Honors Linear Algebra – Midterm 1 Solutions

Wednesday, February 28 2018

Multiple choice

1. The triangle inequality says that $|\mathbf{u}+\mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$, as we have seen in Homework 1 and can easily be visualized geometrically: in a triangle, the length of the longest side of the triangle is less than or equal to the sum of the lengths of the two other sides. Hence, **answer D** is the expression that is not always true.

2. The last row of the matrix in **answer C** subtracts 4 times Row 1 from Row 3, and the first two rows of that matrix keep Row 1 and Row 2 unchanged. The correct answer is **answer C**.

3. The 4×3 matrix with all zeroes does not have rank 1, so the correct answer is **answer E**.

4. The augmented matrix is
$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 4 & 6 & 4 & 2 \\ -1 & 0 & -1 & c \end{bmatrix}$$
Replacing Row 2 with Row 2 - 4 Row 1 leads to
$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ -1 & 0 & -1 & c \end{bmatrix}$$
Replacing Row 3 with Row 3 + Row 1 leads to
$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 2 & 0 & 1+c \end{bmatrix}$$
Replacing Row 3 with Row 3 + Row 2 leads to
$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & c-1 \end{bmatrix}$$
The last row tells us that the system has a solution if $c - 1 = 0$. The correct answer is **answer D**.

5. If A has three pivots and 4 columns, there is one free column, so the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. The correct answer is **answer B**.

True or False

1. $$\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$
so the statement is **False**.

2. The columns of the associated matrix are all colinear with $(2, -1, 3)$. The right-hand side, $(9, 6, 12)$ is not. So the statement is **False**.

3. Let A be an invertible symmetric matrix: $A^T = A$. Taking the inverse on both sides of this equality, we have

$$(A^T)^{-1} = A^{-1} \Leftrightarrow (A^{-1})^T = A^{-1}$$

so A^{-1} is symmetric, and the statement is **True**.

4. Let A and B be two symmetric matrices: $A^T = A$ and $B^T = B$.

$$(A+B)^T = A^T + B^T = A + B$$

The statement is **True**.

5. Let A and B be two symmetric matrices: $A^T = A$ and $B^T = B$.

$$(AB)^T = B^T A^T = BA \neq AB \qquad \text{(in general)}$$

The statement is **False**.

Problem 1

1. The idea is to find a matrix A which is not invertible, and then choose a matrix B such that the columns of B are in the nullspace of A. We may for example take

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad BA = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

2. We know that the row space of a matrix and its null space are orthogonal subspaces. Hence, any vector in the row space must be orthogonal to any vector in the nullspace. $[1\ 2\ 1]^T$ and $[1\ -2\ \ 1]^T$ are not orthogonal, so it is not possible to find such a matrix.

3. Let A be a monotone matrix, and let us assume it is not invertible. Then there exists $\mathbf{x}_0 \neq \mathbf{0}$ such that $A\mathbf{x}_0 = 0$. In that case, we also have $A(-\mathbf{x}_0) = 0$. But either $\mathbf{x}_0$ or $-\mathbf{x}_0$ must have negative components, so we have a contradiction: the matrix A would not be monotone. We conclude that A must be invertible $\square$

Problem 2

Let $\mathbf{x} = (x_1, x_2, x_3, x_4)$. The first and third rows of the system are

$$\begin{aligned} x_1 + 2x_2 - x_4 &= 10 \\ 2x_1 + 4x_2 - 2x_4 = 24 \Leftrightarrow x_1 + 2x_2 - x_4 &= 12 \end{aligned}$$

We can see that the two equations cannot both be satisfied at the same time. So the linear system does not have a solution.

Note that we could also have seen this by proceeding in the normal way. The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 10 \\ -2 & -3 & 4 & 5 & -13 \\ 2 & 4 & 0 & -2 & 24 \end{bmatrix}$$

Replacing Row 2 with Row 2 + 2Row 1 leads to the augmented matrix

$$\left[\begin{array}{ccccc} 1 & 2 & 0 & -1 & 10 \\ 0 & 1 & 4 & 3 & 7 \\ 2 & 4 & 0 & -2 & 24 \end{array}\right]$$

Replacing Row 3 with Row 3 - 2Row 1 leads to the augmented matrix

$$\left[\begin{array}{ccccc} 1 & 2 & 0 & -1 & 10 \\ 0 & 1 & 4 & 3 & 7 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right]$$

We can see that the third row leads to an incompatible condition.

Problem 3

1. We first use operations on the rows and columns of M to write

$$\operatorname{rank}(M) = \operatorname{rank}\begin{pmatrix} A & A \\ A & B \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A & A \\ 0 & B-A \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A & 0 \\ 0 & B-A \end{pmatrix}$$

   Now, consider the matrix on the right-hand side. We see that Gaussian elimination for the first n columns will lead to as many pivot columns as A has pivot columns. Likewise, Gaussian elimination for the last n columns will lead to as many pivot columns as B has pivot columns. Hence, the rank of M is $\operatorname{rank}(A) + \operatorname{rank}(B - A)$.

2. Since $\operatorname{rank}(A) \leq n$ and $\operatorname{rank}(B - A) \leq n$, M is invertible if and only if $\operatorname{rank}(A) = \operatorname{rank}(B - A) = n$, i.e. iff A and $B - A$ are invertible.

3. To compute M^{-1}, we first note that the operations on the rows and columns in the first question correspond to

$$\begin{pmatrix} I_n & 0 \\ -I_n & I_n \end{pmatrix}\begin{pmatrix} A & A \\ A & B \end{pmatrix}\begin{pmatrix} I_n & -I_n \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B-A \end{pmatrix}$$

   Now,

$$\begin{pmatrix} I_n & 0 \\ -I_n & I_n \end{pmatrix}^{-1} = \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix}$$

   and

$$\begin{pmatrix} I_n & -I_n \\ 0 & I_n \end{pmatrix}^{-1} = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$$

   so we finally obtain

$$M^{-1} = \begin{pmatrix} I_n & -I_n \\ 0 & I_n \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & (B-A)^{-1} \end{pmatrix}\begin{pmatrix} I_n & 0 \\ -I_n & I_n \end{pmatrix} = \begin{pmatrix} A^{-1} + (B-A)^{-1} & -(B-A)^{-1} \\ -(B-A)^{-1} & (B-A)^{-1} \end{pmatrix}$$

Problem 4

The vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$ in $\mathbb{R}^4$ which are perpendicular to both $\mathbf{u} = (2, 2, 3, 7)$ and $\mathbf{v} = (4, 4, 5, 12)$ satisfy simultaneously $\mathbf{u} \cdot \mathbf{x} = 0$ and $\mathbf{v} \cdot \mathbf{x} = 0$:

$$\begin{cases} 2x_1 + 2x_2 + 3x_3 + 7x_4 = 0 \\ 4x_1 + 4x_2 + 5x_3 + 12x_4 = 0 \end{cases}$$

We compute the row reduced echelon form of

$$\begin{bmatrix} 2 & 2 & 3 & 7 \\ 4 & 4 & 5 & 12 \end{bmatrix}$$

Replacing Row 2 - 2 Row 1 leads to

$$\begin{bmatrix} 2 & 2 & 3 & 7 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

We divide the two pivot rows by the pivots:

$$\begin{bmatrix} 1 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Replacing Row 1 with Row 1 - 3/2 Row 2 then leads to

$$R = \begin{bmatrix} 1 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The second and the fourth columns are free columns. Setting $x_2 = 1$ and $x_4 = 0$, we can read the first special solution from the second column of R: $\mathbf{s}_1 = (-1, 1, 0, 0)$. Setting $x_2 = 0$ and $x_4 = 1$, we can read the second special solution from the fourth column of R: $\mathbf{s}_2 = (-\frac{1}{2}, 0, -2, 1)$.

We conclude that the vectors in $\mathbf{R}^4$ that are perpendicular to both $(2, 2, 3, 7)$ and $(4, 4, 5, 12)$ have the general form:

$$\mathbf{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -2 \\ 1 \end{bmatrix}, \qquad x_2 \in \mathbb{R},\ x_4 \in \mathbb{R}$$

Problem 5

(A) Replacing Row 2 with Row 2 + Row 1 (multiplier $l_{21} = -1$), we find

$$\begin{bmatrix} 1 & 3 & 7 & 1 \\ 0 & 2 & 5 & 2 \\ 2 & 0 & 0 & -5 \\ 1 & 1 & 4 & -1 \end{bmatrix}$$

Replacing Row 3 with Row 3 - 2Row 1 (multiplier $l_{31} = 2$), we find

$$\begin{bmatrix} 1 & 3 & 7 & 1 \\ 0 & 2 & 5 & 2 \\ 0 & -6 & -14 & -7 \\ 1 & 1 & 4 & -1 \end{bmatrix}$$

Replacing Row 4 with Row 4 - Row 1 (multiplier $l_{41} = 1$), we find

$$\begin{bmatrix} 1 & 3 & 7 & 1 \\ 0 & 2 & 5 & 2 \\ 0 & -6 & -14 & -7 \\ 0 & -2 & -3 & -2 \end{bmatrix}$$

Replacing Row 3 with Row 3 + 3 Row 2 (multiplier $l_{32} = -3$), we find

$$\begin{bmatrix} 1 & 3 & 7 & 1 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & -2 & -3 & -2 \end{bmatrix}$$

Replacing Row 4 with Row 4 + Row 2 (multiplier $l_{42} = -1$), we find

$$\begin{bmatrix} 1 & 3 & 7 & 1 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Replacing Row 4 with Row 4 -2 Row 3 (multiplier $l_{43} = 2$), we find

$$U = \begin{bmatrix} 1 & 3 & 7 & 1 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The matrix L is easily constructed: the multipliers l_{ij} go in the proper ij lower diagonal entry; the diagonal is filled with 1's, and the entries above are all filled with zeroes.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 1 & -1 & 2 & 1 \end{bmatrix}$$

To solve the matrix equations, we first solve $L\mathbf{c} = \mathbf{b}$ where $\mathbf{b}$ is the right-hand side of each matrix equation, and then $U\mathbf{w} = \mathbf{c}$, where $\mathbf{w}$ is the unknown vector ($\mathbf{x}$, $\mathbf{y}$, or $\mathbf{z}$).

- $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 1 & -1 & 2 & 1 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$
  $$\begin{bmatrix} 1 & 3 & 7 & 1 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} -\frac{5}{4} \\ -\frac{1}{4} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$
- $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 1 & -1 & 2 & 1 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \Rightarrow \mathbf{c} = \begin{bmatrix} 2 \\ 4 \\ 10 \\ -16 \end{bmatrix}$$
  $$\begin{bmatrix} 1 & 3 & 7 & 1 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \mathbf{y} = \begin{bmatrix} 2 \\ 4 \\ 10 \\ -16 \end{bmatrix} \Rightarrow \mathbf{y} = \begin{bmatrix} -19 \\ 5 \\ 2 \\ -8 \end{bmatrix}$$
- $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 1 & -1 & 2 & 1 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$
  $$\begin{bmatrix} 1 & 3 & 7 & 1 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \mathbf{z} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \mathbf{z} = \begin{bmatrix} \frac{7}{4} \\ \frac{3}{4} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$