

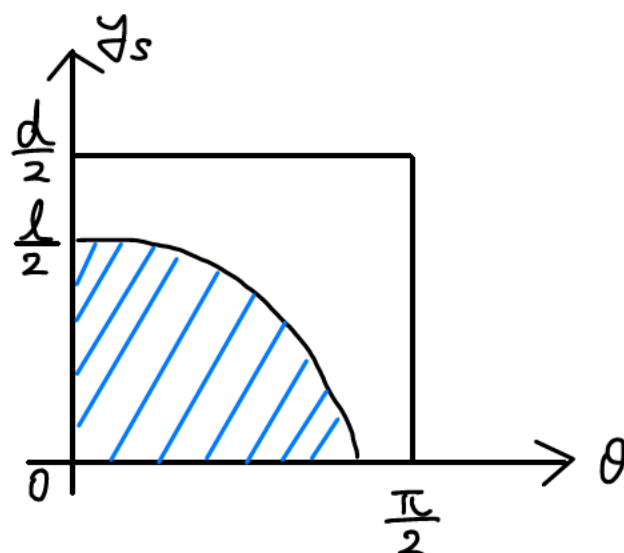
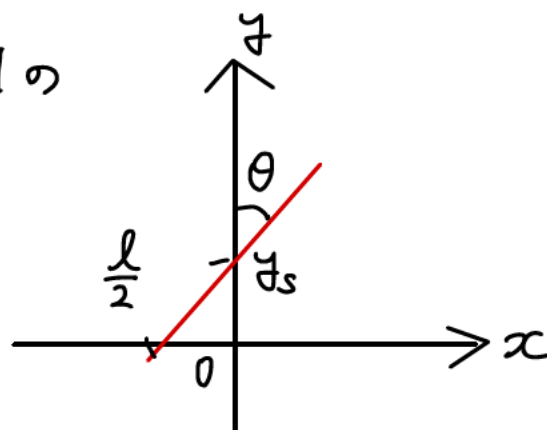
問1

- (1) (y_s, θ) は $y_s: [0, \frac{d}{2}]$, $\theta: [0, \frac{\pi}{2}]$ の範囲でランダム.

x 軸と交わるには、右上図から

$$y_s \leq \frac{l}{2} \cos \theta \quad \text{であればよい.}$$

(右下図青領域)



ランダムに点を1つ取る

\Leftrightarrow 右下図の長方形内の点を取る.

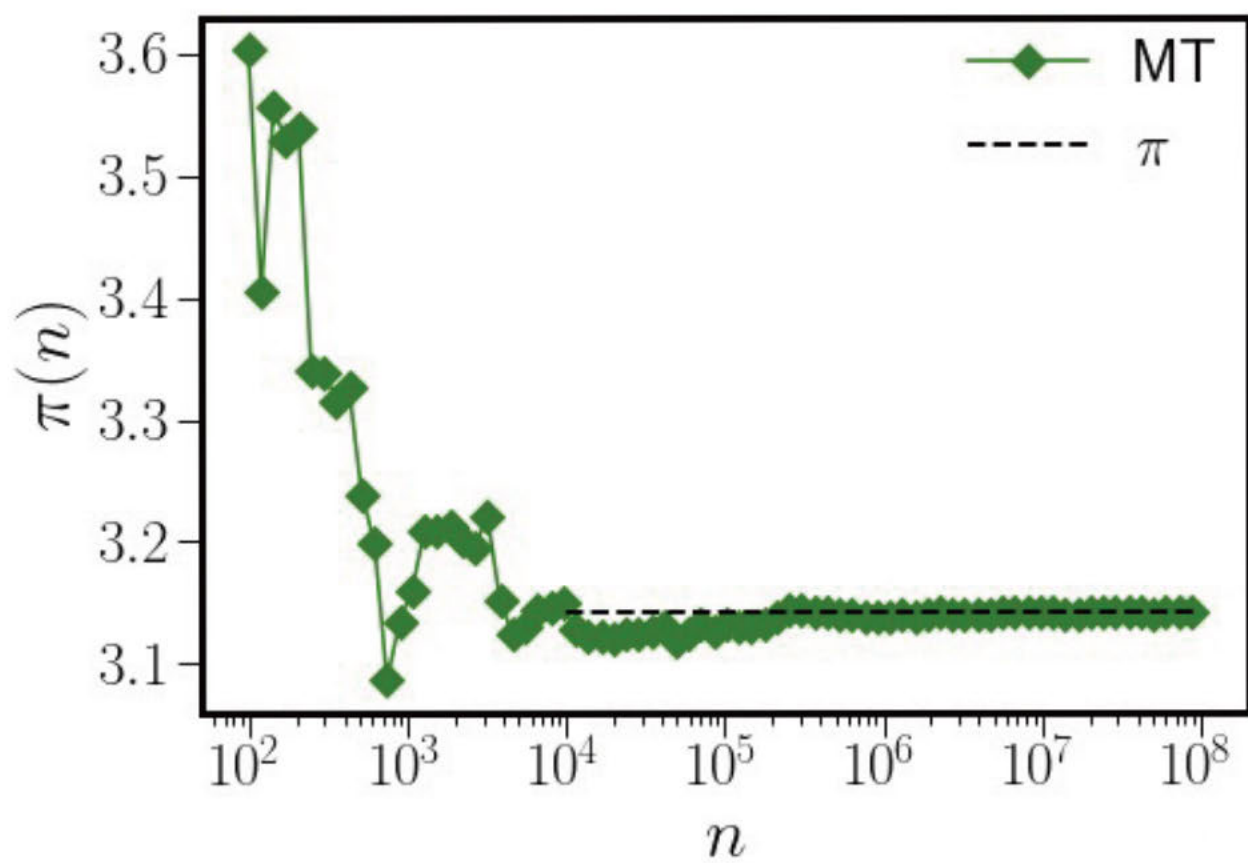
なので,

ランダムに取る全事象は $\frac{\pi d}{4}$

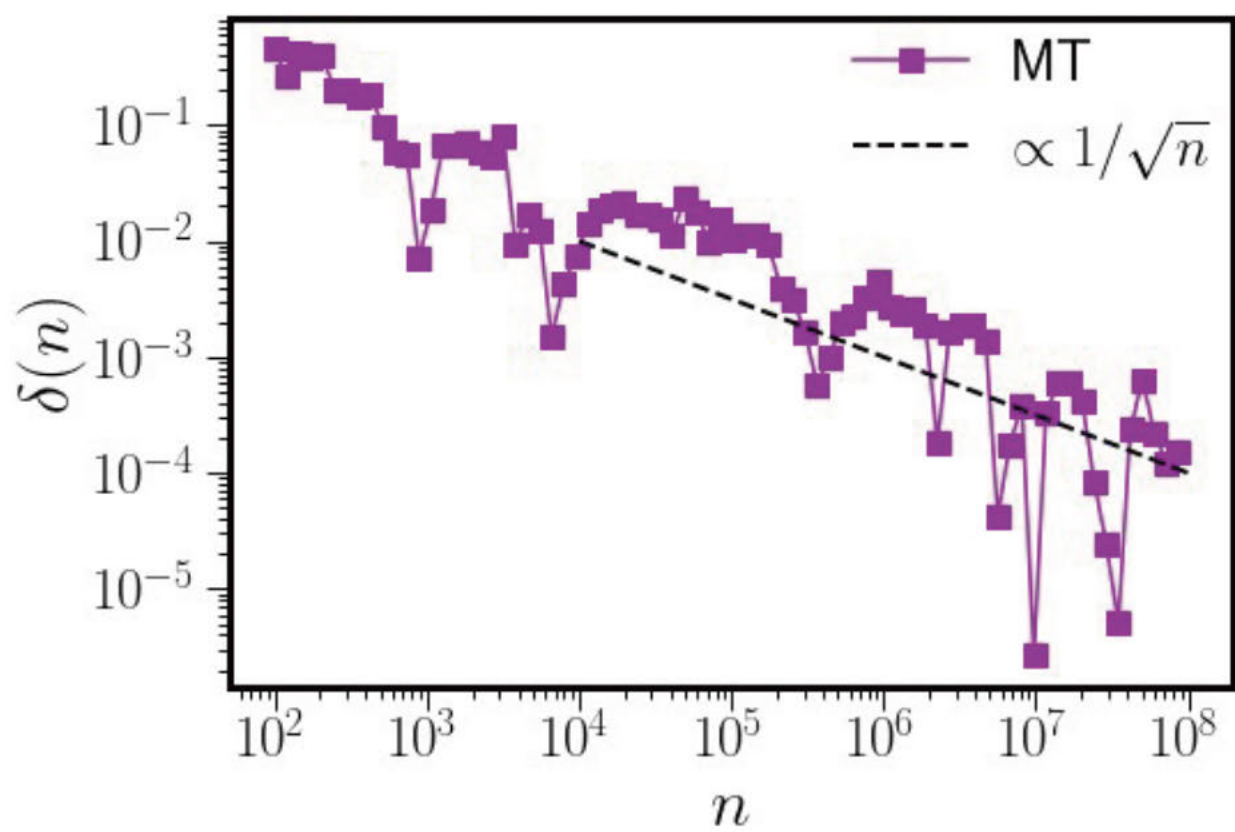
右下図青領域の面積は $\int_0^{\frac{\pi}{2}} \frac{l}{2} \cos \theta d\theta = \frac{l}{2}$ なので

求める確率は $\frac{\frac{l}{2}}{\frac{\pi d}{4}} = \frac{2l}{\pi d}$

(2)



(3)



問2

$$m\ddot{x}(t) = -\zeta\dot{x}(t) - kx(t)$$

(1) $x(t) = e^{\lambda t}$ と仮定して代入すると、 $m\lambda^2 + \zeta\lambda + k = 0 \dots \textcircled{1}$

この二次方程式の判別式 $D = \zeta^2 - 4mk$ の値で運動が変わる.

(i) 減衰振動

$$D = \zeta^2 - 4mk < 0, \text{ i.e. } \zeta^2 < 4mk \text{ のとき}$$

(ii) 過減衰

$$D = \zeta^2 - 4mk > 0, \text{ i.e. } \zeta^2 > 4mk \text{ のとき}$$

(iii) 臨界減衰

$$D = \zeta^2 - 4mk = 0, \text{ i.e. } \zeta^2 = 4mk \text{ のとき}$$

(2)

$$x = a\tilde{x}, t = t_0\tilde{t}, \dot{x} = v = \left(\frac{a}{t_0}\right)\tilde{v} \text{ とすると、}$$

$$\frac{d}{dt} v(t) = \frac{v(t+\Delta t) - v(t)}{\Delta t} \text{ より 離散化して}$$

$$v(t+\Delta t) = v(t) - \frac{\zeta}{m} v(t)\Delta t - \frac{k}{m} x(t)\Delta t$$

無次元化して

$$\frac{a}{t_0} \tilde{v}(\tilde{t} + \Delta\tilde{t}) = \frac{a}{t_0} \left(1 - \frac{\zeta}{m} t_0 \Delta\tilde{t} \right) \tilde{v}(\tilde{t}) - \frac{k a t_0 \Delta\tilde{t}}{m} \tilde{x}(\tilde{t})$$

$$\underline{\tilde{v}(\tilde{t} + \Delta\tilde{t}) = \left(1 - \frac{\zeta}{m} t_0 \Delta\tilde{t} \right) \tilde{v}(\tilde{t}) - \frac{k}{m} t_0^2 \Delta\tilde{t} \tilde{x}(\tilde{t}) \dots \textcircled{2}}$$

また、位置については

$$\tilde{x}(\tilde{t} + \Delta\tilde{t}) = \tilde{x}(\tilde{t}) + \tilde{v}(\tilde{t} + \Delta\tilde{t})\Delta\tilde{t} \quad \text{から}$$

$$\begin{aligned}\tilde{x}(\tilde{t} + \Delta\tilde{t}) &= \tilde{x}(\tilde{t}) + \left(1 - \frac{\gamma}{m} t_0 \Delta\tilde{t}\right) \tilde{v}(\tilde{t}) \Delta\tilde{t} - \frac{k}{m} t_0^2 \Delta\tilde{t}^2 \tilde{x}(\tilde{t}) \\ &= \left(1 - \frac{k}{m} t_0^2 \Delta\tilde{t}^2\right) \tilde{x}(\tilde{t}) + \left(1 - \frac{\gamma}{m} t_0 \Delta\tilde{t}\right) \tilde{v}(\tilde{t}) \Delta\tilde{t}\end{aligned}$$

$$(3) \quad t_d = \frac{m}{\gamma}, \quad t_s = \sqrt{\frac{m}{k}} \quad \text{より,}$$

$$\tilde{v}(\tilde{t} + \Delta\tilde{t}) = \left(1 - \frac{t_0}{t_d} \Delta\tilde{t}\right) \tilde{v}(\tilde{t}) - \left(\frac{t_0}{t_s}\right)^2 \Delta\tilde{t} \tilde{x}(\tilde{t})$$

$$(4) \quad t_s = t_0 \quad \text{とすると}$$

$$\tilde{v}(\tilde{t} + \Delta\tilde{t}) = \left(1 - \frac{t_0}{t_d} \Delta\tilde{t}\right) \tilde{v}(\tilde{t}) - \Delta\tilde{t} \tilde{x}(\tilde{t})$$

$$\text{また、初期条件は } x(0) = 0, \quad \dot{x}(0) = \frac{a}{t_0}$$

$$\begin{aligned}\tilde{x}(\tilde{t} + \Delta\tilde{t}) &= \left(1 - \frac{k}{m} t_0^2 \Delta\tilde{t}^2\right) \tilde{x}(\tilde{t}) + \left(1 - \frac{\gamma}{m} t_0 \Delta\tilde{t}\right) \tilde{v}(\tilde{t}) \Delta\tilde{t} \\ &= \left(1 - \left(\frac{t_0}{t_s}\right)^2 \Delta\tilde{t}^2\right) \tilde{x}(\tilde{t}) + \left(1 - \frac{t_0}{t_d} \Delta\tilde{t}\right) \tilde{v}(\tilde{t}) \Delta\tilde{t} \\ &= \left(1 - \Delta\tilde{t}^2\right) \tilde{x}(\tilde{t}) + \left(1 - \frac{t_0}{t_d} \Delta\tilde{t}\right) \tilde{v}(\tilde{t}) \Delta\tilde{t}\end{aligned}$$

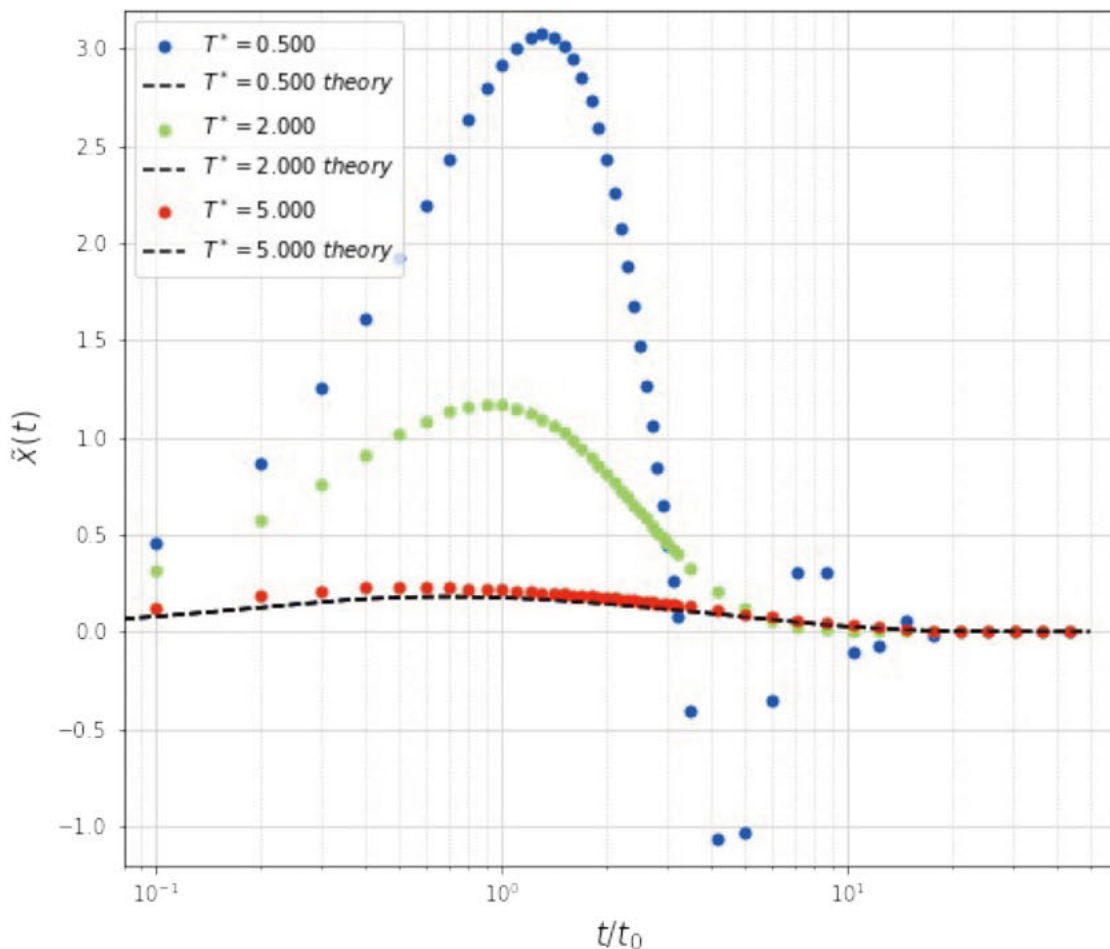
これから $T^* = \frac{t_0}{t_d}$ がパラメータ. 運動が変わる境目を調べる.

(1)より $D = \zeta^2 - 4m$ の正負だったので、

$$\begin{aligned} (3) \text{より } D &= \frac{m^2}{t_d^2} - 4m \frac{m}{t_s^2} = m^2 \left(\frac{1}{t_d^2} + \frac{2}{t_s} \right) \left(\frac{1}{t_d} - \frac{2}{t_s} \right) \\ &= m^2 \left(\frac{1}{t_d} + \frac{2}{t_0} \right) \left(\frac{1}{t_d} - \frac{2}{t_0} \right) \end{aligned}$$

この正負は **青枠** で決まる.

i.e. $t_0 - 2t_d$ の正負 $\Leftrightarrow \frac{t_0}{t_d} - 2$ の正負 ($\because t > 0$)
 $\underbrace{\quad}_{T^*}$



減衰振動と臨界減衰の理論推定値もplotする予定でしたが
計算ミスで上手く一致しませんでした. 今後の課題とします.

以下、理論解.

$$\ddot{x}(t) + \frac{1}{t_d} \dot{x}(t) + \frac{1}{t_s^2} x(t) = 0$$

$x(t) = e^{\lambda t}$ と仮定すれば

$$\lambda^2 + \frac{1}{t_d} \lambda + \frac{1}{t_s^2} = 0 \quad \therefore \lambda_{\pm} = -\frac{1}{2t_d} \pm \frac{\sqrt{t_s^2 - 4t_d^2}}{2t_d t_s}$$

以降、 $T^* = \frac{t_s}{t_d}$, $t_s = t_0$ などとする.

(i) $t_s - 2t_d < 0$ i.e. $T^* < 2$ のとき

$$\lambda'_{\pm} = -\frac{1}{2t_d} \pm i \frac{1}{2t_0} \sqrt{4 - T^{*2}}$$

$$x(t) = A e^{-\frac{1}{2t_d}t} e^{+i \frac{1}{2t_0} \sqrt{4 - T^{*2}}t} + B e^{-\frac{1}{2t_d}t} e^{-i \frac{1}{2t_0} \sqrt{4 - T^{*2}}t}$$

初期条件 $x(0) = A + B = 0$ より

$$\begin{aligned} x(t) &= A e^{-\frac{1}{2t_d}t} (e^{+i \frac{1}{2t_0} \sqrt{4 - T^{*2}}t} - e^{-i \frac{1}{2t_0} \sqrt{4 - T^{*2}}t}) \\ &= A e^{-\frac{1}{2t_d}t} \cdot 2i \sin \frac{\sqrt{4 - T^{*2}}}{2t_0} t \end{aligned}$$

また,

$$\dot{x}(0) = 2A i \frac{\sqrt{4 - T^{*2}}}{2t_0} = \frac{a}{t_0} \quad \therefore A = -i \frac{a}{\sqrt{4 - T^{*2}}}$$

$$\therefore \tilde{x}(\tilde{t}) = \frac{2}{\sqrt{4 - T^{*2}}} e^{-\frac{T^*}{2}\tilde{t}} \cdot \sin \frac{\sqrt{4 - T^{*2}}}{2} \tilde{t}$$

(ii) $t_s - 2t_d = 0$ i.e. $T^* = 2$ のとき

$$x(t) = (A + Bt)e^{\lambda t} \quad (\lambda = -\frac{1}{2t_d})$$

初期条件 $x(0) = A = 0$

$$\dot{x}(0) = B = \frac{a}{t_d} \quad \text{より}$$

$$x(t) = a\tilde{t}e^{-\tilde{t}}$$

無次元化して, $\tilde{x}(\tilde{t}) = \tilde{t}e^{-\tilde{t}}$

(iii) $t_s - 2t_d > 0$ i.e. $T^* > 2$ のとき

$$x(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t}$$

初期条件 $x(0) = A + B = 0$

$$\dot{x}(0) = A(\lambda_+ - \lambda_-) = \frac{a}{t_d} \quad \therefore A = \frac{a}{\sqrt{T^{*2} - 4}} \quad \text{より}$$

$$x(t) = \frac{a}{\sqrt{T^{*2} - 4}} (e^{\lambda_+ t} - e^{\lambda_- t}) \quad (\lambda_{\pm} = (-T^* \pm \sqrt{T^{*2} - 4}) \frac{\tilde{t}}{2})$$

無次元化して,

$$\tilde{x}(\tilde{t}) = \frac{1}{\sqrt{T^{*2} - 4}} (e^{\lambda_+ \tilde{t}} - e^{\lambda_- \tilde{t}}) \quad (\lambda_{\pm} = (-T^* \pm \sqrt{T^{*2} - 4}) \frac{\tilde{t}}{2})$$

以上より

(i) 減衰振動

$$\tilde{x}(\tilde{t}) = \frac{2}{\sqrt{4-T^{*2}}} e^{-\frac{T^*}{2}\tilde{t}} \cdot \sin \frac{\sqrt{4-T^{*2}}}{2} \tilde{t}$$

(ii) 過減衰

$$\tilde{x}(\tilde{t}) = \tilde{t} e^{-\tilde{t}}$$

(iii) 臨界減衰

$$\tilde{x}(\tilde{t}) = \frac{1}{\sqrt{T^{*2}-4}} (e^{\lambda_+ \tilde{t}} - e^{\lambda_- \tilde{t}}) \quad (\lambda_{\pm} = (-T^* \pm \sqrt{T^{*2}-4}) \frac{\tilde{t}}{2})$$