Exam Review/Revision For Math Data Analytics

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Linear equations into matrices

Google search or chatgpt prompt: "How to represent a linear equation as a matrix?"
Video: Matrix equations and systems | Matrices | Precalculus | Khan Academy

Exercises:

Given a system of linear algebraic equations. Write this system in the matrix form $\mathbf{Ax} = \mathbf{b}$.

1) $5x - 2y = -6$
$-x + 5y = 15$

2) $-3x - 4y = 20$
$3x - 5y = 25$

3) $x + 3y - 2z = -11$
$-2x - 5y + 3z = 17$
$4x - z = 1$

4) $-2x - 4y - 5z = 11$
$-x + 4z = -25$
$-3x - 5y + z = -25$

Given in matrix form, write the system of linear algebraic equations

5) $\left[\begin{array}{cc|c} 3 & 4 & 1 \\ -3 & 2 & 23 \end{array}\right]$

6) $\left[\begin{array}{cc|c} -5 & 1 & -16 \\ 1 & 5 & -2 \end{array}\right]$

7) $\left[\begin{array}{ccc|c} 3 & -1 & 1 & 8 \\ 0 & -1 & 2 & -10 \\ -2 & 2 & 2 & -8 \end{array}\right]$

8) $\left[\begin{array}{ccc|c} -5 & -4 & 3 & -8 \\ 1 & 0 & 4 & 0 \\ 3 & -5 & 5 & -10 \end{array}\right]$

Solutions:

1) $5x - 2y = -6$
$-x + 5y = 15$

$$\left[\begin{array}{cc|c} 5 & -2 & -6 \\ -1 & 5 & 15 \end{array}\right]$$

2) $-3x - 4y = 20$
$3x - 5y = 25$

$$\left[\begin{array}{cc|c} -3 & -4 & 20 \\ 3 & -5 & 25 \end{array}\right]$$

3) $x + 3y - 2z = -11$
$-2x - 5y + 3z = 17$
$4x - z = 1$

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & -11 \\ -2 & -5 & 3 & 17 \\ 4 & 0 & -1 & 1 \end{array}\right]$$

4) $-2x - 4y - 5z = 11$
$-x + 4z = -25$
$-3x - 5y + z = -25$

$$\left[\begin{array}{ccc|c} -2 & -4 & -5 & 11 \\ -1 & 0 & 4 & -25 \\ -3 & -5 & 1 & -25 \end{array}\right]$$

5) $\left[\begin{array}{cc|c} 3 & 4 & 1 \\ -3 & 2 & 23 \end{array}\right]$

$3x + 4y = 1$
$-3x + 2y = 23$

6) $\left[\begin{array}{cc|c} -5 & 1 & -16 \\ 1 & 5 & -2 \end{array}\right]$

$-5x + y = -16$
$x + 5y = -2$

7) $\left[\begin{array}{ccc|c} 3 & -1 & 1 & 8 \\ 0 & -1 & 2 & -10 \\ -2 & 2 & 2 & -8 \end{array}\right]$

$3x - y + z = 8$
$-y + 2z = -10$
$-2x + 2y + 2z = -8$

8) $\left[\begin{array}{ccc|c} -5 & -4 & 3 & -8 \\ 1 & 0 & 4 & 0 \\ 3 & -5 & 5 & -10 \end{array}\right]$

$-5x - 4y + 3z = -8$
$x + 4z = 0$
$3x - 5y + 5z = -10$

Adding/multiplying matrices

Google search or chatgpt prompt: “Rules to add and multiply matrices?”
Video: Scalar Multiplication of Matrices and Matrix Operations
How To Multiply Matrices - Quick & Easy!
Defined and undefined matrix operations | Matrices | Precalculus | Khan Academy

Exercises:

Simplify. Write "undefined" for expressions that are undefined.

1) $\begin{bmatrix} 2 & -1 \\ -6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 4 \\ -3 & -5 \end{bmatrix}$

2) $\begin{bmatrix} 2 & 6 \\ -6 & 4 \end{bmatrix} \cdot \left(\begin{bmatrix} 5 & 3 \\ -6 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \right)$

3) $\begin{bmatrix} -1 & 5 \\ 5 & -5 \end{bmatrix} \cdot \begin{bmatrix} -3 & 6 \\ -3 & 0 \end{bmatrix}$

4) $\begin{bmatrix} 1 & -6 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 3 \\ -2 \end{bmatrix}$

5) $\begin{bmatrix} -2 \\ -3 \\ -6 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ 6 \\ 0 \\ -3 \end{bmatrix}$

6) $-4 \cdot \left(\begin{bmatrix} -3 & -6 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 6 \\ -1 & -4 \end{bmatrix} \right)$

7) $\begin{bmatrix} 3 & 1 & 3 \\ 0 & 5 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -6 & -1 \\ 1 & 1 & 4 \end{bmatrix}$

8) $-5\left(\begin{bmatrix} 2 \\ -1 \\ -6 \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix} \right)$

Solutions:

1) $\begin{bmatrix} 2 & -1 \\ -6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 4 \\ -3 & -5 \end{bmatrix}$

$\begin{bmatrix} 11 & 13 \\ -27 & -29 \end{bmatrix}$

2) $\begin{bmatrix} 2 & 6 \\ -6 & 4 \end{bmatrix} \cdot \left(\begin{bmatrix} 5 & 3 \\ -6 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \right)$

$\begin{bmatrix} -12 & 22 \\ -52 & -22 \end{bmatrix}$

3) $\begin{bmatrix} -1 & 5 \\ 5 & -5 \end{bmatrix} \cdot \begin{bmatrix} -3 & 6 \\ -3 & 0 \end{bmatrix}$

$\begin{bmatrix} -12 & -6 \\ 0 & 30 \end{bmatrix}$

4) $\begin{bmatrix} 1 & -6 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 3 \\ -2 \end{bmatrix}$

Undefined

5) $\begin{bmatrix} -2 \\ -3 \\ -6 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ 6 \\ 0 \\ -3 \end{bmatrix}$

$\begin{bmatrix} -6 \\ 3 \\ -6 \\ -1 \end{bmatrix}$

6) $-4 \cdot \left(\begin{bmatrix} -3 & -6 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 6 \\ -1 & -4 \end{bmatrix} \right)$

$\begin{bmatrix} -48 & -24 \\ 24 & 40 \end{bmatrix}$

7) $\begin{bmatrix} 3 & 1 & 3 \\ 0 & 5 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -6 & -1 \\ 1 & 1 & 4 \end{bmatrix}$

$\begin{bmatrix} 6 & 10 & 16 \\ 1 & -30 & -5 \end{bmatrix}$

8) $-5\left(\begin{bmatrix} 2 \\ -1 \\ -6 \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix} \right)$

$\begin{bmatrix} -20 \\ 25 \\ 10 \end{bmatrix}$

Gaussian elimination aka Solving linear equations with a matrix

SEVERAL TOPICS depend on knowing how to do gaussian elimination

Google search or chatgpt prompt: “How to solve a linear equation with a matrix? Or how to do gaussian elimination?”

Video:

Gaussian elimination | Lecture 10 | Matrix Algebra for Engineers

For matrix operations you are able to do on a matrix

How To Perform Elementary Row Operations Using Matrices

https://matrixcalc.org/slu.html

Exercises:

2) $\begin{cases} 3x-y+z=-1 \\ 2x+3y+z=4 \\ 5x+4y+2z=5 \end{cases}$

3) $\begin{cases} 2x+y+z=-2 \\ 2x-y+3z=6 \\ 3x-5y+4z=7 \end{cases}$

4) $\begin{cases} 6x+2y-4z=15 \\ -3x-4y+2z=-6 \\ 4x-6y+3z=8 \end{cases}$

5) $\begin{cases} 3x+3z=0 \\ 2x+2y=2 \\ 3y+3z=3 \end{cases}$

6) $\begin{cases} x+3y-3z=12 \\ 3x-y+4z=0 \\ -x+2y-z=1 \end{cases}$

7) $\begin{cases} x+y+z=3 \\ 2x-y-z=0 \\ x+2y-z=-1 \end{cases}$

1) $\begin{cases} 4x+2y-6z=34 \\ 2x+y+3z=3 \\ 6x+3y-3z=37 \end{cases}$

8) $\begin{cases} 2x-4y+z=10 \\ x+2y-z=1 \\ -x-3y+2z=0 \end{cases}$

9) $\begin{cases} 3x+y-z=4 \\ x+2y+2z=5 \\ 4x+y-z=3 \end{cases}$

10) $\begin{cases} 5x+6y-5z=-1 \\ 3x-4y-3z=7 \\ -2x+5y+z=-4 \end{cases}$

11) $\begin{cases} 2x+3z=1 \\ 3x-5y=10 \\ 4y-3z=13 \end{cases}$

12) $\begin{cases} 3y+4z=6 \\ 3x-5z=3 \\ 2x+5y=2 \end{cases}$

13) $\begin{cases} \frac{1}{2}x+\frac{1}{3}y=4 \\ \frac{1}{2}y-\frac{1}{4}z=1 \\ \frac{1}{4}x+\frac{1}{2}z=5 \end{cases}$

14) $\begin{cases} \frac{1}{3}x-\frac{1}{3}z=-2 \\ \frac{1}{6}y+\frac{1}{3}z=7 \\ \frac{2}{3}x+\frac{1}{4}y=9 \end{cases}$

Solutions:

2) $(-1,1,3)$

3) $(-3,0,4)$

4) $\left(2,-\frac{1}{2},-1\right)$

5) $(0,1,0)$

6) $(3,1,-2)$

7) $(1,0,2)$

1) Infinite solutions

8) $(3,-1,0)$

9) $(-1,5,-2)$

10) $(-2,-1,-3)$

11) $(5,1,-3)$

12) $(6,-2,3)$

13) $(4,6,8)$

14) $(9,12,15)$

Transposing, Inverse, Identity (Only focusing on transposing)

Google search or chatgpt prompt: “how to evaluate an expression with a matrix with a (transpose/inverse/identity)”
Video:

Transpose of a matrix | Matrices | Precalculus | Khan Academy

Rule 4 might show up in test, it's the similar for inverse

Exercises:

(1) $(A^T)^T = A$ apply twice to where y

(2) $(A+B)^T = A^T + B^T$

(3) For a scalar c, $(cA)^T = cA^T$

(4) $(AB)^T = B^T A^T$ To prove th

Hard to find problems for these, solve both sides of = sign and they should be the same answer

For the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ verify that

(i) $3(A+B) = 3A + 3B$ (ii) $(A-B)^T = A^T - B^T$.

2. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 0 & 1 \\ 2 & 7 \end{bmatrix}$ verify that $3(A^T - B) = (3A - 3B^T)^T$.

6. If $A = \begin{bmatrix} 11 & 0 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$ verify that $(AB)^T = \begin{bmatrix} 0 & 1 \\ 11 & 3 \\ 22 & 7 \end{bmatrix} = B^T A^T$

Solutions:

(i) $A+B=\begin{bmatrix} 2 & 1 \\ 2 & 5 \end{bmatrix}$; $3(A+B)=\begin{bmatrix} 6 & 3 \\ 6 & 15 \end{bmatrix}$; $3A=\begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$;

$3B=\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$; $3A+3B=\begin{bmatrix} 6 & 3 \\ 6 & 15 \end{bmatrix}$.

(ii) $A-B=\begin{bmatrix} 0 & 3 \\ 4 & 3 \end{bmatrix}$; $(A-B)^T=\begin{bmatrix} 0 & 4 \\ 3 & 3 \end{bmatrix}$; $A^T=\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$;

$B^T=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$; $A^T-B^T=\begin{bmatrix} 0 & 4 \\ 3 & 3 \end{bmatrix}$.

2. $A^T=\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, $A^T-B=\begin{bmatrix} 2 & 0 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$, $3(A^T-B)=\begin{bmatrix} 6 & 0 \\ 6 & 12 \\ 3 & -3 \end{bmatrix}$

$B^T=\begin{bmatrix} -1 & 0 & 2 \\ 4 & 1 & 7 \end{bmatrix}$, $3A-3B^T=\begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}-\begin{bmatrix} -3 & 0 & 6 \\ 12 & 3 & 21 \end{bmatrix}=\begin{bmatrix} 6 & 6 & 3 \\ 0 & 12 & -3 \end{bmatrix}$

Linear independence/dependance

Uses Gaussian Elimination

Google search or chatgpt prompt: “How to determine if vectors are linearly independent with gaussian elimination?”

Video: How to Determine if a Set of Vectors is Linearly Independent [Passing Linear Alge…

Linear Independence with Gaussian Elimination

Exercises:

6 Determine whether the following set of vectors is linearly independent or linearly dependent. If the set is linearly dependent, express one vector in the set as a linear combination of the others.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 7 \\ 11 \end{bmatrix} \right\}.$$

5. Given a set of vectors.

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$$

Determine whether these vectors are linearly independent. If not, choose from the given set a linearly dependent vector and write its representation as a linear combination of other vectors from the set.

Solutions:

6.

Set is not linearly independent

v_1 , v_2 , v_3 are linearly independent

v_4 is linearly dependant

$v_4 = -v_1 + 2v_2 + 3v_3$

5.

Set is not linearly independent

v_1 , v_2 , v_3 are linearly independent

v_4 is linearly dependent

$v_4 = -v_1 + v_2$

Rank

Uses Gaussian Elimination and based on linear independence
Google search or chatgpt prompt: “How do i find rank of a matrix?”
Video: https://youtu.be/cSj82GG6MX4?si=6LFY4uu26ezTygX8
Row echelon form is how the matrix looks like after gaussian elimination

Exercises:

Make a with these vectors and find the rank, these are from the last two problems\

A= $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 7 \\ 11 \end{bmatrix} \right\}.$ B= $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$

Tons more practice at https://www.math-exercises.com/matrices/rank-of-a-matrix

Solutions:

rank(A) = 3
rank(B) = 3

Matrix decomposition/factorization

Uses Gaussian Elimination, linear independence, and rank
Google search or chatgpt prompt: "How do I do matrix decomposition/factorization?"
Video: Example of matrix factorization

Exercises:

Determine the rank of the following matrices, identify the linearly independent columns.

$$A_1 = \begin{bmatrix} 3 & 4 & 7 & -8 & -5 \\ -2 & -3 & -5 & 5 & 3 \\ 1 & 3 & 4 & -1 & 0 \\ 1 & 1 & 2 & -3 & -2 \\ -2 & -3 & -5 & 5 & 3 \end{bmatrix},$$

Now compose two matrices that are [5 x rank(A_1)] and [rank(A_1) x 5] that multiplied will result to A_1

Determine the rank of the following matrices, identify the linearly independent columns.

$$A_2 = \begin{bmatrix} 2 & -5 & -5 & 3 & -5 \\ 1 & 0 & 0 & 4 & 0 \\ 0 & 2 & 2 & 2 & 2 \\ -2 & 4 & 4 & -4 & 4 \\ -2 & 5 & 5 & -3 & 5 \end{bmatrix}.$$

Now compose two matrices that are [5 x rank(A_2)] and [rank(A_2) x 5] that multiplied will result to A_2

Solutions:

1. Determine the rank of the following matrices, identify the linearly independent columns. Compute the rank decomposition using the previously discussed algorithm;

$$A_1 = \begin{bmatrix} 3 & 4 & 7 & -8 & -5 \\ -2 & -3 & -5 & 5 & 3 \\ 1 & 3 & 4 & -1 & 0 \\ 1 & 1 & 2 & -3 & -2 \\ -2 & -3 & -5 & 5 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -5 & -5 & 3 & -5 \\ 1 & 0 & 0 & 4 & 0 \\ 0 & 2 & 2 & 2 & 2 \\ -2 & 4 & 4 & -4 & 4 \\ -2 & 5 & 5 & -3 & 5 \end{bmatrix}.$$

linear independant

$\mathrm{rank}(A_2) = 3$

$$A_1 = \begin{bmatrix} 1 & 1 & 2 & -3 & -2 \\ 1 & 3 & 4 & -1 & 0 \\ 3 & 4 & 7 & -8 & -5 \\ -2 & -3 & -5 & 5 & 3 \\ -2 & -3 & -5 & 5 & 3 \end{bmatrix} \begin{matrix} \\ -R1 \\ -3R1 \\ +2R1 \\ -R4 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & -3 & -2 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & -3 & -2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \boxed{\mathrm{rank}(A_1) = 2}$$

$[5\times 2]$ $[2\times 5]$

$$\begin{bmatrix} 3 & 4 \\ -2 & -3 \\ 1 & 3 \\ 1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & -4 & -3 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & -8 \\ -2 & -3 & 5 \\ 1 & 3 & -1 \\ 1 & 1 & -3 \\ -2 & -3 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -3 \\ 1 & 3 & -1 \\ 3 & 4 & -8 \\ -2 & -3 & 5 \\ -2 & -3 & -5 \end{bmatrix} \begin{matrix} \\ -R1 \\ 3R1 \\ +2R1 \\ +R4 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & 3 & 0 \\ 3 & 4 & -5 \\ -2 & -3 & 7 \\ -2 & -3 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 2 & 4 & 4 & -4 & 4 \\ 2 & -5 & -5 & 3 & -5 \\ -2 & 5 & 5 & -3 & 5 \end{bmatrix} \begin{matrix} \\ \\ -2R1 \\ -2R1 \\ +R4 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 4 & 4 & -12 & 4 \\ 0 & -5 & -5 & -5 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R2} \begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5×3 3×5

$$\begin{bmatrix} 2 & -5 & 3 \\ 1 & 0 & 4 \\ 0 & 2 & 2 \\ -2 & 4 & -4 \\ -2 & 5 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Probability

There is no easy way to do this but practice and gain intuition.
Google search or chatgpt prompt: “How to do probability? How to do probability with combinatorics?”
Videos:
These videos for without Combinatorics

- Introduction to Probability, Basic Overview - Sample Space, & Tree Diagrams
- Probability Formulas, Symbols & Notations - Marginal, Joint, & Conditional Probabilities
- Probability of Complementary Events & Sample Space
- Conditional Probability With Venn Diagrams & Contingency Tables

With Combinatorics like in class

- Permutations Combinations Factorials & Probability
- Permutations, Combinations & Probability (14 Word Problems)
- Permutations, Combinations, and Probability (15 Word Problems)

Exercises and Solution Worksheets:

Do enough until you feel confident

worksheet 1

worksheet 2

Mutually exclusive and independent events

Deals with probability

Google search or chatgpt prompt: "How do I show an event is mutually exclusive and/or independent?"

Videos: Probability of Mutually Exclusive Events With Venn Diagrams

Probability - Independent and Dependent Events

Multiplication & Addition Rule - Probability - Mutually Exclusive & Independent Events

Exercises:

Determine if events *A* and *B* are mutually exclusive.

9) $P(A) = \frac{3}{10}$ $P(B) = \frac{1}{2}$ $P(A \text{ or } B) = \frac{4}{5}$

10) $P(A) = \frac{7}{20}$ $P(B) = \frac{11}{20}$ $P(A \text{ or } B) = \frac{283}{400}$

11) $P(A) = \frac{7}{20}$ $P(B) = \frac{3}{10}$ $P(A \text{ and } B) = \frac{21}{400}$

12) $P(A) = 0.2$ $P(B) = 0.35$ $P(A \text{ and } B) = 0$

Determine if the scenario involves mutually exclusive events.

1) A spinner has an equal chance of landing on each of its eight numbered regions. After spinning, it lands in region three or six.

2) A bag contains six yellow jerseys numbered one to six. The bag also contains four purple jerseys numbered one to four. You randomly pick a jersey. It is purple or has a number greater than five.

3) A magazine contains twelve pages. You open to a random page. The page number is eight or ten.

4) A box of chocolates contains six milk chocolates and four dark chocolates. Two of the milk chocolates and three of the dark chocolates have peanuts inside. You randomly select and eat a chocolate. It is a milk chocolate or has no peanuts inside.

Determine if events A and B are independent.

9) $P(A) = \frac{2}{5}$ $P(B) = \frac{1}{5}$ $P(A \text{ and } B) = \frac{2}{25}$

10) $P(A) = \frac{2}{5}$ $P(B) = \frac{1}{4}$ $P(A \text{ and } B) = \frac{1}{25}$

11) $P(A) = \frac{9}{20}$ $P(B) = \frac{1}{2}$ $P(A|B) = \frac{27}{50}$

12) $P(\text{not } A) = \frac{3}{4}$ $P(B) = \frac{3}{10}$ $P(A \text{ and } B) = \frac{3}{40}$

Determine whether the scenario involves independent or dependent events.

1) You flip a coin and then roll a fair six-sided die. The coin lands heads-up and the die shows a one.

2) A bag contains eight red marbles and four blue marbles. You randomly pick a marble and then pick a second marble without returning the marbles to the bag. The first marble is red and the second marble is blue.

3) A box of chocolates contains five milk chocolates, five dark chocolates, and five white chocolates. You randomly select and eat three chocolates. The first piece is milk chocolate, the second is dark chocolate, and the third is white chocolate.

4) A cooler contains ten bottles of sports drink: four lemon-lime flavored, three orange flavored, and three fruit-punch flavored. Three times, you randomly grab a bottle, return the bottle to the cooler, and then mix up the bottles. The first time, you get a lemon-lime drink. The second and third times, you get fruit-punch.

Solutions:

9) $P(A) = \frac{3}{10}$ $P(B) = \frac{1}{2}$ $P(A \text{ or } B) = \frac{4}{5}$

Mutually exclusive

10) $P(A) = \frac{7}{20}$ $P(B) = \frac{11}{20}$ $P(A \text{ or } B) = \frac{283}{400}$

Not mutually exclusive

11) $P(A) = \frac{7}{20}$ $P(B) = \frac{3}{10}$ $P(A \text{ and } B) = \frac{21}{400}$

Not mutually exclusive

12) $P(A) = 0.2$ $P(B) = 0.35$ $P(A \text{ and } B) = 0$

Mutually exclusive

1) A spinner has an equal chance of landing on each of its eight numbered regions. After spinning, it lands in region three or six.

Mutually exclusive

2) A bag contains six yellow jerseys numbered one to six. The bag also contains four purple jerseys numbered one to four. You randomly pick a jersey. It is purple or has a number greater than five.

Mutually exclusive

3) A magazine contains twelve pages. You open to a random page. The page number is eight or ten.

Mutually exclusive

4) A box of chocolates contains six milk chocolates and four dark chocolates. Two of the milk chocolates and three of the dark chocolates have peanuts inside. You randomly select and eat a chocoate. It is a milk chocolate or has no peanuts inside.

Not mutually exclusive

Determine if events *A* and *B* are independent.

9) $P(A) = \frac{2}{5}$ $P(B) = \frac{1}{5}$ $P(A \text{ and } B) = \frac{2}{25}$

Independent

10) $P(A) = \frac{2}{5}$ $P(B) = \frac{1}{4}$ $P(A \text{ and } B) = \frac{1}{25}$

Dependent

11) $P(A) = \frac{9}{20}$ $P(B) = \frac{1}{2}$ $P(A|B) = \frac{27}{50}$

Dependent

12) $P(\text{not } A) = \frac{3}{4}$ $P(B) = \frac{3}{10}$ $P(A \text{ and } B) = \frac{3}{40}$

Independent

1) You flip a coin and then roll a fair six-sided die. The coin lands heads-up and the die shows a one.

Independent

2) A bag contains eight red marbles and four blue marbles. You randomly pick a marble and then pick a second marble without returning the marbles to the bag. The first marble is red and the second marble is blue.

Dependent

3) A box of chocolates contains five milk chocolates, five dark chocolates, and five white chocolates. You randomly select and eat three chocolates. The first piece is milk chocolate, the second is dark chocolate, and the third is white chocolate.

Dependent

4) A cooler contains ten bottles of sports drink: four lemon-lime flavored, three orange flavored, and three fruit-punch flavored. Three times, you randomly grab a bottle, return the bottle to the cooler, and then mix up the bottles. The first time, you get a lemon-lime drink. The second and third times, you get fruit-punch.

Independent

Probability distribution

Deals with probability
Google search or chatgpt prompt: “How to construct a probability distribution for a random variable?”
Video: Constructing a probability distribution for random variable | Khan Academy
Finding the Expected Value of a Probability Distribution
Probability Distribution - Sum of Two Dice

Exercises:

Chances are the case will be coins or dice
Again, suppose that we toss 4 unbiased coins. The random variable X takes the value equal to the difference between the number of tails and the number of heads: X = #tails - #heads.
1. Which values does this variable take?

2. Compute the probability distribution for X.

Solutions:

1. Which values does this variable take?

No tails HHHH	One tail HHHT	Two tails HHTT	Three Tails HTTT	Four Tails TTTT	
-4	-2	0	2	4	x=#tails - #heads

2. Compute the probability distribution for X.

No tails HHHH	One tail HHHT	Two tails HHTT	Three Tails HTTT	Four Tails TTTT	
1/16	4/16	6/16	4/16	1/16	probability

Expectation and variance

Deals with distribution

Google search or chatgpt prompt: “How to find expected value and variance of a discrete random distribution?”

Video: Mean (expected value) of a discrete random variable | AP Statistics | Khan Academy

Variance and standard deviation of a discrete random variable | AP Statistics | Khan Academy

Exercises:

https://www.khanacademy.org/math/statistics-probability/random-variables-stats-library/random-variables-discrete/e/mean-expected-value-discrete-random-variable

https://www.khanacademy.org/math/statistics-probability/random-variables-stats-library/random-variables-discrete/e/standard-deviation-discrete-random-variable

Bayes Theorem

Deals with probability
Google search or chatgpt prompt: “How to use Bayes Theorem?”
Video: Bayes' Theorem of Probability With Tree Diagrams & Venn Diagrams
Bayes' Theorem, Clearly Explained!!!!

Exercises and solutions website:

https://math.oxford.emory.edu/site/math117/probSetBayesTheorem/

https://gtribello.github.io/mathNET/bayes-theorem-problems.html

Derivatives for critical points for functions

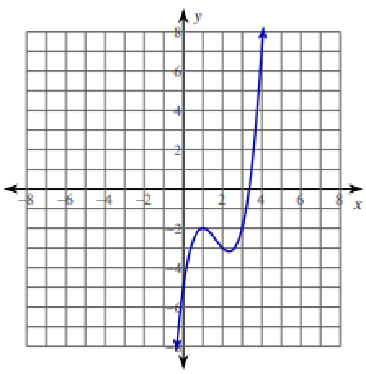
Google search or chatgpt prompt: “How to find critical points for a function using derivatives?”
Video: Watch first 5 mins Basic Differentiation Rules For Derivatives
Finding Critical Numbers

Exercises:

IGNORE THE WORD RELATIVE

For each problem, find all points of relative minima and maxima.

1) $y = x^3 - 5x^2 + 7x - 5$



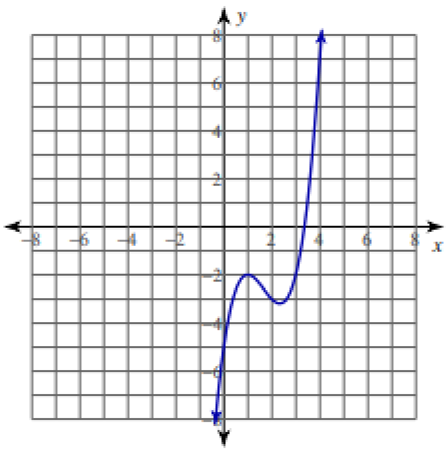
For each problem, find all points of relative minima and maxima.

3) $y = -x^3 - 3x^2 - 1$

4) $y = x^4 - 2x^2 + 3$

Solutions:

1) $y = x^3 - 5x^2 + 7x - 5$



Relative minimum: $\left(\frac{7}{3}, -\frac{86}{27}\right)$

Relative maximum: $(1, -2)$

For each problem, find all points of relative minima and maxima.

3) $y = -x^3 - 3x^2 - 1$

Relative minimum: $(-2, -5)$
Relative maximum: $(0, -1)$

4) $y = x^4 - 2x^2 + 3$

Relative minima: $(-1, 2)$, $(1, 2)$
Relative maximum: $(0, 3)$

Derivatives for critical points for multivariable functions

Uses GAUSSIAN ELIMINATION and derivatives
Google search or chatgpt prompt: "How to find critical points for a multivariable function using derivatives and gaussian elimination?"
Video: watch up to 5 mins Partial Derivatives - Multivariable Calculus

Take partial derivatives of each variable and set equal to 0, then set up in a matrix and then do GAUSSIAN ELIMINATION

Exercises:

Find the critical points of each of the following functions:

b. $g(x, y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

7. Find the critical point of the folowing function of 3 variables:
$f(x, y, z) = x^2 + y^2 + z^2 + xy + yz - 3x + 2z$.

Solutions:

b. Critical point at $f(-1,-1)$

7. Critical point at $f(\ 7/4,-\frac{1}{2},-\frac{3}{4})$

Writing a polynomial given from points

Uses GAUSSIAN ELIMINATION

Google search or chatgpt prompt: "How do I write a polynomial equation from points using a matrix?"
Video: Using a Matrix to find a polynomial Equation

Exercises:

Write a function from the given points (from video)

x	f(x) or y
0	4
1	6
2	22
3	70

Write a function from the given points (from notes)

x	f(x) or y
-2	-1
-1	1
0	2
1	-1

Solutions:

$f(x) = 3x^3 - 2x^2 + x + 4$

$f(x) = ½ x^3 - 2x^2 - ½ x + 2$