Asymptotic dimension, distributed algorithms, and local graph concepts

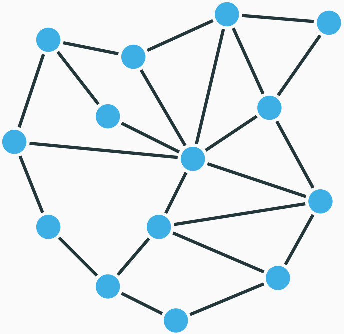
Marthe Bonamy [1] Cyril Gavoille [1] Timothé Picavet [1] Alexandra Wesolek [2]

[1]LaBRI, Bordeaux

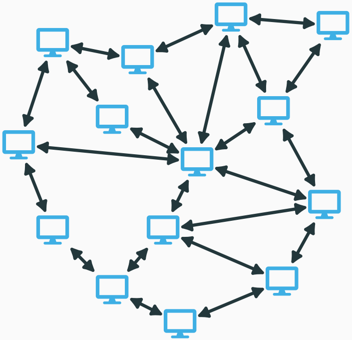
[2]TU Berlin

Distributed algorithms

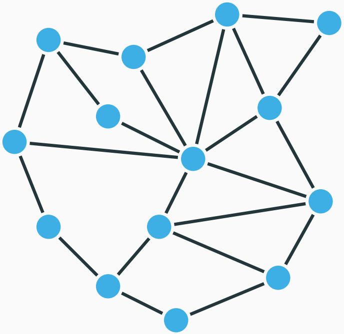
Centralized view



Distributed view

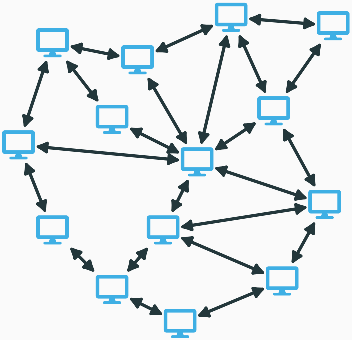


Centralized view



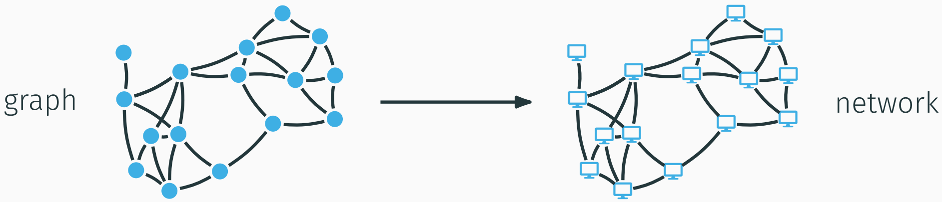
Focused on computing

Distributed view



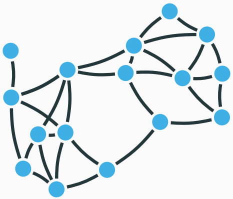
Focused on communication

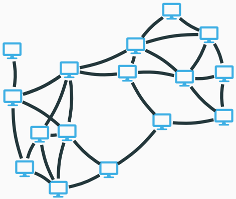
The LOCAL model



The LOCAL model

graph





network

vertices

+

edges

UID UID

computers

+

connections

The LOCAL model

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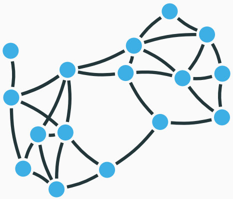

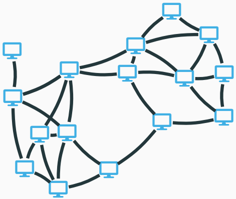

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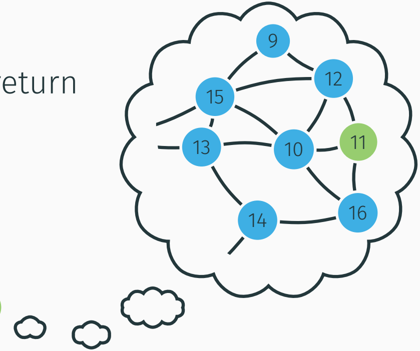
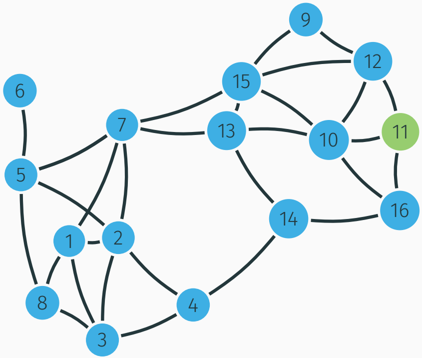
The network is also the input graph!

Running time T

Each vertex sees its distance-T neighborhood and decides its return value.





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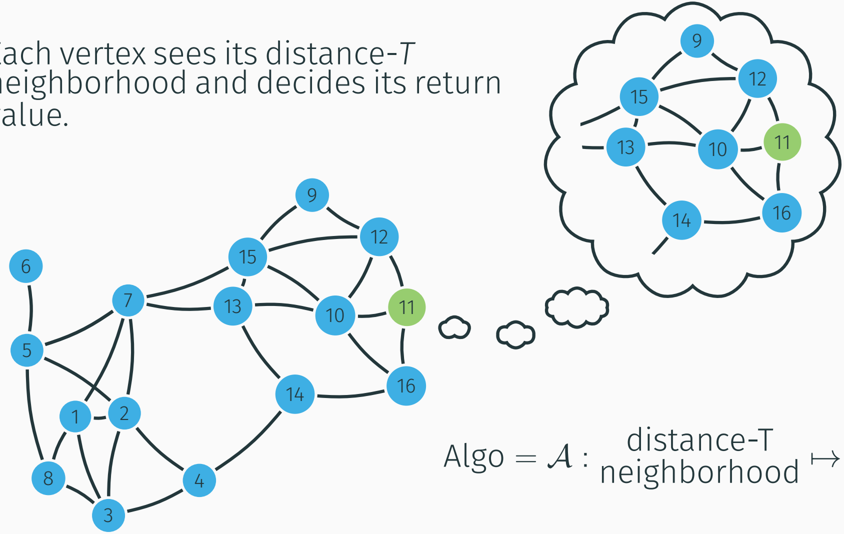
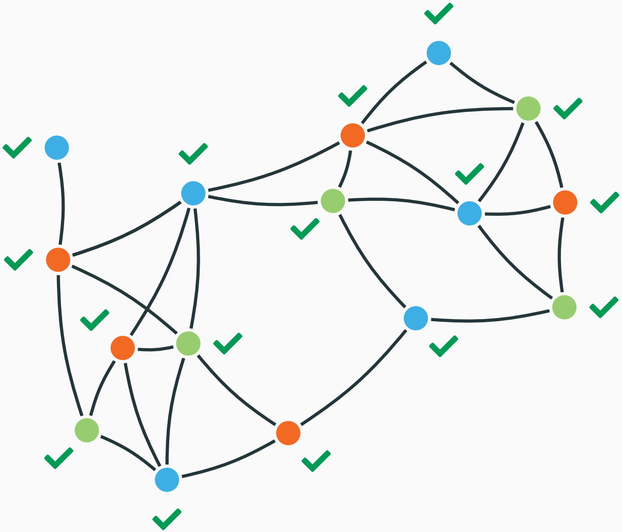


$$\text{Algo} = \mathcal{A} : \begin{array}{c}\text{distance-T}\\\text{neighborhood}\end{array} \mapsto \begin{array}{c}\text{local}\\\text{return value}\end{array}$$

An example: 3-coloring



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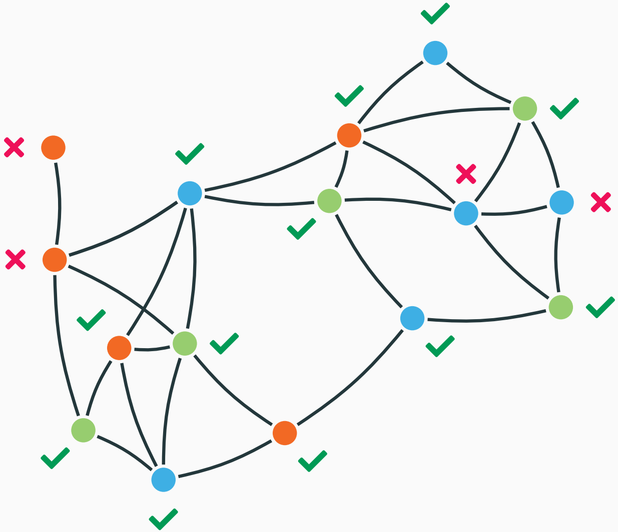
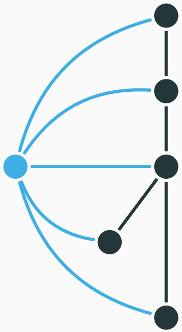


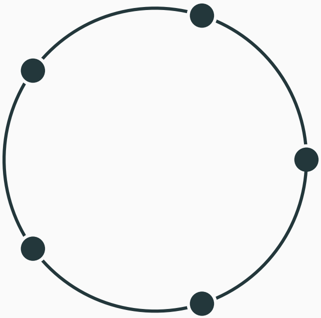
Complexity differences between LOCAL and centralized

Maximum Independent Set
when ∃ universal vertex



Easy in LOCAL
Hard in centralized

Detecting Cycles



Hard in LOCAL
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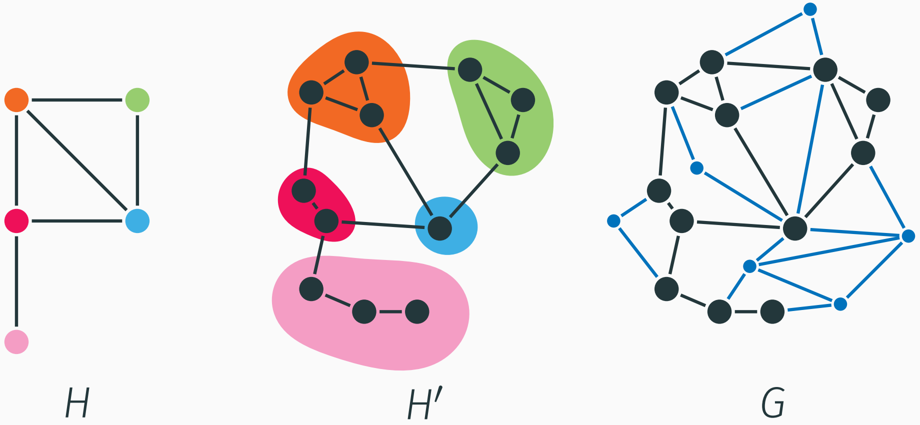
Graph minors



H is a minor of G

State of the art for MDS with $\mathcal{O}(1)$ LOCAL rounds

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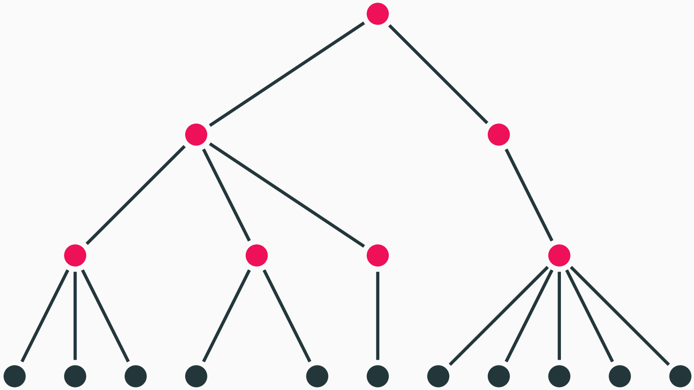
- $K_{2,t}$-minor-free graphs
  - $\mathcal{O}(1)$-approximation

Example 1: trees



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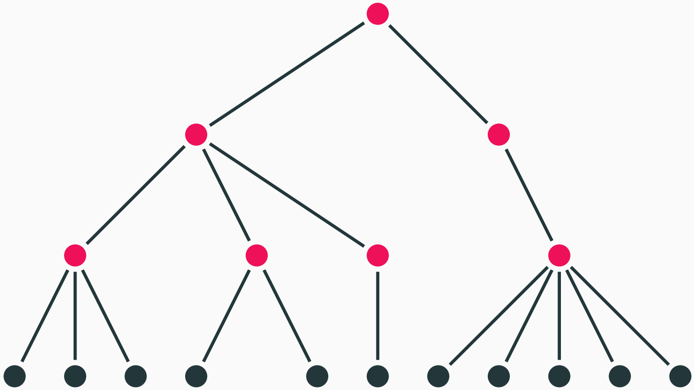
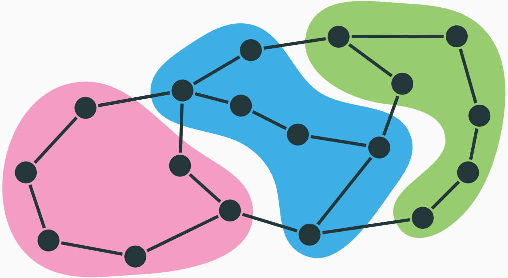


Theorem

$|\{v \in V(T) \mid \mathrm{d}(v) \geq 2\}| \leq 3 \cdot \mathrm{MDS}(T)$

Example 2: high girth graphs



Reuse the analysis of trees?

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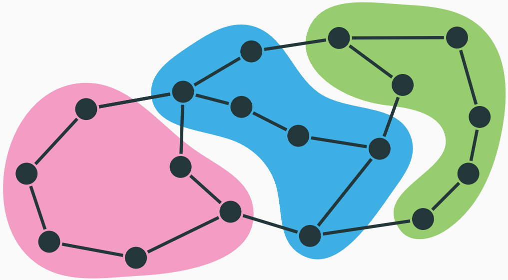

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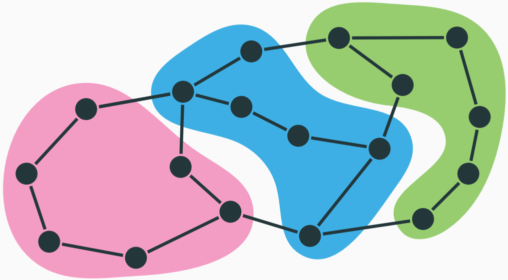

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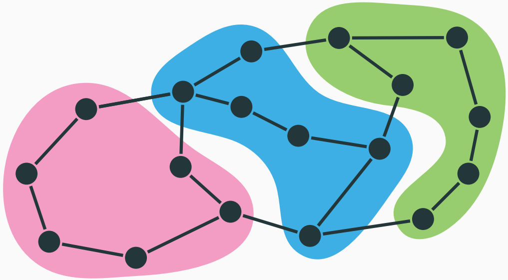

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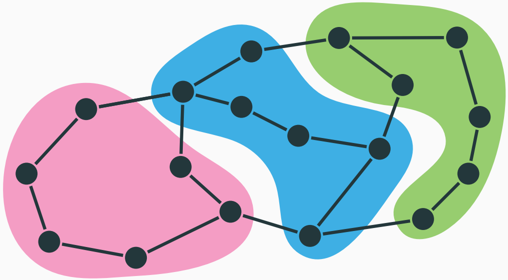

Reuse the analysis of trees?

- Every vertex is in a patato
- Diameter $\leq$ girth
- Spacing between same colors $\implies$ in parallel
- Finite number of colors

Asymptotic dimension

Asymptotic dimension of $\mathcal{C}$ is d if

$\exists f : \mathbb{N} \to \mathbb{N}, \forall G \in \mathcal{C}, \forall r, \exists C_1, C_2, \ldots, C_{d+1} \subseteq \mathcal{P}(V(G))$, such that

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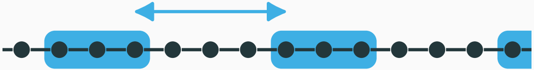

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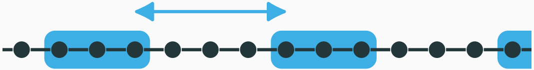
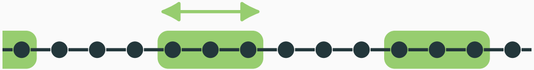
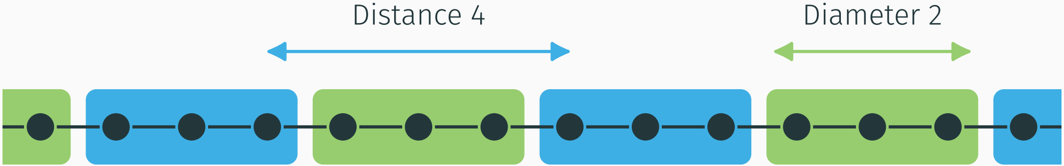


- **Boundedness:** $\forall B \in C_i, \mathrm{diam}_G(B) \leq f(r)$

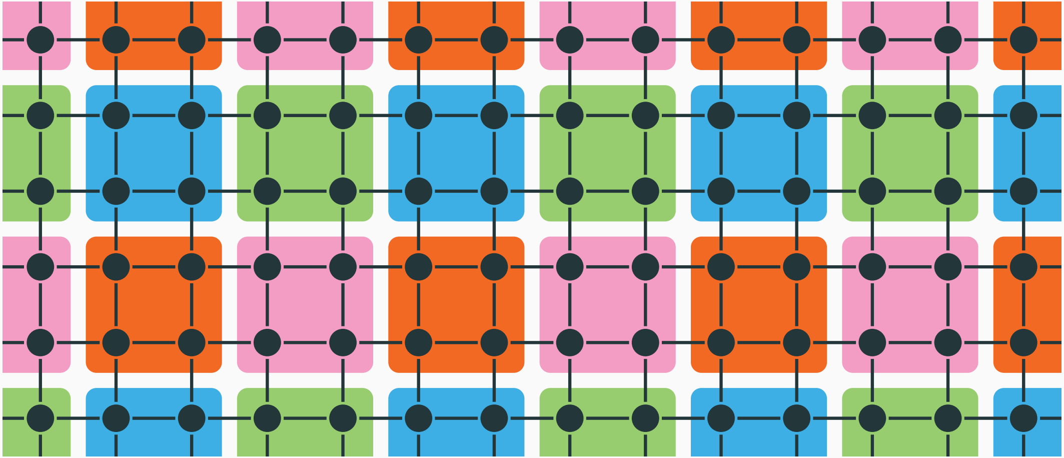


Example 1: the path



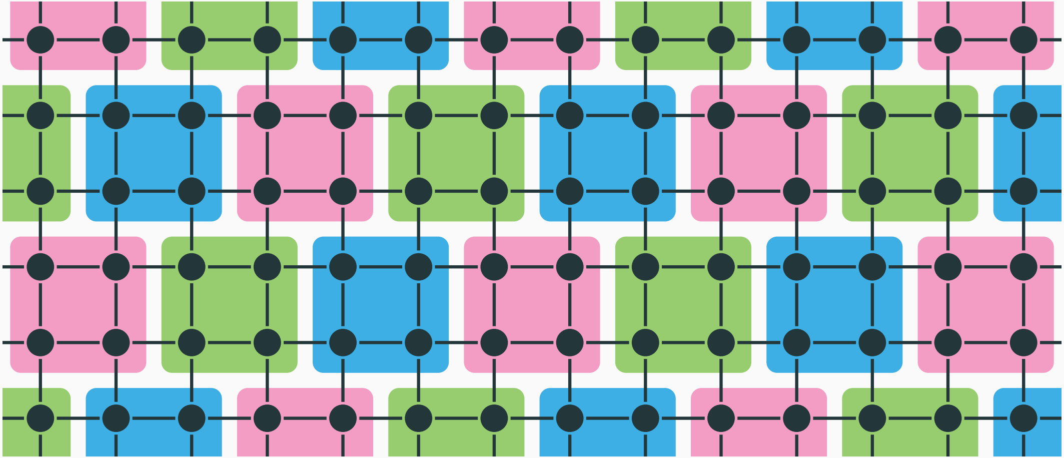
Dimension $= 1$

Example 2: the grid – try 1



Dimension ≤ 3

Example 2: the grid – try 2



Dimension $= 2!$

Asymptotic dimension and graph minors

Theorem (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, Scott, 2020)

Every class forbidding a minor has asymptotic dimension ≤ 2.

Application: distributed algorithms

How to use graph theory in distributed algorithms ?

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Global concept

$\downarrow$

Local concept

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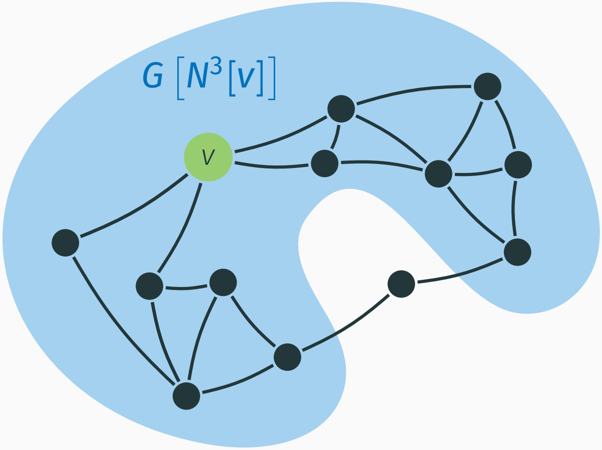
↓

Local concept

Definition

v is a *r-local cutvertex* if v is a cutvertex of $G[N^r[v]]$.



Cutvertices locaux

Theorem

For every graph G, #cutvertices $\leq 3\,\mathrm{MDS}(G)$.

$$\Downarrow$$

Theorem

Let $\mathcal{C}$ be of asymptotic dimension d.

Then $\forall r \geq r(\mathcal{C})$, $\#$***r-local*** *cutvertices* $\leq 3(\boldsymbol{d}+1)\,\mathrm{MDS}(G)$.

Cutvertices locaux

Theorem

For every graph G and $\boldsymbol{S} \subseteq \boldsymbol{V}(\boldsymbol{G})$, $\#$cutvertices $\in \boldsymbol{S} \leq 3\,\mathrm{MDS}(G, \boldsymbol{N}[S])$.

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Let $\mathcal{C}$ be of asymptotic dimension d.
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Proof

Dimension d, function f: sets $C_1, C_2, \ldots, C_{d+1}$ for $r = 5$.

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$v \in S$ a $(f(5) + 2)$-local cutvertex.

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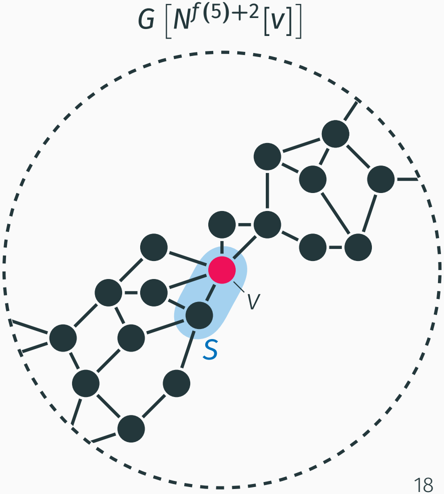

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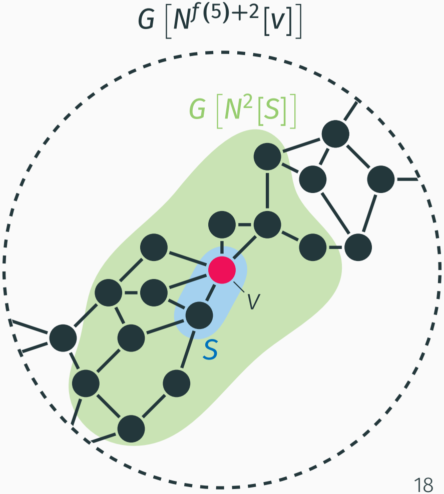
Claim: $N^2[S] \subseteq N^{f(5)+2}[v]$.



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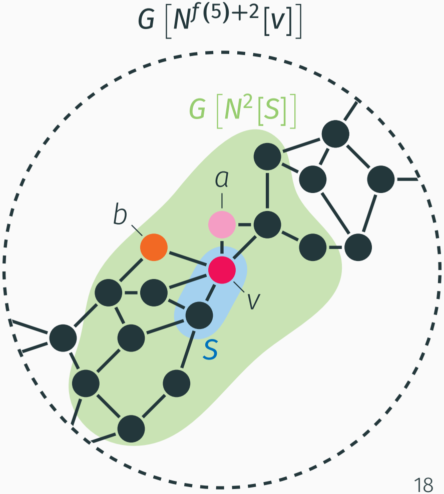

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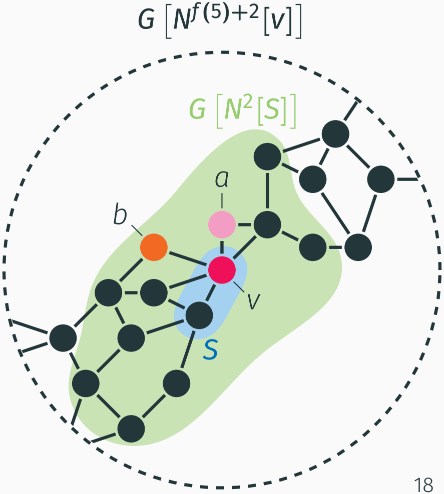

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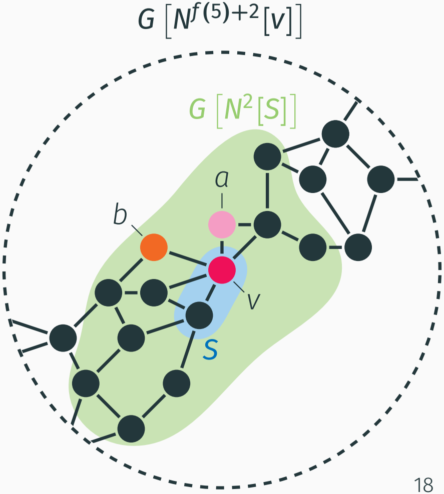

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$(N^2[S]$ are disjoint)

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Applications: local 2-cuts

Theorem

Let $\mathcal{C}$ of asymptotic dimension d.

Then $\forall r \geq r(\mathcal{C})$, $\#\text{vertices} \in r\text{-local } 2\text{-cut} \leq 8(d+1)\,\mathrm{MVC}(G)$.

Applications: approximations on locally-$\mathcal{C}$ classes

Theorem

If there exists a LOCAL algorithm:

- α*-approximation of MDS*
- *on* $\mathcal{C}$
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and if $\mathcal{D}$ *is:*

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Then there exists a LOCAL $\alpha(d+1)$*-approximation of MDS on* $\boldsymbol{\mathcal{D}}$ *in time* r*.*

Applications: MDS in LOCAL model

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*On graphs without the minor $K_{2,t}$, there exists an $\mathcal{O}(1)$-approximation (where the constant is **independant of t**) of Minimum Dominating Set in the LOCAL model, in $f(t)$ rounds.*

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Previous bound on $K_{3,t}$-minor-free graphs: $(2+\varepsilon)\cdot(t+4)$ in $g(\varepsilon, t)$ rounds (Heydt, Kublenz, Ossona de Mendez, Siebertz, Vigny 2022).

Conclusion and perspectives

Recap: 2 steps to go from global to local

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Relativize the result to all subsets S:

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$$\text{local cutvertex} \in S \implies \text{cutvertex in } G[S]$$

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- Apply asymptotic dimension

Asymptotic dimension $\rightarrow$ set S of small diameter

local cutvertex $\in S \implies$ cutvertex in $G[S]$

Union bound to conclude 😁

Conclusion and perspectives

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☻ Thanks! ☻