Lecture 3: Planning by Dynamic Programming

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Outline

What is Dynamic Programming?

Dynamic sequential or temporal component to the problem

Programming optimising a "program", i.e. a policy

- c.f. linear programming

- A method for solving complex problems
- By breaking them down into subproblems
  - Solve the subproblems
  - Combine solutions to subproblems

Requirements for Dynamic Programming

Dynamic Programming is a very general solution method for problems which have two properties:

- Optimal substructure
  - *Principle of optimality* applies
  - Optimal solution can be decomposed into subproblems
- Overlapping subproblems
  - Subproblems recur many times
  - Solutions can be cached and reused

a subproblem that we solve is likely part of consequtive subproblems that we will need to solve, thus we cache

- Markov decision processes satisfy both properties
  - Bellman equation gives recursive decomposition
  - Value function stores and reuses solutions

value function is in essence our cache

Planning by Dynamic Programming

planning is when we are given full info of the MDP dynamics
and we are asked to solve it

- Dynamic programming assumes full knowledge of the MDP
- It is used for *planning* in an MDP
- For prediction:
  - Input: MDP $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$ and policy π
  - or: MRP $\langle \mathcal{S}, \mathcal{P}^\pi, \mathcal{R}^\pi, \gamma \rangle$
  - Output: value function v_π

how much value we will get for any state
in this MDP given the provided policy π

- Or for control:
  - Input: MDP $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$
  - Output: optimal value function v_*
  - and: optimal policy π_*

what is the optimal value we can extract
from the system and how to do so

Other Applications of Dynamic Programming

Dynamic programming is used to solve many other problems, e.g.

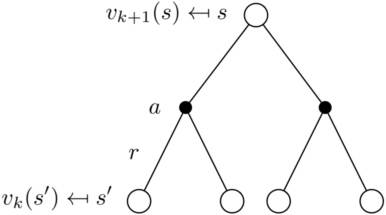
- Scheduling algorithms
- String algorithms (e.g. sequence alignment)
- Graph algorithms (e.g. shortest path algorithms)
- Graphical models (e.g. Viterbi algorithm)
- Bioinformatics (e.g. lattice models)

Iterative Policy Evaluation

problem here consists of evaluating a given policy that we are given
we use Bellman Expectation equation to do policy evaluation

- Problem: evaluate a given policy π
- Solution: iterative application of Bellman expectation backup
- $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_\pi$ we start with an initial assumption for all states of the MDP, v1 = vector of zeros, size equal to # of states
- Using *synchronous* backups,
  - At each iteration $k + 1$
  - For all states $s \in \mathcal{S}$
  - Update $v_{k+1}(s)$ from $v_k(s')$
  - where s' is a successor state of s
- We will discuss *asynchronous* backups later
- Convergence to v_π will be proven at the end of the lecture

Iterative Policy Evaluation (2)

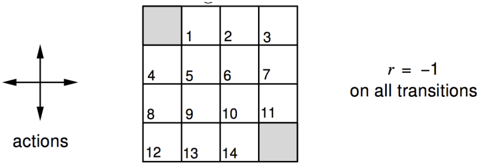


we will be turning the Bellman function into a synchronous iterative update algorithm to apply on the MDP until convergence

$$v_{k+1}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_k(s') \right)$$

$$\mathbf{v}^{k+1} = \boldsymbol{\mathcal{R}}^{\boldsymbol{\pi}} + \gamma \boldsymbol{\mathcal{P}}^{\boldsymbol{\pi}} \mathbf{v}^k$$

Evaluating a Random Policy in the Small Gridworld



- Undiscounted episodic MDP ($\gamma = 1$)
- Nonterminal states $1, \ldots, 14$
- One terminal state (shown twice as shaded squares)
- Actions leading out of the grid leave state unchanged
- Reward is -1 until the terminal state is reached
- Agent follows uniform random policy

$$\pi(n|\cdot) = \pi(e|\cdot) = \pi(s|\cdot) = \pi(w|\cdot) = 0.25$$

we here have a prediction problem where we are given a random uniform policy and we are asked to calculate the value function

Iterative Policy Evaluation in Small Gridworld

v_k for the Random Policy

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

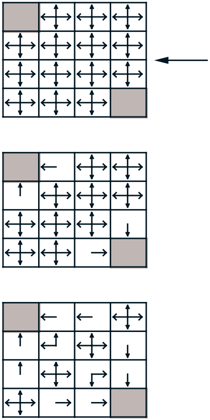
$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

Greedy Policy w.r.t. v_k



random policy

Iterative Policy Evaluation in Small Gridworld (2)

$k = 3$

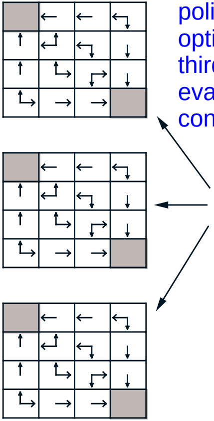
0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0



acting greedily even on a random policy helps finding out better policies. in this instance, the optimal policy converges from the third iteration, even if the policy evaluation has not converged yet.

optimal policy

How to Improve a Policy

the intuition here is starting from a policy π, how can we improve that and return back a better policy(?) if we have that, we can continue iterating until we get the optimal policy

- Given a policy π
  - Evaluate the policy π

$$v_\pi(s) = \mathbb{E}\left[R_{t+1} + \gamma R_{t+2} + ...|S_t = s\right]$$

  - Improve the policy by acting greedily with respect to v_π

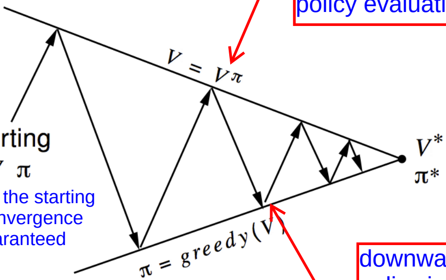
$$\pi' = \text{greedy}(v_\pi)$$

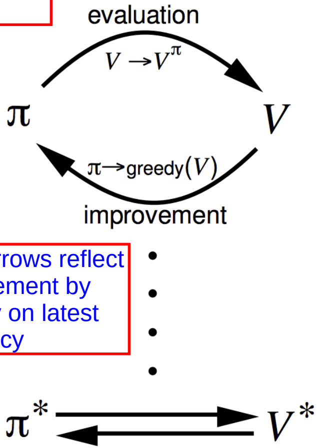
from the previous lecture we know that every MDP has an optimal value function

- In Small Gridworld improved policy was optimal, $\pi' = \pi^*$
- In general, need more iterations of improvement / evaluation
- But this process of policy iteration always converges to $\pi*$

as described in the previous example, we are going to start with a policy, evaluate it, act greedily upon it, evaluate it again, improve further, evaluate again, improve and so on...

Policy Iteration





Policy evaluation Estimate v_π

Iterative policy evaluation

Policy improvement Generate $\pi' \geq \pi$

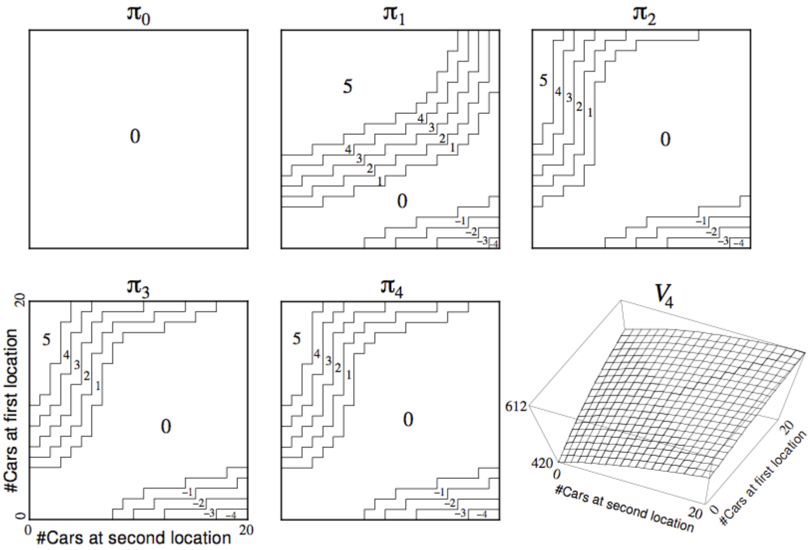
Greedy policy improvement

Jack's Car Rental

- States: Two locations, maximum of 20 cars at each
- Actions: Move up to 5 cars between locations overnight
- Reward: \$10 for each car rented (must be available)
- Transitions: Cars returned and requested randomly
  - Poisson distribution, n returns/requests with prob $\frac{\lambda^n}{n!}e^{-\lambda}$
  - 1st location: average requests = 3, average returns = 3
  - 2nd location: average requests = 4, average returns = 2

here we are doing planning, so the dynamics of the systems are known

Policy Iteration in Jack's Car Rental



Policy Improvement

- Consider a deterministic policy, $a = \pi(s)$
- We can *improve* the policy by acting greedily

follow policy π' for one step

follow policy π thereafter

$$\pi'(s) = \underset{a \in \mathcal{A}}{\operatorname{argmax}}\ q_\pi(s,a)$$

pick actions in a way that gives as the maximum action value

- This improves the value from any state s over one step,

first examine if by acting greedly we can have better outcome for the very next step, ignore future for now

$$q_\pi(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_\pi(s,a) \geq q_\pi(s, \pi(s)) = v_\pi(s)$$

the inequality above tells us that we are better of following π' for one step and π there after vs stick to π

- It therefore improves the value function, $v_{\pi'}(s) \geq v_\pi(s)$

$$\begin{aligned}
v_\pi(s) \leq q_\pi(s, \pi'(s)) &= \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma q_\pi(S_{t+1}, \pi'(S_{t+1})) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 q_\pi(S_{t+2}, \pi'(S_{t+2})) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \ldots \mid S_t = s\right] = v_{\pi'}(s)
\end{aligned}$$

Policy Improvement (2)

> being greedy does not mean only optimizing for the next step, rather, look at the best action
> we can take if we optimize across all possible actions for the one step and then look at our
> value function which summarises all future rewards but under our previous policy

- If improvements stop,

$$q_\pi(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_\pi(s, a) = q_\pi(s, \pi(s)) = v_\pi(s)$$

- Then the Bellman optimality equation has been satisfied

$$v_\pi(s) = \max_{a \in \mathcal{A}} q_\pi(s, a)$$

- Therefore $v_\pi(s) = v_*(s)$ for all $s \in \mathcal{S}$
- so π is an optimal policy

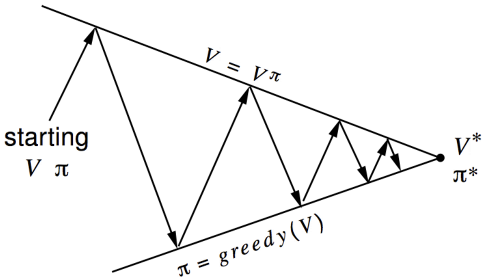
this is essentially telling us that if the process described in the previous slide stops yielding better results that vπ then it means we have reached optimality

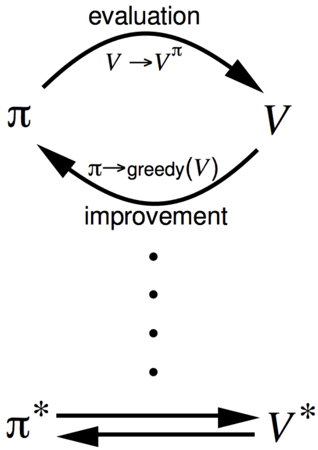
Modified Policy Iteration

intuition here comes from then small gridworld where at k=3 iteration we already had the optimal policy, so the convergence of the policy evaluation was "wasted" time...

- Does policy evaluation need to converge to v_π?
- Or should we introduce a stopping condition
  - e.g. ϵ-convergence of value function
- Or simply stop after k iterations of iterative policy evaluation?
- For example, in the small gridworld $k = 3$ was sufficient to achieve optimal policy
- Why not update policy every iteration? i.e. stop after $k = 1$
  - This is equivalent to *value iteration* (next section)

Generalised Policy Iteration





Policy evaluation Estimate v_π

Any policy evaluation algorithm

Policy improvement Generate $\pi' \geq \pi$

Any policy improvement algorithm

Principle of Optimality

Any optimal policy can be subdivided into two components:

- An optimal first action A_*
- Followed by an optimal policy from successor state S'

Theorem (Principle of Optimality)

A policy $\pi(a|s)$ achieves the optimal value from state s, $v_\pi(s) = v_*(s)$, if and only if

- *For any state s' reachable from s*
- *π achieves the optimal value from state s', $v_\pi(s') = v_*(s')$*

Policy vπ optimises the MDP (gets the max juice) if and only if, for any state that we can reach, this policy is optimal from that state onwards

Deterministic Value Iteration

the idea is that we start at any given state (root) in the MDP. the premise is that we are given the values for all the one step look-ahead states (leafs). We only need to optimize for all possible things we can do from the root onwards.

- If we know the solution to subproblems $v_*(s')$
- Then solution $v_*(s)$ can be found by one-step lookahead

$$v_*(s) \leftarrow \max_{a \in \mathcal{A}} \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_*(s')$$

we need to work the optimal action in reaching to any of the leaves.

leaf nodes value is supposedly given

- The idea of value iteration is to apply these updates iteratively
- Intuition: start with final rewards and work backwards
- Still works with loopy, stochastic MDPs

Example: Shortest Path

in the previous gridworld we were evaluating a random policy π that we were given

Problem

g			

V_1

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

V_2

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1

V_3

0	-1	-2	-2
-1	-2	-2	-2
-2	-2	-2	-2
-2	-2	-2	-2

here we are trying to find the optimal path, from wherever we may find ourselves into

V_4

0	-1	-2	-3
-1	-2	-3	-3
-2	-3	-3	-3
-3	-3	-3	-3

V_5

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-4
-3	-4	-4	-4

V_6

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-5

V_7

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-6

Value Iteration

keep in mind, we are still in the planning context, i.e. someone tells us the dynamics of the environment so we don't have to find them on our own.

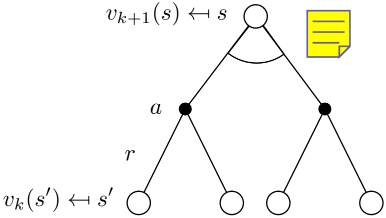
- Problem: find optimal policy π
- Solution: iterative application of Bellman optimality backup
- $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_*$
- Using synchronous backups
  - At each iteration $k+1$
  - For all states $s \in \mathcal{S}$
  - Update $v_{k+1}(s)$ from $v_k(s')$

say that we arbitrary stop at v2. the intermediate values we are getting do not necessarily correspond to any actual policy (like we saw in policy iteration) rather are simply intermediate constrcuts we calculate throughout the process

- Convergence to v_* will be proven later
- Unlike policy iteration, there is no explicit policy
- Intermediate value functions may not correspond to any policy

in the previous part (about Policy iteration) we saw how to use Bellman Expectation equation to find the value function for a given policy. here, we use Bellman Optimality equation to find the optimal policy

Value Iteration (2)



this is the Bellman Optimality equation, though turned into an iterative update format

$$v_{k+1}(s) = \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_k(s') \right)$$

$$\mathbf{v}_{k+1} = \max_{a \in \mathcal{A}} \boldsymbol{\mathcal{R}}^{\boldsymbol{a}} + \gamma \boldsymbol{\mathcal{P}}^{\boldsymbol{a}} \mathbf{v}_k$$

Example of Value Iteration in Practice

http://www.cs.ubc.ca/∼poole/demos/mdp/vi.html

Synchronous Dynamic Programming Algorithms

all of these are planning problems, we are given the MDP and we are trying to solve different questions...

Problem	Bellman Equation	Algorithm
Prediction how much reward you get for a given policy	Bellman Expectation Equation	Iterative Policy Evaluation
Control	Bellman Expectation Equation + Greedy Policy Improvement	Policy Iteration
Control how to get the optimal reward from the MDP	Bellman Optimality Equation	Value Iteration

- Algorithms are based on state-value function $v_\pi(s)$ or $v_*(s)$
- Complexity $O(mn^2)$ per iteration, for m actions and n states
- Could also apply to action-value function $q_\pi(s, a)$ or $q_*(s, a)$
- Complexity $O(m^2 n^2)$ per iteration

Asynchronous Dynamic Programming

- DP methods described so far used *synchronous* backups
- i.e. all states are backed up in parallel
- *Asynchronous DP* backs up states individually, in any order
- For each selected state, apply the appropriate backup
- Can significantly reduce computation
- Guaranteed to converge if all states continue to be selected

Asynchronous Dynamic Programming

the idea here is to improve the algorithm and do smart update instead of updates through full sweeps

Three simple ideas for asynchronous dynamic programming:

- *In-place* dynamic programming
- *Prioritised sweeping*
- *Real-time* dynamic programming

In-Place Dynamic Programming

- Synchronous value iteration stores two copies of value function

  for all s in $\mathcal{S}$

$$v_{new}(s) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_{old}(s') \right)$$

$$v_{old} \leftarrow v_{new}$$

- In-place value iteration only stores one copy of value function

  for all s in $\mathcal{S}$

$$v(s) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v(s') \right)$$

Prioritised Sweeping

- Use magnitude of Bellman error to guide state selection, e.g.

$$\left| \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v(s') \right) - v(s) \right|$$

- Backup the state with the largest remaining Bellman error
- Update Bellman error of affected states after each backup
- Requires knowledge of reverse dynamics (predecessor states)
- Can be implemented efficiently by maintaining a priority queue

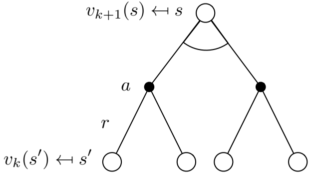
Real-Time Dynamic Programming

- Idea: only states that are relevant to agent
- Use agent's experience to guide the selection of states
- After each time-step S_t, A_t, R_{t+1}
- Backup the state S_t

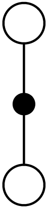
$$v(S_t) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}^a_{S_t} + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}^a_{S_t s'} v(s') \right)$$

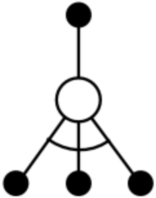
Full-Width Backups

- DP uses *full-width* backups
- For each backup (sync or async)
  - Every successor state and action is considered
  - Using knowledge of the MDP transitions and reward function
- DP is effective for medium-sized problems (millions of states)
- For large problems DP suffers Bellman's *curse of dimensionality*
  - Number of states $n = |\mathcal{S}|$ grows exponentially with number of state variables
- Even one backup can be too expensive



Sample Backups





- In subsequent lectures we will consider *sample backups*
- Using sample rewards and sample transitions $\langle S, A, R, S' \rangle$
- Instead of reward function $\mathcal{R}$ and transition dynamics $\mathcal{P}$
- Advantages:
  - Model-free: no advance knowledge of MDP required
  - Breaks the curse of dimensionality through sampling
  - Cost of backup is constant, independent of $n = |\mathcal{S}|$

Approximate Dynamic Programming

- Approximate the value function
- Using a *function approximator* $\hat{v}(s, \mathbf{w})$
- Apply dynamic programming to $\hat{v}(\cdot, \mathbf{w})$
- e.g. Fitted Value Iteration repeats at each iteration k,
  - Sample states $\tilde{\mathcal{S}} \subseteq \mathcal{S}$
  - For each state $s \in \tilde{\mathcal{S}}$, estimate target value using Bellman optimality equation,

$$\tilde{v}_k(s) = \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a \hat{v}(s', \mathbf{w_k}) \right)$$

  - Train next value function $\hat{v}(\cdot, \mathbf{w_{k+1}})$ using targets $\{\langle s, \tilde{v}_k(s) \rangle\}$

Some Technical Questions

- How do we know that value iteration converges to v_*?
- Or that iterative policy evaluation converges to v_π?
- And therefore that policy iteration converges to v_*?
- Is the solution unique?
- How fast do these algorithms converge?
- These questions are resolved by *contraction mapping theorem*

Value Function Space

- Consider the vector space $\mathcal{V}$ over value functions
- There are $|\mathcal{S}|$ dimensions
- Each point in this space fully specifies a value function $v(s)$
- What does a Bellman backup do to points in this space?
- We will show that it brings value functions *closer*
- And therefore the backups must converge on a unique solution

Value Function ∞-Norm

- We will measure distance between state-value functions u and v by the ∞-norm
- i.e. the largest difference between state values,

$$||u - v||_\infty = \max_{s \in \mathcal{S}} |u(s) - v(s)|$$

Bellman Expectation Backup is a Contraction

- Define the *Bellman expectation backup operator* T^π,

$$T^\pi(v) = \mathcal{R}^\pi + \gamma \mathcal{P}^\pi v$$

- This operator is a γ-contraction, i.e. it makes value functions closer by at least γ,

$$\begin{aligned} ||T^\pi(u) - T^\pi(v)||_\infty &= ||\left(\mathcal{R}^\pi + \gamma \mathcal{P}^\pi u\right) - \left(\mathcal{R}^\pi + \gamma \mathcal{P}^\pi v\right)||_\infty \\ &= ||\gamma \mathcal{P}^\pi (u - v)||_\infty \\ &\leq ||\gamma \mathcal{P}^\pi||u - v||_\infty||_\infty \\ &\leq \gamma||u - v||_\infty \end{aligned}$$

Contraction Mapping Theorem

Theorem (Contraction Mapping Theorem)

For any metric space $\mathcal{V}$ that is complete (i.e. closed) under an operator $T(v)$, where T is a γ-contraction,

- *T converges to a unique fixed point*
- *At a linear convergence rate of γ*

Convergence of Iter. Policy Evaluation and Policy Iteration

- The Bellman expectation operator T^{π} has a unique fixed point
- v_{π} is a fixed point of T^{π} (by Bellman expectation equation)
- By contraction mapping theorem
- Iterative policy evaluation converges on v_{π}
- Policy iteration converges on v_{*}

Bellman Optimality Backup is a Contraction

- Define the *Bellman optimality backup operator* T^*,

$$T^*(v) = \max_{a \in \mathcal{A}} \mathcal{R}^a + \gamma \mathcal{P}^a v$$

- This operator is a γ-contraction, i.e. it makes value functions closer by at least γ (similar to previous proof)

$$||T^*(u) - T^*(v)||_\infty \leq \gamma ||u - v||_\infty$$

Convergence of Value Iteration

- The Bellman optimality operator T^* has a unique fixed point
- v_* is a fixed point of T^* (by Bellman optimality equation)
- By contraction mapping theorem
- Value iteration converges on v_*