ODEs: The Driven Pendulum

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1 Core Task 1: Solving ODEs using RK45 (pendulum_ode.py, pendulum_initial_conditions.py)

The code of pendulum_ode uses the fourth order Runge-Kutta method to integrate the equation of the undamped, undriven pendulum. Plots are presented in Figures 1, 2 for the first and last seconds of the simulation with different total running times. The code also outputs a dataframe shown in Figures 3 containing the time, θ, analytical θ, ω and energy.

Conservation of energy from the numerical method was also investigated in this script, by running the simulation for 10 000 natural periods and plotting energy vs time in Figure 4. Using also the energy column produced by the dataframe, it was observed that the energy decreased by 9.07%.

The program pendulum_initial_conditions investigates how the initial displacement θ_0 affects the period of oscillation. The code is explained with sufficient commenting in the script itself. The result is shown in Figure 5 and the period for $\theta_0 = \frac{\pi}{2}$ was found to be 7.43 seconds.

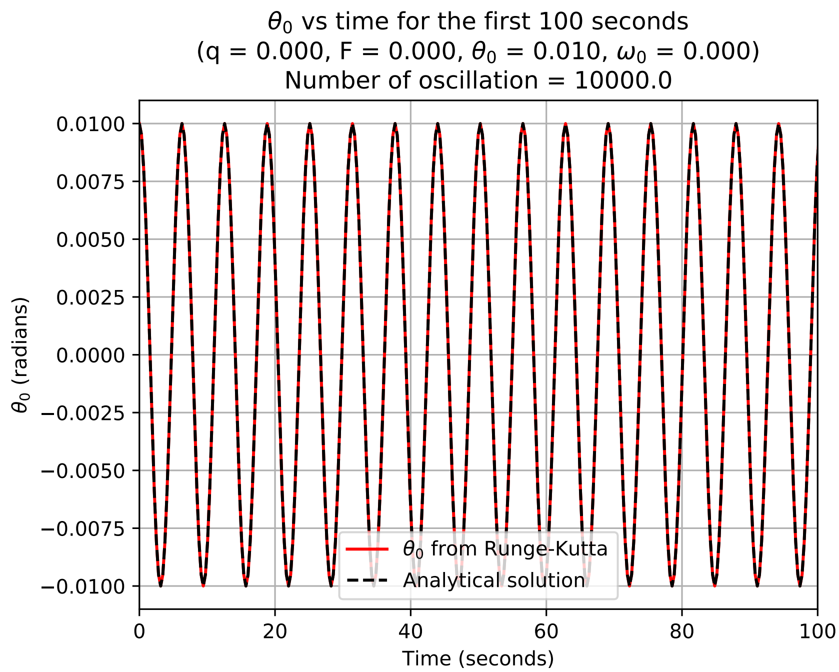

Figure 1: The first 100 seconds of the simulation showing the analytical solution overlapping with the numerical RK solution.

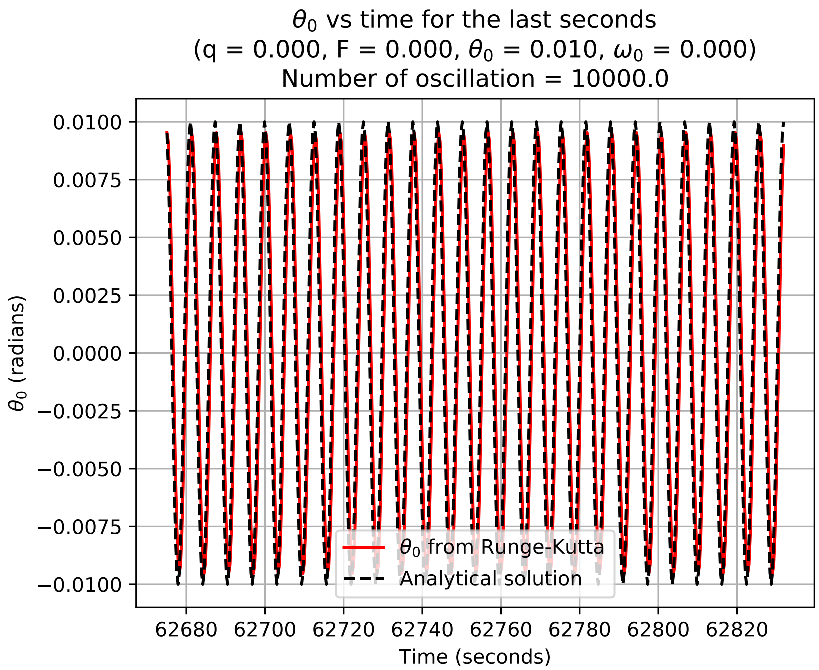

Figure 2: In the last seconds of the simulation and around 10 000 natural periods of oscillation, the numerical solution diverges from the analytical solution by around 10%.

	Time (seconds)	Theta (radians)	Analytical theta (radians)	Omega (rad s^(-1))	Energy (Joules)
0	0.000000	1.000000e-02	1.000000e-02	0.000000e+00	0.004808
1	0.314161	9.510568e-03	9.510560e-03	-3.090138e-03	0.004808
2	0.628322	8.090177e-03	8.090151e-03	-5.877800e-03	0.004808
3	0.942483	5.877864e-03	5.877814e-03	-8.090112e-03	0.004808
4	1.256643	3.090183e-03	3.090110e-03	-9.510511e-03	0.004808
5	1.570804	1.186214e-08	-7.854021e-08	-9.999950e-03	0.004808
199995	62830.596428	-3.090225e-04	3.090110e-03	9.530645e-03	0.004372
199996	62830.910589	2.651249e-03	5.877814e-03	9.159669e-03	0.004372
199997	62831.224750	5.351993e-03	8.090151e-03	7.892075e-03	0.004372
199998	62831.538911	7.528843e-03	9.510560e-03	5.851953e-03	0.004372
199999	62831.853072	8.968714e-03	1.000000e-02	3.239012e-03	0.004372

Figure 3: A dataframe containing all the data of the simulation showing the first and last 5 time points. It is evident that the numerical solution diverges from the analytical towards the end. Energy is also lost in numerical error.

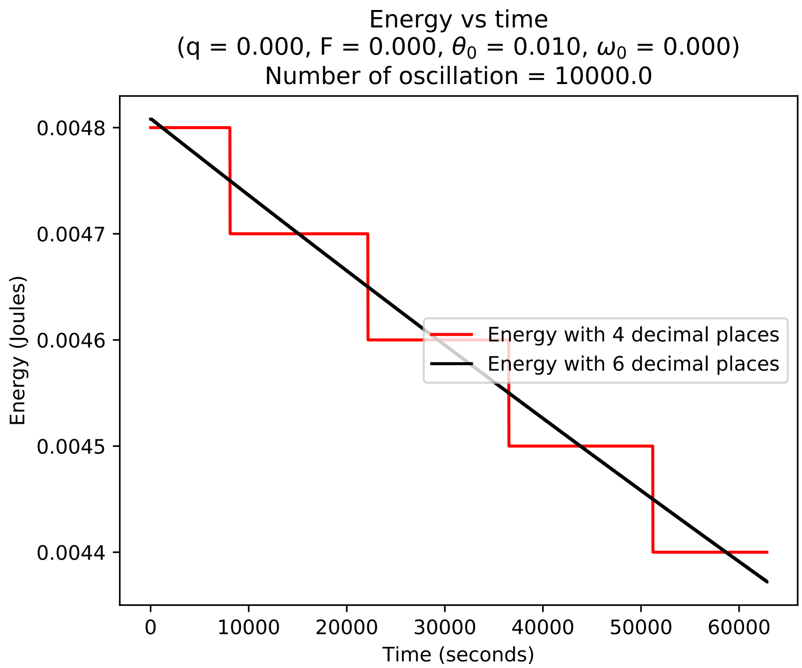

Figure 4: Numerical error in θ and ω cause the energy to slowly decrease and by the end of the simulation the decrease is 9.07%.

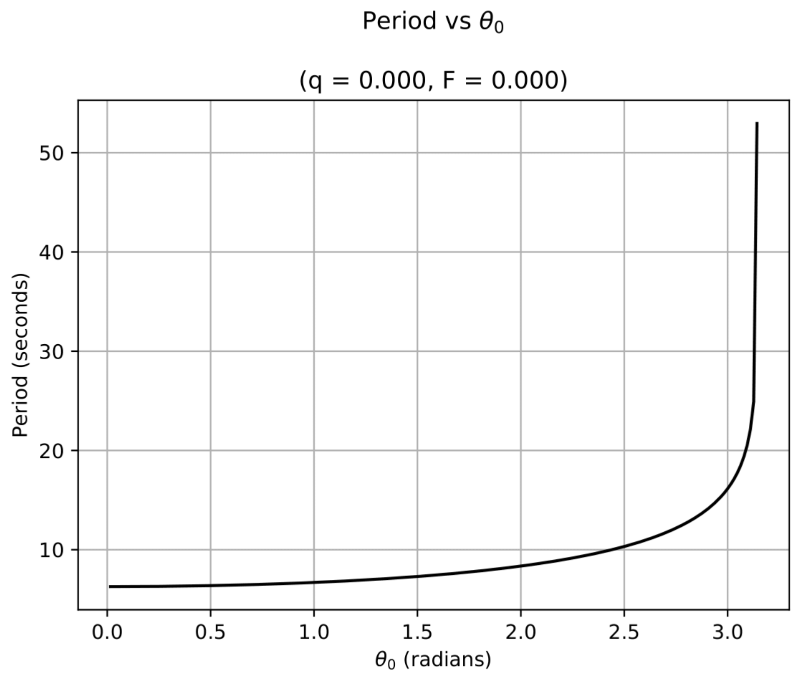

Figure 5: Period vs initial displacement showing a very sharp increase in period after a large initial angle, which implies that the system tends to aperiodicity. Using the data for this graph the period for $\theta_0 = \frac{\pi}{2}$ was found to be 7.43 seconds while the natural period for small oscillations is 6.28 seconds.

2 Supplementary Task 1: Sensitivity to initial conditions (pendulum_stability.py)

The purpose of this code is to investigate how much two solutions diverge when the initial displacement is perturbed very slightly. Of course since this is a chaotic system, the two solutions diverge very fast. Plots are presented in Figures 6 and 7. A dataframe shown in Figure 8 is again produced which shows the two solutions diverging after around 55 seconds into the simulation.

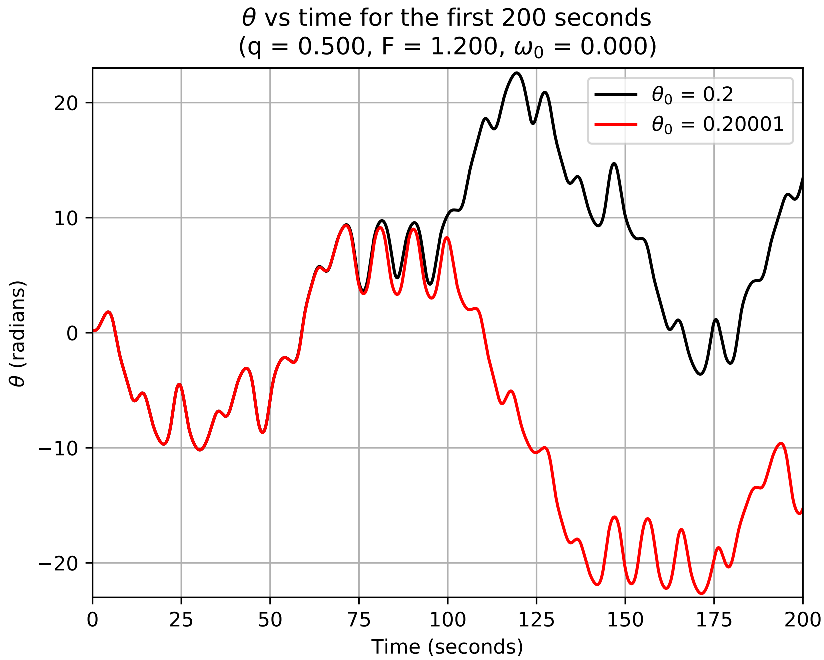

Figure 6: The two solutions are seen to diverge fast as the system exhibits chaos and is very sensitive to initial conditions. The θ is not constrained to a 2π range for clarity.

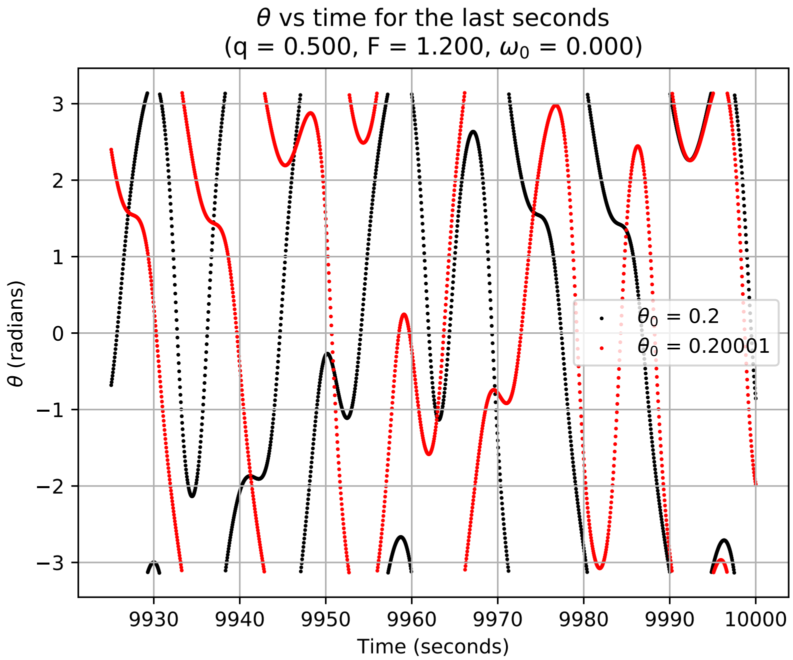

Figure 7: The last seconds of the simulation with corrected θ shows how the solutions evolved to a very different state.

	Time (seconds)	Theta (radians)	Theta perturbed (radians)	Omega (rad s^-1)	Omega perturbed (rad s^-1)	Theta percentage difference (%)	Omega percentage difference (%)
0	0.000000	0.200000	0.200010	0.000000	0.000000	5.000000e-03	NaN
1	0.050000	0.199770	0.199780	-0.008815	-0.008815	4.999668e-03	5.488080e-03
2	0.100001	0.199155	0.199165	-0.015417	-0.015418	4.997023e-03	6.190869e-03
3	0.150001	0.198265	0.198275	-0.019849	-0.019851	4.989609e-03	7.110152e-03
4	0.200001	0.197206	0.197216	-0.022159	-0.022161	4.974981e-03	8.364828e-03
5	0.250001	0.196083	0.196093	-0.022401	-0.022403	4.950720e-03	1.018043e-02
6	0.300002	0.194999	0.195009	-0.020635	-0.020638	4.914466e-03	1.304314e-02
7	0.350002	0.194052	0.194062	-0.016928	-0.016931	4.863983e-03	1.822951e-02
8	0.400002	0.193338	0.193347	-0.011350	-0.011354	4.797239e-03	3.051198e-02
9	0.450002	0.192947	0.192956	-0.003978	-0.003982	4.712506e-03	9.610369e-02
10	0.500003	0.192968	0.192977	0.005109	0.005104	4.608470e-03	8.151392e-02
11	0.550003	0.193485	0.193494	0.015824	0.015820	4.484337e-03	2.835657e-02
12	0.600003	0.194576	0.194585	0.028080	0.028075	4.339929e-03	1.706271e-02
13	0.650003	0.196317	0.196325	0.041784	0.041779	4.175749e-03	1.214824e-02
14	0.700004	0.198777	0.198785	0.056841	0.056836	3.993006e-03	9.397077e-03
15	0.750004	0.202022	0.202030	0.073152	0.073146	3.793593e-03	7.637578e-03
1065	53.250266	-2.290794	-2.294993	0.335433	0.332816	1.833157e-01	7.801653e-01
1066	53.300267	-2.274507	-2.278839	0.316023	0.313333	1.904618e-01	8.512764e-01
1067	53.350267	-2.259193	-2.263662	0.296491	0.293728	1.977878e-01	9.320583e-01
1068	53.400267	-2.244859	-2.249467	0.276855	0.274018	2.052875e-01	1.024630e+00
1069	53.450267	-2.231509	-2.236261	0.257135	0.254225	2.129539e-01	1.131756e+00
1070	53.500268	-2.219146	-2.224046	0.237354	0.234370	2.207801e-01	1.257106e+00
1071	53.550268	-2.207774	-2.212824	0.217536	0.214478	2.287586e-01	1.405660e+00
1072	53.600268	-2.197393	-2.202598	0.197707	0.194574	2.368818e-01	1.584347e+00
1073	53.650268	-2.188003	-2.193367	0.177894	0.174686	2.451423e-01	1.803086e+00
1074	53.700269	-2.179603	-2.185129	0.158126	0.154842	2.535323e-01	2.076597e+00
1075	53.750269	-2.172189	-2.177881	0.138433	0.135072	2.620445e-01	2.427694e+00
1076	53.800269	-2.165758	-2.171620	0.118846	0.115407	2.706717e-01	2.893704e+00
1077	53.850269	-2.160302	-2.166338	0.099397	0.095878	2.794070e-01	3.540180e+00
1078	53.900270	-2.155815	-2.162029	0.080118	0.076518	2.882440e-01	4.493734e+00
1079	53.950270	-2.152287	-2.158683	0.061042	0.057358	2.971769e-01	6.034643e+00
1080	54.000270	-2.149707	-2.156289	0.042202	0.038433	3.062005e-01	8.931323e+00

Figure 8: The above dataframe holds the data for the two solutions with similar initial conditions and shows how the percentage difference in θ and ω grows after 53 seconds indicating that the perturbation starts to diverge. This behaviour is what characterizes chaotic systems.

3 Core Task 2: The driven, damped pendulum (pendulum_ode.py, period_forcing_term.py)

Using pendulum_ode.py the system was investigated by tweaking the damping coefficient q and the forcing term F, while period_forcing_term.py was used to plot the period vs the forcing term F and produce a dataframe for the points in the graph. The latter was also used to produce plots of θ and ω vs time for the first 100 seconds for all values of F. The relevant plots are included in Figures 9 to 35, along with relevant explanations in their caption.

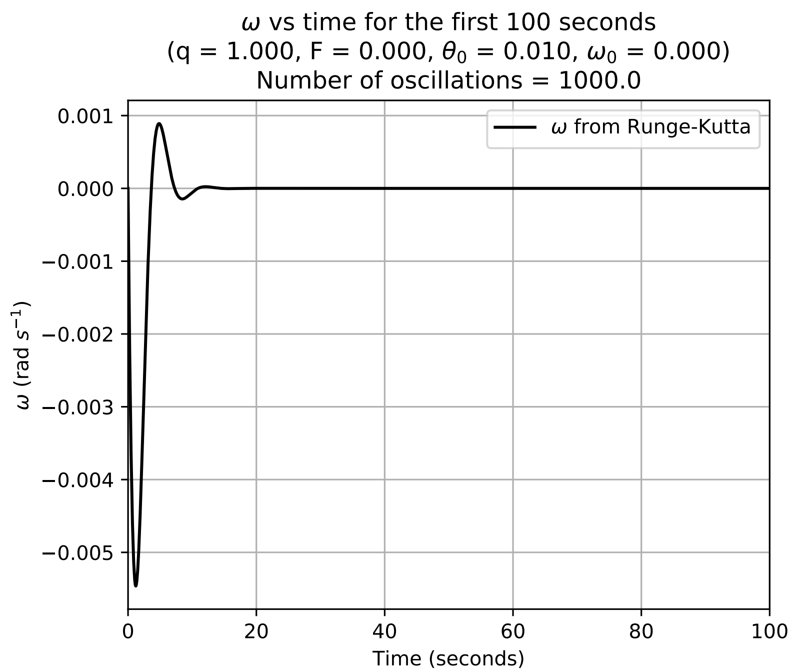

Figure 9: Damping slows the pendulum to a stop very quickly.

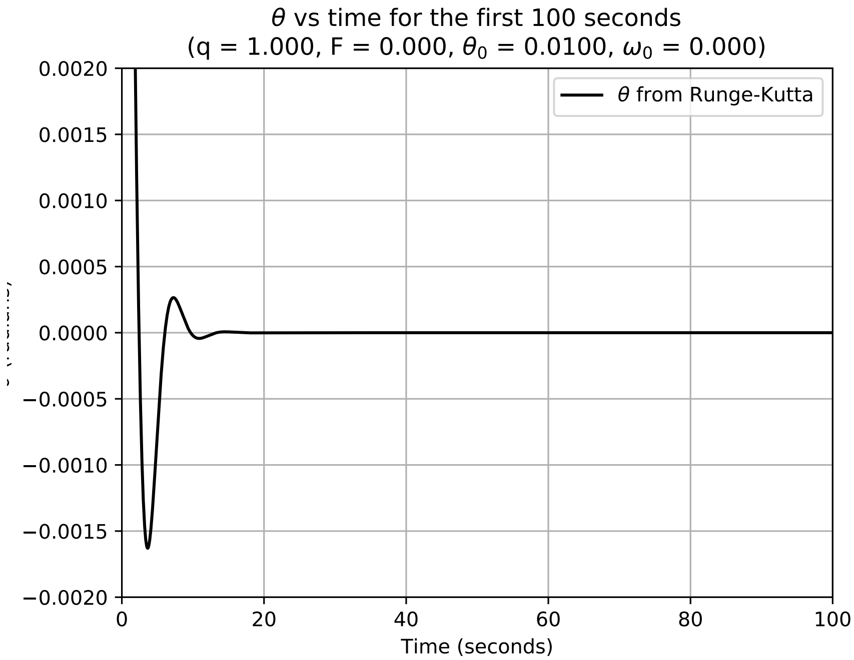

Figure 10: Damping here is considered light since the pendulum oscillates before decaying to vanishing amplitude.

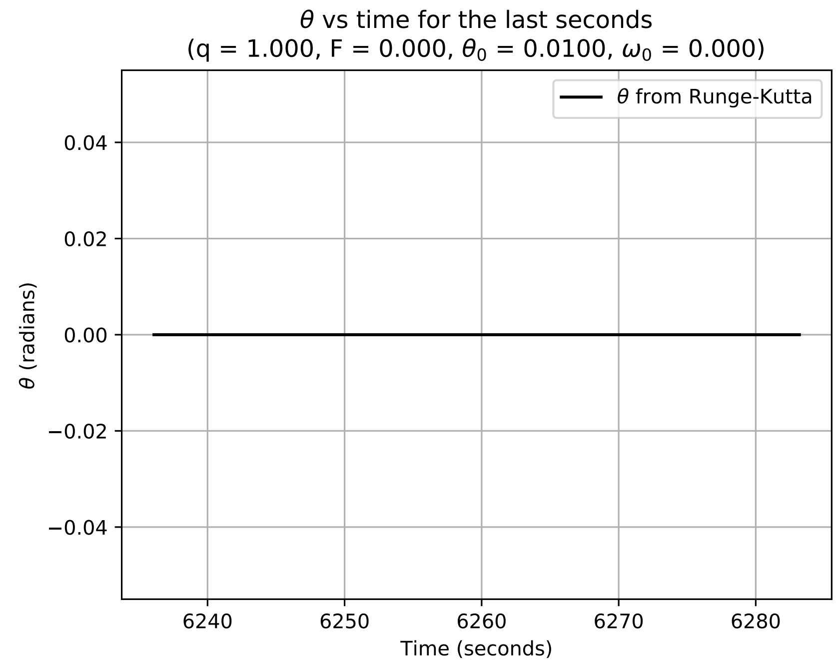

Figure 11: The pendulum stops moving completely by the end of the 10 000 oscillations.

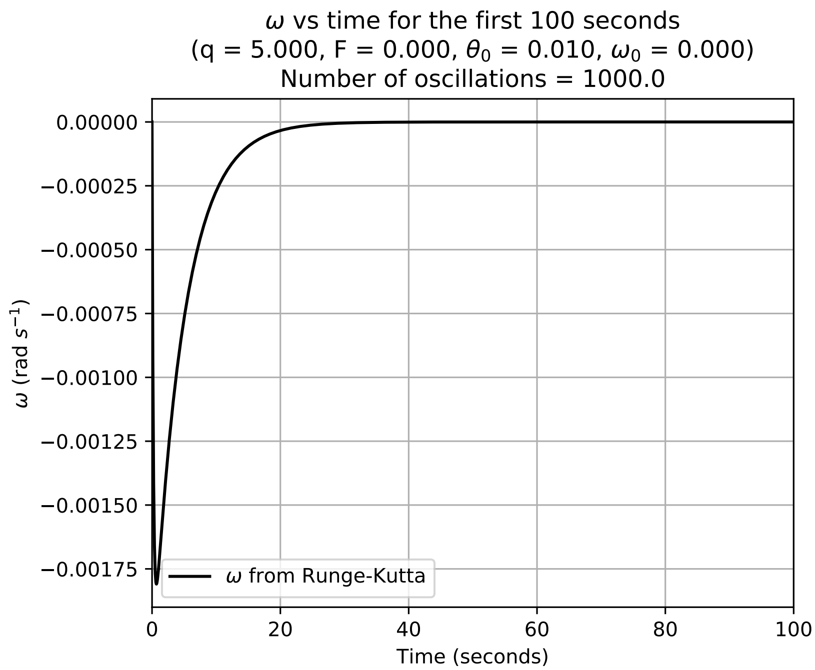

Figure 12: This level of damping reduces angular velocity to vanishing magnitude very fast.

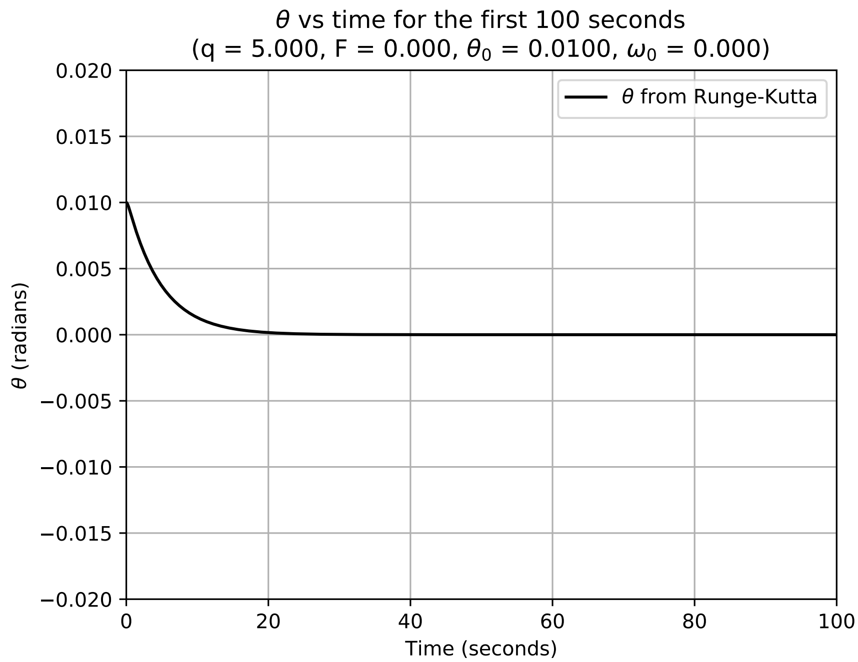

Figure 13: The damping can be considered critical as the pendulum returns to equilibrium without oscillating.

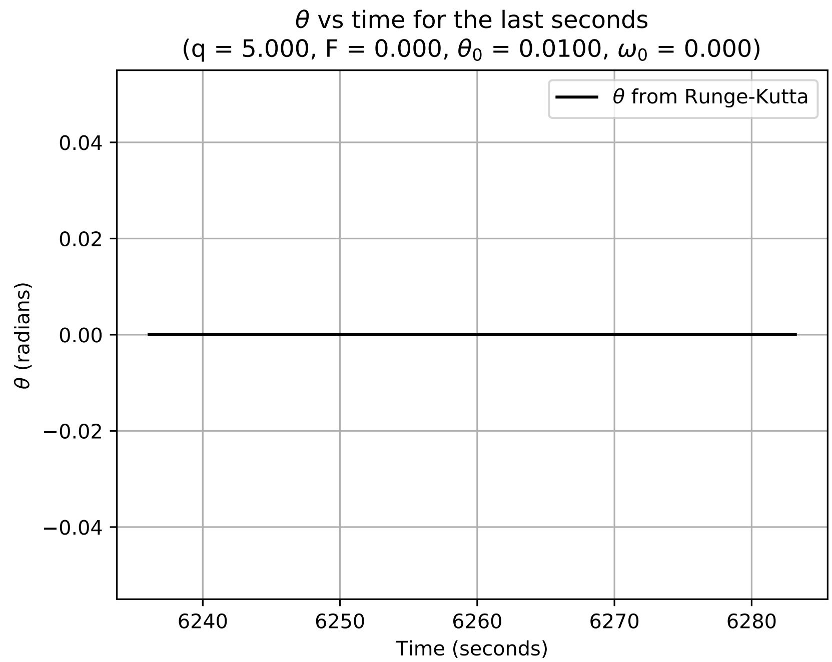

Figure 14: Complete amplitude decay by the end of the simulation.

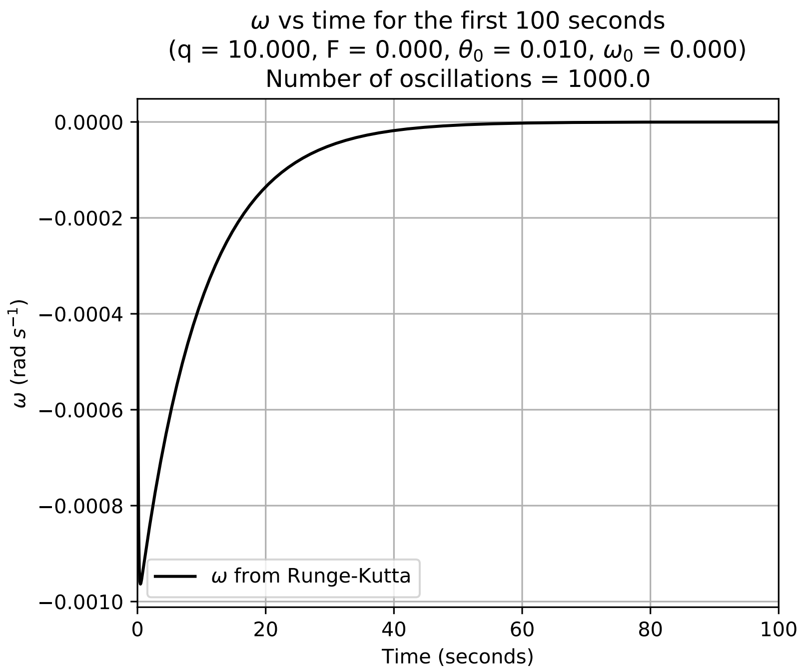

Figure 15: A big damping factor with no forcing term shows fast angular velocity decay.

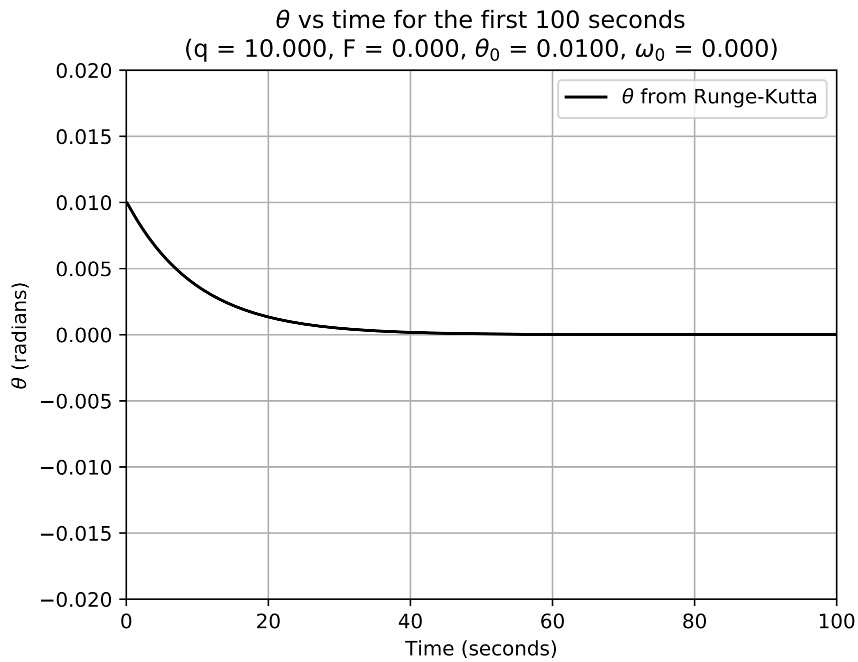

Figure 16: Here heavy damping is experienced by the pendulum as it does not oscillate and returns slowly to equilibrium.

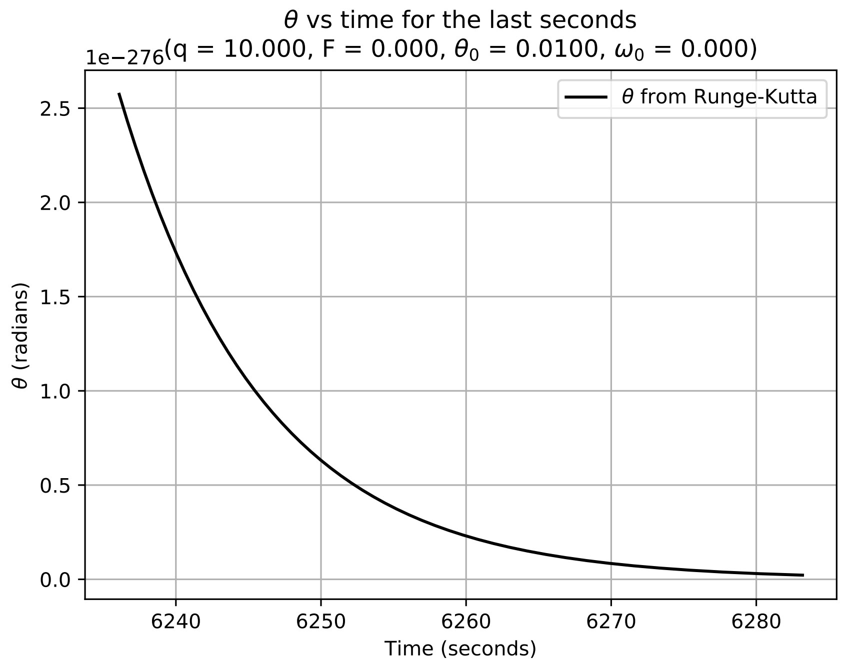

Figure 17: The factor on the y-axis implies that the amplitude has vanished to 0 radians.

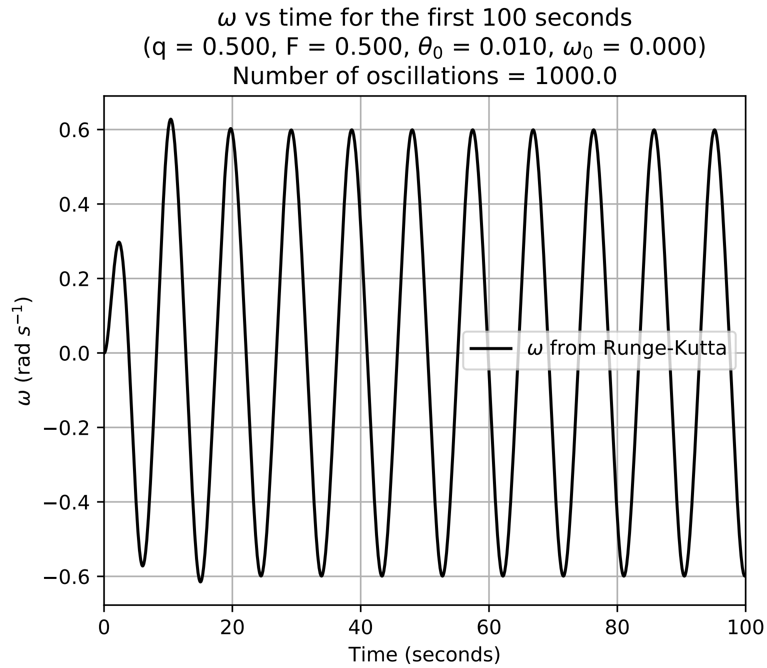

Figure 18: When a force is introduced the system builds up angular velocity to a steady state after 20 seconds.

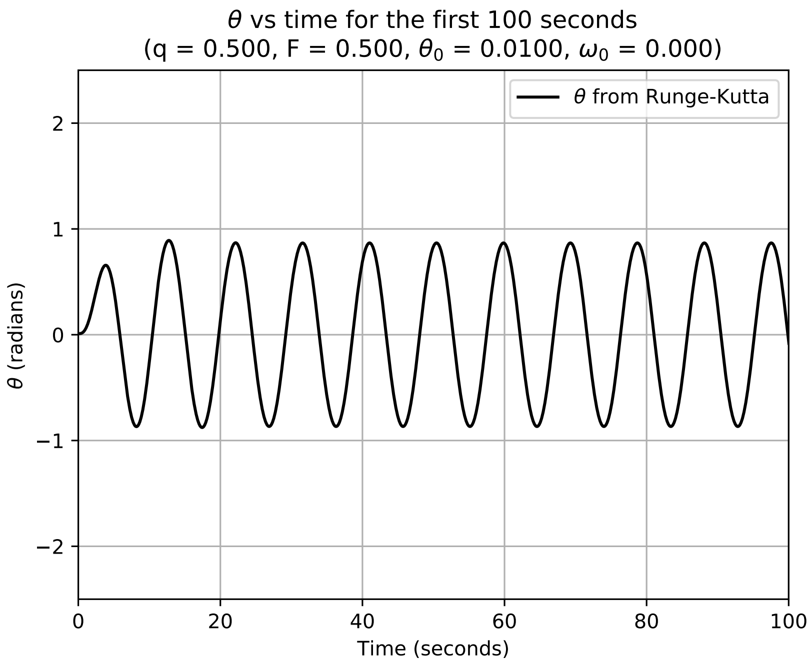

Figure 19: The input energy from the forcing term is equal to the dissipation of energy from the damping factor in the steady state shown after around 20 seconds.

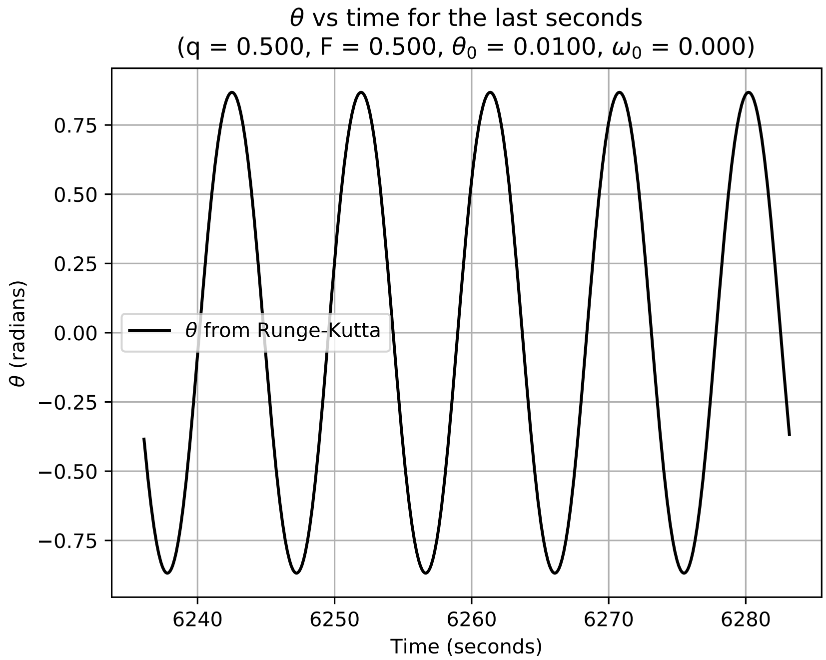

Figure 20: Even after 10 000 oscillations the maximum amplitude is constant and the system remains in steady state.

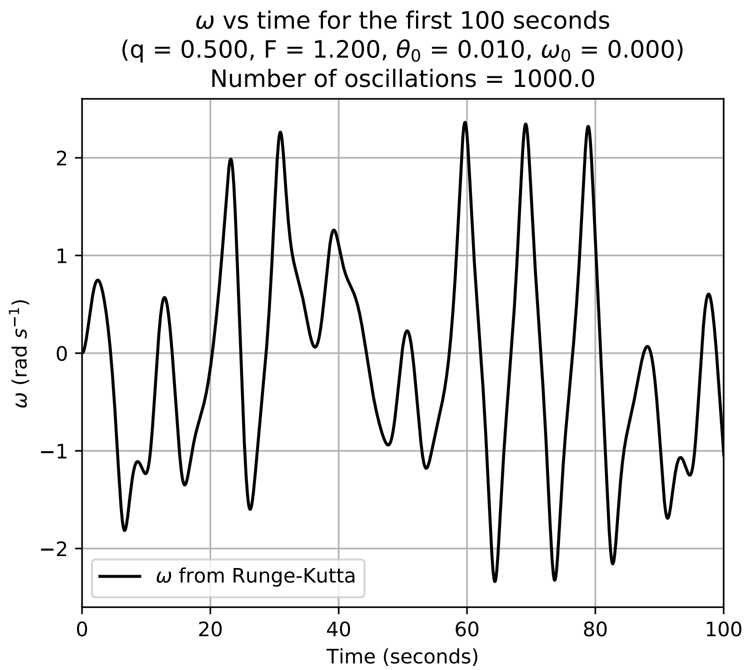

Figure 21: With this combination of parameters the system experiences a very large periodicity and so angular velocity shows no clear pattern in the first 100 seconds.

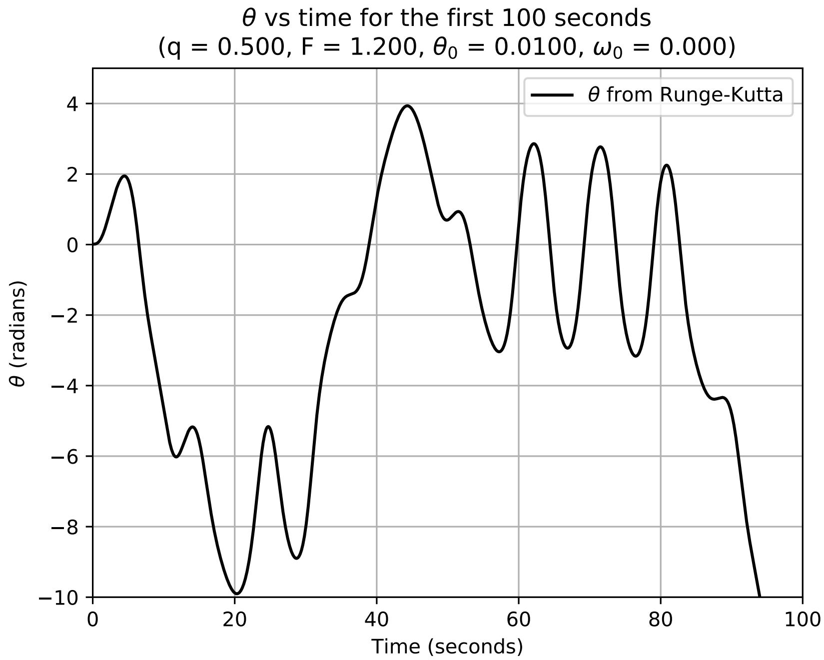

Figure 22: After 100 seconds θ tends towards larger negative values which implies it is mostly spinning in one direction.

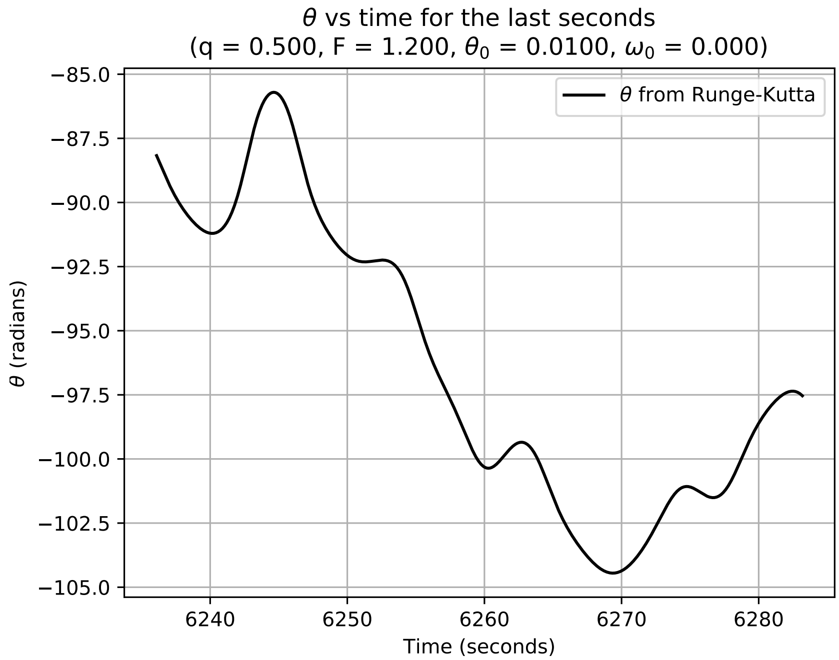

Figure 23: The end of the simulation shows again no clear pattern, indicating that these parameters cause the system to become unstable.

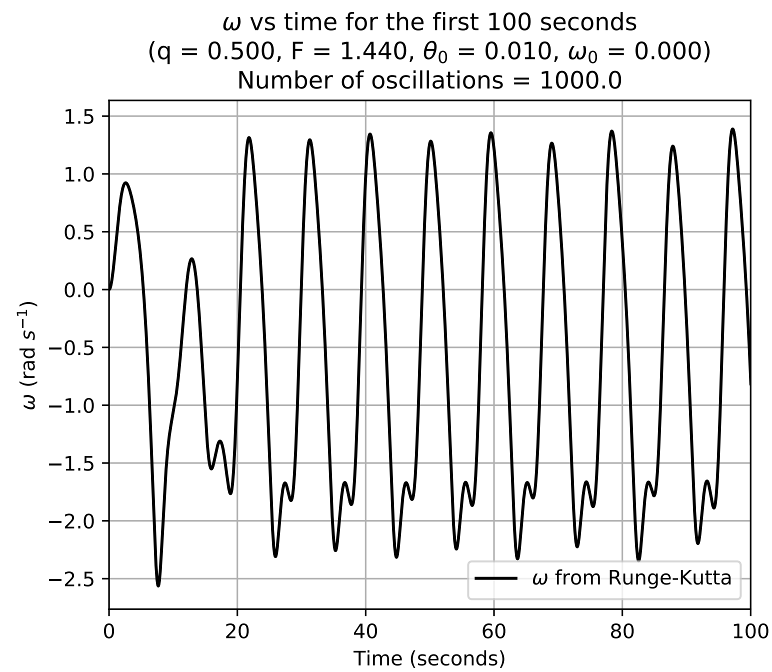

Figure 24: Using this particular forcing term gives a periodic structure to angular velocity in steady state after 20 seconds.

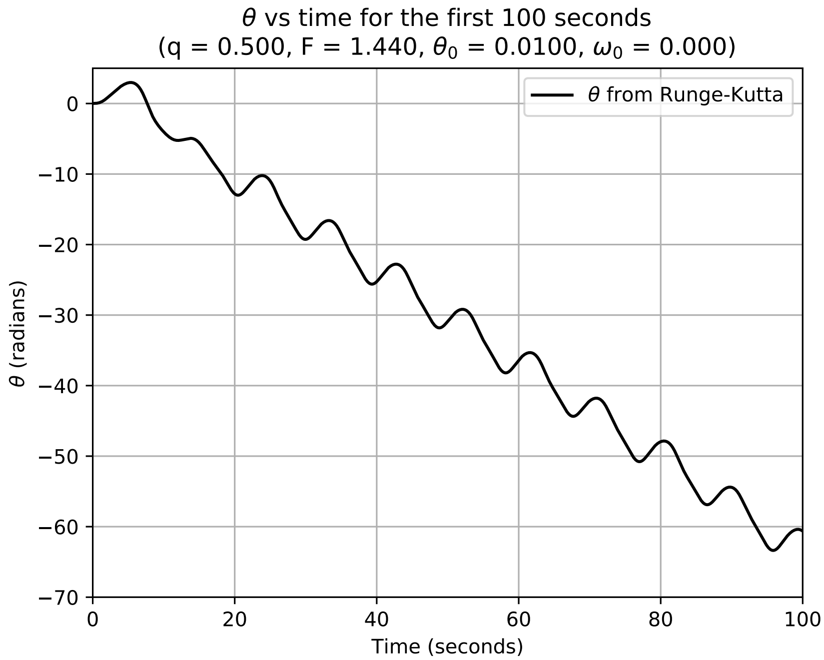

Figure 25: Since angular velocity has a bigger magnitude in one direction, θ grows large only in one direction which again implies the pendulum is spinning only in a specific direction.

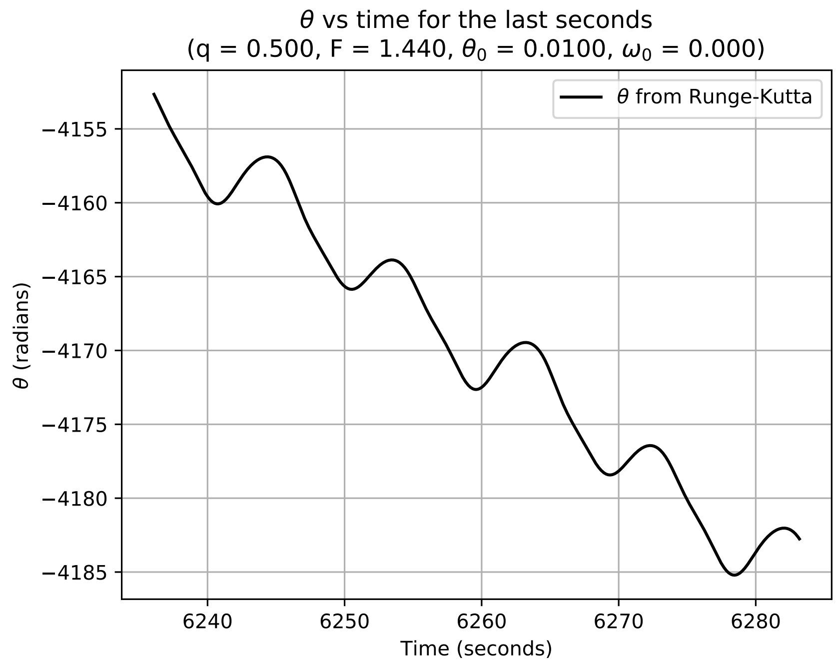

Figure 26: By the end of the simulation θ has become very large and negative with periodic loss of gradient possibly when the pendulum reaches the top of each oscillation as it spins only in one direction.

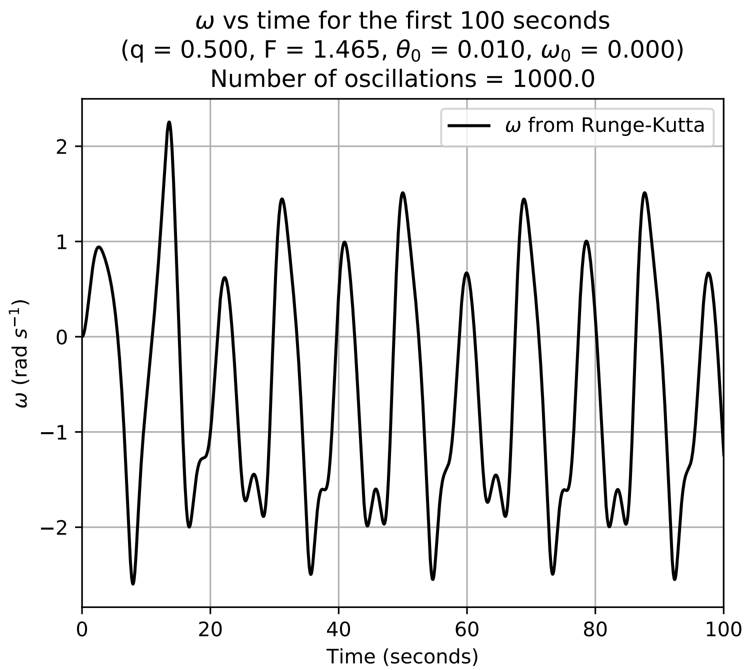

Figure 27: It appears that the period has doubled when using F = 1.465 rather than F = 1.44. In chaos theory this is called period doubling bifurcation and usually occurs when the parameter change causes the system to move from one limit cycle to another limit cycle with twice the period in the phase space.

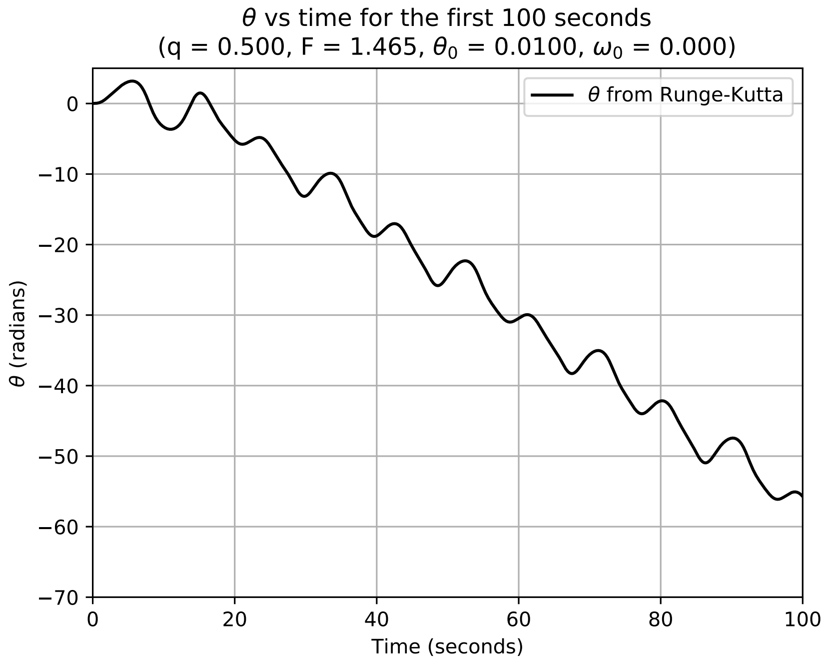

Figure 28: It is evident that as we increase the force, the pendulum will tend to rotate in one direction only.

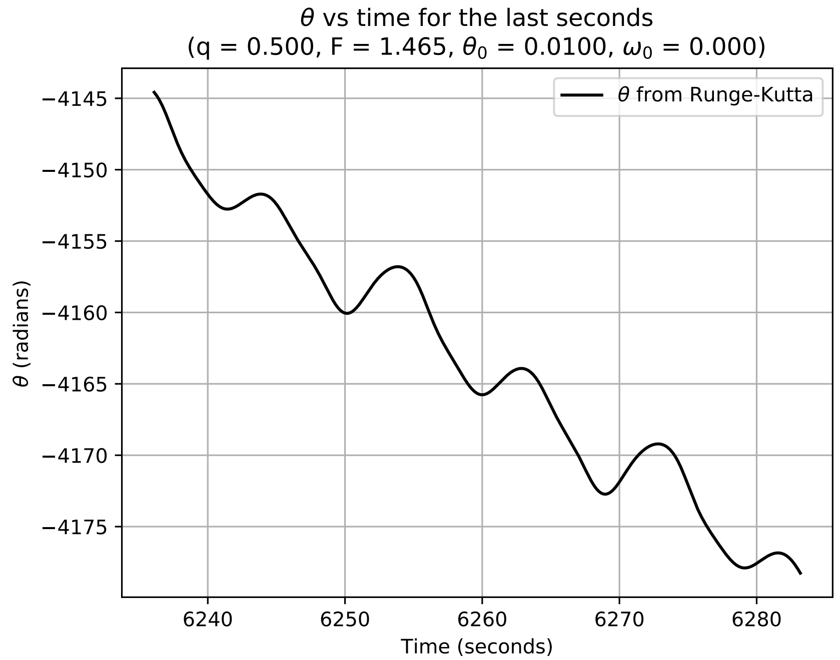

Figure 29: Large negative θ values indicate rotation mostly in one direction.

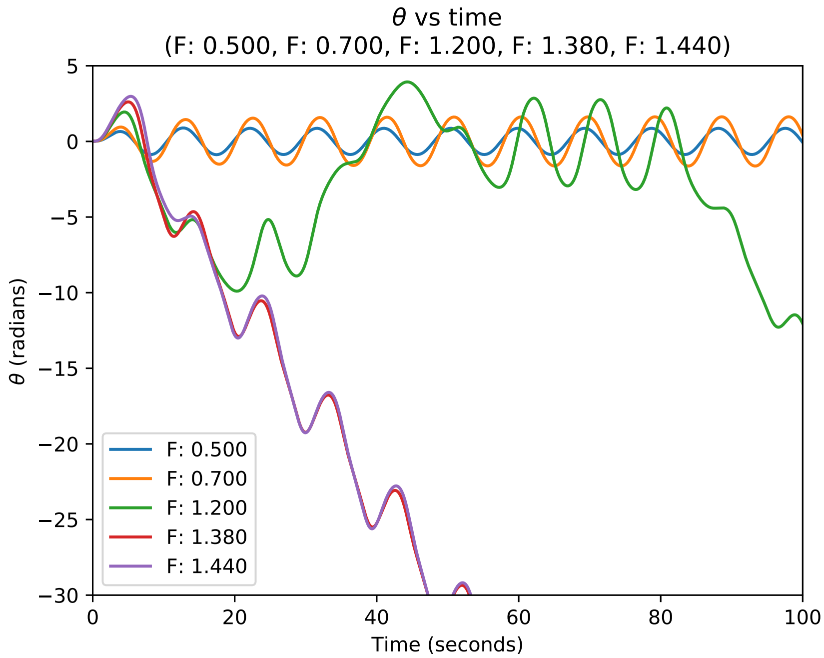

Figure 30: θ vs time for the first 100 seconds using 5 different forcing terms. The behaviour for some is completely different, while for others it is quite similar.

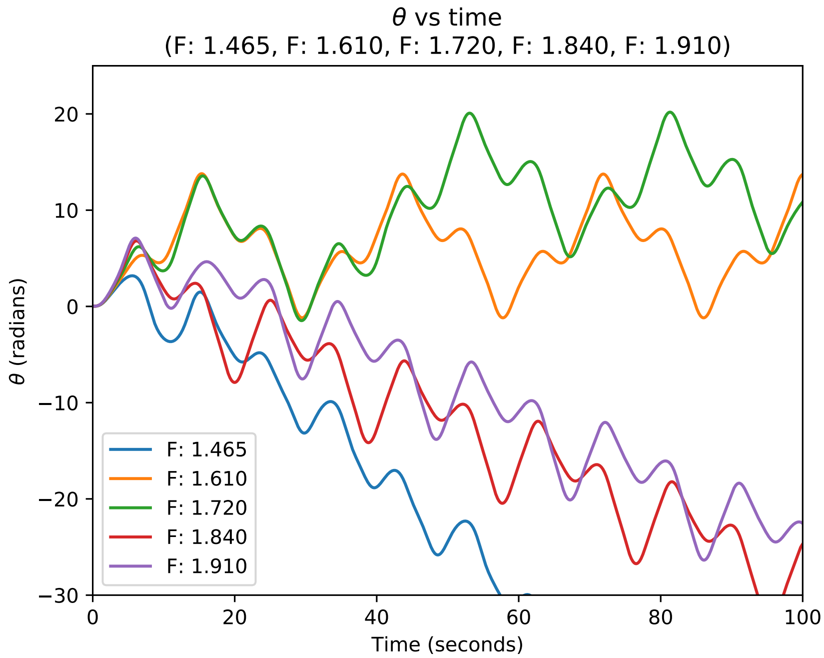

Figure 31: The behaviour diverges for each system with different forcing terms, even when the F values are similar.

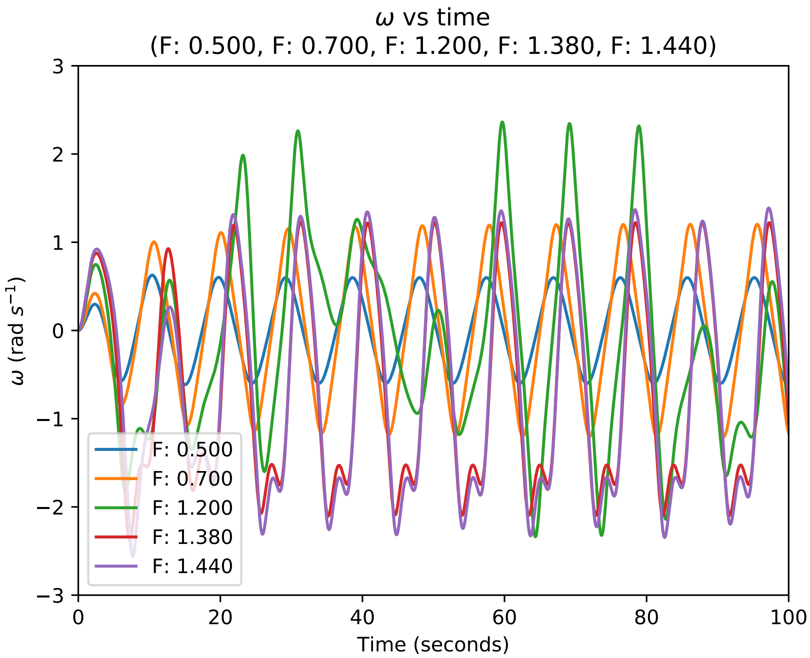

Figure 32: Angular velocity shows some similarities with pairs of F values while for others it diverges to different behaviour.

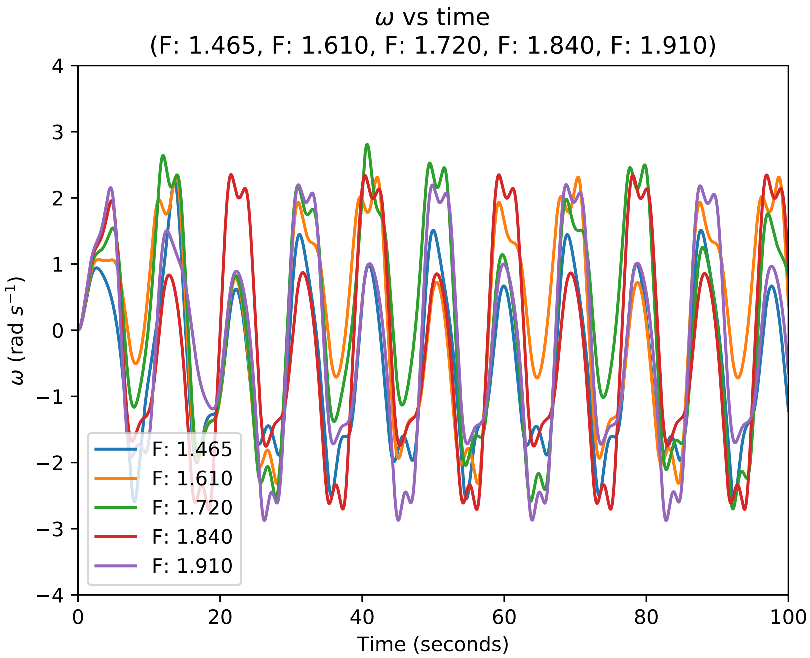

Figure 33: Although some periodicity is visible, it is clear that the system is sensitive to the forcing parameter.

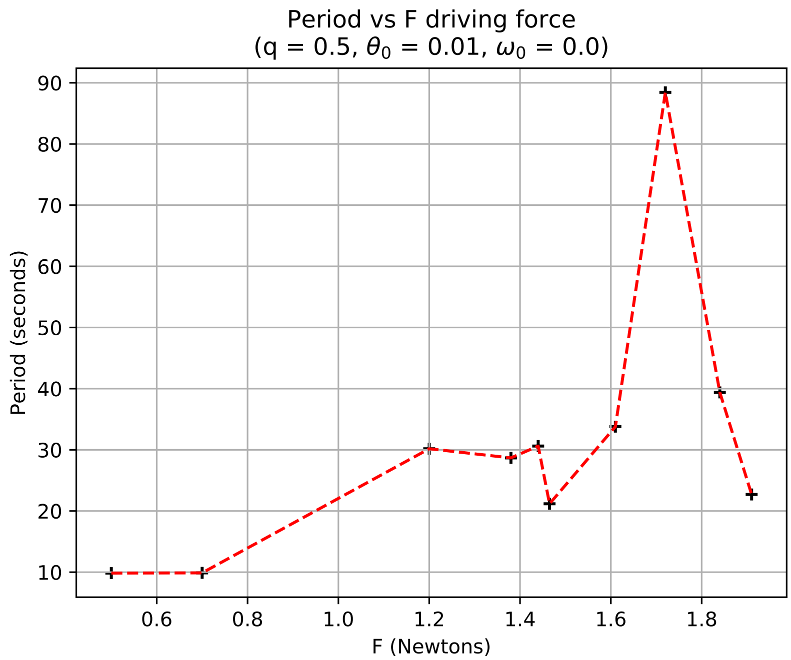

Figure 34: Period vs forcing term shows that some regions of F values cause the system to become unstable and sharply increase the period of oscillation.

	F (Newtons)	Period (seconds)
0	0.500	9.848064
1	0.700	9.885380
2	1.200	30.174814
3	1.380	28.680143
4	1.440	30.620153
5	1.465	21.180995
6	1.610	33.791665
7	1.720	88.443027
8	1.840	39.386864
9	1.910	22.701802

Figure 35: Data showing how F values affect the period.

4 Supplementary Task 2: Phase portrait (pendulum_stability.py)

In this task plots were made in phase space to observe the change in behaviour between two systems with a very slight change in initial displacement. In Figure 36 the plot presents the sensitivity of the pendulum, indicating that it is a chaotic system. In Figure 37 a stable limit cycle is shown and in Figure 38 the trivial attractor point of equilibrium is illustarted. In Figure 39 the Lyapunov instability of a source was plot using a forcing term with 0 damping.

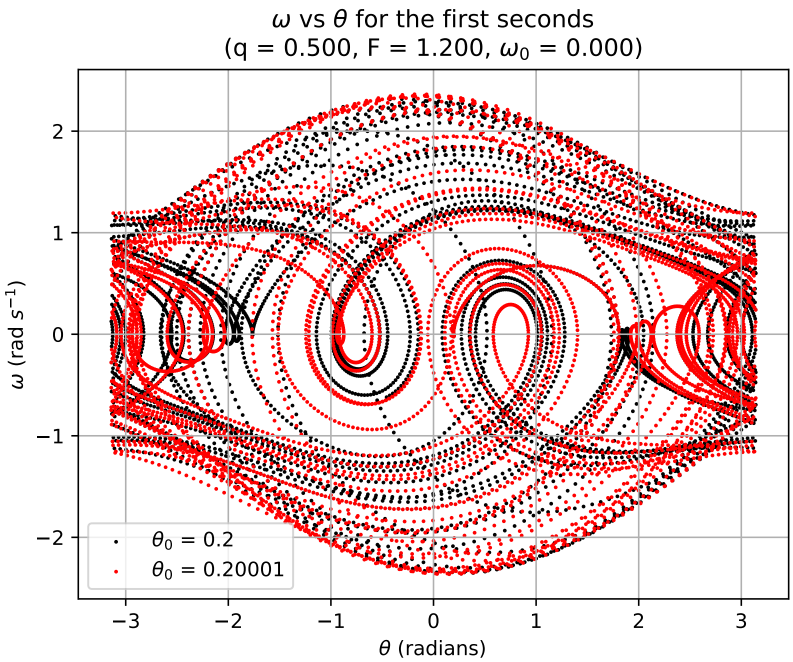

Figure 36: Phase space plots of the pendulum with two almost identical initial conditions, indicating a repeller.

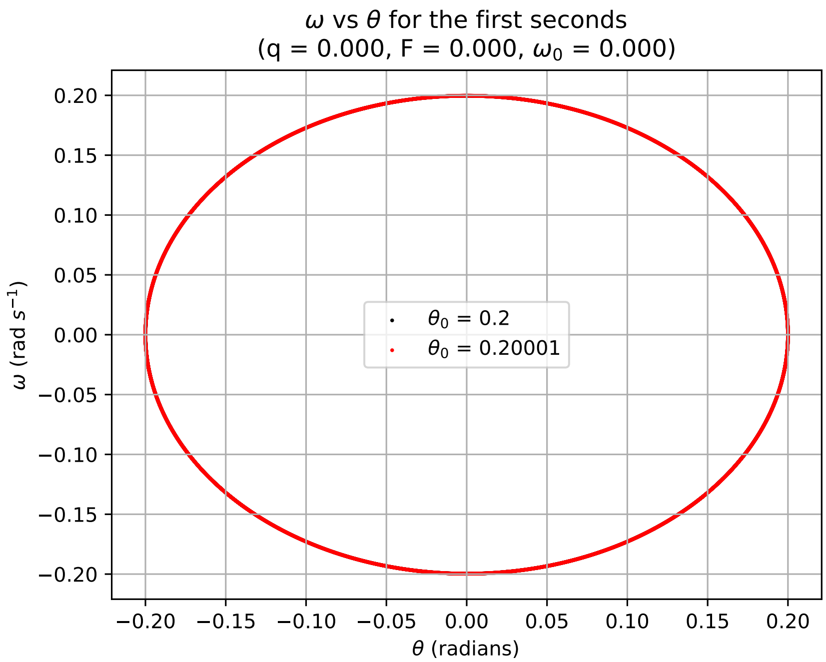

Figure 37: A stable limit cycle which is Lyapunov stable.

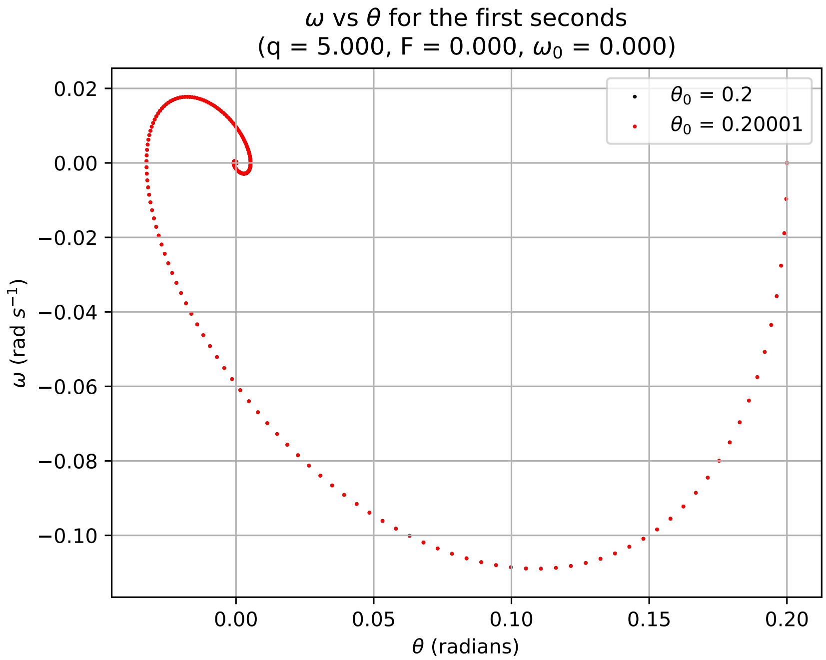

Figure 38: A sink attractor which is Lyapunov asymptotically stable.

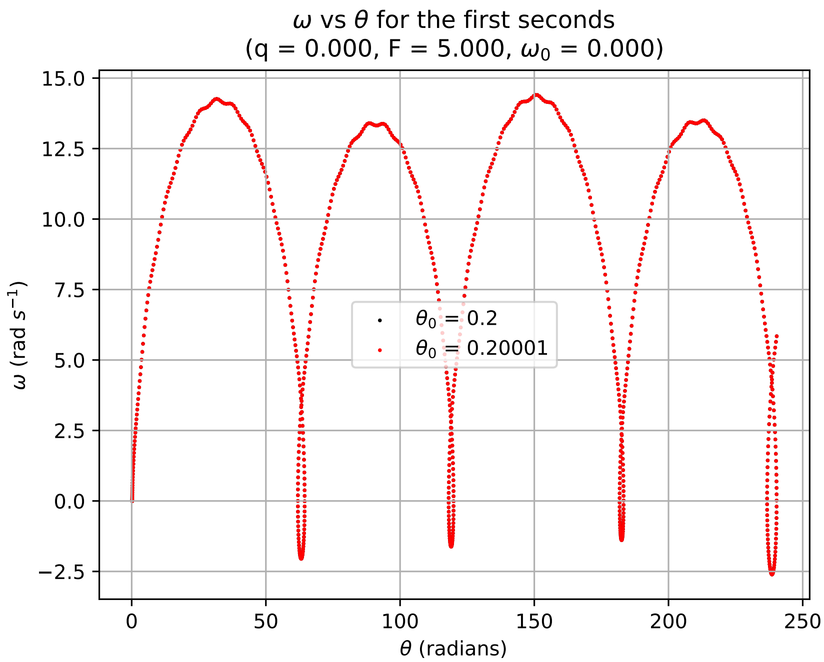

Figure 39: A source which is Lyapunov unstable.