Lecture 2: Markov Decision Processes

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Introduction to MDPs

- *Markov decision processes* formally describe an environment for reinforcement learning
- Where the environment is *fully observable*
- i.e. The current *state* completely characterises the process
- Almost all RL problems can be formalised as MDPs, e.g.
    - Optimal control primarily deals with continuous MDPs
    - Partially observable problems can be converted into MDPs
    - Bandits are MDPs with one state

Markov Property

"The future is independent of the past given the present"

Definition

A state S_t is *Markov* if and only if

$$\mathbb{P}\left[S_{t+1} \mid S_t\right] = \mathbb{P}\left[S_{t+1} \mid S_1, ..., S_t\right]$$

- The state captures all relevant information from the history
- Once the state is known, the history may be thrown away
- i.e. The state is a sufficient statistic of the future

State Transition Matrix

For a Markov state s and successor state s', the *state transition probability* is defined by

$$\mathcal{P}_{ss'} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s\right]$$

State transition matrix $\mathcal{P}$ defines transition probabilities from all states s to all successor states s',

$$\mathcal{P} = \begin{array}{cc} & \textit{to} \\ \textit{from} & \begin{bmatrix} \mathcal{P}_{11} & \dots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{n1} & \dots & \mathcal{P}_{nn} \end{bmatrix} \end{array}$$

where each row of the matrix sums to 1.

Markov Process

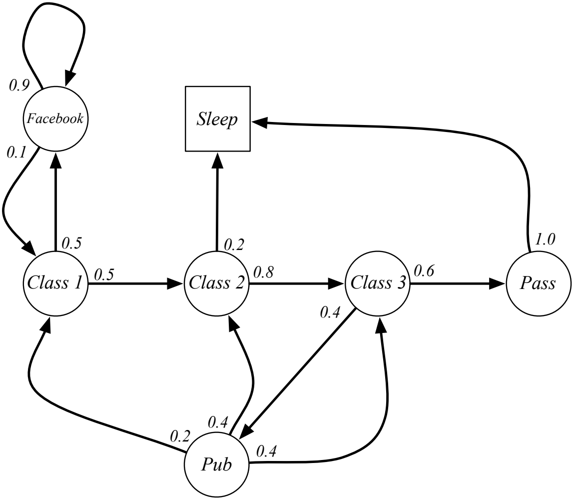
A Markov process is a memoryless random process, i.e. a sequence of random states $S_1, S_2, ...$ with the Markov property.

Definition

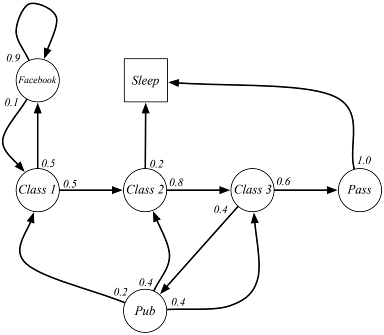
A *Markov Process* (or *Markov Chain*) is a tuple $\langle \mathcal{S}, \mathcal{P} \rangle$

- $\mathcal{S}$ is a (finite) set of states
- $\mathcal{P}$ is a state transition probability matrix, $\mathcal{P}_{ss'} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s\right]$

Example: Student Markov Chain



Example: Student Markov Chain Episodes



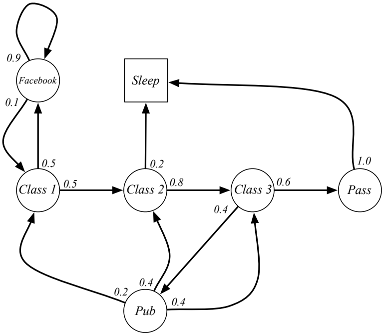
Sample episodes for Student Markov Chain starting from $S_1 = C1$

$$S_1, S_2, ..., S_T$$

samples drawn from the MC

- C1 C2 C3 Pass Sleep
- C1 FB FB C1 C2 Sleep
- C1 C2 C3 Pub C2 C3 Pass Sleep
- C1 FB FB C1 C2 C3 Pub C1 FB FB FB C1 C2 C3 Pub C2 Sleep

Example: Student Markov Chain Transition Matrix



to

from

$$\mathcal{P} = \begin{array}{cc} & \begin{array}{ccccccc} C1 & C2 & C3 & \mathit{Pass} & \mathit{Pub} & \mathit{FB} & \mathit{Sleep} \end{array} \\ \begin{array}{c} C1 \\ C2 \\ C3 \\ \mathit{Pass} \\ \mathit{Pub} \\ \mathit{FB} \\ \mathit{Sleep} \end{array} & \left[\begin{array}{ccccccc} & 0.5 & & & & 0.5 & \\ & & 0.8 & & & & 0.2 \\ & & & 0.6 & 0.4 & & \\ & & & & & & 1.0 \\ 0.2 & 0.4 & 0.4 & & & & \\ 0.1 & & & & & 0.9 & \\ & & & & & & 1 \end{array} \right] \end{array}$$

Markov Reward Process

A Markov reward process is a Markov chain with values.

Definition

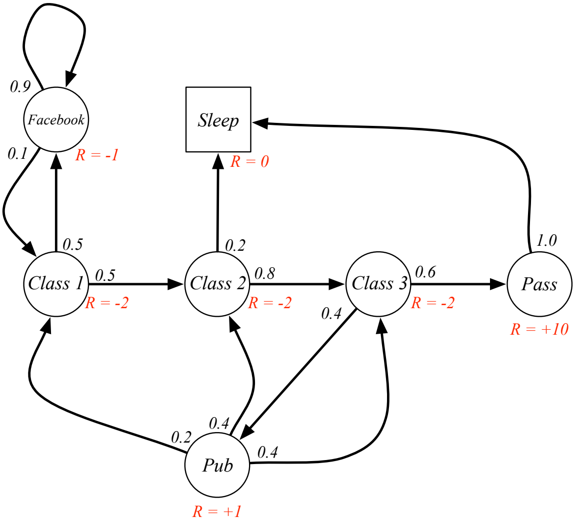
A *Markov Reward Process* is a tuple $\langle \mathcal{S}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{P}$ is a state transition probability matrix,
  $\mathcal{P}_{ss'} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s\right]$
- $\mathcal{R}$ is a reward function, $\mathcal{R}_s = \mathbb{E}\left[R_{t+1} \mid S_t = s\right]$
- γ is a discount factor, $\gamma \in [0, 1]$

reward from a single state

ultimately we are after maximizing the cumulative sum of these rewards

Example: Student MRP



Return

Definition

The *return* G_t is the total discounted reward from time-step t.

$$G_t = R_{t+1} + \gamma R_{t+2} + ... = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

G is a random sample from the MRP, hence there is no Expectation operator

- The *discount* $\gamma \in [0, 1]$ is the present value of future rewards
- The value of receiving reward R after $k + 1$ time-steps is $\gamma^k R$.
- This values immediate reward above delayed reward.
  - γ close to 0 leads to "myopic" evaluation
  - γ close to 1 leads to "far-sighted" evaluation

Why discount?

Most Markov reward and decision processes are discounted. Why?

- Mathematically convenient to discount rewards
- Avoids infinite returns in cyclic Markov processes
- Uncertainty about the future may not be fully represented
- If the reward is financial, immediate rewards may earn more interest than delayed rewards
- Animal/human behaviour shows preference for immediate reward
- It is sometimes possible to use *undiscounted* Markov reward processes (i.e. $\gamma = 1$), e.g. if all sequences terminate.

Uncertainty also reflects the trust that we have to our model, especially when it comes to future rewards

Value Function

The value function $v(s)$ gives the long-term value of state s

Definition

The *state value function* $v(s)$ of an MRP is the expected return starting from state s

$$v(s) = \mathbb{E}\left[G_t \mid S_t = s\right]$$

Example: Student MRP Returns

Sample returns for Student MRP:
Starting from $S_1 = \text{C1}$ with $\gamma = \frac{1}{2}$

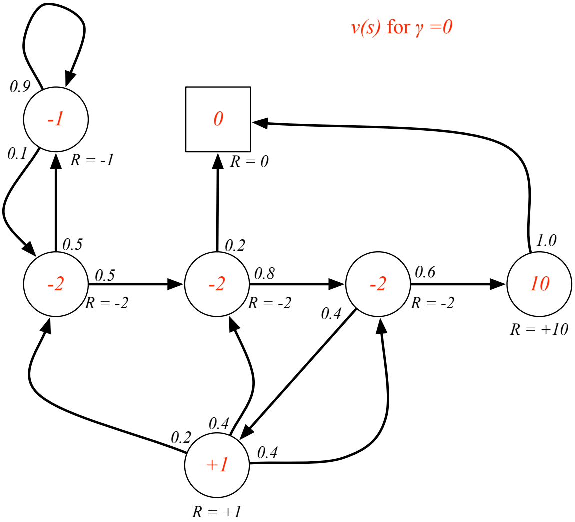
$$G_1 = R_2 + \gamma R_3 + ... + \gamma^{T-2} R_T$$

Different samples from MC

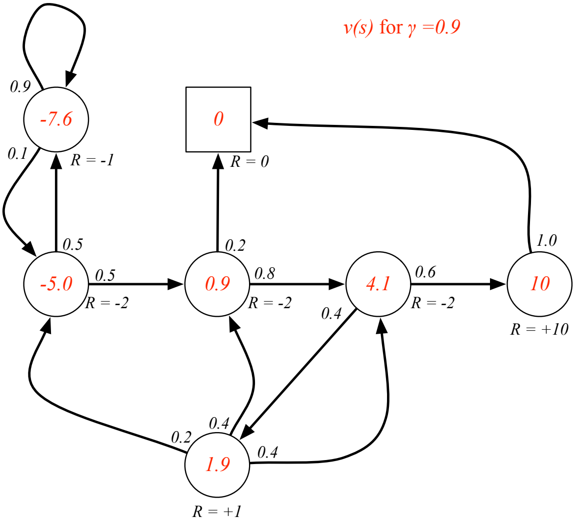
C1 C2 C3 Pass Sleep	$v_1 = -2 - 2 * \frac{1}{2} - 2 * \frac{1}{4} + 10 * \frac{1}{8}$	$=$	-2.25
C1 FB FB C1 C2 Sleep	$v_1 = -2 - 1 * \frac{1}{2} - 1 * \frac{1}{4} - 2 * \frac{1}{8} - 2 * \frac{1}{16}$	$=$	-3.125
C1 C2 C3 Pub C2 C3 Pass Sleep	$v_1 = -2 - 2 * \frac{1}{2} - 2 * \frac{1}{4} + 1 * \frac{1}{8} - 2 * \frac{1}{16} \ldots$	$=$	-3.41
C1 FB FB C1 C2 C3 Pub C1 ...	$v_1 = -2 - 1 * \frac{1}{2} - 1 * \frac{1}{4} - 2 * \frac{1}{8} - 2 * \frac{1}{16} \ldots$	$=$	-3.20
FB FB FB C1 C2 C3 Pub C2 Sleep			

one way to estimate the value of a particular state (~i.e. Expected Return) of an MRP is to draw samples starting from that state and average their values

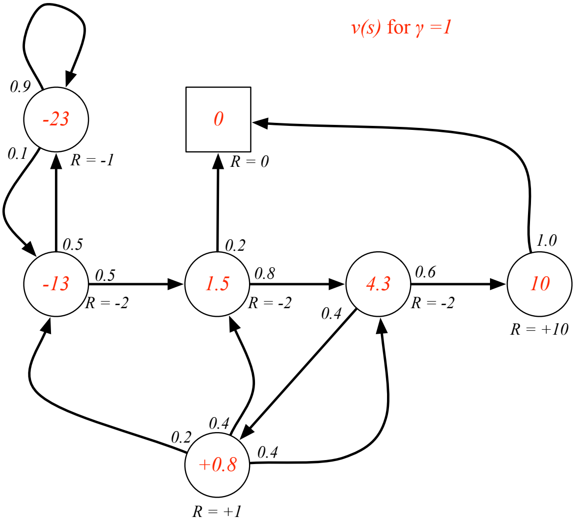
Example: State-Value Function for Student MRP (1)



Example: State-Value Function for Student MRP (2)



Example: State-Value Function for Student MRP (3)



Bellman Equation for MRPs

The value function can be decomposed into two parts:

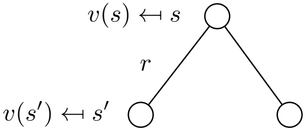
- immediate reward R_{t+1}
- discounted value of successor state $\gamma v(S_{t+1})$

breaks into two parts, immediate reward
and the value from where I end up

$$
\begin{aligned}
v(s) &= \mathbb{E}\left[G_t \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma \left(R_{t+2} + \gamma R_{t+3} + \ldots\right) \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma G_{t+1} \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma v(S_{t+1}) \mid S_t = s\right]
\end{aligned}
$$

Bellman Equation for MRPs (2)

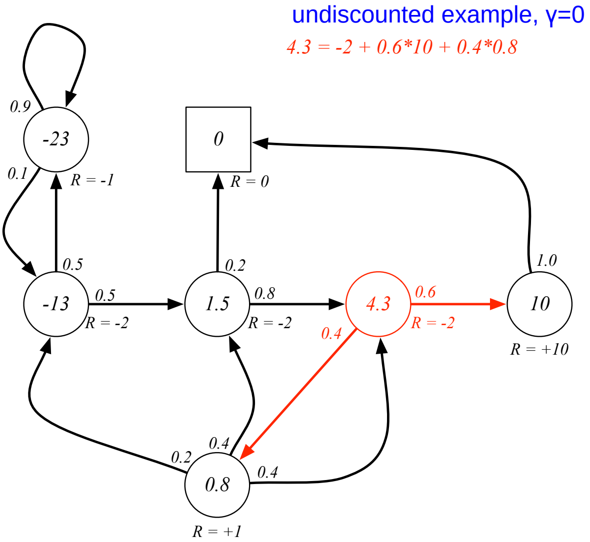
$$v(s) = \mathbb{E}\left[R_{t+1} + \gamma v(S_{t+1}) \mid S_t = s\right]$$



one step ahead lookup process

$$v(s) = \mathcal{R}_s + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'} v(s')$$

Example: Bellman Equation for Student MRP



Bellman Equation in Matrix Form

The Bellman equation can be expressed concisely using matrices,

$$v = \mathcal{R} + \gamma \mathcal{P} v$$

where v is a column vector with one entry per state

dot product

$$\begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix} = \begin{bmatrix} \mathcal{R}_1 \\ \vdots \\ \mathcal{R}_n \end{bmatrix} + \gamma \begin{bmatrix} \mathcal{P}_{11} & \dots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{11} & \dots & \mathcal{P}_{nn} \end{bmatrix} \begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix}$$

Solving the Bellman Equation

- The Bellman equation is a linear equation
- It can be solved directly:

Bellman is preferably used to validate MC,
rather and not calculating the values

$$
\begin{aligned}
v &= \mathcal{R} + \gamma \mathcal{P} v \\
(I - \gamma \mathcal{P})\, v &= \mathcal{R} \\
v &= (I - \gamma \mathcal{P})^{-1} \mathcal{R}
\end{aligned}
$$

- Computational complexity is $O(n^3)$ for n states
- Direct solution only possible for small MRPs
- There are many iterative methods for large MRPs, e.g.
  - Dynamic programming
  - Monte-Carlo evaluation
  - Temporal-Difference learning

Markov Decision Process

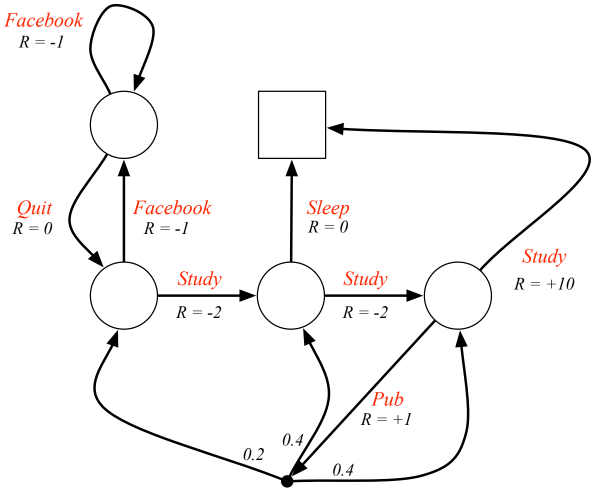
A Markov decision process (MDP) is a Markov reward process with decisions. It is an *environment* in which all states are Markov.

Definition

A *Markov Decision Process* is a tuple $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{A}$ is a finite set of actions
- $\mathcal{P}$ is a state transition probability matrix,
  $\mathcal{P}_{ss'}^{a} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s, A_t = a\right]$
- $\mathcal{R}$ is a reward function, $\mathcal{R}_s^a = \mathbb{E}\left[R_{t+1} \mid S_t = s, A_t = a\right]$
- γ is a discount factor $\gamma \in [0, 1]$.

Example: Student MDP



Policies (1)

Definition

A *policy* π is a distribution over actions given states,

$$\pi(a|s) = \mathbb{P}\left[A_t = a \mid S_t = s\right]$$

- A policy fully defines the behaviour of an agent
- MDP policies depend on the current state (not the history)
- i.e. Policies are *stationary* (time-independent), $A_t \sim \pi(\cdot|S_t), \forall t > 0$

  policy only depends on state s and not on previous states (history) nor time stamp (time-independent)

Policies (2)

- Given an MDP $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$ and a policy π
- The state sequence $S_1, S_2, ...$ is a Markov process $\langle \mathcal{S}, \mathcal{P}^\pi \rangle$
- The state and reward sequence $S_1, R_2, S_2, ...$ is a Markov reward process $\langle \mathcal{S}, \mathcal{P}^\pi, \mathcal{R}^\pi, \gamma \rangle$
- where

$$\mathcal{P}^\pi_{s,s'} = \sum_{a \in \mathcal{A}} \pi(a|s) \mathcal{P}^a_{ss'}$$

$$\mathcal{R}^\pi_{s} = \sum_{a \in \mathcal{A}} \pi(a|s) \mathcal{R}^a_{s}$$

we can always flatten our MDP (into either MP or MRP) given a certain policy by averaging out all the things that happen under that policy

Value Function

Definition

The *state-value function* $v_\pi(s)$ of an MDP is the expected return starting from state s, and then following policy π

$$v_\pi(s) = \mathbb{E}_\pi\left[G_t \mid S_t = s\right]$$

expectation when sampling all actions based on policy π, starting from state s

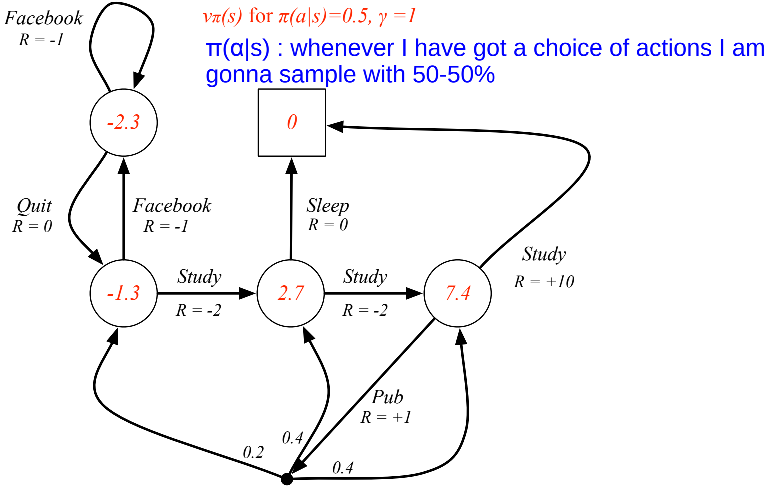
Definition

The *action-value function* $q_\pi(s, a)$ is the expected return starting from state s, taking action a, and then following policy π

$$q_\pi(s, a) = \mathbb{E}_\pi\left[G_t \mid S_t = s, A_t = a\right]$$

state-value v is telling us how good is to be on a particular state
action-value q is telling us how good it to take a particular action from a particular state

Example: State-Value Function for Student MDP



Bellman Expectation Equation

The state-value function can again be decomposed into immediate reward plus discounted value of successor state,

$$v_\pi(s) = \mathbb{E}_\pi \left[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s\right]$$

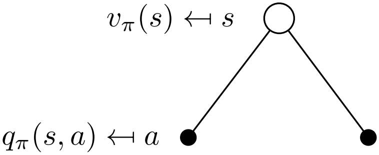
immediate reward + future reward if we are to follow that policy

The action-value function can similarly be decomposed,

$$q_\pi(s,a) = \mathbb{E}_\pi \left[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) \mid S_t = s, A_t = a\right]$$

Bellman Expectation Equation for V^{π}

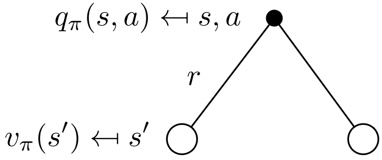
we here assume that selecting an action (black dots)
is based on the policy distribution π(α|s)



$$v_{\pi}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) q_{\pi}(s,a)$$

therefore, to determine the value of the state,
we average across the possible action values (q)

Bellman Expectation Equation for Q^π

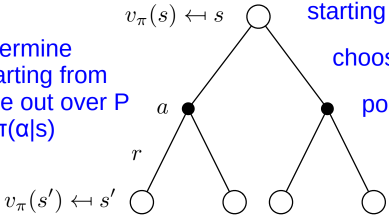


$$q_\pi(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s')$$

we here assume that having taken an action (black dot) we may end up to different states due to the environment dynamics. To estimate q, we average out the possible state values times their transition probabilities (P)

Bellman Expectation Equation for v_π (2)

two step look-ahead to determine beginning state's value. starting from the bottom, we first average out over P and the average out over π(α|s)



starting state

choose action based off π(α|s)

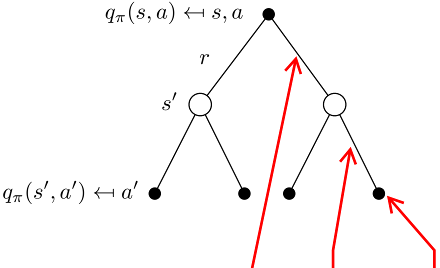
possible actions

environment dynamics P

possible future states

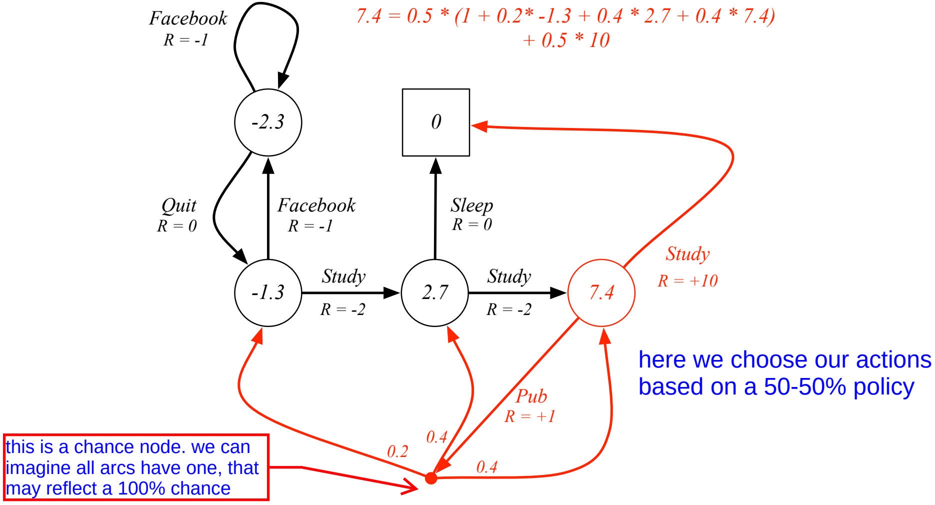
$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s') \right)$$

Bellman Expectation Equation for q_π (2)



$$q_\pi(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \boxed{\mathcal{P}_{ss'}^a} \sum_{a' \in \mathcal{A}} \boxed{\pi(a'|s')} \boxed{q_\pi(s',a')}$$

Example: Bellman Expectation Equation in Student MDP



Bellman Expectation Equation (Matrix Form)

The Bellman expectation equation can be expressed concisely using the induced MRP,

$$v_\pi = \mathcal{R}^\pi + \gamma \mathcal{P}^\pi v_\pi$$

with direct solution

$$v_\pi = \left(I - \gamma \mathcal{P}^\pi\right)^{-1} \mathcal{R}^\pi$$

Optimal Value Function

Definition

The *optimal state-value function* $v_*(s)$ is the maximum value function over all policies

the maximum value you can can expect to extract from the system

$$v_*(s) = \max_\pi v_\pi(s)$$

The *optimal action-value function* $q_*(s, a)$ is the maximum action-value function over all policies

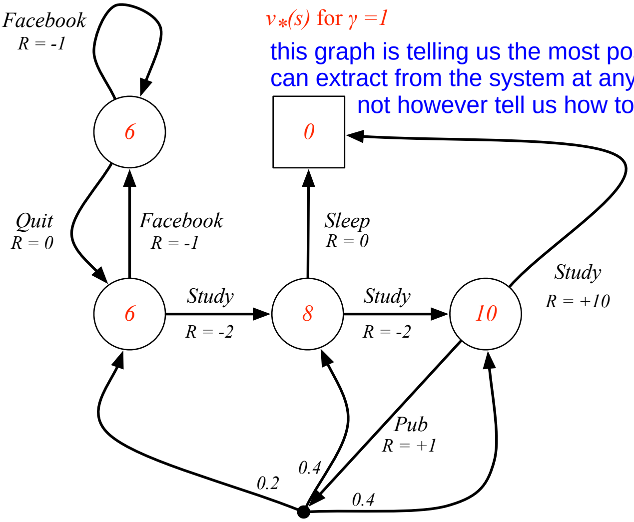
the maximum reward you can expect to extract given a certain action and state

$$q_*(s, a) = \max_\pi q_\pi(s, a)$$

if you find q* then problem is solved, you have got the quantities to behave optimally in the MDP

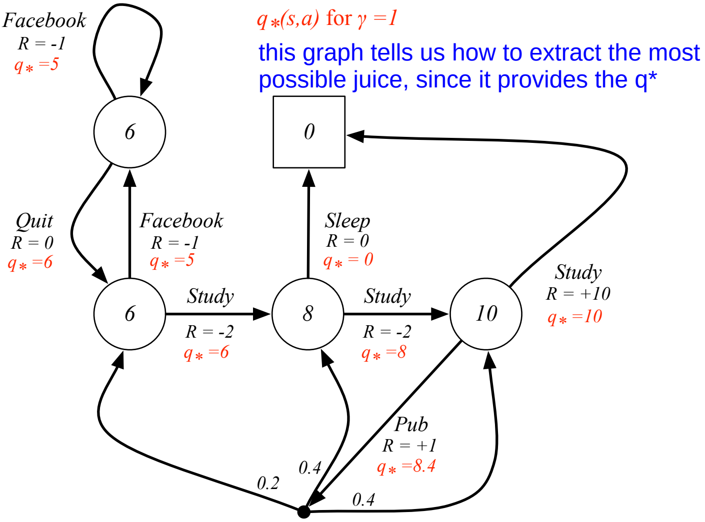
- The optimal value function specifies the best possible performance in the MDP.
- An MDP is "solved" when we know the optimal value fn.

Example: Optimal Value Function for Student MDP



this graph is telling us the most possible juice you can extract from the system at any state. it does not however tell us how to do so.

Example: Optimal Action-Value Function for Student MDP



Optimal Policy

Define a partial ordering over policies

$$\pi \geq \pi' \text{ if } v_\pi(s) \geq v_{\pi'}(s), \forall s$$

Theorem

For any Markov Decision Process

- *There exists an optimal policy π_* that is better than or equal to all other policies, $\pi_* \geq \pi, \forall \pi$*
- *All optimal policies achieve the optimal value function, $v_{\pi_*}(s) = v_*(s)$*
- *All optimal policies achieve the optimal action-value function, $q_{\pi_*}(s, a) = q_*(s, a)$*

all the above tells us that for any MDP, there is an optimal policy that is better than or equal to all other policies

Finding an Optimal Policy

you solve for q* and you pick the action that maximizes q*, with probability (1)

An optimal policy can be found by maximising over $q_*(s,a)$,

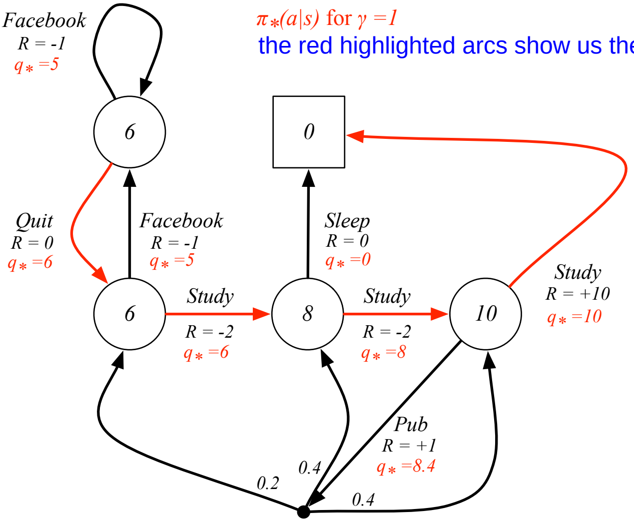
$$\pi_*(a|s) = \begin{cases} 1 & \text{if } a = \underset{a \in \mathcal{A}}{\operatorname{argmax}}\ q_*(s,a) \\ 0 & \textit{otherwise} \end{cases}$$

- There is always a deterministic optimal policy for any MDP
- If we know $q_*(s,a)$, we immediately have the optimal policy

Example: Optimal Policy for Student MDP

$\pi_*(a|s)$ for $\gamma = 1$

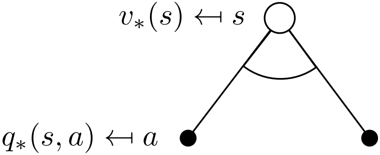
the red highlighted arcs show us the optimal policy



Bellman Optimality Equation for v_*

The optimal value functions are recursively related by the Bellman optimality equations:

this basically says, look at the actions that you can take and take the action that gives the max values



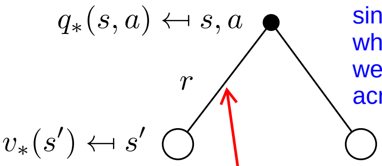
where we were previously averaging, here we take the max of the values

$$v_*(s) = \max_a q_*(s, a)$$

here we have the Bellman Optimality Equation as opposed to Bellman Expectation equation that we reviewed in the previous slides

Bellman Optimality Equation for Q^*

over here we examine the environment dynamics, i.e.
we don't get to pick rather it's a matter of the system dice



since we don't choose at
which state we will end up,
we average the values
across all possible states

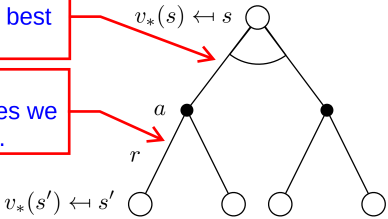
$$q_*(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \boxed{\mathcal{P}_{ss'}^a} v_*(s')$$

Bellman Optimality Equation for V^* (2)

two step look-ahead

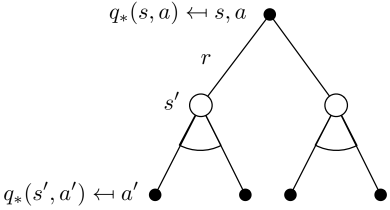
we first get to pick, and we take the action with the best value

we average the values across all possible states we may find ourselves into.



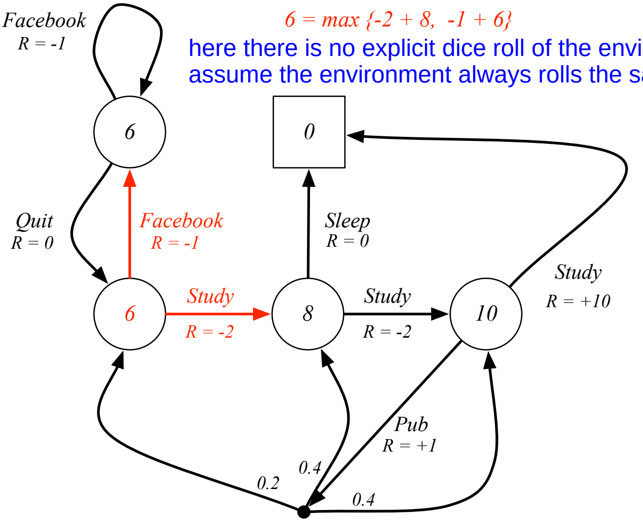
$$v_*(s) = \max_a \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_*(s')$$

Bellman Optimality Equation for Q^* (2)



$$q_*(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a \max_{a'} q_*(s',a')$$

Example: Bellman Optimality Equation in Student MDP



Solving the Bellman Optimality Equation

Bellman Expectation Equation is linear and we
could solve them through matrix inversion for example.

- Bellman Optimality Equation is non-linear
- No closed form solution (in general)
- Many iterative solution methods
  - Value Iteration
  - Policy Iteration
  - Q-learning
  - Sarsa

Extensions to MDPs (no exam)

- Infinite and continuous MDPs
- Partially observable MDPs
- Undiscounted, average reward MDPs

Infinite MDPs (no exam)

The following extensions are all possible:

- Countably infinite state and/or action spaces
  - Straightforward
- Continuous state and/or action spaces
  - Closed form for linear quadratic model (LQR)
- Continuous time
  - Requires partial differential equations
  - Hamilton-Jacobi-Bellman (HJB) equation
  - Limiting case of Bellman equation as time-step $\rightarrow 0$

POMDPs (no exam)

A Partially Observable Markov Decision Process is an MDP with hidden states. It is a hidden Markov model with actions.

Definition

A *POMDP* is a tuple $\langle \mathcal{S}, \mathcal{A}, \mathcal{O}, \mathcal{P}, \mathcal{R}, \mathcal{Z}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{A}$ is a finite set of actions
- $\mathcal{O}$ is a finite set of observations
- $\mathcal{P}$ is a state transition probability matrix,
  $\mathcal{P}_{ss'}^{a} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s, A_t = a\right]$
- $\mathcal{R}$ is a reward function, $\mathcal{R}_{s}^{a} = \mathbb{E}\left[R_{t+1} \mid S_t = s, A_t = a\right]$
- $\mathcal{Z}$ is an observation function,
  $\mathcal{Z}_{s'o}^{a} = \mathbb{P}\left[O_{t+1} = o \mid S_{t+1} = s', A_t = a\right]$
- γ is a discount factor $\gamma \in [0, 1]$.

Belief States (no exam)

Definition

A *history* H_t is a sequence of actions, observations and rewards,

$$H_t = A_0, O_1, R_1, ..., A_{t-1}, O_t, R_t$$

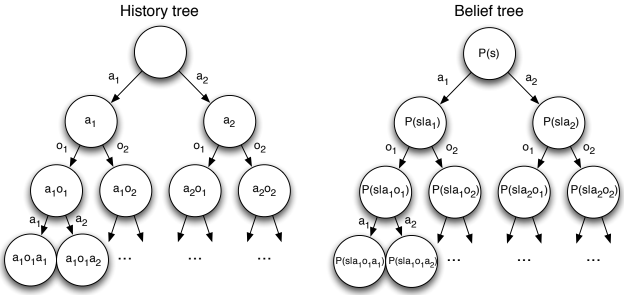
Definition

A *belief state* $b(h)$ is a probability distribution over states, conditioned on the history h

$$b(h) = \left(\mathbb{P}\left[S_t = s^1 \mid H_t = h\right], ..., \mathbb{P}\left[S_t = s^n \mid H_t = h\right]\right)$$

Reductions of POMDPs (no exam)

- The history H_t satisfies the Markov property
- The belief state $b(H_t)$ satisfies the Markov property



- A POMDP can be reduced to an (infinite) history tree
- A POMDP can be reduced to an (infinite) belief state tree

Ergodic Markov Process (no exam)

An ergodic Markov process is

- *Recurrent*: each state is visited an infinite number of times
- *Aperiodic*: each state is visited without any systematic period

Theorem

An ergodic Markov process has a limiting stationary distribution $d^{\pi}(s)$ with the property

$$d^{\pi}(s) = \sum_{s' \in \mathcal{S}} d^{\pi}(s') \mathcal{P}_{s's}$$

Ergodic MDP (no exam)

Definition

An MDP is ergodic if the Markov chain induced by any policy is ergodic.

For any policy π, an ergodic MDP has an *average reward per time-step* ρ^π that is independent of start state.

$$\rho^\pi = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\sum_{t=1}^{T} R_t\right]$$

Average Reward Value Function (no exam)

- The value function of an undiscounted, ergodic MDP can be expressed in terms of average reward.
- $\tilde{v}_\pi(s)$ is the extra reward due to starting from state s,

$$\tilde{v}_\pi(s) = \mathbb{E}_\pi \left[\sum_{k=1}^{\infty} \left(R_{t+k} - \rho^\pi\right) \mid S_t = s \right]$$

There is a corresponding average reward Bellman equation,

$$\begin{aligned}\tilde{v}_\pi(s) &= \mathbb{E}_\pi \left[\left(R_{t+1} - \rho^\pi\right) + \sum_{k=1}^{\infty} \left(R_{t+k+1} - \rho^\pi\right) \mid S_t = s \right] \\ &= \mathbb{E}_\pi \left[\left(R_{t+1} - \rho^\pi\right) + \tilde{v}_\pi(S_{t+1}) \mid S_t = s \right]\end{aligned}$$

Questions?

The only stupid question is the one you were afraid to ask but never did.
-Rich Sutton