TWISTED BILAYER GRAPHENE IN COMMENSURATE ANGLES

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ABSTRACT. The recent discovery of "magic angles" in twisted bilayer graphene (TBG) has spurred extensive research into its electronic properties. The primary tool for studying this thus far has been the famous Bistritzer-MacDonald model (BM model), which relies on several approximations. This work aims to build the first steps in studying magic angles without using this model. Thus, we study a 2d model for TBG in both AA and AB stacking *without* the approximations of the BM model in the continuum setting, using two copies of a potential with the symmetries of graphene, either sharing a common origin (in AA stacking) or with shifted origins (in AB stacking), and twisted with respect to each other. Our results hold for a wide class of potentials in both stacking types. We describe the angles for which the two twisted lattices are commensurate and prove the existence of Dirac cones in the vertices of the Brillouin zone for such angles. Furthermore, we show that for small potentials, the slope of the Dirac cones is small for commensurate angles that are close to incommensurate angles. This work is the first to establish the existence of Dirac cones for twisted bilayer graphene (for either stacking) in the continuum setting without relying on the BM model. This work is the first in a series of works to build a more fundamental understanding of the phenomenon of magic angles.
Keywords: Twisted Bilayer Graphene, Commensurate angles, AA stacking, AB stacking, Honeycomb lattice potential, Periodic operators, Dirac cones.

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1. INTRODUCTION

1.1. **Motivation and main results.**

Recently, the discovery of "magic angles" in Twisted Bilayer Graphene (which we will abbreviate as TBG) [8, 10] led to a wave of studies about its electronic properties both in the physics community, see for example [8, 10, 16, 18, 21, 22, 27, 39], and in the mathematical community, for example [2, 3, 4, 5, 9, 26, 25, 37, 38].

This system was famously studied theoretically by Bistritzer and MacDonald in their seminal paper [8]- where they considered two layers of graphene, one on top of the other, shifted with respect to each other by a vector K_0, and twisted one layer with respect to the other, at some angle θ. Bistritzer and MacDonald, in their work, created an effective model for TBG, which is periodic at all twisting angles- which we will refer to as the BM model. They derived this model from several successive approximations and argued that the resulting operator's spectrum should contain a degenerate Dirac cone, as defined in Section 2, at certain angles. Later work [35] even found that under additional assumptions, the so-called "chiral limit" of the BM model, one gets a flat Floquet band. In other words, they claimed that there is some $E \in \mathbb{R}$, which is an eigenvalue of the approximate operator *for all* quasimomentum in the Brillouin zone, which implies that it is an eigenvalue of infinite multiplicity. Since then, there has been much effort to establish these results rigorously (e.g., [9, 37]). In many cases, the focus was on the above-mentioned chiral limit of this model; see, for example, [2, 3]. Despite these efforts, some aspects of the BM approximations are still not well understood, especially in the continuum setting.

One of the significant difficulties in studying this phenomenon is that most angles are incommensurate twist angles. A *commensurate angle* is an angle for which the twisted system is still periodic (we use a slightly different definition in the following, see remark 2.14). The general theory of second-order elliptic periodic operators (see, for example, [19]) shows that there are no eigenvalues for commensurate angles- thus, if the magic angles contain a flat band, then they must be incommensurate. For incommensurate angles, one can not use many tools (most importantly, the tools provided by Floquet theory) available in the commensurate case. In particular, the standard definition of Dirac cones relies on Floquet theory and thus is not naturally applicable to incommensurate angles. In the BM model, one gains periodicity via the approximation used- as the resulting operator is periodic for all twisting angles- and thus, the Dirac cones can be defined.

This work is the first in a series aiming to build an understanding of magic angles without the BM model. We begin with a foundational understanding of the commensurate case. Similar to irrational and rational rotation on the torus, the commensurate angles are dense in the incommensurate angles. Thus, in later works, we would aim to push our understanding of the behavior in commensurate angles to understand the behavior in the incommensurate case. Thus, a better understanding of the commensurate is the crucial first step in understanding magic angles without the BM model.

This work will focus on the continuum setting- though a similar analysis can be carried out for the discrete operators, known as the tight binding model.

Let V be a honeycomb potential, as defined in [14] (see Section 2 for precise definition)- a periodic potential with the honeycomb lattice symmetries. We will denote by R_θ the matrix that represents a rotation by θ. Then, we will consider the following Hamiltonian acting on $L^2(\mathbb{R}^2)$

$$H^\theta(\lambda) = -\Delta + \lambda W^\theta$$

for $\lambda \in \mathbb{R}$, the amplitude of the potential, and W^θ, a potential generated from $V(R_\theta x)$ and $V(R_{-\theta} x)$ through a general family of possible admissible interacting operators, defined in Section 2. In this work, we will consider both AA stacking (twisting with a common origin) and AB stacking (twisting where the origins are shifted). We remark that superconductivity was first observed in AB stacking, also known as Bernal stacking; for more details, see [21], for example. As a representative example of the class of potentials considered here, one may consider

$$W^\theta_{0,AA} = \frac{1}{2}(V(R_\theta x) + V(R_{-\theta} x)) \tag{1.1.1}$$

$$W^\theta_{0,AB} = \frac{1}{6}\sum_{j=-1}^{1}(V(R_\theta(x + \frac{1}{2}R^j_{\frac{2\pi}{3}}K_0)) + V(R_{-\theta}(x - \frac{1}{2}R^j_{\frac{2\pi}{3}}K_0))) \tag{1.1.2}$$

Remark 1.1. One can immediately note that the AB stacking potential requires more symmetry than the AA stacking case. This is because there is some ambiguity when shifting an infinite system to AB stacking; see remark 2.5 for more details.

Remark 1.2. We note that the model suggested here is a two-dimensional model and not a full three-dimensional model. Similarly to the BM model and its derivations [8, 9, 37], here we model the system as essential two dimensional- with the third dimension entering through the coupling constant between the layers- or in our case by the choice of interaction operator. The evolution of the systems happens mostly along the layers of graphene; the interaction between the layers is crucial, of course, but the qualitative results depend rather weakly on the distance parameter, as for small enough distances. For example, the BM model is a 2-dimensional vector valued tight binding model- so the vertical distance appears only as a parameter, see [8]. So, the analysis so far, and thus the analysis leading to the existence of magic angles, is essentially two-dimensional. In [9], a three-dimensional continuum model was considered- but, as emphasized there as well, it is compared to an effective two-dimensional model- the BM model.

In addition, this assumption is useful technically as it allows us to prove the existence of Dirac cones using the results for graphene, which apply to two-dimensional operators. Later works will try to establish the precise connection between a full three-dimensional model and the two-dimensional effective model.

With this, we may state our main results (more precise statements will appear in Section 2- after some more technical notations will be introduced):

(1) Theorem 2.15 describes the set of commensurate angles- the set of angles for which the two lattices intersect non-trivially, denoted by $\mathcal{C}$. Furthermore, we show that, for a commensurate angle, the new potential is periodic with respect to a scaled honeycomb lattice with a scaling factor of N, defined by the arithmetic properties of the angle θ.
(2) Theorem 2.21 extend the main results of [14] to include different technical conditions. When considering some cases of admissible interacting potentials, such as the representative example given by (1.1.1), one of the difficulties encountered is that the results in [14] do not apply, as one of the technical conditions of the theorem fails. This condition is required to show some separation of eigenvalues and comes from the perturbation theory of simple eigenvalues. Thus, we extend these results by going to higher-order terms in the perturbation theory. This extension will allow us to get a different condition that we could apply to such potentials. We will get that, under some technical conditions, we have for every $\lambda \in \mathbb{R}$ except for a discrete set, at the edges of the new Brillouin zone- the $K^\theta, (K^\theta)'$ points- there is a Dirac cone at the bottom of the spectrum.
(3) Theorem 2.23 shows that for a small amplitude of the potential with respect to the reciprocal of the scaling, that is $\lambda \lesssim \frac{1}{N^2}$, the slope of the Dirac cone v_d is proportional to $\frac{1}{N}$. This result may hint at vanishing Floquet bands for *all* incommensurate angles for small enough λ and gives a quantitative flattening of the Dirac cones, albeit only in the perturbative regime, for commensurate approximations to incommensurate angles.

1.2. **Graphene and twisted bilayer graphene - overview.**

1.2.1. *Single layer graphene.* Graphene is a two-dimensional material made of a single layer of carbon atoms arranged in a hexagonal formation. Though theoretical studies of graphene can be dated back to the mid-19th century (see remarks in [28] for example), only in 2004 did Geim and Novoselov [29] manage, in a work that got them the Noble prize, to produce an isolated layer of graphene. In the following years, the new existing material attracted much attention due to its many exciting properties, including its electronic properties (see [27] for more details about these properties).

There are several ways of studying such materials. One such way is to examine the associated Schördinger operator in the continuum setting, i.e., as an operator acting on $L^2(\mathbb{R}^2)$. Another way is to study these operators through the tight-binding approximation. In this approximation, the full dynamics is approximated by an operator that acts on $\ell^2(\Lambda)$, where Λ is the graph of the periodic lattice of atoms. For a discussion of the tight-binding model, see, for example, [1, 13].

For periodic potentials, such as the operator modeling graphene, Floquet theory allows one to move from the spectrum of the full Hamiltonian, H - which will have absolutely continuous spectrum - to studying a family of operators $H(k)$, each with only pure point spectrum [19]. The eigenfunctions of each $H(k)$, denoted $E_n(k)$, are called bands.

One of the remarkable properties of graphene, which was demonstrated all the way back in Wallace's work [36] in the 40s, is that it has Dirac points. Dirac points are points where two bands - two different eigenfunctions $E_1(k), E_2(k)$ of $H(k)$ - touch conically. In other words, we will say that (E_0, k_0) is a Dirac point in the energy-quasimomentum plane if there is some $\delta > 0$, such that for all $k \in \mathbb{T}^*$ such that $|k - k_0| < \delta$, we have

$$|E_1(k) - E_0| \approx |v_d||k - k_0| \text{ and } \qquad |E_2(k) - E_0| \approx -|v_d||k - k_0|,$$

for some v_d - called the Dirac velocity, see Section 2 for more precise definition. This means that a wave packet localized in momentum space around that point will disperse approximately according to a two-dimensional Dirac equation, the equation of evolution for massless relativistic fermions (see [15] for more details about the dispersion near Dirac points)- and hence the name.

Since the Dirac equation is relativistic, one can use wave packets localized around these Dirac points to see relativistic effects in non-relativistic velocities. Dirac points are also connected to other electric properties of graphene; see [27] for more details.

The existence of these Dirac points was shown first in tight binding setting in the physics literature in [34, 36], and in a richer model that was considered in the mathematics literature in [20]. Later, it was proven for the continuous setting in the seminal work of Fefferman and Weinstein [14]- which the present work draws inspiration from.

Fefferman and Wieinstien modeled a single-layer graphene by a Schrödinger operator with a honeycomb potential acting on $L^2(\mathbb{R}^2)$. A honeycomb lattice, defined here in Section 2, is, roughly speaking, a potential with the same symmetries as graphene. In [14], they showed that this model has, under some mild assumptions, Dirac points at the vertices of the Brillouin zone. Finally, they have shown that these points persist under a broad class of perturbations.

Later, in [7], Berkolaiko and Comech gave a different proof to the results in [14], which made the role of symmetry in the arguments of [14] more evident by using more abstract arguments based on representation theory. Thus, they could generalize the results to many more applications and simplify some of the more technical aspects of that work.

1.2.2. *Twisted bilayer graphene.* As mentioned above, the celebrated model of twisted bilayer graphene was conceived in 2011 by Bistritzer and MacDonald [8]- and predicted the existence of “magic angles”- angles for which the spectrum contains a degenerate Dirac cone. The existence of magic angles implies that some exotic transport properties may occur due to the rise in importance of electron-electron interaction (neglected in the initial Schördinger operator). For more details, see [8, 27]. Later, in 2018, superconductivity was observed in these magic angles by Cao and collaborators [10].

As mentioned above, one of the significant difficulties in this analysis is to study incommensurate angles where the potential is no longer periodic. For this, Bistritzer and MacDonald restricted their attention to the quasimomentum close to the Dirac cones of the single-layer model. They could approximate the evolution with the evolution of periodic operator, *regardless of whether the angle is commensurate or incommensurate.*

This discovery of superconductivity led to the discovery of other configurations of twisting and stacking where these magic angles occur (see, for example, [18, 23]), as well as the ability to “tune” the superconductivity by adjusting the angle (see [39]). These discoveries gave rise to a new field in physics of ”Twistronics” or electronic properties of twisted periodic materials [16].

At the same time, a more rigorous study of TBG started in the mathematical community. The breakthrough work of Becker, Embree, Wittsten, and Zworski [3] as well as an alternate proof given by Watson and Luskin [38], showed that the chiral approximation of the BM Hamiltonian, given by Tarnopolsky, Kruchkov, and Vishwanath [35] does have flat bands. Then, in a series of papers, Backer and collaborators [2, 4, 6] studied this model in greater detail, giving even spectral descriptions of the flat bands. In a very recent work, Becker, Quinn, Tao, Watson, and Yang [5] established the existence of Dirac cones and the existence of magic angels with degenerate Dirac cones for the full BM model.

In the last couple of years, there has been an attempt to understand better the approximations leading to the BM Hamiltonian by Cancès, Garrigue and Gontier [9] in the continuum setting, and by Watson, Kong, Macdonald, and Luskin [37], in the tight binding setting. These studies have rigorously estimated the error terms from the derivation of BM Hamiltonian from the original Dirac equation. Thus, their result bound the error when comparing the evolution of the full operator with the evolution of the approximate operator. This bound is time-dependent. In addition, much work has been dedicated to the numerical modeling of the dynamics of TBG (see, for example, [17, 26]) or more generally about bilayer materials, e.g., [25].

As mentioned, all these works focus on the BM model, and many further restrict the study of the Tarnopolsky, Kruchkov, and Vishwanath chiral approximation. For a recent survey of the results in this field, see [40].

There are still aspects of the BM model that are not well understood, especially the approximation in the continuum setting (for example, defining the Kohan-Sham potential for incommensurate, see [9]). Moreover, superconductivity usually arises from some spectral phenomenon in the single particle theory. Generally speaking, approximations to the evolution in certain time scales, such as the BM model to the full model, do not allow one to get information about the spectral properties. For that, a different notion of convergence is usually required. Thus, we aim to build a more fundamental understanding of the magic angle phenomenon without turning to this model's assumptions. Specifically, in this work, we establish the existence of Dirac cones for commensurate angles without going through the approximate BM model but rather directly from a more general description.

1.3. **Outline of the paper.** This paper is organized as follows:

Section 2 will introduce the basic setting and notation, as well as the main tools of Floquet theory, which allow us to state our main results precisely.

Section 3 will prove Theorem 2.21, which allows us to conclude the existence of Dirac points for honeycomb potentials with slightly different conditions than the main theorems in [7, 14]- thus enabling us to use them for a larger family of twisted bilayer potentials.

Section 4 will prove our main results regarding the twisted bilayer potentials with commensurate angles. First, we prove Theorem 2.15, which shows that our potential is periodic with respect to a scaling of a honeycomb potential. Then, we show that the conditions established in Theorem 2.21 hold for the example of potential given by (1.1.1)- Lemma 4.9. Finally, we show that for a small enough coupling constant, the Dirac velocity decays like the reciprocal of the scaling factor in Theorem 2.23.

Section 5 will give examples of a twisted potential of the type (1.1.1) such that for all angles, the technical condition of Theorem 2.21 holds.

Finally, Appendix A will collect this paper's relevant notation.

2. The setting and results

2.1. **Geometry.** We start with some definitions relating to honeycomb potentials and lattices. We will mostly follow the notations conventions set in [14]. We recall the honeycomb lattice[1] is given by:

$$v_1 = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, v_2 = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}, \qquad \Lambda = v_1\mathbb{Z} \oplus v_2\mathbb{Z}$$

We would also need to consider the reciprocal lattice, defined by

$$k_1 = \frac{4\pi}{\sqrt{3}} \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, k_2 = \frac{4\pi}{\sqrt{3}} \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}, \qquad \Lambda^* = k_1\mathbb{Z} \oplus k_2\mathbb{Z}$$

It will be convenient to define the following matrices

$$\nu = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \qquad \kappa = \begin{pmatrix} k_1 & k_2 \end{pmatrix} = \frac{4\pi}{\sqrt{3}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

[1] This is, in fact, a triangular lattice, but it turns out that for this analysis, this is enough- see discussion in [14].

then we have that we can write both lattices in the following form

$$\Lambda = \nu\mathbb{Z}^2, \Lambda^* = \kappa\mathbb{Z}^2$$

and we note that for any $u_1, u_2 \in \mathbb{Z}^2$ we have that

$$\langle \kappa u_1, \nu u_2 \rangle = 2\pi \langle u_1, u_2 \rangle$$

We will distinguish between the Euclidean inner product, which we will denote by $\langle \cdot, \cdot \rangle$, and the inner product on Hilbert spaces, which we will denote by $(\cdot, \cdot)$, for clarity.

Throughout this paper, quantities with a tilde above them, such as $\tilde{\nu}$, will denote quantities related to a honeycomb lattice, $\tilde{\Lambda}$, without explicit dependence on its base vectors. Λ and Λ^* will always refer to the above choices of base vectors, and quantities with the upper script of θ will refer to quantities related to the new lattice generated by the intersection of twisted lattices by commensurate angle θ.

For any honeycomb lattice, $\tilde{\Lambda}$, with base matrix $\tilde{\nu}$, and dual matrix $\tilde{\kappa}$, we may define the unit cell $\tilde{\Omega}$, and the Brillouin zone, $\tilde{\mathcal{B}}$ by

$$\tilde{\Omega} = \tilde{\nu}[0,1]^2, \qquad \tilde{\mathcal{B}} = \{k \in \mathbb{R}^2 \mid \forall a \in \tilde{\Lambda}^*, |k| \leq |k-a|\}$$

Next, we denote the rotation matrix by angle θ, by

$$R_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

and we will denote the corresponding operator by $\mathcal{R}_\theta$, that is

$$\mathcal{R}_\theta f(x) = f(R_{-\theta} x)$$

We will single out the rotation by $\frac{2\pi}{3}$, by denoting $R = R_{\frac{2}{3}\pi}$, and the corresponding operator we will denote by $\mathcal{R}$.

The points of high symmetry in the Brillouin zone will be of particular importance- these are points where rotation by R results in a shift by the dual lattice:

$$\tilde{\mathbb{P}} = \{\vec{k} \in \tilde{\mathcal{B}} \mid (R - \mathrm{id})\vec{k} \in \tilde{\kappa}\mathbb{Z}^2\}$$

Moreover, we can decompose it into three disjoint orbits:

$$\tilde{K} = \frac{1}{3}\begin{cases} \kappa \begin{pmatrix} 1 \\ -1 \end{pmatrix}, & \tilde{\Lambda} = \Lambda \\ \nu \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \tilde{\Lambda} = \Lambda^* \end{cases}, \qquad \tilde{K}' = -\tilde{K}$$

$$\tilde{\mathbb{P}} = \{\tilde{K}, R\tilde{K}, R^2\tilde{K}\} \bigsqcup \{\tilde{K}', R\tilde{K}', R^2\tilde{K}'\} \bigsqcup \{0\}$$

We will consider the shift operators by the appropriate point of high symmetry. For $\vec{a} \in \mathbb{R}^2$, we may consider the translation operator $\mathcal{T}_{\vec{a}}$ given by

$$\mathcal{T}_{\vec{a}} f(x) = f(x - \vec{a})$$

And of specific interest will be the shift in $K_0 = \frac{1}{3}\nu \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}$, and its dual point $K_0^* = \frac{1}{3}\kappa \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{4\pi}{\sqrt{3}} \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}$, and their associated shift operators $\mathcal{T}_{K_0}$, and $\mathcal{T}_{K_0^*}$. These shifts will allow us to define the AB stacking potentials[2].

[2]We remark that one can write the hexagonal lattice of graphene as $(\Lambda - \frac{1}{2}K_0) \cup (\Lambda + \frac{1}{2}K_0)$.

Remark 2.1. Naturally, one may also consider the opposite shift - which leads to BA stacking. All the analysis below will be valid for this configuration as well.

With this in hand, we will recall the definition of a honeycomb potential given in [14]- extended to treat Λ and Λ^* on equal footing:

Definition 2.2. If $U \in C^\infty(\mathbb{R}^2)$ is a real-valued potential, and $\tilde{\Lambda} \in \{\Lambda, \Lambda^*\}$, such that

(1) For the triangular lattice we have $\forall a \in \tilde{\Lambda}, x \in \mathbb{R}^2, U(x+a) = U(x)$.
(2) It is even: $F[U](x) = U(-x) = U(x)$.
(3) It is symmetric under rotation by R, i.e. $\forall x \in \mathbb{R}^2, \mathcal{R}[U](x) = U(R^{-1}x) = U(x)$.

Then U is a honeycomb potential.

For a list of examples of honeycomb lattices, we refer the reader to [14].

In order to define the twisted potential, we will have to define the set of admissible interaction operators

Definition 2.3. $G : (C^\infty \cap L^\infty) \times (C^\infty \cap L^\infty) \to C^\infty \cap L^\infty$ will be called an admissible interaction operator if it has the following properties, for any $f, g \in C^\infty \cap L^\infty$

(1) It is bounded by the arguments in the sense that there are some $C_g, C_{g'} > 0$ and $\gamma, \gamma' > 0$ such that
$$\|G(f,h)\|_\infty \le C_g(\|f\|_\infty \|h\|_\infty)^\gamma$$
$$\|\nabla G(f,h)\|_\infty \le C_g(\|\nabla f\|_\infty \|\nabla h\|_\infty)^{\gamma'}$$
(2) G respects rotations in the following sense
$$\mathcal{R}_\alpha G(f,h) = G(\mathcal{R}_\alpha f, \mathcal{R}_\alpha h)$$
(3) G respects translations in the following sense
$$\mathcal{T}_{\mathbf{a}} G(f,h) = G(\mathcal{T}_{\mathbf{a}} f, \mathcal{T}_{\mathbf{a}} h)$$

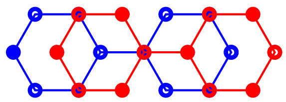
We will call G^* an admissible interaction operator for AB stacking if, on top of the above, we have that, for any $f, g \in C^\infty \cap L^\infty$:
$$G^*(f,g) = G^*(g,f)$$

Remark 2.4. Note that Properties 2 and 3 are quite natural and can be expressed as the fact that spatial position dependence comes from f, g and not from the definition of G.

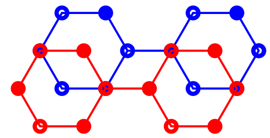
Remark 2.5. We note that there is some intrinsic ambiguity in AB stacking. Since the shift is taking place before the twist, shifting by K_0, RK_0, or $R^{-1}K_0$ all result in the same configuration. This is demonstrated in Figure 1. So, we will average over the three options, thus ensuring that the rotational symmetry will be preserved.

With these definitions in hand, we define the twisted bilayer potential of angle θ, which we will denote by W^θ, in AA stacking and AB stacking:

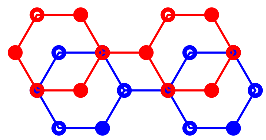
Definition 2.6. Let V be a honeycomb potential with Λ as a lattice, and let G be an admissible interaction operator, and G^* an admissible interaction operator for AB stacking, then the corresponding twisted bilayer potential in AA stacking of angle θ is defined by
$$W^\theta_{AA} = G(\mathcal{R}_\theta V, \mathcal{R}_{-\theta} V)$$
The twisted bilayer potential in AB stacking of angle θ is defined by
$$W^\theta_{AB} = G^*(\frac{1}{3}\sum_{j=-1}^{1} \mathcal{T}_{-\frac{1}{2}R^j K_0} \mathcal{R}_\theta V, \frac{1}{3}\sum_{j=-1}^{1} \mathcal{T}_{\frac{1}{2}R^j K_0} \mathcal{R}_{-\theta} V)$$



(A) Illustration of AB stacking with a shift by K_0.



(B) Illustration of AB stacking with a shift by RK_0.



(C) Illustration of AB stacking with a shift by R^2K_0.

FIGURE 1. Illustration of AB stacking of graphene with the different possible shifts. AN Illustration of shift by K_0 in 1a, shift by RK_0 in 1b, and a shift by R^2K_0 in 1c. In all cases, the red hexagon corresponds to the upper layer, and the blue corresponds to the lower layer

As will be shown later- for AB stacking, even for commensurate angles, we will not have that W^θ_{AB} is a honeycomb potential by the definition above, but almost honeycomb potential, so we will define:

Definition 2.7. If $U^\theta \in C^\infty(\mathbb{R}^2)$ is a real-valued potential, and $\tilde{\Lambda} \in \{\Lambda, \Lambda^*\}$, such that

(1) For the triangular lattice we have $\forall a \in \tilde{\Lambda}, x \in \mathbb{R}^2, U^\theta(x+a) = U^\theta(x)$.
(2) It is invariant under flips: $F^*[U](x) = U^{-\theta}(-x) = U^\theta(x)$.
(3) It is symmetric under rotation by R, i.e. $\forall x \in \mathbb{R}^2, \mathcal{R}[U^\theta](x) = U^\theta(R^{-1}x) = U^\theta(x)$.

Then U^θ is an almost honeycomb potential.

For the convenience of the reader, we collect some examples of admissible interaction operators:

Example 2.8. One may simply take a simple model for G:

$$G(f,g) = \frac{1}{2}(f+g)$$

which obviously commutes with rotation and translations, is symmetric, and is a bounded operator in the above sense. and get that

$$W^\theta_{AA}(x) = \frac{1}{2}(\mathcal{R}_\theta V + \mathcal{R}_{-\theta} V)$$
$$W^\theta_{AB}(x) = \frac{1}{6}\sum_{j=-1}^{1} \mathcal{T}_{\frac{1}{2}R^jK_0}\mathcal{R}_\theta V + \frac{1}{6}\sum_{j=-1}^{1} \mathcal{T}_{\frac{1}{2}R^jK_0}\mathcal{R}_{-\theta} V$$

This will be a prime example in Section 4.

Example 2.9. One can also take an averaging-type operator. Since G is defined for L^∞ functions, we will need to choose some decaying function $w \in L^1(\mathbb{R}^2)$ and define

$$G_w(f,h)(x) = f *_w h(x) = \int\limits_{\mathbb{R}^2} f(z)h(z)w(\|z-x\|)\,dz$$

It is easy to see that for rotations, we have

$$\mathcal{R}_\alpha G_w(f,h)(x) = \int\limits_{\mathbb{R}^2} f(z)h(z)w(\|z - R_{-\alpha}x\|)\,dz = \int\limits_{R_\alpha\mathbb{R}^2} f(R_{-\alpha}y)h(R_{-\alpha}y)w(\|R_{-\alpha}y - R_{-\alpha}x\|)\,dy$$
$$= G_w(\mathcal{R}_\alpha f, \mathcal{R}_\alpha h)(x)$$

as needed. It is naturally symmetric, and commutes with translations, and we have

$$\|G(f,h)\|_\infty \le \|f\|_\infty \|h\|_\infty \|w\|_1$$

Example 2.10. More generally, one may choose $w \in L^1(\mathbb{R}^2)$, and let $p(x,y)$ be a symmetric polynomial in two variables, with actions $(+,\cdot,*_w)$ and real coefficients. Then, we may take

$$G(f,h) = p(f,h)$$

Again, it is easy to check that all the properties will hold.

Remark 2.11. This model is a simplification of the setting of TBG in which magic angles are expected: first, both layers are considered laying in the same plane- rather than one on top of the other, as they are arranged in experiments, see remark 1.2 for further discussion. In addition, in an actual system, mechanical relaxation effects will also change the stacking type (AA, AB, and BA) over the period.

Deriving an explicit, effective two-dimensional model from the full three-dimensional model could prove significant- as suggested by dependence on the distance between the layers in the proofs of magic angles in the chiral limit of the BM model. See, for example, [37, 38] for more details. The author is planning to address both of these assumptions in upcoming works.

Our first result concerns describing the set of commensurate angles. For this, we start by denoting by $\mathcal{C}$ the set of θ for which $\Lambda^\theta = R_\theta\Lambda \cap R_{-\theta}\Lambda$ has a nonzero element. We first note that this is enough to get that Λ^θ contains a lattice:

Proposition 2.12. *If $0 \neq \mathbf{a} \in \Lambda^\theta$, then we have Λ^θ contains a non-degenerate lattice.*

Proof. We note that if $\mathbf{a} \in \Lambda^\theta = R_\theta\Lambda \cap R_{-\theta}\Lambda$, then we have that

$$R\mathbf{a} \in RR_\theta\Lambda \cap RR_{-\theta}\Lambda = R_\theta(R\Lambda) \cap R_{-\theta}(R\Lambda) = \Lambda^\theta$$

Since $\mathbf{a} \neq 0$, we have that $R\mathbf{a}, \mathbf{a}$ are two linearly independent vectors, and so they generate a non-degenerate lattice. And naturally, we will have

$$\forall c \in \mathbb{Z}, c\mathbf{a}, cR\mathbf{a} \in \Lambda^\theta$$

So we conclude that Λ^θ contains a non-degenerate lattice- as needed. $\square$

We note that, in general, we have that

$$\mathcal{T}_\mathbf{a}\mathcal{R}_\alpha = \mathcal{R}_\alpha\mathcal{T}_{R_{-\alpha}\mathbf{a}}$$

and so we have the following proposition:

Proposition 2.13. *We have that W^θ_x, for $x \in \{AA, AB\}$ is periodic with respect to Λ^θ.*

Proof. We note that for any $\mathbf{a} \in \Lambda^\theta$, we have that

$$\begin{aligned}\mathcal{T}_\mathbf{a}W^\theta_{AA} &= G(\mathcal{T}_\mathbf{a}\mathcal{R}_\theta V, \mathcal{T}_\mathbf{a}\mathcal{R}_{-\theta}V) = G(\mathcal{R}_\theta\mathcal{T}_{R_{-\theta}\mathbf{a}}V, \mathcal{R}_{-\theta}\mathcal{T}_{R_\theta\mathbf{a}}V)\\ &= G(\mathcal{R}_\theta V, \mathcal{R}_{-\theta}V) = W^\theta_{AA}\end{aligned}$$

since $R_\theta\mathbf{a}, R_{-\theta}\mathbf{a} \in \Lambda$.

For AB stacking we note that for any vector $\mathbf{b} \in \mathbb{R}^2$ we have that

$$\begin{aligned} G^*(\mathcal{T}_{\mathbf{a}}\mathcal{T}_{-\mathbf{b}}\mathcal{R}_\theta V, \mathcal{T}_{\mathbf{a}}\mathcal{T}_{\mathbf{b}}\mathcal{R}_{-\theta} V) &= G^*(\mathcal{T}_{-\mathbf{b}}\mathcal{T}_{\mathbf{a}}\mathcal{R}_\theta V, \mathcal{T}_{\mathbf{b}}\mathcal{T}_{\mathbf{a}}\mathcal{R}_{-\theta} V) \\ = G^*(\mathcal{T}_{-\mathbf{b}}\mathcal{R}_\theta \mathcal{T}_{R_{-\theta}\mathbf{a}} V, \mathcal{T}_{\mathbf{b}}\mathcal{R}_{-\theta}\mathcal{T}_{R_\theta \mathbf{a}} V) &= G^*(\mathcal{T}_{-\mathbf{b}}\mathcal{R}_\theta V, \mathcal{T}_{\mathbf{b}}\mathcal{R}_{-\theta} V) \end{aligned}$$

since translations commute. From the linearity of the translation operator, we get that $\mathcal{T}_{\mathbf{a}} W^\theta_{AB} = W^\theta_{AB}$. In particular, both potentials will be periodic with respect to Λ^θ- as claimed. □

Remark 2.14. One can also define the set of angles that generate commensurate potentials $\tilde{\mathcal{C}}$- that is the set of all θ such that exists a $0 \neq \mathbf{a} \in \mathbb{R}^2$ such that

$$W^\theta(x + \mathbf{a}) = W^\theta(x)$$

It is easy to see that

$$\mathcal{C} \subset \tilde{\mathcal{C}}$$

We believe that $\mathcal{C} = \tilde{\mathcal{C}}$- though we will not try to prove it here.

With this notation, we will prove the following

Theorem 2.15. *We have*

$$\theta \in \mathcal{C} \cap (0, \frac{\pi}{3}) \iff \exists 0 < b < a, \gcd(b, a) = 1, \tan(\theta) = \frac{\sqrt{3}b}{a}$$

And any other $\tilde{\theta} \in \mathcal{C}$ can be reduced via the potential symmetries to some $\theta \in \mathcal{C} \cap [0, \frac{\pi}{3})$.

Furthermore, if we denote

$$\alpha = \begin{cases} 8\pi, & 3 \mid a \text{ and } 2 \nmid ab \\ 2, & 3 \nmid a \text{ and } 2 \nmid ab \\ 4\pi, & 3 \mid a \text{ and } 2 \mid ab \\ 1, & 3 \nmid a \text{ and } 2 \mid ab \end{cases}, \qquad N = \frac{1}{\alpha}\sqrt{a^2 + 3b^2}$$

then we have that

$$\Lambda^\theta = N \begin{cases} \Lambda, & 3 \nmid a \\ \Lambda^* & 3 \mid a \end{cases}$$

Remark 2.16. Even though the geometry of commensurate angles has been previously considered, see for example [12, 24, 32, 33], and similar rationality conditions have been considered, to the best of our knowledge, none of the previous results explicitly state the new lattice is a scaled version of the honeycomb lattice (or the dual of such lattice).

Throughout most of this paper, we will consider the following operator

$$H^\theta(\lambda) = -\Delta + \lambda W^\theta_x \tag{2.1.1}$$

for $\lambda \in \mathbb{R}$, $x \in \{AA, AB\}$, and W^θ_x a twisted bilayer potential of angle θ, for either AA or AB stacking, that corresponds to some honeycomb potential V.

An immediate corollary of Theorem 2.15 is

Corollary 2.17. *Let W^θ_{AA} (W^θ_{AB}) be a twisted bilayer potential in AA (AB) stacking of angle θ, for $\theta \in \mathcal{C} \cap (0, \frac{\pi}{3})$, then W^θ_{AA} (W^θ_{AB}) is a(n almost) honeycomb potential, with respect to lattice denoted by $\Lambda^\theta \in \{N\Lambda, N\Lambda^*\}$, for N as defined in Theorem 2.15.*

Proof. By Theorem 2.15 and Proposition 2.13, we have that W_x^θ for $x \in \{AA, AB\}$ is periodic with respect to scaled version of Λ or Λ^*. So we just need to check the symmetries for AA stacking we have that

$$\begin{aligned}
\mathcal{R}W_{AA}^\theta &= \mathcal{R}G(\mathcal{R}_\theta V, \mathcal{R}_{-\theta} V) = G(\mathcal{R}\mathcal{R}_\theta V, \mathcal{R}\mathcal{R}_{-\theta} V)\\
&= G(\mathcal{R}_\theta \mathcal{R} V, \mathcal{R}_{-\theta}\mathcal{R} V) = G(\mathcal{R}_\theta V, \mathcal{R}_{-\theta} V) = W_{AA}^\theta\\
\mathcal{R}_\pi W_{AA}^\theta &= \mathcal{R}_\pi G(\mathcal{R}_\theta V, \mathcal{R}_{-\theta} V) = G(\mathcal{R}_\pi\mathcal{R}_\theta V, \mathcal{R}_\pi\mathcal{R}_{-\theta} V)\\
&= G(\mathcal{R}_\theta \mathcal{R}_\pi V, \mathcal{R}_{-\theta}\mathcal{R}_\pi V) = G(\mathcal{R}_\theta V, \mathcal{R}_{-\theta} V) = W_{AA}^\theta
\end{aligned}$$

For AB stacking, we have that

$$\begin{aligned}
&\mathcal{R}W_{AB}^\theta = \mathcal{R}G^*(\frac{1}{3}\sum_{j=-1}^{1}\mathcal{T}_{-\frac{1}{2}R^jK_0}\mathcal{R}_\theta V, \frac{1}{3}\sum_{j=-1}^{1}\mathcal{T}_{\frac{1}{2}R^jK_0}\mathcal{R}_{-\theta} V)\\
&= G^*(\frac{1}{3}\sum_{j=-1}^{1}\mathcal{T}_{-\frac{1}{2}RR^jK_0}\mathcal{R}_\theta\mathcal{R} V, \frac{1}{3}\sum_{j=-1}^{1}\mathcal{T}_{\frac{1}{2}RR^jK_0}\mathcal{R}_{-\theta}\mathcal{R} V)\\
&= G^*(\frac{1}{3}\sum_{j=-1}^{1}\mathcal{T}_{-\frac{1}{2}R^jK_0}\mathcal{R}_\theta V, \frac{1}{3}\sum_{j=-1}^{1}\mathcal{T}_{\frac{1}{2}R^jK_0}\mathcal{R}_{-\theta} V)\\
&= W_{AB}^\theta
\end{aligned}$$

since we have that $R^2 = R^{-1}$. We will also have that

$$\begin{aligned}
&\mathcal{R}_\pi W_{AB}^{-\theta} = \mathcal{R}_\pi G^*(\frac{1}{3}\sum_{j=-1}^{1}\mathcal{T}_{-\frac{1}{2}R^jK_0}\mathcal{R}_{-\theta} V, \frac{1}{3}\sum_{j=-1}^{1}\mathcal{T}_{\frac{1}{2}R^jK_0}\mathcal{R}_{\theta} V)\\
&= G^*(\frac{1}{3}\sum_{j=-1}^{1}\mathcal{T}_{\frac{1}{2}R^jK_0}\mathcal{R}_{-\theta}\mathcal{R}_\pi V, \frac{1}{3}\sum_{j=-1}^{1}\mathcal{T}_{-\frac{1}{2}R^jK_0}\mathcal{R}_{\theta}\mathcal{R}_\pi V)\\
&= G^*(\frac{1}{3}\sum_{j=-1}^{1}\mathcal{T}_{-\frac{1}{2}R^jK_0}\mathcal{R}_\theta V, \frac{1}{3}\sum_{j=-1}^{1}\mathcal{T}_{\frac{1}{2}R^jK_0}\mathcal{R}_{-\theta} V)\\
&= W_{AB}^\theta
\end{aligned}$$

where we used that G^* is symmetric. □

2.2. **Floquet theory.** Next, we will need to introduce some key notions in Floquet's theory for Schrödinger operator with honeycomb potentials. This section will consider an arbitrary potential U, periodic with respect to a honeycomb lattice $\tilde{\Lambda}$, and corresponding unit cell $\tilde{\Omega}$. We will consider the operator

$$\tilde{H} = -\Delta + U.$$

Define the following spaces

$$L^2_k(\tilde{\Omega}) = \{f \in L^2(\tilde{\Omega}) \mid \forall a \in \tilde{\Lambda}, f(x+a) = e^{-i\langle k,a\rangle} f(x)\}.$$

the spaces of pseudo-periodic functions on the unit cell $\tilde{\Omega}$, for $k \in \tilde{\mathcal{B}}$. These spaces are equipped with natural inner product

$$\forall f, g \in L^2_k(\tilde{\Omega}),\ (f,g) = \frac{1}{|\tilde{\Omega}|}\int\limits_{\tilde{\Omega}} \bar{f}(x) g(x)\, dx$$

where $|\cdot|$ means the Lebesgue measure of the set. Usually, we suppress unit cell dependence, which should be inferred from the context.

Define for $f \in L^2(\mathbb{R}^2)$ the *Floquet transform*

$$(\mathcal{U}f)(k,y) = \sum_{\vec{n}\in\mathbb{Z}^2} e^{-i\langle k,\nu\vec{n}\rangle} f(y+\nu\vec{n})$$

for $y \in \mathbb{R}^2$ and $k \in \mathcal{B}$. As an $L^2(\mathcal{B}) \otimes L^2(\tilde{\Omega})$ convergent sum, the Floquet transform defines a bounded map from $L^2(\mathbb{R}^2)$ to $L^2(\mathcal{B}) \otimes L^2(\tilde{\Omega})$. The following properties of the Floquet transform are standard. See, for instance, Sections 4 and 5 of [19]:

Proposition 2.18. *The map $f \mapsto \mathcal{U}f$ has the following properties:*

1. *$\mathcal{U}$ is a unitary map from $L^2(\mathbb{R}^2)$ to $L^2(\mathcal{B}) \otimes L^2(\tilde{\Omega})$.*
2. *We have the unitary equivalence*

$$\mathcal{U}\tilde{H}\mathcal{U}^* = \int_{\mathcal{B}}^{\oplus} \tilde{H}(k)\,\frac{dk}{|\mathcal{B}|},$$

where

$$\tilde{H}(k) = -\Delta + U,$$

acts on L^2_k is a self-adjoint operator.

3. *For any $k \in \mathbb{T}^*$, $\tilde{H}(k)$ is bounded from below and has only pure point spectrum- so we have*

$$E_1(k) \le E_2(k) \le \dots$$

Where $E_n(k) \xrightarrow{n\to\infty} \infty$.

The reader may find the necessary background on direct integrals of Hilbert spaces in [31].

We note that for periodic function, i.e., $f \in L^2_0 = L^2_{per}$, we also have the following Fourier representation

$$f(y) = \sum_{\vec{m}\in\mathbb{Z}^2} \hat{f}_{\vec{m}} e^{i\langle\kappa\vec{m},y\rangle} \qquad \hat{f}_{\vec{m}} = \frac{1}{|\Omega|}\int_{\Omega} e^{-i\langle\kappa\vec{m},y\rangle} f(y)\,dy$$

This representation will used mostly in the context of the potential.

2.2.1. *Rotational symmetry.* On top of the translation symmetry (which allows for the use of Floquet transform), we also have symmetry with respect to rotation by R, as we have that for honeycomb potential U

$$\mathcal{R}[U](x) = U(R^{-1}x) = U(x)$$

Representation theory for R- invariant Hamitonains allows us to do an isotypic decomposition of the space; see [7] for more details. So, we define

$$L^2_{k,\sigma} = \{f \in L^2_k \mid \mathcal{R}f = \sigma f\}$$

for $\sigma \in \{1, \tau, \bar{\tau}\}$, where $\tau = e^{\frac{2\pi}{3}i} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$- the cubic root of unity. Moreover, we have that for $\tilde{K}_* \in \tilde{\mathbb{P}}$, one of the high symmetry points, the operator $\tilde{H}(\tilde{K}_*)$ maps $L^2_{\tilde{K}_*,\sigma}$ to itself- and thus allow us to reduce our study of $\tilde{H}(\tilde{K}_*)$ to its action on each $L^2_{\tilde{K}_*,\sigma}$.

It will be convenient to introduce the following notation for $\tilde{K}_* \in \tilde{\mathbb{P}}$, and $\vec{m} \in \mathbb{Z}^2$:

$$\tilde{K}_*(\vec{m}) = \tilde{K}_* + \tilde{\kappa}\vec{m}$$

Then, we can define

$$B = \tilde{\kappa}^{-1} R \tilde{\kappa} \qquad \varrho_1 = \tilde{\kappa}^{-1}(R - \mathrm{id})\tilde{K}_*$$
$$\varrho_{-1} = \tilde{\kappa}^{-1}(R^{-1} - \mathrm{id})\tilde{K}_* \qquad \varrho_0 = 0$$

Then, we can write that

$$R\tilde{K}_*(\vec{m}) = \tilde{K}_*(B\vec{m} + \varrho_1)$$
$$R^2\tilde{K}_*(\vec{m}) = \tilde{K}_*(B^{-1}\vec{m} + \varrho_{-1}),$$

And, similarly to [14] we define the equivalence $\approx$ that identifies the orbit of $\vec{m}$ under $B^j\vec{m} + \varrho_j, j \in \mathbb{Z}_3$ (throughout this paper we will take $\mathbb{Z}_3 = \{\pm 1, 0\}$), and we denote $\mathcal{S} = \mathbb{Z}^2/\approx$.

Remark 2.19. We would suppress the dependence of $\varrho_{\pm 1}$, B, and $\mathcal{S}$, on the exact choice of $\tilde{\kappa}$, which should be inferred from context.

We also note that if U is a honeycomb potential, we will have that

$$\forall \vec{m} \in \mathbb{Z}^2, \hat{U}_{B\vec{m}} = \hat{U}_{\vec{m}}$$

For the convenience of the reader, we show the explicit forms of these in the case of W^θ; when we recall that, then we will have two cases when $\kappa^\theta = \frac{1}{N}\kappa$ and when $\kappa^\theta = \frac{1}{N}\nu$:

$$(\kappa^\theta)^{-1}R(\kappa^\theta) = B = \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, & \kappa^\theta = \frac{1}{N}\kappa \\ \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, & \kappa^\theta = \frac{1}{N}\nu \end{cases}, \varrho_0 = 0$$

$$\varrho_1 = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \kappa^\theta = \frac{1}{N}\kappa \\ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, & \kappa^\theta = \frac{1}{N}\nu \end{cases}, \varrho_{-1} = \begin{cases} \begin{pmatrix} -1 \\ 0 \end{pmatrix}, & \kappa^\theta = \frac{1}{N}\kappa \\ \begin{pmatrix} 0 \\ -1 \end{pmatrix}, & \kappa^\theta = \frac{1}{N}\nu \end{cases}$$

when we considered $\tilde{K}_* = K$, for $\tilde{K}_* = K'$, one should take $\varrho_j' = -\varrho_j$ for $j \in \mathbb{Z}_3$.

2.3. **Main theorems.** To better understand the statement of our main theorem, we recall the main theorems from [7, 14] regarding the existence of the Dirac cones can be written as:

Theorem 2.20 ([7] -Theorems 2.4-2.5, [14] -Theorem 5.1)**.** *Let $H = -\Delta + \lambda U$, for $\lambda \in \mathbb{R}$ and U a honeycomb potential with $\tilde{\Lambda} = \Lambda$, be such that*

$$\hat{U}_{-\varrho_{-1}} = \frac{1}{|\tilde{\Omega}|} \int\limits_{\tilde{\Omega}} e^{-i\langle \kappa\varrho_{-1}, x\rangle} U \, dx \neq 0 \tag{2.3.1}$$

Then, for all $\lambda \in \mathbb{R}$ except possibly on a discrete set, we have that, for $\tilde{K}_* \in \{\tilde{K}, \tilde{K}'\}$

(1) There exists an eigenvalue $E_0(\lambda, \tilde{K}_*)$ of multiplicity exactly 2 in $L^2_{\tilde{K}_*}$, with eigenfunctions $\Phi_1(\lambda, x) \in L^2_{\tilde{K}_*,\tau}$, and $\Phi_2(\lambda, x) = \bar{\Phi}_1(\lambda, -x) \in L^2_{\tilde{K}_*,\bar{\tau}}$.

(2) There is some $\delta_k > 0$, and two pairs $(E_+(\lambda, k), \Phi_+(\lambda, k))$, $(E_-(\lambda, k), \Phi_-(\lambda, k))$ - which are Lipscitz continuous in k, such that for all $|k - \tilde{K}_*| < \delta$ we have

$$|E_\pm(\lambda, k) - E_0(\lambda, \tilde{K}^*)|^2 = |v_d(\lambda)|^2 |k - \tilde{K}_*|^2 + O(|k - \tilde{K}_*|^3)$$

So, there is a Dirac cone at $(\tilde{K}_*, E_0(\tilde{K}_*))$.

(3) *The slope of the cone, v_d, is given by*

$$v_d(\lambda) = -2i(\Phi_1(\lambda,\cdot), \partial_{x_1}\Phi_2(\lambda,\cdot))$$

It is easy to see that condition (2.3.1) will not hold in the case of twisted bilayer potentials of the type give in (1.1.1): this condition requires that mode denoted by ϱ_{-1} will not be 0, with respect to the new lattice Λ^θ. In other words, we want that the Fourier mode corresponding to k_j^θ will be non-zero, for $j \in \{1,2\}$, depending on whether the new periodic lattice is Λ or Λ^*. By duality scaling, one get that this correspond to $\frac{1}{N}\tilde{k}_j$, where $\tilde{k} \in \{k, v\}$, depending on the underlying lattice. Conversely, W_{AA}^θ contains twisted copies of the potential (which only twist the Fourier coefficients). The potential first non-zero mode will, in the best-case scenario, correspond to k_j, and thus the lowest frequency W_{AA}^θ could have will be some rotation of k_j, and in particular, we will have that $\frac{1}{N}\tilde{k}_j$ will not be in its support. See the full proof in the proof of Proposition 4.6.

We note that there are two differences between theorem 2.20 and our setting: one, we will deal with potentials that are periodic with respect to Λ^* and not only Λ, and second, we will also have to deal with almost honeycomb potential, that is not even but invariant under flips (which are given by swapping $(x,\theta) \to (-x,-\theta)$). These changes will be addressed in the proof of Theorem 3.1- which is the heart of the use of symmetries.

Thus, we get that we need to extend these results by pushing to the next order, and so we will prove the following statement

Theorem 2.21. *Let $\tilde{H} = -\Delta + \lambda U$, for $\lambda \in \mathbb{R}$ and U a honeycomb potential, with $\tilde{\Lambda} \in \{\Lambda, \Lambda^*\}$, be such that for any $\vec{m} \in \mathcal{S}$, there is some $\ell \in \mathbb{Z}_3$*

$$\hat{U}_{\vec{m}-\varrho_\ell} = 0 \tag{2.3.2}$$

Then we may choose $\mathcal{S}$ such that $\vec{m} - \varrho_{-1} \notin \operatorname{supp}\hat{U}$, with this choice, if we have

$$\sum_{\vec{m}\in\mathcal{S}\setminus\{\vec{0}\}} \frac{\hat{U}_{\vec{m}}\hat{U}_{\vec{m}-\varrho_{-1}}}{|\tilde{K}_*|^2 - |\tilde{K}_*(\vec{m})|^2} \neq 0 \tag{2.3.3}$$

then for all $\lambda \in \mathbb{R}$ except possible on a discrete set, we have that, for $\tilde{K}_* \in \{\tilde{K}, \tilde{K}'\}$ that the conclusions (1) - (3) of Theorem 2.20 hold.

Furthermore, even if condition (2.3.2) does not hold, we have that there is some $C > 0$ such that

$$|v_d(\lambda)|^2 \leq C(|\tilde{K}_*|^2 + \lambda\|U\|_\infty + \lambda^2\|\nabla U\|_\infty^2 \sum_{\vec{m}\in\mathcal{S}\setminus\{\vec{0}\}} \frac{1}{|\tilde{K}_* + \tilde{\kappa}\vec{m}|^4}) + O(\lambda^3\|U\|^3) \tag{2.3.4}$$

as $\lambda \to 0$.

The above theorem will allow us to conclude our main theorem:

Theorem 2.22. *Let $H^\theta = -\Delta + \lambda W^\theta$, for $\lambda \in \mathbb{R}$ and twisted bilayer potential with respect to honeycomb potential V, and angle $\theta \in \mathcal{C} \cap (0, \frac{\pi}{3})$, in either AA or AB stacking. W^θ is periodic with respect to Λ^θ. Let $K_*^\theta \in \mathbb{P}^\theta$- one of the points of high symmetry, then if we have*

$$\hat{W}^\theta_{-\varrho_{-1}} \neq 0 \tag{2.3.5}$$

or

$$\forall \vec{m} \in \mathcal{S} \exists \ell \in \mathbb{Z}_3, \hat{W}^\theta_{\vec{m}-\varrho_\ell} = 0 \text{ and } \sum_{\vec{m}\in\mathcal{S}\setminus\{\vec{0}\}} \frac{\hat{W}^\theta_{\vec{m}}\hat{W}^\theta_{\vec{m}-\varrho_{-1}}}{|K_*^\theta(\vec{m})|^2 - |K_*^\theta|^2} \neq 0 \tag{2.3.6}$$

then for all $\lambda \in \mathbb{R}$ except possible on a discrete set, we have that the conclusions (1) - (3) of Theorem 2.20 hold.

As a result of the proofs above, we get the following result about the vanishing of the Dirac points for small potentials:

Theorem 2.23. *We have for $\theta \in \mathcal{C} \cap (0, \frac{\pi}{3})$, that for any $\delta > 0$, if $|\lambda| < \frac{\delta}{N^2}$, then there is some constant $0 < C = C(\delta, V, G)$ such that*

$$|v_d(\lambda)| \leq \frac{C}{N} + O(N^{-3})$$

Finally, we will show that a set of examples for which condition (2.3.6) holds:

Proposition 2.24. *Define the equivalence relation $\sim_B$ by*

$$\vec{m} \sim_B \vec{n} \iff \exists \ell \in \mathbb{Z}_3, B^\ell \vec{m} = \vec{n}$$

Then denote $\tilde{\mathcal{S}} = \mathbb{Z}^2 / \sim_B$.

Let $(a_{\vec{m}})_{\vec{m} \in \tilde{\mathcal{S}}}$ be exponentially decaying sequence such that

$$\forall \vec{m} \in \tilde{\mathcal{S}}, a_{\vec{m}} > 0$$

We define

$$V(x) = \pm \sum_{\vec{m} \in \tilde{\mathcal{S}}} a_{\vec{m}} \sum_{\ell \in \mathbb{Z}_3} \cos(\langle \kappa B^\ell \vec{m}, x \rangle)$$

Then V is a honeycomb potential. And if we define the twisted potential as in (1.1.1), in AA stacking, that is

$$W^\theta = \frac{1}{2}(\mathcal{R}_\theta V + \mathcal{R}_{-\theta} V)$$

Then we have that for any $\theta \in \mathcal{C} \cap (0, \frac{\pi}{3})$ we have that

$$\sum_{\vec{m} \in \mathcal{S} \setminus \{\vec{0}\}} \frac{\hat{W}^\theta_{\vec{m}} \hat{W}^\theta_{\vec{m} - \varrho_{-1}}}{|K^\theta_*(\vec{m})|^2 - |K^\theta_*|^2} \neq 0, \text{ and } \forall \vec{m} \in \mathcal{S} \exists \ell \in \mathbb{Z}_3, \hat{W}^\theta_{\vec{m} - \varrho_\ell} = 0$$

holds.

3. Existence of Dirac points

In this section, we will prove Theorem 2.21- about the existence of Dirac cones under different technical conditions than in [14]. We start by noting that, as mentioned in the theorem, we may fix the choice of $\mathcal{S}$ in such a way that we will have for any $\vec{m} \in \mathcal{S}$

$$\hat{U}_{\vec{m} - \varrho_1} = 0$$

for $\vec{m} = 0$, this implies that $\hat{U}_{0 - \varrho_1} = 0$, and so we get that condition (2.3.1) does not hold.

Proof of Theorem 2.21. We recall that we consider

$$\tilde{H} = -\Delta + \lambda U$$

Where U is periodic with respect to $\tilde{\Lambda} \in \{\Lambda, \Lambda^*\}$, and we have $\tilde{\kappa} \in \{\kappa, \nu\}$- the reciprocal lattice matrix.

Theorem 2.4 from [7] allows one to extract the rule of symmetries and gives an explicit condition for the existence of Dirac cones in the form of an eigenvalue of multiplicity two. We will need the following slightly different statements:

Theorem 3.1. *[[7] - a version of Theorem 2.4] Let $\tilde{H}$ be a self-adjoint operator that is periodic with respect to Λ or Λ^* and invariant under the rotation R, and is invariant under F or F^*. Let $\tilde{K}_* \in \tilde{\mathbb{P}}$ be one of the high symmetry points. Then we have that*

$$L^2_{\tilde{K}_*} = L^2_{\tilde{K}_*,1} \oplus L^2_{\tilde{K}_*,\perp}$$

where the splitting is $\tilde{H}$ invariant. Since $\tilde{H}$ is also invariant under F or F^*, we have that all the eigenvalues restricted to $L^2_{\tilde{K}_*,\perp}$ have even multiplicity. If the multiplicity of some eigenvalue E_0 is exactly 2, we have that

$$|E_\pm(\lambda,k) - E_0(\lambda,\tilde{K}_*)|^2 = |v_d(\lambda)|^2|k-\tilde{K}_*|^2 + O(|k-\tilde{K}_*|^3)$$

for some $v_d \in \mathbb{C}$.

Proof. The proof of Theorem 2.4 in [7] relies only on two steps: First, they show that if E is a double eigenvalue in $L^2_{\tilde{K}_*}$, the conclusion holds (Lemma 3.1 there), and the second step shows the splitting and the evenness of the multiplicity (Lemma 4.3). Both lemmas rely only on the symmetries of the Hamiltonian and the restriction to the points of high symmetry subspace (that is, the space $L^2_{\tilde{K}_*}$ is invariant under rotation). So, this theorem can apply to the case where the U is periodic with respect to Λ^*, with its high symmetry points (irrespective of the choice of base vectors).

To conclude our version of the theorem, we need to address the change between F (which was dealt with in [7]) and F^*. In the case of F^*, we should consider the symmetry group as acting on (x,θ). We note that F^*, when combined with complex conjugation, plays the same rule at $\overline{V}$ in the proof in [7], as it has the same relation with the rotation and is an involution. Thus, it will induce the same co-representation structure, and the proof will proceed exactly the same. □

So, to prove there is a Dirac cone around a point $(\tilde{K}_*, E)$ (or in other words, Theorem 2.21), we need to show that E has a double eigenvalue and that $v_d \neq 0$.

Using Lemma 5.3 in [7] or Proposition 4.1 in [14], we can conclude that

$$v_d = -2i(\Phi_1, \partial_{x_1}\Phi_2)_{L^2_{\tilde{K}_*}}$$

We would follow the proof of Theorem 2.5 in [7] (which is similar to Proposition 6.3 in [14]), for $\lambda = 0$, the free Laplacian, the energy $E = |\tilde{K}_*|^2$ is of multiplicity 3 - where each of the spaces $\{L^2_{\tilde{K}_*,\sigma}\}_{\sigma\in\{1,\tau,\bar{\tau}\}}$ has a simple eigenvalue. The perturbation theory of simple eigenvalues gives that each of the eigenvalues extends to an analytic function $E_\sigma(\lambda)$, see [7, 14] for more details. Thus it will be enough to show that $E_\tau = E_{\bar{\tau}} \neq E_1$, as functions. By the above, it will suffice to show that $E_\tau(\lambda) \neq E_1(\lambda)$ for some λ (as the remaining eigenvalues must remain of even multiplicity, and thus have to be of multiplicity 2). Then, these functions may only intersect in a discrete set.

For this, we consider small λ and energies close to $|\tilde{K}_*|^2$, as mentioned above we have some smooth function $E_\sigma(\lambda)$ such that

$$(-\Delta + \lambda U)\Phi_\sigma = E_\sigma(\lambda)\Phi_\sigma$$

We recall that for $\lambda = 0$, we have that the eigenfunctions in $L^2_{\tilde{K}^*.\sigma}$, for $\sigma \in \{1,\tau,\bar{\tau}\}$ are given by

$$\psi_0^\sigma = \frac{1}{\sqrt{3}}\sum_{\ell\in\mathbb{Z}^3} \sigma^{-\ell} e^{i\langle \tilde{K}_* + \tilde{\kappa}\rho_\ell, x\rangle}$$

$$\psi_{\vec{m}}^\sigma = \frac{1}{\sqrt{3}}\sum_{\ell\in\mathbb{Z}^3} \sigma^{-\ell} e^{i\langle \tilde{K}_* + \tilde{\kappa}(B^\ell\vec{m}+\rho_\ell), x\rangle}$$

Using second-order perturbation theory, or the Rayleigh-Schrödinger coefficients (see, for example, [31] -page 7), we get that

$$E_\sigma(\lambda) = E_\sigma(0) + \lambda E_\sigma^{(1)} + \lambda^2 E_\sigma^{(2)} + O(\lambda^3)$$

where

$$E_\sigma^{(1)} = (\psi_0^\sigma, U\psi_0^\sigma)_{L^2_{\tilde{K}_*}}$$
$$E_\sigma^{(2)} = \sum_{\vec{m}\in\mathcal{S}\setminus\{\vec{0}\}} \frac{|(\psi_{\vec{m}}^\sigma, U\psi_0^\sigma)_{L^2_{\tilde{K}_*}}|^2}{|\tilde{K}_*|^2 - |\tilde{K}_*(\vec{m})|^2}$$

Using the estimate in Theorem 2.1 [11], we see that, in fact, we have that

$$E_\sigma(\lambda) = E_\sigma(0) + \lambda E_\sigma^{(1)} + \lambda^2 E_\sigma^{(2)} + O(\lambda^3\|U\|^3) \tag{3.0.1}$$

So we compute:

$$\begin{aligned}
&(\psi_{\vec{m}}^\sigma, U\psi_0^\sigma)_{L^2_{\tilde{K}_*}} \\
&= \frac{1}{3|\tilde{\Omega}|}\int_{\tilde{\Omega}} \sum_{\ell,\ell'\in\mathbb{Z}^3} \bar{\sigma}^{-\ell} e^{-i\langle \tilde{K}_*+\tilde{\kappa}B^\ell\vec{m}+\tilde{\kappa}\varrho_\ell, x\rangle} U(x)\sigma^{-\ell'} e^{i\langle \tilde{K}_*+\tilde{\kappa}\varrho_{\ell'}, x\rangle}\,dx \\
&= \frac{1}{3|\tilde{\Omega}|}\int_{\tilde{\Omega}} \sum_{\ell,\ell'\in\mathbb{Z}^3} \sigma^{\ell-\ell'} e^{i\langle \tilde{\kappa}(\varrho_{\ell'}-B^\ell\vec{m}-\varrho_\ell), x\rangle} U(x)\,dx \\
&= \frac{1}{3}\sum_{\ell,\ell'\in\mathbb{Z}^3} \sigma^{\ell-\ell'}\hat{U}_{\varrho_{\ell'}-B^\ell\vec{m}-\varrho_\ell} = \frac{1}{3}\sum_{\ell,\ell'\in\mathbb{Z}^3} \sigma^{\ell-\ell'}\hat{U}_{B^{-\ell}\varrho_{\ell'}-\vec{m}-B^{-\ell}\varrho_\ell} \\
&= \frac{1}{3}\sum_{\ell,\ell'\in\mathbb{Z}^3} \sigma^{\ell-\ell'}\hat{U}_{\varrho_{\ell'-\ell}-\vec{m}} = \frac{1}{3}\sum_{\ell,\ell'\in\mathbb{Z}^3} \sigma^{\ell-\ell'}\hat{U}_{\vec{m}-\varrho_{\ell'-\ell}} \\
&= \sum_{\ell\in\mathbb{Z}^3} \sigma^{-\ell}\hat{U}_{\vec{m}-\varrho_\ell} = \hat{U}_{\vec{m}} + \sigma\hat{U}_{\vec{m}-\varrho_{-1}}
\end{aligned}$$

where we used that $\hat{U}_{B\vec{m}} = \hat{U}_{\vec{m}}$ for all $\vec{m}\in\mathbb{Z}^2$, and we recall that we chose that $\mathcal{S}$ in such a way that $\hat{U}_{\vec{m}-\varrho_1} = 0$, for all $\vec{m}\in\mathcal{S}$.

In particular we get that

$$E_\sigma^{(1)} = (\psi_0^\sigma, U\psi_0^\sigma)_{L^2_{\tilde{K}_*}} = \hat{U}_0 + \sigma\hat{U}_{0-\varrho_{-1}}$$

Note

$$B^{-1}\varrho_1 = -\varrho_{-1} \implies \hat{U}_{-\varrho_{-1}} = \hat{U}_{-\varrho_1} = 0$$

since $\hat{U}_{B\vec{m}} = \hat{U}_{-\vec{m}}$ for all $\vec{m}\in\mathbb{Z}^2$. So we got that

$$E_\sigma^{(1)} = \hat{U}_{\vec{0}}$$

So $E_\sigma^{(1)}$ is independent of σ - so we see that the eigenvalues do not separate in the first order (as expected from this argument in [7] or [14]).

So, we compute the next order, and we start by noting that

$$|(\psi_{\vec{m}}^\sigma, U\psi_0^\sigma)_{\tilde{\Omega}}|^2 = |\hat{U}_{\vec{m}}|^2 + |\hat{U}_{\vec{m}-\varrho_{-1}}|^2 + (\sigma+\sigma^{-1})\hat{U}_{\vec{m}}\hat{U}_{\vec{m}-\varrho_{-1}}$$

So we have that

$$E_\sigma^{(2)} = \sum_{\vec{m}\in\mathcal{S}\setminus\{\vec{0}\}} \frac{|\hat{U}_{\vec{m}}|^2 + |\hat{U}_{\vec{m}-\varrho_{-1}}|^2 + (\sigma+\sigma^{-1})\hat{U}_{\vec{m}}\hat{U}_{\vec{m}-\varrho_{-1}}}{|\tilde{K}_*|^2 - |\tilde{K}_*(\vec{m})|^2}$$

Now by assumption

$$\sum_{\vec{m}\in\mathcal{S}\setminus\{\vec{0}\}} \frac{\hat{U}_{\vec{m}}\hat{U}_{\vec{m}-\varrho_{-1}}}{|\tilde{K}_*|^2-|\tilde{K}_*(\vec{m})|^2} \neq 0$$

Thus,

$$E_\tau(\lambda) \neq E_1(\lambda)$$

as they differ in the second-order term. By Theorem 3.1 we conclude that the multiplicity is even in $L^2_{\tilde{K}_*,\perp}$, and so we can conclude that

$$E_{\bar{\tau}}(\lambda) = E_\tau(\lambda) \neq E_1(\lambda)$$

as needed.

Finally, we need to show that v_d is not 0 except for finitely many points. So, we recall from the proof of Theorem 2.5 in [7] that $v_d(\lambda)$ is analytic. So we may show that $v_d(\lambda)\neq 0$ for $\lambda = 0$, thus concluding the proof. So we can compute

$$\begin{aligned}
(\psi_0^\tau, \partial_{x_1}\psi_0^{\bar{\tau}})_{L^2_{\tilde{K}_*}} &= \frac{1}{3|\tilde{\Omega}|}\int\limits_{\tilde{\Omega}} \sum_{\ell,\ell'\in\mathbb{Z}^3} \tau^{\ell} e^{-i\langle \tilde{K}_*+\tilde{\kappa}\varrho_\ell, x\rangle}\partial_{x_1}\tau^{\ell'} e^{i\langle \tilde{K}_*+\tilde{\kappa}\varrho_{\ell'}, x\rangle}\,dx \\
&= \frac{i}{3|\tilde{\Omega}|}\int\limits_{\tilde{\Omega}} \sum_{\ell,\ell'\in\mathbb{Z}^3} \tau^{\ell+\ell'} e^{-i\langle \tilde{K}_*+\tilde{\kappa}\varrho_\ell, x\rangle}\langle \tilde{K}_*+\tilde{\kappa}\varrho_{\ell'}, \begin{pmatrix}1\\0\end{pmatrix}\rangle e^{i\langle \tilde{K}_*+\tilde{\kappa}\varrho_{\ell'}, x\rangle}\,dx \\
&= i\sum_{\ell,\ell'\in\mathbb{Z}^3} \tau^{\ell+\ell'}\langle \tilde{K}_*+\tilde{\kappa}\varrho_{\ell'}, \begin{pmatrix}1\\0\end{pmatrix}\rangle \frac{1}{3|\tilde{\Omega}|}\int\limits_{\tilde{\Omega}} e^{i\langle \tilde{\kappa}(\varrho_{\ell'}-\varrho_\ell), x\rangle}\,dx \\
&= \sum_{\ell\in\mathbb{Z}^3} \tau^{2\ell}\langle \tilde{K}_*+\tilde{\kappa}\varrho_{\ell}, \begin{pmatrix}1\\0\end{pmatrix}\rangle \frac{1}{3} \\
&= i\frac{1}{3}(1+\tau^2+\tau^{-2})\langle \tilde{K}_*, \begin{pmatrix}1\\0\end{pmatrix}\rangle + \frac{1}{3}(\tau^2\langle\tilde{\kappa}\varrho_1, \begin{pmatrix}1\\0\end{pmatrix}\rangle + \tau^{-2}\langle\tilde{\kappa}\varrho_{-1}, \begin{pmatrix}1\\0\end{pmatrix}\rangle)
\end{aligned}$$

Noting that

$$(1+\tau^2+\tau^{-2}) = 1+\tau^{-1}+\tau = 0$$

We got that

$$v_d(0) = 2i(\psi_0^\tau, \partial_{x_1}\psi_0^{\bar{\tau}})_{L^2_{\tilde{K}_*}} = \frac{2i}{3}(\tau^{-1}\langle\tilde{\kappa}\varrho_1, \begin{pmatrix}1\\0\end{pmatrix}\rangle + \tau\langle\tilde{\kappa}\varrho_{-1}, \begin{pmatrix}1\\0\end{pmatrix}\rangle) \tag{3.0.2}$$

So we have that

$$(\tau^{-1}\langle\tilde{\kappa}\varrho_1, \begin{pmatrix}1\\0\end{pmatrix}\rangle + \tau\langle\tilde{\kappa}\varrho_{-1}, \begin{pmatrix}1\\0\end{pmatrix}\rangle) = \begin{cases} -2\pi i, & \tilde{\kappa}=\kappa \\ \frac{\sqrt{3}}{2}, & \tilde{\kappa}=\nu \end{cases}$$

And we conclude that

$$v_d(0) = -2i(\psi_0^\tau, \partial_{x_1}\psi_0^{\bar{\tau}})_{L^2_{\tilde{K}_*}} = \begin{cases} -4\pi, & \tilde{\kappa}=\kappa \\ -\sqrt{3}i, & \tilde{\kappa}=\nu \end{cases} \neq 0$$

as needed.

For the last part of the theorem, we recall that

$$v_d(\lambda) = -2i(\Phi_1, \partial_{x_1}\Phi_2)_{L^2_{\tilde{K}_*}}$$

where Φ_j are the eigenfunctions which have

$$(-\Delta + U)\Phi_j = E_\sigma(\lambda)\Phi_j$$

for $j \in \{1, 2\}$, where $\sigma \in \{\tau, \bar{\tau}\}$. So we can write

$$|v_d(\lambda)|^2 = 4|(\Phi_1, \partial_x \Phi_2)_{L^2_{\tilde{K}_*}}|^2 \le 4\|\Phi_1\|^2 \|\partial_x \Phi_2\|^2$$

With the normalization of the eigenfunctions ($\|\Phi_1\| = 1 = \|\Phi_2\|$), we get

$$|v_d(\lambda)|^2 \le 4\|\partial_x \Phi_2\|^2 \le 4\|\nabla \Phi_2\|^2$$

Recalling that $E_\tau = E_{\bar{\tau}} = E$, we write

$$\begin{aligned}
\|\nabla \Phi_2\|^2 &= (\Phi_2, (-\Delta)\Phi_2)_{L^2_{\tilde{K}_*}} = (\Phi_2, (E - \lambda U)\Phi_2)_{L^2_{\tilde{K}_*}} \\
&= E\|\Phi_2\|^2 - (\Phi_2, \lambda U \Phi_2)_{L^2_{\tilde{K}_*}} \le E + |\lambda| \|U\|_\infty \|\Phi_2\|^2 \\
&= E + 2|\lambda| \|U\|_\infty
\end{aligned}$$

So, using Equation (3.0.1), we get the following expansion:

$$\begin{aligned}
&|v_d(\lambda)|^2 \le 4(E + 2|\lambda| \|U\|_\infty) \\
&= 4(|E_\tau(0)| + |\lambda||E_\tau^{(1)}| + \lambda^2 |E_\tau^{(2)}| + O(\lambda^3 \|U\|^3) + 2|\lambda| \|U\|_\infty) \\
&= 4(|\tilde{K}_*|^2 + |\lambda|(|\hat{U}_0| + 2\|U\|_\infty) + \lambda^2 \sum_{\vec{m} \in \mathcal{S} \setminus \{\vec{0}\}} \frac{|\hat{U}_{\vec{m}} + \tau \hat{U}_{\vec{m} - \varrho_{-1}}|^2}{|\tilde{K}_*|^2 - |\tilde{K}_*(\vec{m})|^2} \\
&+ O(\lambda^3 \|U\|^3))
\end{aligned}$$

We note that for any $\vec{m} \in \mathcal{S}$, since U is smooth, we have

$$|\hat{U}_{\vec{m}}| \le \frac{\|\nabla U\|_\infty}{|\tilde{K}_* + \kappa \vec{m}|}$$

And so we have that

$$|\hat{U}_{\vec{m}} + \tau \hat{U}_{\vec{m} - \varrho_{-1}}|^2 \le (|\hat{U}_{\vec{m}}| + |\hat{U}_{\vec{m} - \varrho_{-1}}|)^2 \le 4 \frac{\|\nabla U\|_\infty^2}{|\tilde{K}_* + \kappa \vec{m}|^2}$$

combining all the above, we have

$$\begin{aligned}
|v_d(\lambda)|^2 \le & 4(|\tilde{K}_*|^2 + 3|\lambda| \|U\|_\infty + \lambda^2 4 \|\nabla U\|_\infty^2 \sum_{\vec{m} \in \mathcal{S} \setminus \{\vec{0}\}} \frac{1}{|\tilde{K}_* + \tilde{\kappa} \vec{m}|^4} \\
& + O(\lambda^3 \|U\|^3))
\end{aligned}$$

To conclude, we have

$$\begin{aligned}
|v_d(\lambda)|^2 \le & 16(|\tilde{K}_*|^2 + |\lambda| \|U\|_\infty + \lambda^2 \|\nabla U\|_\infty^2 \sum_{\vec{m} \in \mathcal{S} \setminus \{\vec{0}\}} \frac{1}{|\tilde{K}_* + \tilde{\kappa} \vec{m}|^4}) \\
& + O(\lambda^3 \|U\|^3)
\end{aligned}$$

as claimed.

In the more general case, where we do not assume that for any $\vec{m} \in \mathcal{S}$

$$\hat{U}_{\vec{m} - \varrho_1} = 0$$

We will still have that

$$\begin{aligned}
&|v_d(\lambda)|^2 \le 4(E + 2|\lambda| \|U\|_\infty) \\
&\le 4(|E_\tau(0)| + |\lambda||E_\tau^{(1)}| + \lambda^2 |E_\tau^{(2)}| + O(\lambda^3 \|U\|^3) + 2|\lambda| \|U\|_\infty)
\end{aligned}$$

Only this time we will have that

$$|E_\tau^{(1)}| = |(\psi_0^\tau, U\psi_0^\tau)_{L^2_{\tilde{K}_*}}| = |\hat{U}_0 + (\tau+\bar{\tau})\hat{U}_{0-\varrho_{-1}}| \le 3\|U\|_\infty$$

And

$$\begin{aligned}|E_\tau^{(2)}| &= |\sum_{\vec{m}\in\mathcal{S}\setminus\{\vec{0}\}} \frac{|(\psi_{\vec{m}}^\tau, U\psi_0^\tau)_{L^2_{\tilde{K}_*}}|^2}{|\tilde{K}_*|^2 - |\tilde{K}_*(\vec{m})|^2}| \le \sum_{\vec{m}\in\mathcal{S}\setminus\{\vec{0}\}} \frac{|\sum_{\ell\in\mathbb{Z}^3}\tau^{-\ell}\hat{U}_{\vec{m}-\varrho_\ell}|^2}{||\tilde{K}_*|^2 - |\tilde{K}_*(\vec{m})|^2|} \\ &\le 9\|\nabla U\|^2 \sum_{\vec{m}\in\mathcal{S}\setminus\{\vec{0}\}} \frac{1}{|\tilde{K}_*(\vec{m})|^4}\end{aligned}$$

So we get that there is some constant $C>0$ such that

$$\begin{aligned}|v_d(\lambda)|^2 \le& C(|\tilde{K}_*|^2 + |\lambda|\|U\|_\infty + \lambda^2\|\nabla U\|_\infty^2 \sum_{\vec{m}\in\mathcal{S}\setminus\{\vec{0}\}} \frac{1}{|\tilde{K}_* + \tilde{\kappa}\vec{m}|^4}) \\ &+ O(\lambda^3\|U\|^3)\end{aligned}$$

as needed. $\square$

Remark 3.2. We note that we could have that $\hat{U}_{\vec{m}}\hat{U}_{\vec{m}-\varrho_{-1}} = 0$ for any $\vec{m}\in\operatorname{supp}\hat{U}^\theta$. As the proof above shows, one must go to higher-order terms in the perturbation series to get a sufficient non-degeneracy condition for such cases. We will not develop the other terms in this work. Such consideration might also affect the asymptotic results for $v_d(\lambda)$.

4. Twisted bilayer potential

This section will prove two of our main results: Theorem 2.15, which describes the commensurate angles. Then we will focus on the representative example of 1.1.1, and prove Lemma 4.9 describing the Fourier support of W^θ- which together with Theorem 2.21 will allow us to prove Theorem 2.22, about the existence of Dirac cones for twisted potentials, in full.

4.1. **Proof of Theorem 2.15.** We start by providing a full description of the commensurate angles. We mention that a different approach to finding the new lattice vectors can be found in [32] using Clifford algebras. However, their results are hard to read- as they give a different base for each case (depending on the parity and whether or not $3\mid a$ - in the notation below)- and they get a different set of spanning vectors. So, we will provide complete proof that the new lattice will be periodic with respect to a scaled version of the honeycomb lattice.

First, We show that we can reduce our problem to the range $\theta\in[0,\frac{\pi}{3})$:

Proposition 4.1. *Let $\theta\in[0,2\pi)$, then we have some $\tilde{\theta}\in[0,\frac{\pi}{3}]$ such that $H^\theta = H^{\tilde{\theta}}$.*

Proof. Let $\theta\in[0,2\pi)$. We start by noting that

$$V(R_{-\theta-\frac{\pi}{3}}x) = V(R_{-\frac{\pi}{3}}R_{-\theta}x) = V(RR_\pi^{-1}R_{-\theta}x) = V(-R_{-\theta}x) = V(R_{-\theta}x)$$

since $R_\pi = -\operatorname{id}$, and $R = R_{\frac{2\pi}{3}}$.

Similarly

$$V(R_{\theta+\frac{\pi}{3}}x) = V(R_{\frac{\pi}{3}}R_\theta x) = V(R^{-1}R_\pi R_\theta x) = V(-R_\theta x) = V(R_\theta x)$$

So we get that

$$
\begin{aligned}
W_{AA}^{\theta+\frac{\pi}{3}}(x) &= G(\mathcal{R}_{\theta+\frac{\pi}{3}}V, \mathcal{R}_{-\theta-\frac{\pi}{3}}V) = G(\mathcal{R}_{\theta}V, \mathcal{R}_{-\theta}V) = W_{AA}^{\theta}(x) \\
W_{AB}^{\theta+\frac{\pi}{3}}(x) &= G^{*}(\mathcal{R}_{\theta+\frac{\pi}{3}}V, \frac{1}{3}\sum_{j=-1}^{1}\mathcal{T}_{R^{j}K_{0}}\mathcal{R}_{-\theta-\frac{\pi}{3}}V) \\
&= G^{*}(\mathcal{R}_{\theta}V, \frac{1}{3}\sum_{j=-1}^{1}\mathcal{T}_{R^{j}K_{0}}\mathcal{R}_{-\theta}V) = W_{AB}^{\theta}(x)
\end{aligned}
$$

as the potentials are the same. So we conclude it is enough to take $\theta \in [0, \frac{\pi}{3})$. □

Now, we will give a better description of the commensurate lattice $\Lambda^{\theta} = R_{\theta}\Lambda \cap R_{-\theta}\Lambda$

Lemma 4.2. *Let $\theta \in \mathcal{C} \cap (0, \frac{\pi}{3})$, and $\Lambda^{\theta} = R_{\theta}\Lambda \cap R_{-\theta}\Lambda$. Then we have the following*

(1) *First*

$$\tan(\theta) = \frac{\sqrt{3}b}{a}$$

$a = b = 1$

$N = \frac{1}{2}\sqrt{4} = 1$

for $0 < b < a$, and a and b are co-primes.

(2) *Denoting*

$$\alpha = \begin{cases} 8\pi, & 3 \mid a \text{ and } 2 \nmid ab \\ 2, & 3 \nmid a \text{ and } 2 \nmid ab \\ 4\pi, & 3 \mid a \text{ and } 2 \mid ab \\ 1, & 3 \nmid a \text{ and } 2 \mid ab \end{cases}, \qquad N = \frac{1}{\alpha}\sqrt{a^{2} + 3b^{2}}$$

then we have that

$$\Lambda^{\theta} = N \begin{cases} \Lambda, & 3 \nmid a \\ \Lambda^{*} & 3 \mid a \end{cases}$$

$R_\theta = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

(3) *And we have that*

$$R_{\theta} = \frac{1}{\alpha N}\begin{pmatrix} a & -\sqrt{3}b \\ \sqrt{3}b & a \end{pmatrix}$$

Before we prove this claim, we will need the following identity:

Proposition 4.3. *We have that*

$$\bigcup_{r \in \{0, \pm 1\}} (4\pi)(\mathbb{Z}^{2} + \frac{r}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \nu^{-1}\kappa\mathbb{Z}^{2}$$

Proof. We recall that we have

$$\frac{1}{4\pi}\kappa = \begin{pmatrix} \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \qquad \nu^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 1 \\ \frac{1}{\sqrt{3}} & -1 \end{pmatrix}$$

We compute

$$
\begin{aligned}
\frac{1}{4\pi}\nu^{-1}\kappa &= \begin{pmatrix} \frac{1}{\sqrt{3}} & 1 \\ \frac{1}{\sqrt{3}} & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \\
4\pi(\nu^{-1}\kappa)^{-1} &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}
\end{aligned}
$$

So, we need to show that

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \bigcup_{r\in\{0,\pm1\}} (\mathbb{Z}^2 + \frac{r}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \mathbb{Z}^2$$

Then we note that

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \frac{1}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Naturally, we have that

$$\bigcup_{r\in\{0,\pm1\}} (\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbb{Z}^2 + r\begin{pmatrix} 1 \\ 1 \end{pmatrix}) \subset \mathbb{Z}^2$$

On the other hand, let $\begin{pmatrix} m \\ n \end{pmatrix} \in \mathbb{Z}^2$, then we take $r \in \mathbb{Z}_3$ such that

$$r \equiv 2m - n \mod 3$$

Define

$$\tilde{m} = \frac{2m-n-r}{3}, \qquad \tilde{n} = n - m + \tilde{m} = \frac{2n-m-r}{3}$$

We note that $\tilde{m}, \tilde{n} \in \mathbb{Z}^2$, and so we have that

$$\begin{aligned} 2\tilde{m} + \tilde{n} + r &= \frac{1}{3}(4m - 2n - 2r + 2n - m - r + 3r) = m \\ 2\tilde{n} + \tilde{m} + r &= \frac{1}{3}(4n - 2m - 2r + 2m - n - r + 3r) = n \end{aligned}$$

so we can write

$$\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \tilde{m} \\ \tilde{n} \end{pmatrix} + r\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which gives us the reverse inclusion and allows us to conclude

$$\bigcup_{r\in\{0,\pm1\}} (\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbb{Z}^2 + r\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \mathbb{Z}^2$$

which allows us to conclude that

$$\bigcup_{r\in\{0,\pm1\}} (4\pi)(\mathbb{Z}^2 + \frac{r}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \nu^{-1}\kappa\mathbb{Z}^2$$

as claimed. □

Now we can prove Lemma 4.2 describing the new lattice generated by a commensurate angle:

Proof of Lemma 4.2. We recall that

$$R_\theta + R_{-\theta} = 2\cos(\theta)\mathrm{Id}$$

So if $x \in R_\theta\Lambda \cap R_{-\theta}\Lambda$ we have some $u, v \in \Lambda$ such that, since $0 < \theta < \frac{\pi}{3}$, so $\cos(\theta) \neq 0$

$$\begin{aligned} x &= R_\theta u = R_{-\theta}v = 2\cos(\theta)v - R_\theta v \\ v &= \frac{1}{2\cos(\theta)}R_\theta(u+v) \\ R_\theta x &= \frac{1}{2\cos(\theta)}R_\theta(u+v) \\ x &= \frac{1}{2\cos(\theta)}(u+v) \in \frac{1}{2\cos(\theta)}\Lambda \end{aligned}$$

In particular, we get that

$$R_\theta\Lambda \cap R_{-\theta}\Lambda \subset \frac{1}{2\cos(\theta)}\Lambda \cap R_\theta\Lambda$$

Denoting $A = \nu^{-1}R_\theta\nu$ we get that

$$(R_\theta\Lambda \cap R_{-\theta}\Lambda) \subset \nu A(\frac{1}{2\cos(\theta)}A^{-1}\mathbb{Z}^2 \cap \mathbb{Z}^2)$$

So, we may compute

$$A = \begin{pmatrix} \cos(\theta) + \frac{\sin(\theta)}{\sqrt{3}} & \frac{2}{\sqrt{3}}\sin(\theta) \\ -\frac{2}{\sqrt{3}}\sin(\theta) & \cos(\theta) - \frac{\sin(\theta)}{\sqrt{3}} \end{pmatrix}$$
$$\frac{1}{\cos(\theta)}A^{-1} = \mathrm{Id} - \frac{\tan(\theta)}{\sqrt{3}}\begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} = \mathrm{Id} - \frac{\tan(\theta)}{\sqrt{3}}\mathcal{I}$$

Thus, we get that

$$R_\theta\Lambda \cap R_{-\theta}\Lambda \subset \nu A((\frac{1}{2}\mathrm{Id} - \frac{\tan(\theta)}{2\sqrt{3}}\mathcal{I})\mathbb{Z}^2 \cap \mathbb{Z}^2)$$

We note that since $R_\theta\Lambda \cap R_{-\theta}\Lambda \neq \{0\}$, then we have some $(\mathrm{Id} + \frac{\tan(\theta)}{2\sqrt{3}}\mathcal{I})\mathbb{Z}^2 \cap \mathbb{Z}^2 \neq \{0\}$, so we have some $u, v \in \mathbb{Z}^2$ such that

$$\mathbb{Z}^2 \ni v = \frac{1}{2}u - \frac{\tan(\theta)}{2\sqrt{3}}\mathcal{I}u \implies \frac{\tan(\theta)}{\sqrt{3}}\mathcal{I}u \in \mathbb{Z}^2$$

So, we conclude that

$$\mathbb{Z}^2 \cap \frac{\tan(\theta)}{\sqrt{3}}\mathcal{I}\mathbb{Z}^2 \neq \{0\}$$

Thus we conclude that, in particular, $\frac{\tan(\theta)}{\sqrt{3}} \in \mathbb{Q}$. So we write that

$$\frac{\tan(\theta)}{\sqrt{3}} = \frac{b}{a}$$

where $a, b \in \mathbb{Z}$ are co-prime. Since $0 < \theta < \frac{\pi}{3}$, we get that

$$0 < \frac{b}{a} = \frac{\tan(\theta)}{\sqrt{3}} < 1$$

and we can choose $a, b > 0$, and $b < a$, thus proving part 1 of the Lemma.

So, we get that

$$R_\theta\Lambda \cap R_{-\theta}\Lambda \subset \nu A((\frac{1}{2}\mathrm{Id} - \frac{b}{2a}\mathcal{I})\mathbb{Z}^2 \cap \mathbb{Z}^2)$$

In particular we get that if $v \in (\mathrm{Id} - \frac{b}{2a}\mathcal{I})\mathbb{Z}^2 \cap \mathbb{Z}^2$ we have some $u \in \mathbb{Z}^2$ such that

$$\begin{aligned} v &= (\frac{1}{2}\mathrm{Id} - \frac{b}{2a}\mathcal{I})u \\ 2av &= au - b\mathcal{I}u \end{aligned}$$

From this last equality, since $\mathcal{I} \cong \mathrm{Id} \mod 2$ and $\gcd(a,b) = 1$, we can get the following equations:

$$0 \cong (a+b)u \mod 2 \tag{4.1.1}$$

$$0 \cong \mathcal{I}u \mod a \tag{4.1.2}$$

Equation (4.1.1) implies that if we denote

$$\epsilon = \begin{cases} 1, & 2 \nmid ab \\ 0, & 2 \mid ab \end{cases}$$

We get that $2^{\epsilon-1}u \in \mathbb{Z}^2$. Equation (4.1.2) implies when writing $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$$0 \cong 2^{\epsilon-1}\begin{pmatrix} u_1 + 2u_2 \\ -2u_1 - u_2 \end{pmatrix} \mod a \implies a \mid 3 \cdot 2^{\epsilon-1}(u_1 + u_2)$$

Define

$$\rho = \begin{cases} 1, & 3 \mid a \\ 0, & 3 \nmid a \end{cases}$$

writing $a = 3^\rho c$, then we will have that $c \mid 3^{1-\rho}2^{\epsilon-1}(u_1+u_2), 2^{\epsilon-1}(2u_1+u_2)$ which implies that $c \mid 3^{1-\rho}2^{\epsilon-1}u_1, 3^{1-\rho}2^{\epsilon-1}u_2$. Since $c \nmid 3^{1-\rho}$, we get that

$$c \mid 2^{\epsilon-1}u_1, 2^{\epsilon-1}u_2$$

so $c^{-1}2^{\epsilon-1}u \in \mathbb{Z}^2$. Then we have that

$$\mathbb{Z}^2 \ni \frac{b}{a}\mathcal{I}u = \frac{2^{1-\epsilon}b}{3^\rho}\mathcal{I}(2^{\epsilon-1}c^{-1}u)$$

If $\rho = 0$, it is evident that $\frac{b}{a}\mathcal{I}u \in \mathbb{Z}^2$. If $\rho = 1$, we need in particular that

$$\frac{1}{3}\mathcal{I}2^{\epsilon-1}c^{-1}u \in \mathbb{Z}^2$$

as $3 \nmid b$. Thus, we need that $3 \mid u_1 + 2u_2, 2u_1 + u_2$ which implies that $u_1 \cong u_2 \mod 3$, so we can write,

$$\begin{aligned} 2^{\epsilon-1}c^{-1}u &= 3p + r\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ u &= 2^{1-\epsilon}c(3p + r\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = 2^{1-\epsilon}a(p + \frac{r}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix}) \end{aligned}$$

for $r \in \mathbb{Z}_3$. Combining both cases, we get

$$u = 2^{1-\epsilon}a(p + \frac{\rho r}{3^\rho}\begin{pmatrix} 1 \\ 1 \end{pmatrix})$$

To recap, we have shown that

$$\begin{aligned} &v \in \frac{1}{2\cos(\theta)}A^{-1}\mathbb{Z}^2 \cap \mathbb{Z}^2 \implies \\ &\exists r \in \mathbb{Z}_3, p \in \mathbb{Z}^2, v = \frac{1}{2\cos(\theta)}A^{-1}2^{1-\epsilon}a(p + \frac{\rho r}{3^\rho}\begin{pmatrix} 1 \\ 1 \end{pmatrix}) \end{aligned}$$

and we had

$$(R_\theta\Lambda \cap R_{-\theta}\Lambda) \subset \nu A(\frac{1}{2\cos(\theta)}A^{-1}\mathbb{Z}^2 \cap \mathbb{Z}^2)$$

So we get that if $\mathbf{a} \in (R_\theta\Lambda \cap R_{-\theta}\Lambda)$, then we have that there is some $r \in \mathbb{Z}_3$, and $u \in \mathbb{Z}^2$ such that

$$\mathbf{a} = \nu A\frac{1}{2\cos(\theta)}A^{-1}2^{1-\epsilon}a(u + \frac{\rho r}{3^\rho}\begin{pmatrix}1\\1\end{pmatrix}) = \nu\frac{1}{2^\epsilon\cos(\theta)}a(u + \frac{\rho r}{3^\rho}\begin{pmatrix}1\\1\end{pmatrix})$$

We note that since $0 < \theta < \frac{\pi}{3}$, so that $\cos(\theta) > 0$, we can write

$$\cos(\theta) = \frac{1}{\sqrt{1+\tan^2(\theta)}} = \frac{1}{\sqrt{1+\frac{3b^2}{a^2}}} = \frac{a}{\sqrt{a^2+3b^2}}$$

Denote $N = \sqrt{a^2+3b^2}2^{-\epsilon}(4\pi)^{-\rho}$, we get that

$$A\mathbb{Z}^2 \cap A^{-1}\mathbb{Z}^2 \subset \bigcup_{r\in\mathbb{Z}_3}(4\pi)^\rho N(\mathbb{Z}^2 + \frac{\rho r}{3^\rho}\begin{pmatrix}1\\1\end{pmatrix})$$

We will show the opposite containment: Let $p \in \mathbb{Z}^2, r \in \mathbb{Z}_3$, and let

$$v = N(4\pi)^\rho(p + \frac{\rho r}{3^\rho}\begin{pmatrix}1\\1\end{pmatrix})$$

We note that we have that

$$A = \cos(\theta)(\mathrm{Id} + \frac{\tan(\theta)}{\sqrt{3}}\mathcal{I}) = \frac{a}{N2^\epsilon(4\pi)^\rho}\begin{pmatrix}a+b & 2b\\ -2b & a-b\end{pmatrix}$$

So we have that

$$\begin{aligned}Av &= \frac{a}{2^\epsilon}\begin{pmatrix}a+b & 2b\\ -2b & a-b\end{pmatrix}(p + \frac{\rho r}{3^\rho}\begin{pmatrix}1\\1\end{pmatrix})\\ &= a\begin{pmatrix}2^{-\epsilon}(a+b) & 2^{1-\epsilon}b\\ -2^{1-\epsilon}b & 2^{-\epsilon}(a-b)\end{pmatrix}p + a3^{-\rho}\rho r\begin{pmatrix}2^{-\epsilon}(a+3b)\\ 2^{-\epsilon}(a-3b)\end{pmatrix}\end{aligned}$$

Noting that $2^{-\epsilon}(a\pm b), 2^{-\epsilon}(a\pm 3b), a3^{-\rho}, 2^{1-\epsilon} \in \mathbb{Z}^2$ we conclude that $Av \in \mathbb{Z}^2$, similar computation (up to changing $b \mapsto -b$) implies that $A^{-1}v \in \mathbb{Z}^2$, which give the opposite containment.

Thus, we may conclude

$$A^{-1}\mathbb{Z}^2 \cap A\mathbb{Z}^2 = (4\pi)^\rho N\bigcup_{r\in\mathbb{Z}_3}(\mathbb{Z}^2 + \frac{\varrho r}{3^\rho}\begin{pmatrix}1\\1\end{pmatrix})$$

Using the identity in Proposition 4.3, we can conclude that

$$A^{-1}\mathbb{Z}^2 \cap A\mathbb{Z}^2 = N\begin{cases}\mathbb{Z}^2, & \rho = 0\\ \nu^{-1}\kappa\mathbb{Z}^2, & \rho = 1\end{cases} \tag{4.1.3}$$

Applying ν to both sides of the Equation (4.1.3) allows us to conclude

$$R_\theta\Lambda \cap R_{-\theta}\Lambda = N\begin{cases}\Lambda, & 3 \nmid a\\ \Lambda^*, & 3 \mid a\end{cases}$$

and we have shown part 2 of the Lemma, for $\alpha = 2^\epsilon(4\pi)^\rho$.

Finally, we note that

$$R_\theta = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \frac{1}{\sqrt{a^2+3b^2}} \begin{pmatrix} a+b & 2b \\ -2b & a-b \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & 1 \\ \frac{1}{\sqrt{3}} & -1 \end{pmatrix}$$
$$= \frac{1}{2^\epsilon (4\pi)^\rho N} \begin{pmatrix} a & -\sqrt{3}b \\ \sqrt{3}b & a \end{pmatrix}$$

as claimed- and concluding the proof of the lemma. □

This description allows us to conclude that there are no rational rotations in $\mathcal{C} \cap (0, \frac{\pi}{3})$, other than $\frac{\pi}{6}$:

Corollary 4.4. *Let $\theta \in (0, \frac{\pi}{3}) \setminus \{\frac{\pi}{6}\}$ such that $\frac{\tan(\theta)}{\sqrt{3}} \in \mathbb{Q}$ Then $\theta \notin \pi\mathbb{Q}$.*

Proof. Since we have that

$$\frac{\tan(\theta)}{\sqrt{3}} \in \mathbb{Q} \implies \tan^2(\theta) \in \mathbb{Q}$$

by the generalization of Niven's Theorem found in [30], we have that $\theta \in \mathbb{Q}\pi$ only if θ is a integer multiple of $\frac{\pi}{4}, \frac{\pi}{6}$, which is not in the domain above. □

Remark 4.5. We note that for $\theta = \frac{\pi}{6}$ we have that for AA stacking we get that

$$W_{AA}^{\frac{\pi}{6}}(x) = G(\mathcal{R}_{\frac{\pi}{6}} V, \mathcal{R}_{-\frac{\pi}{6}} V) = G(\mathcal{R}_{\frac{\pi}{6}} V, \mathcal{R}_{\frac{\pi}{6}} V) = \mathcal{R}_{\frac{\pi}{6}} G(V, V) = \mathcal{R}_{\frac{\pi}{6}} W_{AA}^0$$

So we have that $H_{AA}^{\frac{\pi}{6}}$ is uniterily equivalent (by rotation by $\frac{\pi}{6}$) to H_{AA}^0. Moreover, we have that $R_{\frac{\pi}{6}} \Lambda = \frac{\sqrt{3}}{4\pi} \Lambda^*$, as predicted by Lemma 4.2.

For AB stacking, we have that

$$W_{AB}^{\frac{\pi}{6}}(x) = G^*(\frac{1}{3}\sum_{j=-1}^{1} \mathcal{T}_{-\frac{1}{2}R^j K_0} \mathcal{R}_{\frac{\pi}{6}} V, \frac{1}{3}\sum_{j=-1}^{1} \mathcal{T}_{\frac{1}{2}R^j K_0} \mathcal{R}_{-\frac{\pi}{6}} V)$$
$$= G^*(\frac{1}{3}\sum_{j=-1}^{1} \mathcal{T}_{-\frac{1}{2}R^j K_0} \mathcal{R}_{\frac{\pi}{6}} V, \frac{1}{3}\sum_{j=-1}^{1} \mathcal{T}_{\frac{1}{2}R^j K_0} \mathcal{R}_{\frac{\pi}{6}} V)$$
$$= G^*(\mathcal{R}_{\frac{\pi}{6}} \frac{1}{3}\sum_{j=-1}^{1} \mathcal{T}_{-\frac{1}{2}R^j R_{-\frac{\pi}{6}} K_0} V, \mathcal{R}_{\frac{\pi}{6}} \frac{1}{3}\sum_{j=-1}^{1} \mathcal{T}_{\frac{1}{2}R^j R_{-\frac{\pi}{6}} K_0})$$

And we note that

$$R_{-\frac{\pi}{6}} K_0 = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2\sqrt{3}} \end{pmatrix}$$
$$= \frac{1}{4\pi\sqrt{3}}(2k_1 + 1k_2) = \frac{\sqrt{3}}{4\pi}(\frac{1}{3}(k_2 - k_1) + k_1) \in \frac{\sqrt{3}}{4\pi}(\Lambda^* + K_0^*)$$

Thus we get that a twist of $\frac{\pi}{6}$ of AB stacking results in AB stacking of the dual lattice, up to a scaling factor of $\frac{\sqrt{3}}{4\pi}$, as predicted by Lemma 4.2.

4.2. **Existence of Dirac points for additive twisted bilayer potentials.** In the following section, we will consider specifically

$$W^{\theta}_{0,AA} = \frac{1}{2}(\mathcal{R}_\theta V + \mathcal{R}_{-\theta} V)$$
$$W^{\theta}_{0,AB} = \frac{1}{2}(\frac{1}{3}\sum_{j=-1}^{1} \mathcal{T}_{-\frac{1}{2}R^j K_0}\mathcal{R}_\theta V + \frac{1}{3}\sum_{j=-1}^{1} \mathcal{T}_{\frac{1}{2}R^j K_0}\mathcal{R}_{-\theta} V)$$

For this potential, we will establish some results relating to the support of $(\hat{W}^{\theta}_{0,AA})_{\vec{m}}, (\hat{W}^{\theta}_{0,AB})_{\vec{m}}$. This section will consider $W^{\theta}_{0,AA}$ a twisted bilayer potential for $\theta \in \mathcal{C} \cap (0, \frac{\pi}{3})$, and will denote

$$\mathcal{A}_1 = (N\kappa^\theta)^{-1} R_\theta \kappa, \qquad\qquad \mathcal{A}_{-1} = (N\kappa^\theta)^{-1} R_{-\theta}\kappa$$

where we recall that $N\kappa^\theta \in \{\kappa, \nu\}$.

We start by computing $\mathcal{A}_1$ explicitly, getting $\mathcal{A}_{-1}$ will be done by replacing $b \mapsto -b$. For that, we first note that, for α as in Proposition 4.2

$$R_\theta\kappa = \frac{4\pi}{\sqrt{3}\alpha N}\begin{pmatrix} a & -\sqrt{3}b \\ \sqrt{3}b & a\end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}\end{pmatrix} = \frac{4\pi}{\sqrt{3}\alpha N}\begin{pmatrix} \frac{a-3b}{2} & \frac{a+3b}{2} \\ \frac{\sqrt{3}(a+b)}{2} & \frac{\sqrt{3}(b-a)}{2}\end{pmatrix}$$

So, we compute if $N\kappa^\theta = \kappa$

$$(N\kappa^\theta)^{-1}R_\theta\kappa = \frac{1}{\alpha N}\begin{pmatrix} a-b & 2b \\ -2b & a+b\end{pmatrix} = (A^{-1})^T$$

then we have that $\det \mathcal{A} = 1$.

And, if $N\kappa^\theta = \nu$

$$(N\kappa^\theta)^{-1}R_\theta\kappa = \frac{4\pi}{\sqrt{3}\alpha N}\begin{pmatrix} \frac{2a}{\sqrt{3}} & \frac{-a+3b}{\sqrt{3}} \\ \frac{-a-3b}{\sqrt{3}} & \frac{2a}{\sqrt{3}}\end{pmatrix} = \frac{1}{N3\cdot 2^\epsilon}\begin{pmatrix} 2a & -a+3b \\ -a-3b & 2a\end{pmatrix}$$

we note the last expression is, up to a factor of N, an integer matrix, as $3 \mid a$. And we have that $\det \mathcal{A} = (\frac{4\pi}{\sqrt{3}})^2$.

The above notation will allow us to provide more details on the Fourier support of $W^{\theta}_{0,AA}, W^{\theta}_{0,AB}$.

Proposition 4.6. *We have that for $W^{\theta}_{0,AA}, W^{\theta}_{0,AB}$ as above, for $\theta \in \mathcal{C} \cap (0, \frac{\pi}{3})$*

$$\operatorname{supp} \hat{W}^{\theta}_{0,AA} = \{\vec{m} \mid (\hat{W}_{0,AA})^{\theta}_{\vec{m}} \neq 0\} \subset N(\mathcal{A}_1\mathbb{Z}^2 \cup \mathcal{A}_{-1}\mathbb{Z}^2)$$
$$\operatorname{supp} \hat{W}^{\theta}_{0,AB} = \{\vec{m} \mid (\hat{W}_{0,AB})^{\theta}_{\vec{m}} \neq 0\} \subset N(\mathcal{A}_1\mathbb{Z}^2 \cup \mathcal{A}_{-1}\mathbb{Z}^2)$$

Proof. We will start be focusing on the AA stacking, and we note that we know that

$$V(x) = \sum_{\vec{p}\in\mathbb{Z}^2} \hat{V}_{\vec{p}} e^{i\langle\kappa\vec{p}, x\rangle}$$

So we have that

$$W^{\theta}_{0,AA}(x) = \frac{1}{2}(\sum_{p\in\mathbb{Z}^2} \hat{V}_{\vec{p}} e^{i\langle\kappa\vec{p}, R_{-\theta}x\rangle} + \sum_{p\in\mathbb{Z}^2} \hat{V}_{\vec{p}} e^{i\langle\kappa\vec{p}, R_{\theta}x\rangle})$$
$$= \frac{1}{2}(\sum_{p\in\mathbb{Z}^2} \hat{V}_{\vec{p}} e^{i\langle R_\theta\kappa\vec{p}, x\rangle} + \sum_{p\in\mathbb{Z}^2} \hat{V}_{\vec{p}} e^{i\langle R_{-\theta}\kappa\vec{p}, x\rangle})$$

We note that

$$R_{\pm\theta}\kappa\vec{p} = N\kappa^\theta(N\kappa^\theta)^{-1}R_{\pm\theta}\kappa\vec{p} = N\kappa^\theta\mathcal{A}_{\pm 1}\vec{p}$$

Inserting this, we write

$$W^{\theta}_{0,AA}(x) = \frac{1}{2}\Big(\sum_{\vec{p}\in\mathbb{Z}^2} \hat{V}_{\vec{p}} e^{i\langle \kappa^{\theta} N\mathcal{A}_1 \vec{p}, x\rangle} + \sum_{\vec{p}\in\mathbb{Z}^2} \hat{V}_{\vec{p}} e^{i\langle \kappa^{\theta} N\mathcal{A}_{-1} \vec{p}, x\rangle}\Big)$$

So we got that

$$W^{\theta}_{0,AA}(x) = \frac{1}{2}\Big(\sum_{\vec{q}\in N\mathcal{A}_1\mathbb{Z}^2} \hat{V}_{\frac{1}{N}\mathcal{A}_1^{-1}\vec{q}} e^{i\langle \kappa^{\theta} \vec{q}, x\rangle} + \sum_{\vec{q}\in N\mathcal{A}_{-1}\mathbb{Z}^2} \hat{V}_{\frac{1}{N}\mathcal{A}_{-1}^{-1}\vec{q}} e^{i\langle \kappa^{\theta} \vec{q}, x\rangle}\Big) \tag{4.2.1}$$

On the other hand, we have that, as a function periodic with respect to Λ^{θ}:

$$W^{\theta}_{0,AA}(x) = \sum_{\vec{m}\in\mathbb{Z}^2} (\hat{W}^{\theta}_{0,AA})_{\vec{m}} e^{i\langle \kappa^{\theta}\vec{m}, x\rangle}$$

So, we may conclude

$$\operatorname{supp} \hat{W}^{\theta}_{0,AA} \subset N(\mathcal{A}_1\mathbb{Z}^2 \cup \mathcal{A}_{-1}\mathbb{Z}^2)$$

as claimed.

The argument for AB stacking is identical since a shift in real space corresponds to multiplying $V_{\vec{p}}$ by the phase, e.g., $e^{i\kappa\vec{p},K_0}$- which does not change the Fourier support. □

Now we will show that ϱ_{-1} can be decomposed into the two lattices:

Proposition 4.7. *We have that*

$$\varrho_{-1} \in N\mathcal{A}_1\mathbb{Z}^2 + N\mathcal{A}_{-1}\mathbb{Z}^2$$

Proof. We start in the case where $N\kappa^{\theta} = \kappa$. We need to show that

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} \in N\mathcal{A}_1\mathbb{Z}^2 + N\mathcal{A}_{-1}\mathbb{Z}^2$$

In this case, we have that $3 \nmid a$. Then, we note that we have that (a, b) and $(a, 3)$, are both pairs of co-prime numbers. So we have some numbers $\tilde{p}, \tilde{q}, m, n \in \mathbb{Z}$ such that

$$a\tilde{p} + b\tilde{q} = 1 \tag{4.2.2}$$

$$3m + an = 1 \tag{4.2.3}$$

by Bézout's identity theorem. Denote $q = \tilde{q} + a(\tilde{p} + \tilde{q}), p = \tilde{p} - b(\tilde{p} + \tilde{q})$, we note that then we have that

$$ap + bq = 1 \tag{4.2.4}$$

We denote

$$v_1 = 2^{\epsilon}\begin{pmatrix} -\frac{p+q(4m-1)}{2} \\ nqb - mq \end{pmatrix} \qquad v_{-1} = 2^{\epsilon}\begin{pmatrix} \frac{q(4m-1)-p}{2} \\ mq + nqb \end{pmatrix}$$

First we will show that $v_{\pm 1} \in \mathbb{Z}^2$: If $\epsilon = 1$, the above is evidently in $\mathbb{Z}^2$. If $\epsilon = 0$, then we note that

$$p \pm q = \tilde{p} \pm \tilde{q} - b(\tilde{p} + \tilde{q}) \pm a(\tilde{p} + \tilde{q}) = (1 - b \pm a)\tilde{p} - \tilde{q}(b \mp 1 \mp a)$$

noting that if $\epsilon = 0$ both expression above are divisible by 2, so we get that

$$p + q(4m-1) \cong p - q \cong 0 \mod 2$$
$$p - q(4m-1) \cong p - q \cong 0 \mod 2$$

as needed.

Then, we can compute

$$\begin{aligned} N\mathcal{A}_1 v_1 + N\mathcal{A}_{-1} v_{-1} &= 2^{-\epsilon}(a\mathrm{Id} + b\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix})v_1 + 2^{-\epsilon}(a\mathrm{Id} - b\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix})v_{-1} \\ &= a\begin{pmatrix} -p \\ 2qbn \end{pmatrix} + b\begin{pmatrix} q(4m-1) - 4mq \\ 2q(4m-1) - 2mq \end{pmatrix}) \\ &= a\begin{pmatrix} -p \\ 2qbn \end{pmatrix} + b\begin{pmatrix} -q \\ 6qm - 2q \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 2qb(an+3m) - 2qb \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \varrho_{-1} \end{aligned}$$

as needed.

In the case where $N\kappa^\theta = \nu$, We need to show that

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} \in N\mathcal{A}_1\mathbb{Z}^2 + N\mathcal{A}_{-1}\mathbb{Z}^2$$

We note that Equation (4.2.4) still holds, with the same p, q which are defined as above, then we consider

$$v_1 = 2^\epsilon \begin{pmatrix} \frac{q-p}{2} \\ -p \end{pmatrix} \qquad v_{-1} = 2^\epsilon \begin{pmatrix} -\frac{p+q}{2} \\ -p \end{pmatrix}$$

We have that $v_{\pm 1} \in \mathbb{Z}$, as above. So, we compute

$$\begin{aligned} &N\mathcal{A}_1 v_1 + N\mathcal{A}_{-1} v_{-1} \\ &= 2^{-\epsilon}(\frac{a}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}(v_1 + v_{-1}) + b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(v_1 - v_{-1})) \\ &= 2^{-\epsilon}(\frac{a}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}2^\epsilon\begin{pmatrix} -p \\ -2p \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}2^\epsilon\begin{pmatrix} q \\ 0 \end{pmatrix}) \\ &= \frac{a}{3}\begin{pmatrix} 0 \\ -3p \end{pmatrix} + b\begin{pmatrix} 0 \\ -q \end{pmatrix} = \begin{pmatrix} 0 \\ -ap - bq \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \varrho_{-1} \end{aligned}$$

as needed. □

An immediate consequence of this is the following proposition that will allow us to understand condition (2.3.3) better:

Proposition 4.8. *There are $v_{\pm 1} \in \mathbb{Z}^2$ such that*

$$\kappa^\theta \varrho_{-1} = R_\theta \kappa v_1 + R_{-\theta}\kappa v_1$$

Proof. By Proposition 4.7 we have that there is some $v_{\pm 1} \in \mathbb{Z}^2$ such that

$$N\mathcal{A}_1 v_1 + N\mathcal{A}_{-1} v_{-1} = \varrho_{-1}$$

Recalling that $\mathcal{A}_t = (N\kappa^\theta)^{-1} R_\theta^t \kappa$, for $t \in \{\pm 1\}$ we can apply $N\kappa^\theta$ to both sides to get

$$\begin{aligned} N\kappa^\theta \varrho_{-1} &= NR_\theta \kappa v_1 + NR_{-\theta}\kappa v_{-1} \\ \kappa^\theta \varrho_{-1} &= R_\theta \kappa v_1 + R_{-\theta}\kappa v_{-1} \end{aligned}$$

as claimed. □

With this, we get the following result:

Lemma 4.9. *Let $H_x^\theta = -\Delta + \lambda W_{0,x}^\theta$, for $x \in \{AA, AB\}$, and $W_{0,x}^\theta$ defined in (1.1.1) or (1.1.2) for $\lambda \in \mathbb{R}$ and twisted bilayer potential with respect to honeycomb potential V, and angle $\theta \in \mathcal{C} \cap (0, \frac{\pi}{6})$. Then we have that for any $\vec{m} \in \mathcal{S}$, then we have for some $\ell \in \mathbb{Z}_3$*

$$(\hat{W}_{0,x}^\theta)_{\vec{m}-\varrho_\ell} = 0$$

Furthermore, we have that

$$(\hat{W}_{0,x}^\theta)_{\vec{m}}(\hat{W}_{0,x}^\theta)_{\vec{m}-\varrho_{-1}} \neq 0 \implies \exists t \in \{\pm 1\}, \kappa^\theta \vec{m} = R_\theta^t \kappa v_t + N(\Lambda^\theta)^*$$

where v_t are as in Proposition 4.8.

Proof of Lemma 4.9. Let $W_{0,x}^\theta$ be as above, and assume that

$$\vec{m}, \vec{m} - \varrho_1, \vec{m} - \varrho_{-1} \in \operatorname{supp}(\hat{W}_{0,x}^\theta)$$

Since

$$\operatorname{supp} \hat{W}_{0,x}^\theta \subset N(\mathcal{A}_1\mathbb{Z}^2 \cup \mathcal{A}_{-1}\mathbb{Z}^2)$$

then, by the pigeonhole principle, we have that two of the three vectors are in either $N\mathcal{A}_1\mathbb{Z}^2$ or $N\mathcal{A}_1^{-1}\mathbb{Z}^2$. In other words, we have that there are some $\ell, \ell' \in \mathbb{Z}_3$ and $t \in \{\pm 1\}$ such that

$$\vec{m} - \varrho_\ell, \vec{m} - \varrho_{\ell'} \in N\mathcal{A}_t\mathbb{Z}^2$$

In particular, this implies that

$$\varrho_{\ell'} - \varrho_\ell \in N\mathcal{A}_t\mathbb{Z}^2$$

But direct computation show that $\varrho_{\pm 1}, \varrho_{\pm 1} - \varrho_{\mp 1} \notin \operatorname{supp}(\hat{W}_{0,x}^\theta)$:

$$\frac{1}{N}\mathcal{A}_{\pm 1}^{-1}\varrho_1 = \frac{1}{N^2 2^\epsilon}\begin{cases} \begin{pmatrix} \mp 2b \\ a \mp b \end{pmatrix}, & N\kappa^\theta = \kappa \\ \frac{1}{4\pi}\begin{pmatrix} -2a \\ -a \mp 3b \end{pmatrix} & N\kappa^\theta = \nu \end{cases} \notin \mathbb{Z}^2$$

$$\frac{1}{N}\mathcal{A}_{\pm 1}^{-1}\varrho_{-1} = \frac{1}{N^2 2^\epsilon}\begin{cases} \begin{pmatrix} -a \mp b \\ \mp 2b \end{pmatrix}, & N\kappa^\theta = \kappa \\ \frac{1}{4\pi}\begin{pmatrix} -a \pm 3b \\ -2a \end{pmatrix}, & N\kappa^\theta = \nu \end{cases} \notin \mathbb{Z}^2$$

$$\frac{1}{N}\mathcal{A}_{\pm 1}^{-1}(\varrho_1 - \varrho_{-1}) = \frac{1}{N^2 2^\epsilon}\begin{cases} \begin{pmatrix} a \mp b \\ a \pm b \end{pmatrix}, & N\kappa^\theta = \kappa \\ \frac{1}{4\pi}\begin{pmatrix} -a \mp 3b \\ a \mp 3b \end{pmatrix}, & N\kappa^\theta = \nu \end{cases} \notin \mathbb{Z}^2$$

So we conclude that at least one of $\vec{m}, \vec{m} - \varrho_1, \vec{m} - \varrho_{-1}$ are not $\operatorname{supp}(\hat{W}_{0,x}^\theta)$ - as claimed.

If we know that $\vec{m}, \vec{m} - \varrho_{-1} \in \operatorname{supp} \hat{W}_{0,x}^\theta$, by the above we get that there is $t \in \{\pm 1\}$ such that

$$\vec{m} \in N\mathcal{A}_t\mathbb{Z}^2, \qquad \vec{m} - \varrho_{-1} \in N\mathcal{A}_{-t}\mathbb{Z}^2$$

writing $\varrho_{-1} = N\mathcal{A}_1 v_1 + N\mathcal{A}_{-1}v_{-1}$ then we get that

$$\vec{m} - N\mathcal{A}_t v_t \in N\mathcal{A}_{-t}\mathbb{Z}^2$$

but since $\vec{m} \in N\mathcal{A}_t\mathbb{Z}^2$ we may conclude that

$$\begin{aligned}
&\vec{m} - N\mathcal{A}_t v_t \in N\mathcal{A}_t\mathbb{Z}^2 \cap N\mathcal{A}_{-t}\mathbb{Z}^2 \\
&N\kappa^\theta(\vec{m} - N\mathcal{A}_t v_t) \in NR_\theta\kappa\mathbb{Z}^2 \cap NR_{-\theta}\kappa\mathbb{Z}^2 \\
&\kappa^\theta(\vec{m} - N\mathcal{A}_t v_t) \in R_\theta\Lambda^* \cap R_{-\theta}\Lambda^* = N\begin{cases}\Lambda^*, & a \nmid 3 \\ \Lambda, & a \mid 3\end{cases} = N^2\kappa^\theta\mathbb{Z}^2
\end{aligned}$$

The second to last equality comes from the proof of Lemma 4.2 when applied to Λ^*. So we have that

$$\vec{m} - N\mathcal{A}_t v_t \in N^2\mathbb{Z}^2$$

Applying κ^θ to both sides yields the result as claimed. □

Now we get as an immediate consequence Theorem 2.22

Proof of Theorem 2.22. By [14], if we have that $\hat{W}^\theta_{\varrho_1} \neq 0$, we get the wanted result. In the other case, Theorem 2.21 holds for W^θ. Thus completing the proof. □

4.3. **Flattening of the Dirac cones for weak potential.** Finally, we prove our statement about the flatting of the cone for small potentials and angles close to incommensurate angles, Theorem 2.23. It is important to recall that this Theorem holds for *all* twisted potentials, not only for potentials of the type of (1.1.1):

Proof of Theorem 2.23. Equation (2.3.4), in the context of twisted potential, will have the form of, for some $C > 0$

$$\begin{aligned}
|v_d(\lambda)|^2 \leq &C(|K_*^\theta|^2 + \lambda\|W^\theta\|_\infty + \lambda^2\|\nabla W^\theta\|_\infty^2 \sum_{\vec{m}\in\mathcal{S}\setminus\{\vec{0}\}} \frac{1}{|K_*^\theta + \kappa^\theta\vec{m}|^4}) \\
&+ O(\lambda^3\|W^\theta\|^3)
\end{aligned}$$

We note that for any $\vec{m} \in \mathcal{S} \setminus \{0\}$ we have some constant $c > 0$ such that

$$|K_*^\theta + \kappa^\theta\vec{m}| > c|k_1^\theta|$$

Using the fact that the sum above can be treated as a Riemann sum, we have that for some constant $C > 0$, whose exact value may change between inequalities

$$\sum_{\vec{m}\in\mathcal{S}\setminus\{\vec{0}\}} \frac{1}{|K_*^\theta + \kappa^\theta\vec{m}|^4} \leq C \int\limits_{|x|>c|k_1^\theta|} \frac{1}{|x|^4}\,dx \leq C \int\limits_{c|k_1^\theta|} \frac{1}{r^3}\,dr \leq \frac{C}{c|k_1^\theta|^2}$$

We note that using the soundness of G, we can write

$$\|W^\theta\|_\infty \leq C_g\|V\|_\infty^{2\gamma}, \qquad \|\nabla W^\theta\|_\infty \leq C_{g'}\|\nabla V\|_\infty^{2\gamma'}$$

So we get that for some constant $C > 0$, we have

$$\begin{aligned}
|v_d(\lambda)|^2 &\leq C(|K_*^\theta|^2 + \lambda\|V\|_\infty^{2\gamma} + \lambda^2\|\nabla V\|_\infty^{4\gamma'}|k_1^\theta|^{-2}) + O(\lambda^3\|V\|_\infty^{6\gamma}) \\
&\leq C(\frac{1}{N^2}|N^2K_*^\theta|^2 + \lambda\|V\|_\infty^{2\gamma} + \lambda^2\|\nabla V\|_\infty^{4\gamma'}2N^2|Nk_1^\theta|^{-2}) + O(\lambda^3\|V\|_\infty^{6\gamma})
\end{aligned}$$

Recalling that $N\kappa^\theta \in \{\kappa, \nu\}$, and so is independent of N in terms of sizes (up to a factor of $\frac{4\pi}{3}$), so we have that

$$|NK_*^\theta|^2 = O(1)$$

as $N \to \infty$. Thus, we get that for some $C > 0$ depending only on $\|V\|_\infty$, $\|\nabla V\|_\infty$ such that

$$|v_d(\lambda)|^2 \leq C(\frac{1}{N^2} + \lambda + \lambda^2 N^2) + O(\lambda^3 \|V\|_\infty^{6\gamma})$$

In particular, we get that if

$$|\lambda| < \frac{\delta}{N^2}$$

for some $\delta > 0$, we have that

$$\lambda + \lambda^2 N^2 < \frac{(\delta+1)^2}{N^2}$$

So, we may conclude that if

$$|\lambda| < \frac{\delta}{N^2}$$

Since $\|V\|_\infty^{6\gamma}$ is independent of N, we have some constant $0 < C = C(\delta, V, G)$ such that

$$|v_d(\lambda)|^2 \leq \frac{C}{N^2} + O(\lambda^3)$$

So, we may conclude that we have that

$$|\lambda| < \frac{\delta}{N^2} \implies |v_d| \leq \frac{C(\delta, V, G)}{N} + O(N^{-3})$$

for some $\delta, C(\delta, V, G) > 0$ as claimed. ◻

5. Examples

In this section, we will construct a set of examples of potentials of the type of $W^\theta_{0,AA}$ for which the above theorems hold. We recall the proposition:

Proposition 2.24. *Define the equivalence relation $\sim_B$ by*

$$\vec{m} \sim_B \vec{n} \iff \exists \ell \in \mathbb{Z}_3, B^\ell \vec{m} = \vec{n}$$

Then denote $\tilde{\mathcal{S}} = \mathbb{Z}^2 / \sim_B$.

Let $(a_{\vec{m}})_{\vec{m} \in \tilde{\mathcal{S}}}$ be exponentially decaying sequence such that

$$\forall \vec{m} \in \tilde{\mathcal{S}}, a_{\vec{m}} > 0$$

We define

$$V(x) = \pm \sum_{\vec{m} \in \tilde{\mathcal{S}}} a_{\vec{m}} \sum_{\ell \in \mathbb{Z}_3} \cos(\langle \kappa B^\ell \vec{m}, x \rangle)$$

Then V is a honeycomb potential. And if we define the twisted potential as in (1.1.1), in AA stacking, that is

$$W^\theta = \frac{1}{2}(\mathcal{R}_\theta V + \mathcal{R}_{-\theta} V)$$

Then we have that for any $\theta \in \mathcal{C} \cap (0, \frac{\pi}{3})$ we have that

$$\sum_{\vec{m} \in \mathcal{S} \setminus \{\vec{0}\}} \frac{\hat{W}^\theta_{\vec{m}} \hat{W}^\theta_{\vec{m} - \varrho_{-1}}}{|K^\theta_*(\vec{m})|^2 - |K^\theta_*|^2} \neq 0, \text{ and } \forall \vec{m} \in \mathcal{S} \exists \ell \in \mathbb{Z}_3, \hat{W}^\theta_{\vec{m} - \varrho_\ell} = 0$$

holds.

Proof. First, the fact that V defined above is a honeycomb lattice is immediate as it is periodic with respect to Λ, real and even. Having that

$$\hat{V}_{\vec{m}} = \hat{V}_{B^{\pm 1}\vec{m}}$$

implies the symmetry with respect to R on the space side. Finally, since $a_{\vec{m}}$ is exponentially decaying, this implies that $V \in C^\infty$, as needed.

Now, we recall that

$$\hat{W}^\theta_{\vec{m}} \hat{W}^\theta_{\vec{m}-\varrho_{-1}} \neq 0 \implies \exists t \in \{\pm 1\}, \vec{m} \in N\mathcal{A}_t v_t + N^2\mathbb{Z}^2$$

and so we have that

$$\sum_{\vec{m}\in\mathcal{S}\setminus\{\vec{0}\}} \frac{\hat{W}^\theta_{\vec{m}} \hat{W}^\theta_{\vec{m}-\varrho_{-1}}}{|K^\theta_*(\vec{m})|^2 - |K^\theta_*|^2} = \sum_{t\in\{\pm 1\}, u\in\mathbb{Z}^2} \frac{\hat{W}^\theta_{N\mathcal{A}_t v_t + N^2 u} \hat{W}^\theta_{N\mathcal{A}_{-t} v_{-t} + N^2 u}}{|K^\theta_*(N\mathcal{A}_t v_t + N^2 u)|^2 - |K^\theta_*|^2} =$$
$$= \sum_{t\in\{\pm 1\}, u\in\mathbb{Z}^2} \frac{\hat{V}_{v_t + N\mathcal{A}_{-t} u} \hat{V}_{v_{-t} + N\mathcal{A}_t u}}{|K^\theta_*(N\mathcal{A}_t v_t + N^2 u)|^2 - |K^\theta_*|^2}$$

where we used the fact that $N\mathcal{A}_t v_t \in N\mathcal{A}_t\mathbb{Z}^2 \setminus (N\mathcal{A}_{-t}\mathbb{Z}^2)$, and the identification between the Fourier coefficients implied by Equation (4.2.1). If V is defined with $+$, then each summand is strictly positive for the sum above, and if it is defined with a $-$, then each summand is strictly negative, so in both cases, we get

$$\sum_{\vec{m}\in\mathcal{S}\setminus\{\vec{0}\}} \frac{\hat{W}^\theta_{\vec{m}} \hat{W}^\theta_{\vec{m}-\varrho_{-1}}}{|K^\theta_*(\vec{m})|^2 - |K^\theta_*|^2} \neq 0$$

and we conclude that condition (2.3.6) holds. □

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Appendix A. Notation

- We will denote by $\langle\cdot,\cdot\rangle$ the Euclidean inner product on vectors in $\mathbb{R}^2$ or $\mathbb{Z}^2$, and the size of these vector will be denoted by $|\cdot|$.
- Throughout the paper $\tilde{\Lambda}$ will denote a generic honeycomb lattice (without explicit reference to its base vectors), Λ will denote a honeycomb lattice with base vectors defined by

$$v_1 = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, v_2 = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}$$

  Λ^* will denote the dual to Λ, and Λ^θ will denote the lattice with respect to which W^θ is periodic.
- $\tilde{\nu}$ is the base matrix of $\tilde{\Lambda}$, and $\tilde{\kappa}$ is the base matrix of $\tilde{\Lambda}^*$.
- R_θ will denote the rotation matrix by θ, and we will denote $R = R_{\frac{2\pi}{3}}$, their corresponding operators will be denoted by $\mathcal{R}_\theta$ and $\mathcal{R}$ respectively.
- We will denote by $\mathcal{T}_{\vec{a}}$ the translation operator by the vector $\vec{a}$, for any $\vec{a} \in \mathbb{R}^2$.
- We will denote by $\tilde{\Omega} = \tilde{\nu}[0,1]^2$- the unit cell, by $\tilde{\mathcal{B}} = \{k \in \mathbb{R}^2 \mid \forall a \in \tilde{\Lambda}^*, |k| \leq |k-a|\}$ the Brillouin zone, and the points of high symmetry by

$$\begin{aligned} \tilde{\mathbb{P}} &= \{\vec{k} \in \tilde{\mathcal{B}} \mid (R - \mathrm{id})\vec{k} \in \tilde{\kappa}\mathbb{Z}^2\} \\ &= \{\tilde{K}, R\tilde{K}, R^2\tilde{K}\} \bigsqcup \{\tilde{K}', R\tilde{K}', R^2\tilde{K}'\} \bigsqcup \{0\} \end{aligned}$$

- We will distinguish one of the points of high symmetry

$$K_0 = \frac{1}{3}\nu \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix},$$

  and its dual-point

$$K_0^* = \frac{1}{3}\kappa \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{4\pi}{\sqrt{3}} \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}$$

- Throughout the paper, V will denote a honeycomb potential used to define the twisted bilayer potential

$$\begin{aligned} W^\theta_{AA} &= G(\mathcal{R}_\theta V, \mathcal{R}_{-\theta} V) \\ W^\theta_{AB} &= G^*(\frac{1}{3}\sum_{j=-1}^{1} \mathcal{T}_{-\frac{1}{2}R^j K_0} \mathcal{R}_\theta V, \frac{1}{3}\sum_{j=-1}^{1} \mathcal{T}_{\frac{1}{2}R^j K_0} \mathcal{R}_{-\theta} V) \end{aligned}$$

  for G and G^* admissible interaction operators, respectively, and we will use U to denote a generic honeycomb potential, which will be periodic with respect to $\tilde{\Lambda}$.
- We will denote $\tau = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{-\frac{2\pi}{3}i}$ the cubic root of unity.
- $\mathcal{C}$ will be the set of commensurate angles.
- We will consider the following spaces, for $k \in \tilde{\mathcal{B}}$

$$\begin{aligned} L^2_k(\tilde{\Omega}) &= \{f \in L^2(\tilde{\Omega}) \mid \forall a \in \tilde{\Lambda}, f(x+a) = e^{-i\langle k,a\rangle} f(x)\} \\ L^2_{k,\sigma}(\tilde{\Omega}) &= \{f \in L^2_k(\tilde{\Omega}) \mid \mathcal{R}f = \sigma f\} \end{aligned}$$

  for $\sigma \in \{1, \tau, \bar{\tau}\}$.

- For $f \in L^2_0 = L^2_{per}$, we have the following Fourier representation:

$$\hat{f}_{\vec{m}} = \frac{1}{|\Omega|} \int\limits_{\Omega} e^{-i\langle \kappa\vec{m}, y\rangle} f(y)\, dy$$
$$f(y) = \sum_{\vec{m}\in\mathbb{Z}^2} \hat{f}_{\vec{m}} e^{i\langle \kappa\vec{m}, y\rangle}$$

- We will denote by $(\cdot,\cdot)$ the inner product on $L^2_k(\Omega)$ spaces, and norms will be denoted by $\|\cdot\|$.
- We denote the following

$$B = \tilde{\kappa}^{-1} R \tilde{\kappa}$$
$$\varrho_1 = \tilde{\kappa}^{-1}(R - \mathrm{id})\tilde{K}_*$$
$$\varrho_{-1} = \tilde{\kappa}^{-1}(R^{-1} - \mathrm{id})\tilde{K}_*$$
$$\varrho_0 = 0$$

- We define the equivalence $\approx$ that identifies the orbit of $\vec{m}$ under $B^j\vec{m} + \varrho_j, j \in \mathbb{Z}_3$, and we denote $\mathcal{S} = \mathbb{Z}^2/\approx$.
- For $\theta \in \mathcal{C} \cap (0, \frac{\pi}{3})$, we have $\tan(\theta) = \frac{\sqrt{3}b}{a}$ for some co-prime $a, b \in \mathbb{Z}$, such that $0 < b < \frac{a}{b}$, and we denote

$$\alpha = \begin{cases} 8\pi, & 3 \mid a \text{ and } 2 \nmid ab \\ 2, & 3 \nmid a \text{ and } 2 \nmid ab \\ 4\pi, & 3 \mid a \text{ and } 2 \mid ab \\ 1, & 3 \nmid a \text{ and } 2 \mid ab \end{cases}$$
$$N = \frac{1}{\alpha}\sqrt{a^2 + 3b^2}$$

- We denote

$$\mathcal{A}_1 = (N\kappa^{\theta})^{-1} R_{\theta}\kappa$$
$$\mathcal{A}_{-1} = (N\kappa^{\theta})^{-1} R_{-\theta}\kappa$$