E}_3, \ldots, \mathbf{E}_n) + \ldots + a\det(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \ldots, \mathbf{C})$$

Now, the first determinant is just $\prod_{j=1}^{n} \lambda_j$ since the matrix is diagonal. The second determinant is $\prod_{j=2}^{n} \lambda_j$, using the fact that the matrix is lower triangular. The last determinant is $\prod_{j=1}^{n-1} \lambda_j$, using the fact that the matrix is upper triangular. For the other determinants, one uses a cofactor expansion along the row whose only nonzero entry is 1 in the (k,k) position, to find that the result is $\prod_{j=1, j\neq k}^{n} \lambda_j$. Thus, the desired result is

$$\det H = \prod_{j=1}^{n} \lambda_j + a \sum_{k=1}^{n} \prod_{j=1, j\neq k}^{n} \lambda_j$$

Problem 4

1. Consider the matrix

$$A = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix}$$

where a, b, c, and d are real numbers.

$$\begin{aligned}
A^T A &= \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix} \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} \\
&= \begin{bmatrix} a^2+b^2+c^2+d^2 & 0 & 0 & 0 \\ 0 & a^2+b^2+c^2+d^2 & 0 & 0 \\ 0 & 0 & a^2+b^2+c^2+d^2 & 0 \\ 0 & 0 & 0 & a^2+b^2+c^2+d^2 \end{bmatrix}
\end{aligned}$$

Hence, $(a^2+b^2+c^2+d^2)^4 = \det(A^T A) = \det(A)^2$, from which we conclude that

$$\det A = \pm(a^2+b^2+c^2+d^2)^2$$

Now, if we were to expand the determinant of A, we would be able to write

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix} + \ldots \text{(many terms)}$$

Hence, the term which is quartic in a is $+a^4$, from which we conclude that

$$\det A = (a^2+b^2+c^2+d^2)^2$$

2. Let $(a,b,c,d,a',b',c',d') \in \mathbb{Z}^8$. We define

$$A = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} \quad , \quad A' = \begin{bmatrix} a' & b' & c' & d' \\ -b' & a' & -d' & c' \\ -c' & d' & a' & -b' \\ -d' & -c' & b' & a' \end{bmatrix}$$

and

$$A'' = AA' = \begin{bmatrix} a'' & b'' & c'' & d'' \\ -b'' & a'' & -d'' & c'' \\ -c'' & d'' & a'' & -b'' \\ -d'' & -c'' & b'' & a'' \end{bmatrix}$$

with

$$\begin{cases} a'' = aa' - bb' - bb' - dd' \\ b'' = ab' + a'b + cd' - c'd \\ c'' = ac' - bd' + a'c + db' \\ d'' = ad' + bc' - cb' + a'd \end{cases}$$

We have

$$(a''^2 + b''^2 + c''^2 + d''^2)^4 = \det(A'') = \det(AA') = \det(A)\det(A') = (a^2+b^2+c^2+d^2)^4(a'^2+b'^2+c'^2+d'^2)^4$$
$$\Rightarrow \; a''^2 + b''^2 + c''^2 + d''^2 = (a^2+b^2+c^2+d^2)(a'^2+b'^2+c'^2+d'^2)$$

and from the definition of a'', b'', c'', and d'' above, it is clear that these 4 numbers are integers when a, b, c, d, a', b', c', and d' are, which completes our proof.

Problem 5

1. Using direct formulae for C_1 and C_2, and cofactor expansions along the first column for C_3 and the first column and then the first row for C_4, we readily find:

$$C_1 = 0 \;\; , \;\; C_2 = -1 \;\; , \;\; C_3 = 0 \;\; , \;\; C_4 = -C_2 = 1$$

More generally, for any C_n,

$$C_n = \begin{vmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 1 & 0 & 1 \\ 0 & \dots & \dots & 0 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 1 & 0 & 1 \\ 0 & \dots & \dots & 0 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 1 & 0 & 1 \\ 0 & \dots & \dots & 0 & 1 & 0 \end{vmatrix} = -C_{n-2}$$

where we have used a cofactor expansion along the first column for the first equality, leading to an $(n-1)\times(n-1)$ determinant, and then a cofactor expansion along the first row for the second equality, leading to an $(n-2)\times(n-2)$ determinant equal to C_{n-2}.

We thus have the general relationship $C_n = -C_{n-2}$. In particular, $C_{14} = -C_{12} = C_{10} = -C_8 = C_6 = -C_4 = -1$.

2. $D_1 = 3$, $D_2 = 3\times 3 - 1\times 1 = 8$, and using a cofactor expansion, we have $D_3 = 3D_2 - 1\times 3 = 21$.

The Fibonacci sequence are the numbers in the following integer sequence: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$. D_1 is the 4th number in the sequence, D_2 the 6th number, and D_3 the 8th number. We therefore expect D_4 to be the 10th number in the sequence, i.e. 55. Let us verify this by direct calculation:

$$D_4 = \begin{vmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{vmatrix} = 3D_3 - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} = 3D_3 - D_2 = 63 - 8 = 55$$