Technical University of Denmark

Written exam, August 13, 2020.

Course name: Graph Theory.

Course number: 01227.

Aids allowed: All materials allowed by DTU including internet access.

Exam duration: 4 hours

Weighting: Question 1: 30% - Question 2: 24% - Question 3: 22% - Question 4: 12% - Question 5: 12%.

The weighting is only an approximative weighting. Your answers will be judged as a whole.

It is important that you justify your answers. An answer with no justification gives no credit.

You can answer the exam in either Danish or English or a mixture.

Question 1 Consider the transport network in Figure 1 where every edge has a capacity and an orientation.

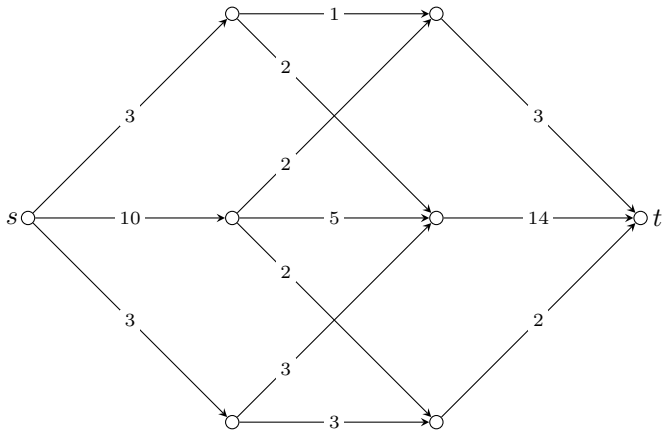

Figure 1: Transport network

(a) Find a maximal flow from s to t, and find a minimal cut. Find the value of the flow and the capacity of the cut.

(b) Find the critical edges. How many are there? Find the edges of optimum capacity. How many are there?

(c) How many integer maximal flows are there? How many minimal cuts are there?

(d) Which edges have the following property: If the capacity of the edge is decreased by 1, then the value of the maximum flow is decreased by 1?

(e) Which edges have the following property: If the capacity of the edge is increased by 1, then the value of the maximum flow is increased by 1?

(f) Which edges have the following property: If the capacity of the edge is increased by 2, then the value of the maximum flow is increased by 2?

Question 2 Consider a school with 5 teachers and 10 classes. Figure 2 shows how many hours each teacher must teach each class during a week. We wish to minimize the number of hours we need to use the school. We assume that the number of classrooms is unlimited.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
t_1	3	3	0	3	3	3	3	3	3	3
t_2	3	3	0	0	3	0	3	0	3	0
t_3	3	3	0	3	3	0	3	3	0	3
t_4	3	3	0	3	0	3	0	3	3	3
t_5	4	3	3	3	0	3	3	3	3	0

Figure 2: Teaching plan

(a) How many hours must the school be used during a week? What is the number of classrooms needed for this?

(b) Now we add the additional condition that classes c_1 and c_2 are never allowed to be taught at the same time. How many hours must the school be used during a week? What is the number of classrooms needed for this?

(You need not show the schedule. But, you must justify your answer by a graph argument.)

Question 3 The numbers in Figure 3 are edge-weights.

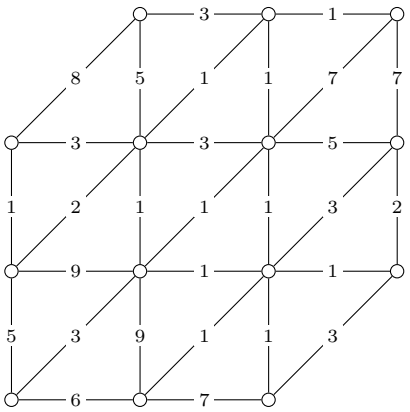

Figure 3: A graph with edge-weights

(a) Among all spanning trees containing all horizontal edges, find one of maximum weight. What is the weight of such a tree? How many such trees are there?

(b) Among all spanning trees containing all horizontal edges, find one of minimum weight. What is the weight of such a tree? How many such trees are there?

Question 4 A random walk starts in s. Find the probability that the walk gets to a before it gets to b.

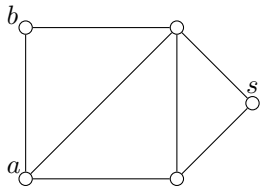

Figure 4: A random walk

Question 5 The graph in Figure 4 is now an electrical network where the edge ab has resistance x Ohm, and the two edges incident with s have resistance y Ohm (where x and y are positive real numbers) and all other edges have resistance 1 Ohm.

Find the network determinant.