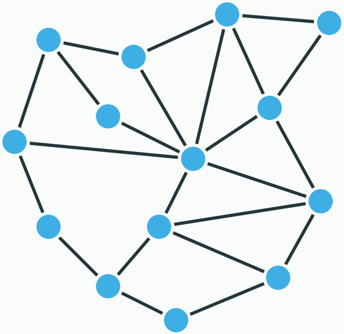
Local Constant Approximation for Dominating Set on Graphs Excluding Large Minors

Marthe Bonamy [1] Cyril Gavoille [1] Timothé Picavet [1] Alexandra Wesolek [2]

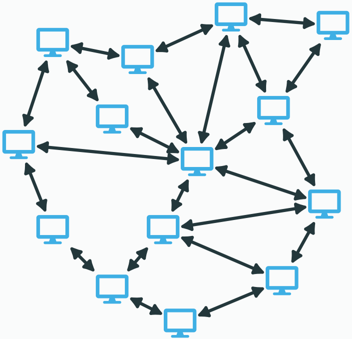
[1]LaBRI, U. Bordeaux

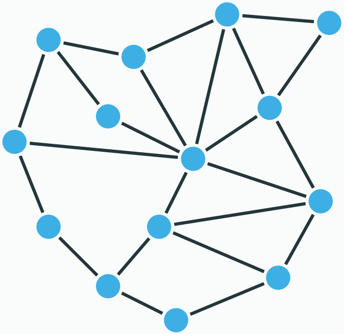
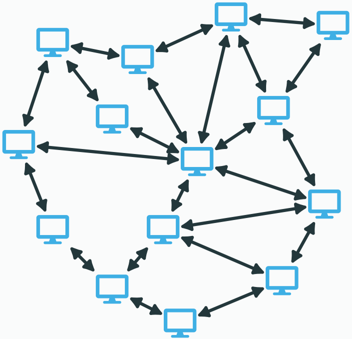
[2]TU Berlin

Centralized view

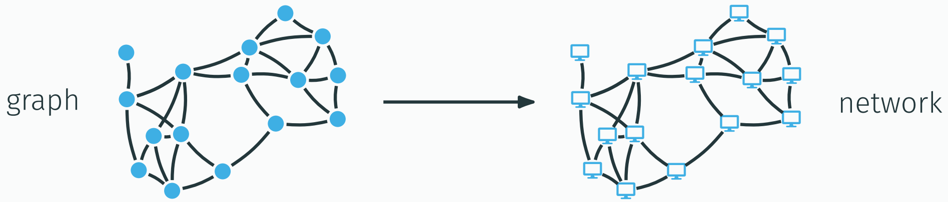


Distributed view



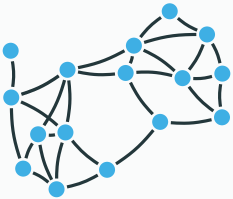
Centralized view	Distributed view
	
Focused on computing	Focused on communication

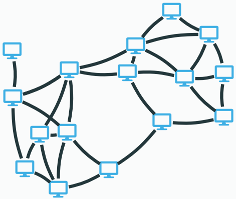
The LOCAL model



The LOCAL model

graph





network

vertices

+

edges

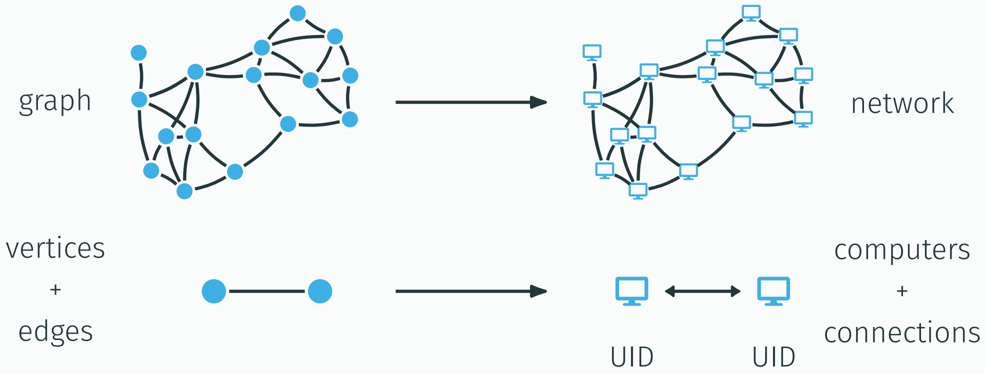
UID UID

computers

+

connections

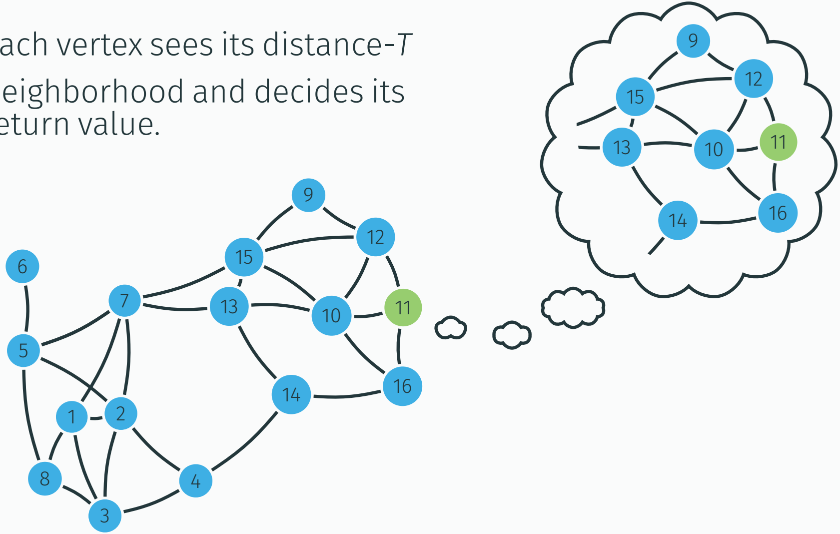
The LOCAL model



The network is also the input graph!

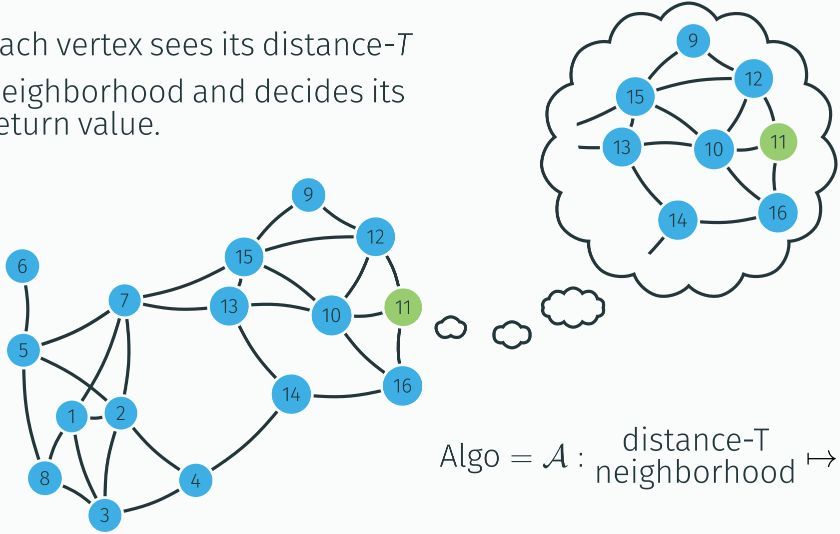
Equivalence with number of rounds T

Each vertex sees its distance-T neighborhood and decides its return value.



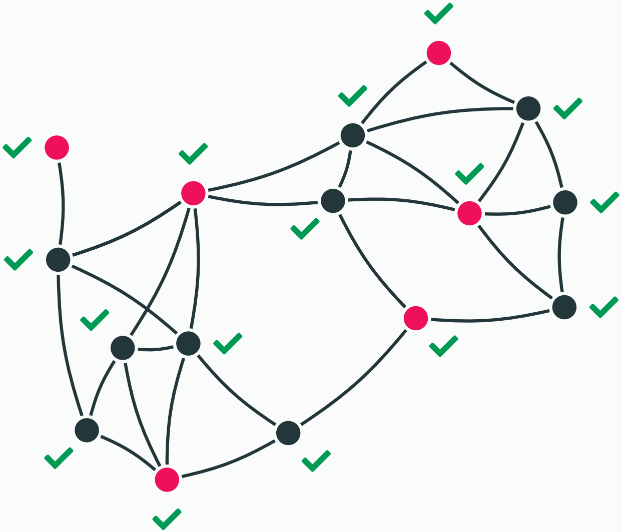
Equivalence with number of rounds T

Each vertex sees its distance-T neighborhood and decides its return value.

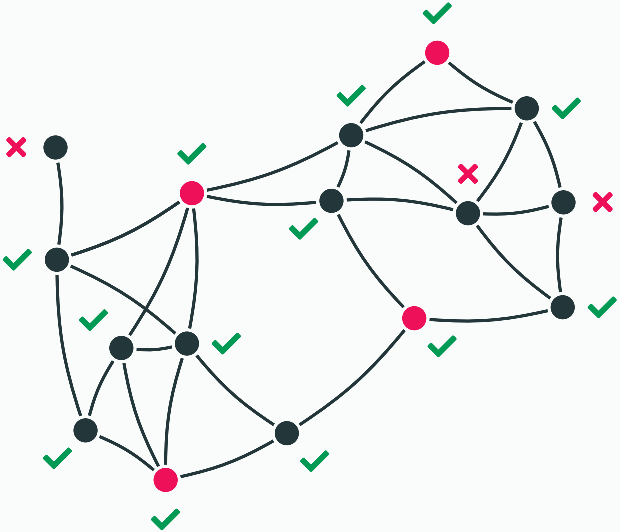


$$\text{Algo} = \mathcal{A} : \begin{matrix}\text{distance-T}\\ \text{neighborhood}\end{matrix} \mapsto \begin{matrix}\text{local}\\ \text{return value}\end{matrix}$$

An example: Minimum Dominating Set

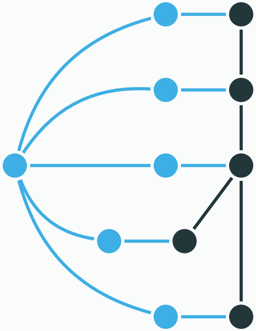


An example: Minimum Dominating Set



Small diameter allows bruteforcing

Minimum Dominating Set
when ∃ universal vertex



Easy in LOCAL
Hard in centralized

Is all hope lost?

Theorem (Kuhn, Moscibroda, and Wattenhofer, 2016)

*It is **impossible** to approximate MINIMUM DOMINATING SET with a constant number of rounds and constant approximation ratio on **general graphs**.*

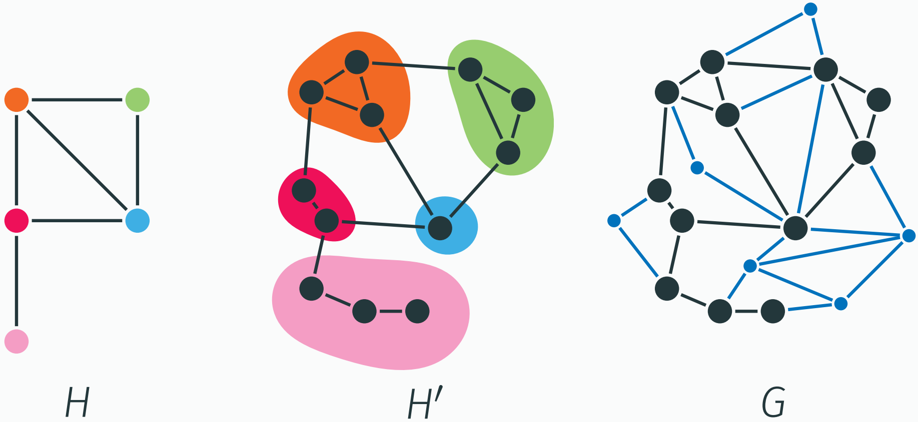
Is all hope lost?

Theorem (Kuhn, Moscibroda, and Wattenhofer, 2016)

It is ***impossible*** *to approximate* MINIMUM DOMINATING SET *with a constant number of rounds and constant approximation ratio on* ***general graphs****.*

Restricting the graph class!

Graph minors



H is a minor of G

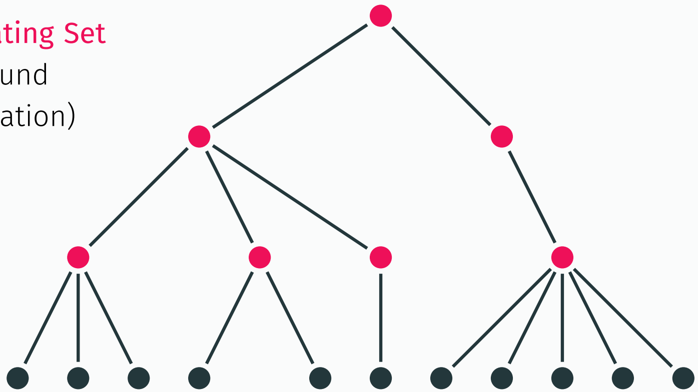
State of the art for MDS with $\mathcal{O}(1)$ LOCAL rounds

MINOR-FREE GRAPHS	APPROX. RATIO lower	APPROX. RATIO upper	#rounds
trees (K_3)	3[2]	3[2]	2
outerplanar (K_4, $K_{2,3}$)	5[3]	5[3]	2
planar (K_5, $K_{3,3}$)	7[1]	$11+\varepsilon$[4]	$\mathcal{O}_\varepsilon(1)$
$K_{2,t}$-minor-free	5[3]	**$2t-1$**	**3**
	5[3]	**50**	**$\mathcal{O}_t(1)$**
$K_{3,t}$-minor-free	7[1]	$(2+\varepsilon)\cdot(t+4)$[4]	$\mathcal{O}_{\varepsilon,t}(1)$
$K_{s,t}$-minor-free	7[1]	$t^{\mathcal{O}(st\sqrt{\log s})}$ [4]	$\mathcal{O}_t(1)$

[1] M. Hilke, C. Lenzen, and J. Suomela. Brief announcement: local approximability of minimum dominating set on planar graphs. PODC 2014.
[2] Folklore
[3] M. Bonamy, L. Cook, C. Groenland, and A. Wesolek. A tight local algorithm for the minimum dominating set problem in outerplanar graphs. DISC 2021.
[4] O. Heydt, S. Kublenz, P. Ossona de Mendez, S. Siebertz, and A. Vigny. Distributed domination on sparse graph classes. European Journal of Combinatorics, 2025.

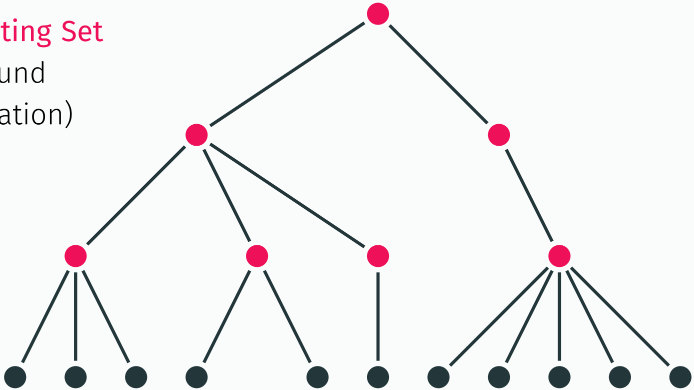
Example 1: trees

- Dominating Set (one-round computation)



Example 1: trees

- Dominating Set (one-round computation)

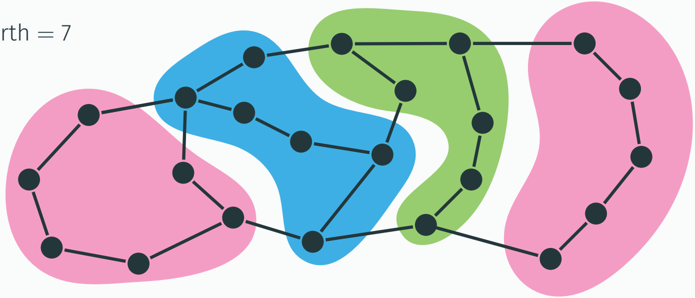


Theorem (folklore)

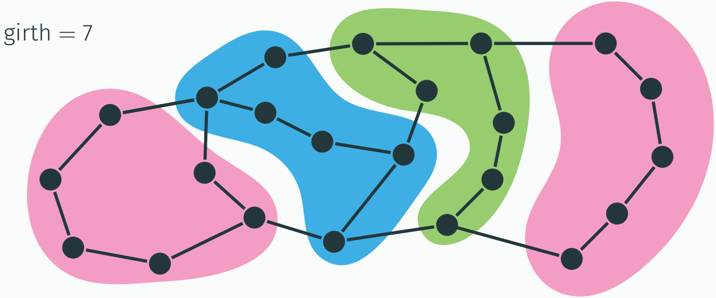
$\#cutvertices = |\{v \in V(T) \mid \mathrm{deg}(v) \geq 2\}| \leq 3 \cdot \mathrm{MDS}(T)$

Example 2: high girth graphs

girth $= 7$

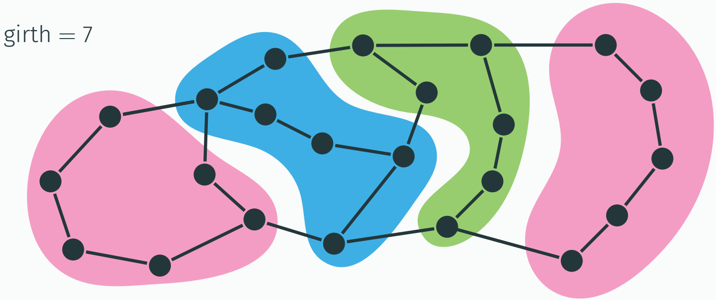


Reusing the algorithm of trees?



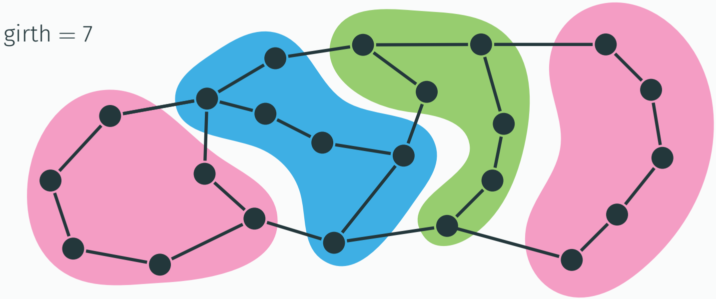
Reusing the algorithm of trees?

- Every vertex is in a bag ($\implies$ covering)



Reusing the algorithm of trees?

- Every vertex is in a bag ($\implies$ covering)
- Diameter $\leq$ girth/2 ($\implies$ to see a tree)

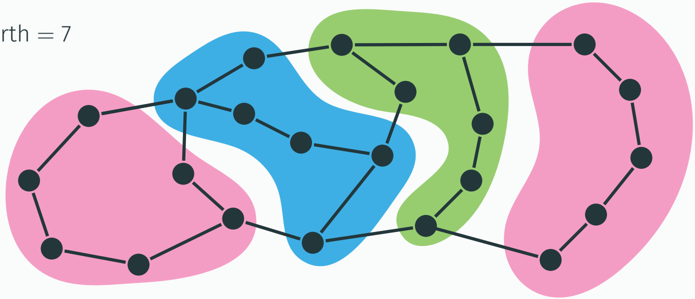


Reusing the algorithm of trees?

- Every vertex is in a bag ($\implies$ covering)
- Diameter $\leq$ girth/2 ($\implies$ to see a tree)
- Spacing clusters of same color ($\implies$ no overcounting for fixed color)

Example 2: high girth graphs

girth $= 7$



Reusing the algorithm of trees?

- Every vertex is in a bag ($\implies$ covering)
- Diameter $\leq$ girth/2 ($\implies$ to see a tree)
- Spacing clusters of same color ($\implies$ no overcounting for fixed color)
- Few colors ($\implies$ to limit overcounting)

Asymptotic dimension

Asymptotic dimension of $\mathcal{C}$ is d if $\exists f : \mathbb{N} \to \mathbb{N}, \forall G \in \mathcal{C}, \forall r \in \mathbb{N}, \exists B_1, B_2, \dots \subseteq V(G)$, such that

Asymptotic dimension

Asymptotic dimension of $\mathcal{C}$ is d if $\exists f : \mathbb{N} \to \mathbb{N}, \forall G \in \mathcal{C}, \forall r \in \mathbb{N}, \exists B_1, B_2, \dots \subseteq V(G)$, such that

- **Cover:** $V(G)$ is partitionned by the B_i's

Asymptotic dimension

Asymptotic dimension of $\mathcal{C}$ is d if $\exists f : \mathbb{N} \to \mathbb{N}, \forall G \in \mathcal{C}, \forall r \in \mathbb{N}, \exists B_1, B_2, \dots \subseteq V(G)$, such that

- **Cover:** $V(G)$ is partitionned by the B_i's

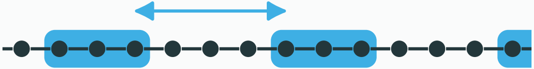
- **Colors:** each B_i's receive a color $\alpha(B_i) \in \{0, 1, \dots, d\}$

Asymptotic dimension

Asymptotic dimension of $\mathcal{C}$ is d if $\exists f : \mathbb{N} \to \mathbb{N}, \forall G \in \mathcal{C}, \forall r \in \mathbb{N}, \exists B_1, B_2, \cdots \subseteq V(G)$, such that

- **Cover:** $V(G)$ is partitionned by the B_i's

- **Colors:** each B_i's receive a color $\alpha(B_i) \in \{0, 1, \ldots, d\}$
- **Disjointness:** if $c(B_i) = c(B_j)$, then $\mathbf{dist}(B_i, B_j) > r$

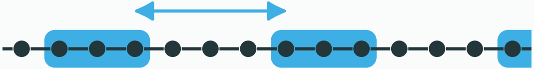


Asymptotic dimension

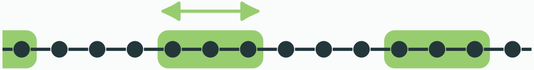
Asymptotic dimension of $\mathcal{C}$ is d if $\exists f : \mathbb{N} \to \mathbb{N}, \forall G \in \mathcal{C}, \forall r \in \mathbb{N}, \exists B_1, B_2, \dots \subseteq V(G)$, such that

- **Cover:** $V(G)$ is partitionned by the B_i's

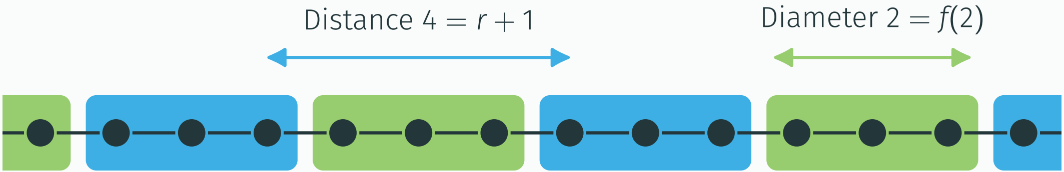
- **Colors:** each B_i's receive a color $\alpha(B_i) \in \{0, 1, \dots, d\}$
- **Disjointness:** if $c(B_i) = c(B_j)$, then $\mathsf{dist}(B_i, B_j) > r$



- **Boundedness:** $\forall i, \mathsf{diam}_G(B_i) \leq f(r)$



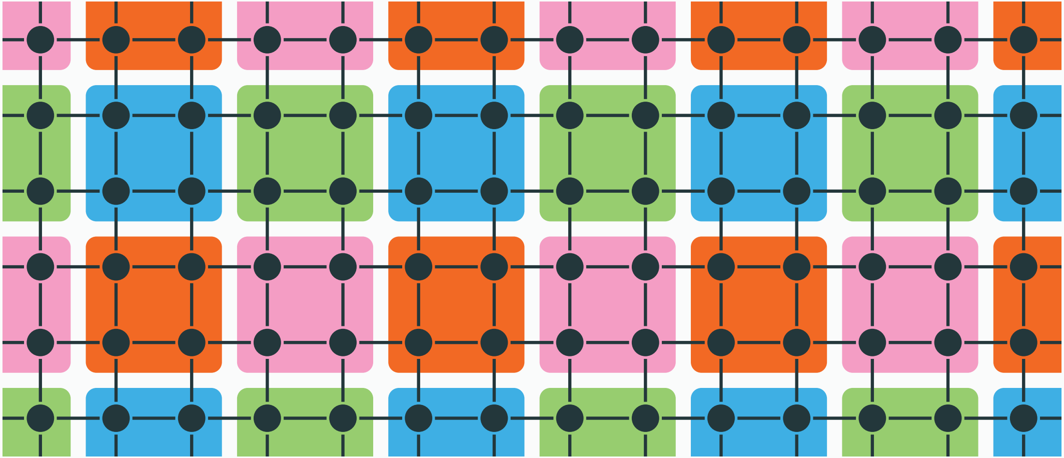
Example 1: the path



Dimension $= 1$ (2 colors)

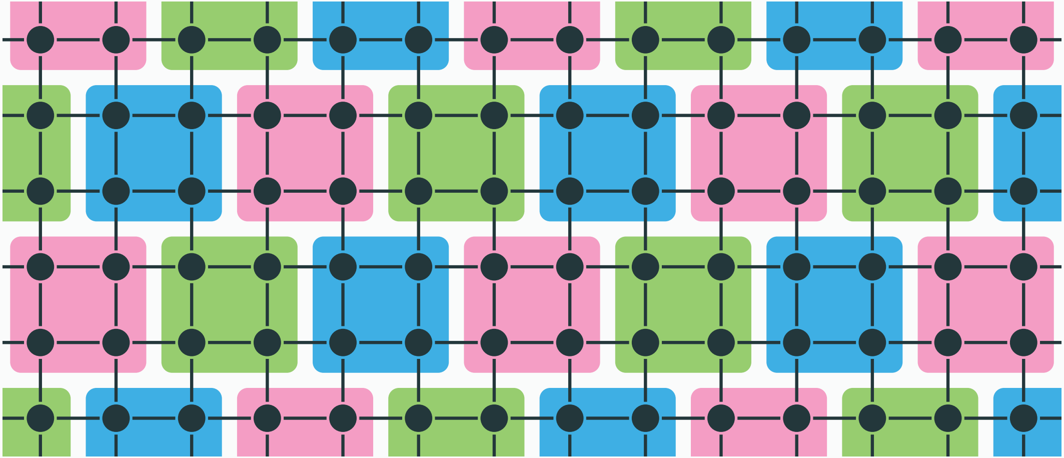
with $f(r) = r - 2$

Example 2: the grid – attempt 1 ($r = 2$)



Dimension ≤ 3

Example 2: the grid – attempt 2 ($r = 2$)



Dimension $= 2!$

Asymptotic dimension and graph minors

Theorem (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, Scott, 2020)

Every class excluding a fixed minor has asymptotic dimension ≤ 2.

Application: distributed algorithms

How to use graph theory in distributed algorithms?

How to use graph theory in distributed algorithms?

Global concept

↓

Local concept

Application: distributed algorithms

How to use graph theory in distributed algorithms?

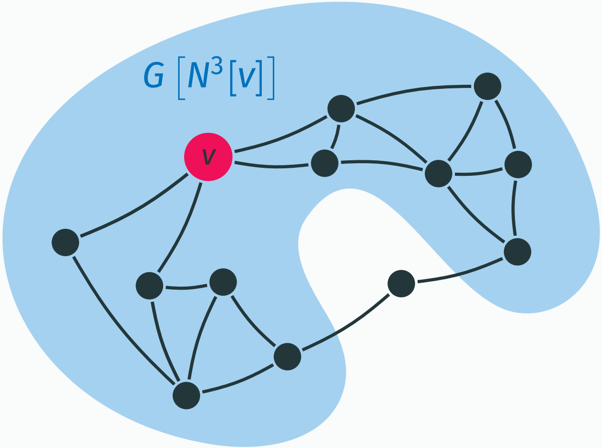
Global concept

$\downarrow$

Local concept

Definition

v is a *r-local cutvertex* if v is a cutvertex of $G[N^r[v]]$.



Applications: MDS in LOCAL model

Theorem (main)

*On graphs excluding $K_{2,t}$ as a minor, there exists a constant-approximation (where the constant is **independent of t**) of MINIMUM DOMINATING SET in the LOCAL model, in $f(t)$ rounds.*

Applications: MDS in LOCAL model

Theorem (main)

*On graphs excluding $K_{2,t}$ as a minor, there exists a constant-approximation (where the constant is **independent of t**) of* MINIMUM DOMINATING SET *in the LOCAL model, in $f(t)$ rounds.*

Previous bound on H-minor-free graphs had $\Omega(|V(H)|)$ in $g(H)$ rounds (Heydt, Kublenz, Ossona de Mendez, Siebertz, Vigny 2022).

Theorem (main)

*On graphs excluding $K_{2,t}$ as a minor, there exists a constant-approximation (where the constant is **independent of t**) of MINIMUM DOMINATING SET in the LOCAL model, in $f(t)$ rounds.*

Previous bound on H-minor-free graphs had $\Omega(|V(H)|)$ in $g(H)$ rounds (Heydt, Kublenz, Ossona de Mendez, Siebertz, Vigny 2022).

Asymptotic dimension only used in the analysis!

The algorithm

$$Y_t(G) = \bigcup\{18t\text{-local cuts of size } \leq 2\} \setminus \{\text{non-interesting vertices}\}$$

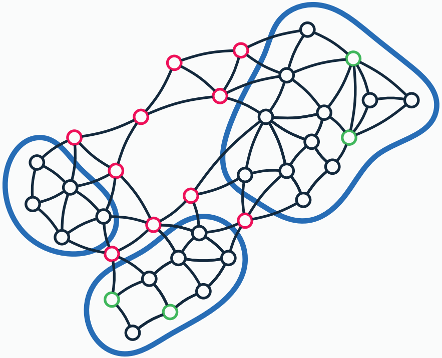
$$\text{Algorithm} = Y_t(G) \cup [\text{brute-force on } G - Y_t(G)]$$

Lemma 1

$|Y_t(G)| = \mathcal{O}(d) \cdot \mathrm{MDS}(G)$.

Lemma 2

If G is $K_{2,t}$-minor-free, every connected component of $G - Y_t(G)$ has diameter $\mathcal{O}_t(1)$.



Local cutvertices

Lemma 1a (folklore)

For every graph G, #cutvertices $\leq 3\,\mathrm{MDS}(G)$.

$$\Downarrow$$

Theorem 1

Let $\mathcal{C}$ be of asymptotic dimension d.

Then $\forall r \geq r_0$, #**r-local** cutvertices $\leq 3(\boldsymbol{d+1})\,\mathrm{MDS}(G)$.

Local cutvertices

Lemma 1a (folklore)

For every graph G, #cutvertices $\leq 3\,\mathrm{MDS}(G)$.

$\Downarrow$

Lemma 1b

For every graph G and $\boldsymbol{S} \subseteq \boldsymbol{V(G)}$, #cutvertices $\in \boldsymbol{S} \leq 3\,\mathrm{MDS}(G, \boldsymbol{N}[\boldsymbol{S}])$.

$\Downarrow$

Theorem 1

Let $\mathcal{C}$ be of asymptotic dimension d.

Then $\forall r \geq r_0$, #**r-local** cutvertices $\leq 3(\boldsymbol{d}+\mathbf{1})\,\mathrm{MDS}(G)$.

Proof of Lemma 1b $\implies$ Theorem 1

Dimension d, function f: sets $B_1, B_2, \ldots$ for $r = 5$.

Proof of Lemma 1b $\implies$ Theorem 1

Dimension d, function f: sets $B_1, B_2, \ldots$ for $r = 5$.

Let S be of weak-diameter $f(5)$.

Proof of Lemma 1b $\implies$ Theorem 1

Dimension d, function f: sets $B_1, B_2, \ldots$ for $r = 5$.

Let S be of weak-diameter $f(5)$.

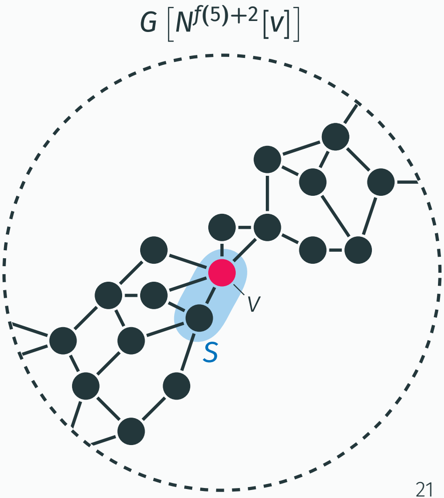
$v \in S$ a $(f(5) + 2)$-local cutvertex.

Proof of Lemma 1b $\implies$ Theorem 1

Dimension d, function f: sets $B_1, B_2, \ldots$ for $r = 5$.

Let S be of weak-diameter $f(5)$.

$v \in S$ a $(f(5) + 2)$-local cutvertex.



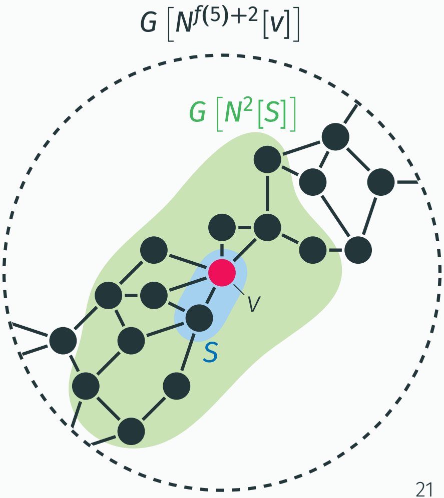
Proof of Lemma 1b $\implies$ Theorem 1

Dimension d, function f: sets $B_1, B_2, \ldots$ for $r = 5$.

Let S be of weak-diameter $f(5)$.

$v \in S$ a $(f(5)+2)$-local cutvertex.

Claim: $N^2[S] \subseteq N^{f(5)+2}[v]$.



Proof of Lemma 1b $\implies$ Theorem 1

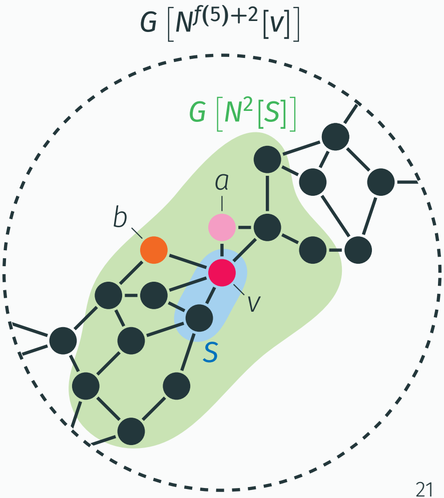
Dimension d, function f: sets $B_1, B_2, \ldots$ for $r = 5$.

Let S be of weak-diameter $f(5)$.

$v \in S$ a $(f(5)+2)$-local cutvertex.

Claim: $N^2[S] \subseteq N^{f(5)+2}[v]$.

Claim: v is a cutvertex of $G[N^2[S]]$ (separates a and b).



Proof of Lemma 1b $\implies$ Theorem 1

Dimension d, function f: sets $B_1, B_2, \ldots$ for $r = 5$.

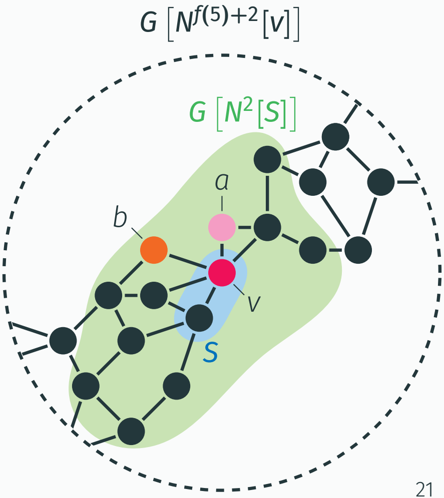
Let S be of weak-diameter $f(5)$.

$v \in S$ a $(f(5)+2)$-local cutvertex.

Claim: $N^2[S] \subseteq N^{f(5)+2}[v]$.

Claim: v is a cutvertex of $G[N^2[S]]$ (separates a and b).

Claim: $\#\text{cutvertex in } S \leq 3\,\mathrm{MDS}(G[N^2[S]], N[S]) \leq 3\,\mathrm{MDS}(G, N[S])$.



Proof of Lemma 1b $\implies$ Theorem 1

Dimension d, function f: sets $B_1, B_2, \ldots$ for $r = 5$.

Let S be of weak-diameter $f(5)$.

$v \in S$ a $(f(5)+2)$-local cutvertex.

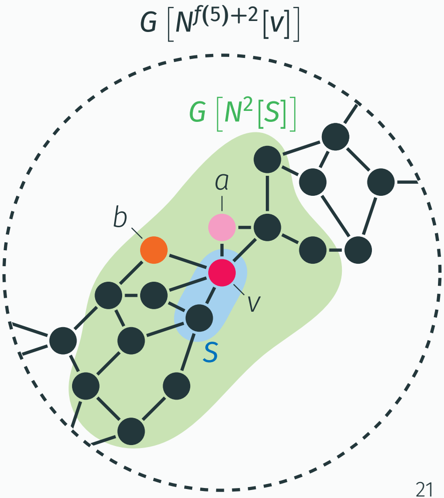
Claim: $N^2[S] \subseteq N^{f(5)+2}[v]$.

Claim: v is a cutvertex of $G[N^2[S]]$ (separates a and b).

Claim: $\#$cutvertex in $S \leq 3\,\mathrm{MDS}(G[N^2[S]], N[S]) \leq 3\,\mathrm{MDS}(G, N[S])$.

$$\Downarrow$$

$$\#(f(5)+2)\text{-local cutvertex} \leq \sum_{c=1}^{d+1} \sum_{\substack{i, \\ \alpha(B_i)=c}} 3\cdot\mathrm{MDS}(G, N[B_i])$$



End of the proof

$$\#(f(5)+2)\text{-local cutvertex} \leq \sum_{c=1}^{d+1} \sum_{\substack{i, \\ \alpha(B_i)=c}} 3 \cdot \mathrm{MDS}(G, \underbrace{N[B_i]}_{\text{at distance 3}})$$

End of the proof

$$\#(f(5)+2)\text{-local cutvertex} \leq \sum_{c=1}^{d+1} \sum_{\substack{i,\\ \alpha(B_i)=c}} 3 \cdot \mathrm{MDS}(G, \underbrace{N[B_i]}_{\text{at distance } 3})$$

$(N^2[B_i]$'s are disjoint)

$$\#(f(5)+2)\text{-local cutvertex} \leq \sum_{i=1}^{d+1} 3 \cdot \mathrm{MDS}(G) = 3(d+1) \cdot \mathrm{MDS}(G).$$

What about local 2-cuts?

Theorem

Let $\mathcal{C}$ of asymptotic dimension d.

Then $\forall r \geq r_0, \#$vertices $\in r$-local 2-cut $\leq 8(d+1)\,\mathrm{MVC}(G)$.

Conclusion and perspectives

Follow-up works:

- H-minor-free graphs admit a 50-approximation if $\mathsf{pathwidth}(H) = 2$.
- Constant factor approximations in locally-nice graphs, e.g. bounded genus.
- Transform algorithms from a class $\mathcal{C}$ to a locally-$\mathcal{C}$ class.

Conclusion and perspectives

Follow-up works:

- H-minor-free graphs admit a 50-approximation if $\mathsf{pathwidth}(H) = 2$.
- Constant factor approximations in locally-nice graphs, e.g. bounded genus.
- Transform algorithms from a class $\mathcal{C}$ to a locally-$\mathcal{C}$ class.

? Without minor $H \rightarrow$ LOCAL $\mathcal{O}(\mathsf{pathwidth}(H))$-approximation in constant time ?

Conclusion and perspectives

Follow-up works:

- H-minor-free graphs admit a 50-approximation if $\mathsf{pathwidth}(H) = 2$.
- Constant factor approximations in locally-nice graphs, e.g. bounded genus.
- Transform algorithms from a class $\mathcal{C}$ to a locally-$\mathcal{C}$ class.

❓ Without minor $H \rightarrow$ LOCAL $\mathcal{O}(\mathsf{pathwidth}(H))$-approximation in constant time ?

☻ Thanks! ☻