Discrete Structures

Relations

Today's Lecture

- Basic concepts of relation
- Types of relations
- Relation on a set
- The inverse of a relation
- Representing relations using Digraphs
- N-*ary* Relations

Relations

If we want to describe a relationship between elements of two sets A and B, we can use **ordered pairs** with their first element taken from A and their second element taken from B.
Since this is a relation between **two sets**, it is called a **binary relation**.

Definition: Let A and B be sets. A binary relation R from A to B is a subset of $A \times B$.

In other words, for a binary relation R we have $R \subseteq A \times B$. We use the notation aRb to denote that $(a, b) \in R$ and $a\underline{R}b$ to denote that $(a, b) \notin R$.

Relations

If we have two sets

$A=\{1,2,3,4,5\}$ and $B=\{5,6,7,8,9\}$

The cartesian product of A and B is

$$A \times B = \{ (1,5), (1,6), (1,7), (1,8), (1,9), (2,5), (2,6), (2,7), (2,8), (2,9), (3,5), (3,6), (3,7), (3,8), (3,9), (4,5), (4,6), (4,7), (4,8), (4,9), (5,5), (5,6), (5,7), (5,8), (5,9) \}.$$

The rule is to add 4:

$$R = \{ (1,5), (2,6), (3,7), (4,8), (5,9) \}.$$

R is subset of $A \times B$.

Relations

One to One Relations

The Rule is 'ADD 4'

Relations

Many to Many relation

If we have two sets

A={Ahmad, Peter, Ali, Jaweria, Hamad} and B={Paris, London, Dubai, New York, Cyprus}

The cartesian product of A and B is

$A \times B$ = { (Ahmad,Paris), …., (Hamad,Cyprus) }.

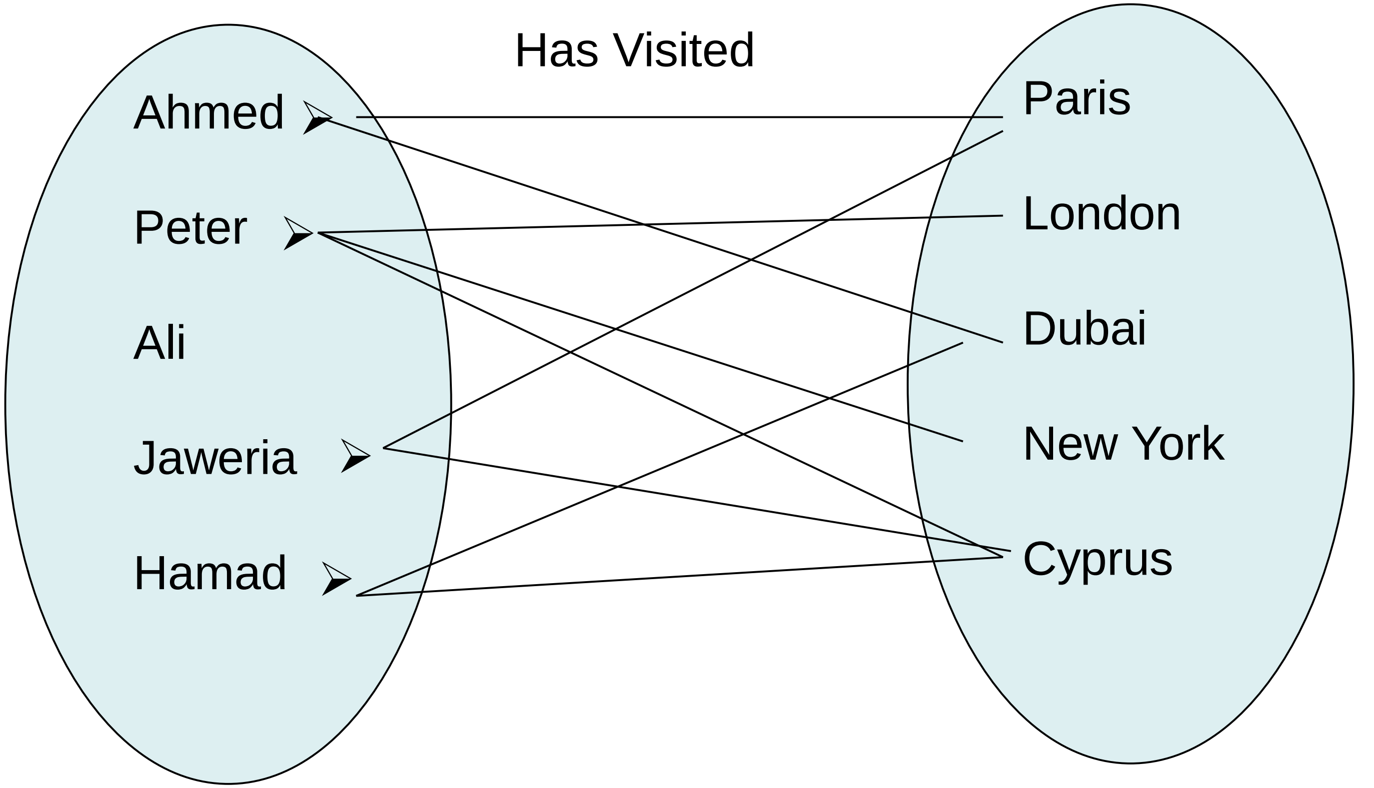
The rule is "Has visited":

R = { (Ahmad,Paris), (Ahmad,Dubai), (Peter,London), (Peter,New York), (Peter,Cyprus), (Jaweria,Paris), (Jaweria,Cyprus), (Hamad,Dubai), (Hamad, Cyprus) }.

R is subset of $A \times B$.

Relations

Many to Many relation



There are *MANY* arrows from each person and each place is related to *MANY* People.

Relations

Many to Many relation

If we have two sets

A={Ahmad, Peter, Ali, Jaweria, Hamad} and B={Paris, London, Dubai, New York, Cyprus}

The cartesian product of A and B is

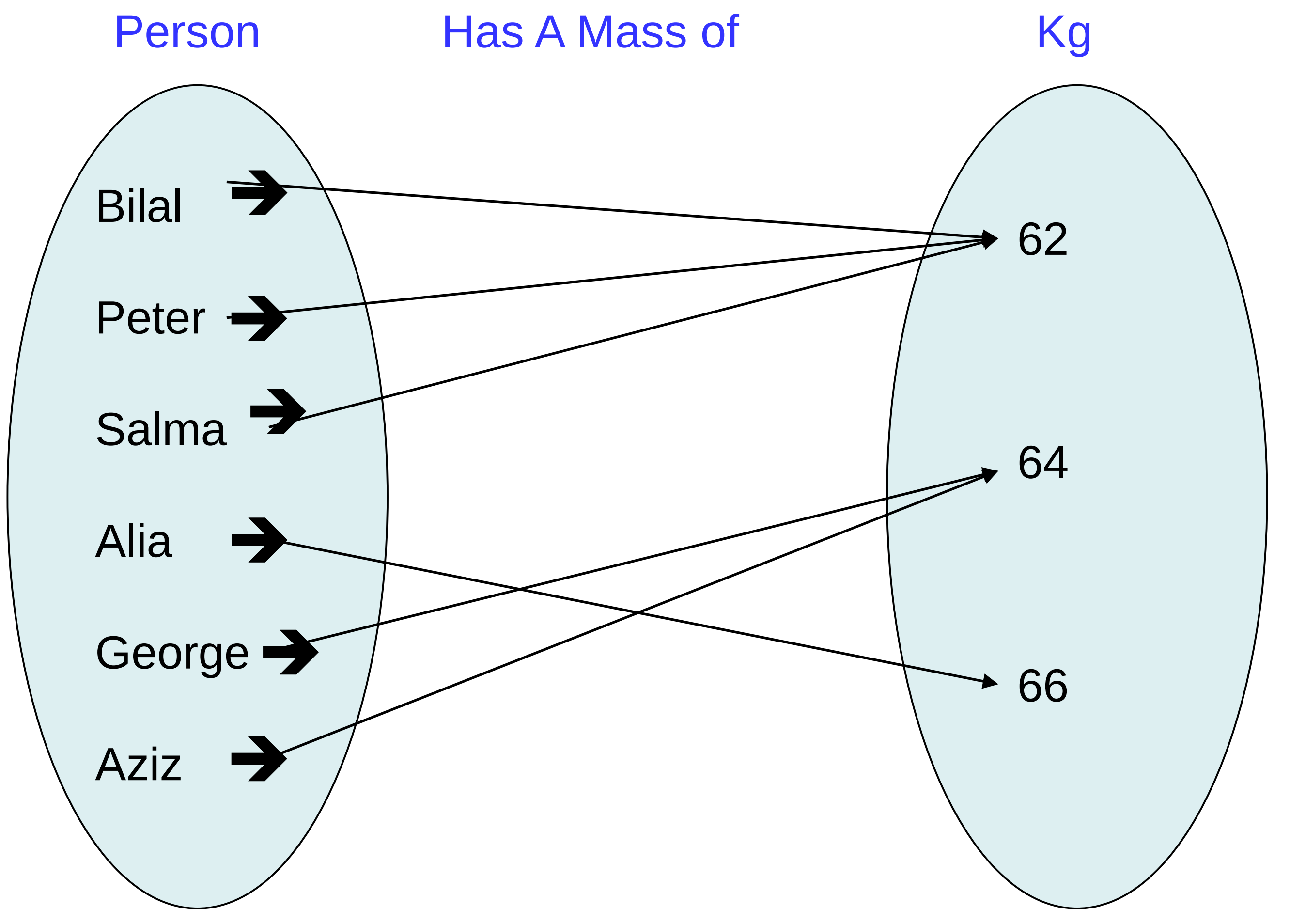
A × B = { (Ahmad,Paris), …., (Hamad,Cyprus) }.

The rule is "Has visited":

R = { (Ahmad,Paris), (Ahmad,Dubai), (Peter,London), (Peter,New York), (Peter,Cyprus), (Jaweria,Paris), (Jaweria,Cyprus), (Hamad,Dubai), (Hamad, Cyprus) }.

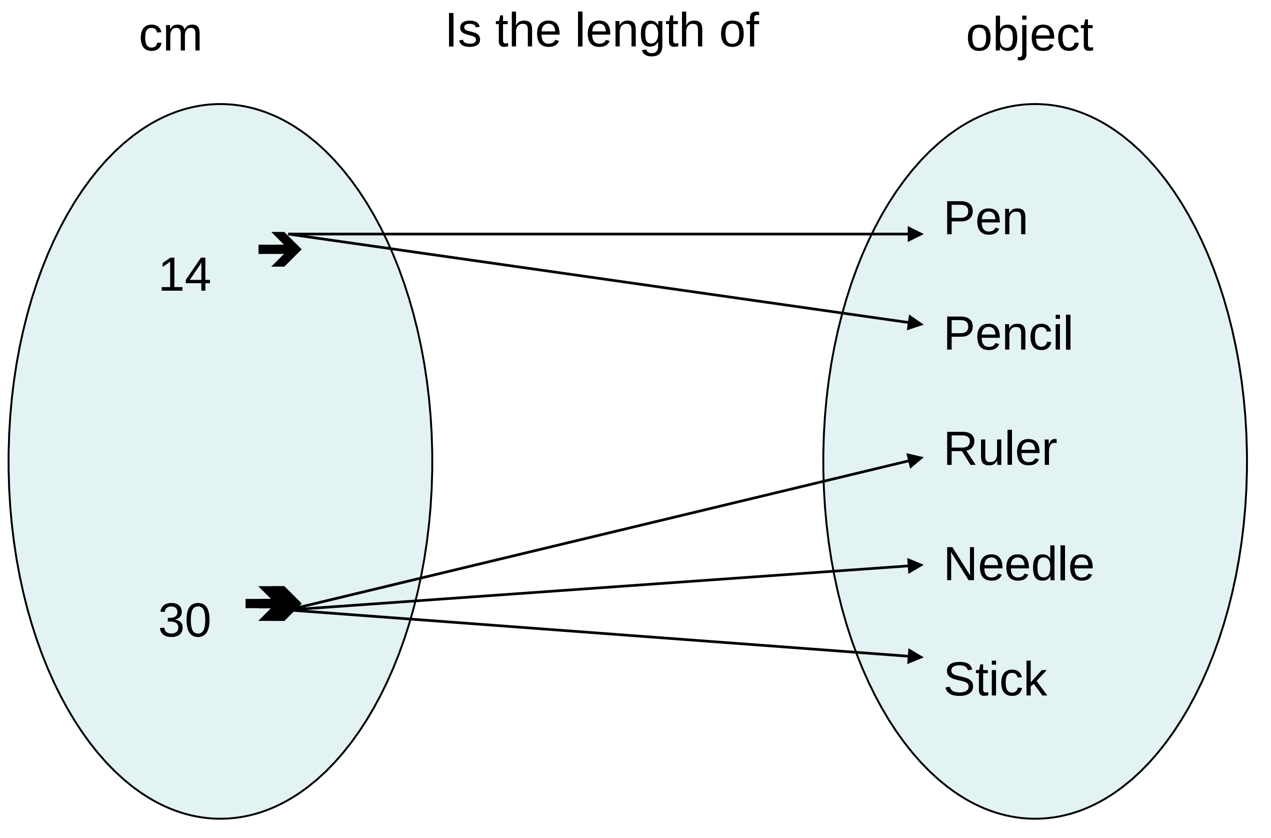
R is subset of A × B.

Relations (Many to One relation)



In this case each person has only one mass, yet several people have the same Mass.

Relations (One to Many relation)

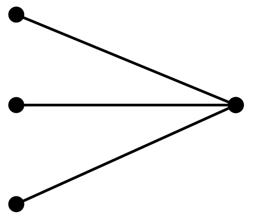


Here one amount is the length of many objects.

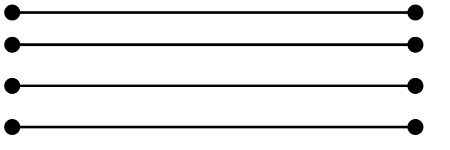
This is a ***ONE to MANY*** relationship

Relations

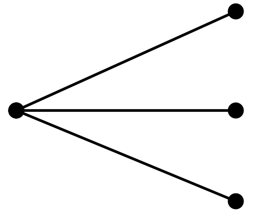
Many to One Relationship



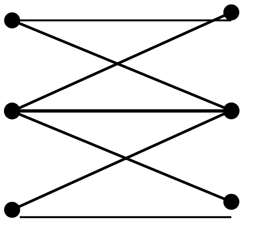
One to One Relationship



One to Many Relationship



Many to Many Relationship



Relation on a Set

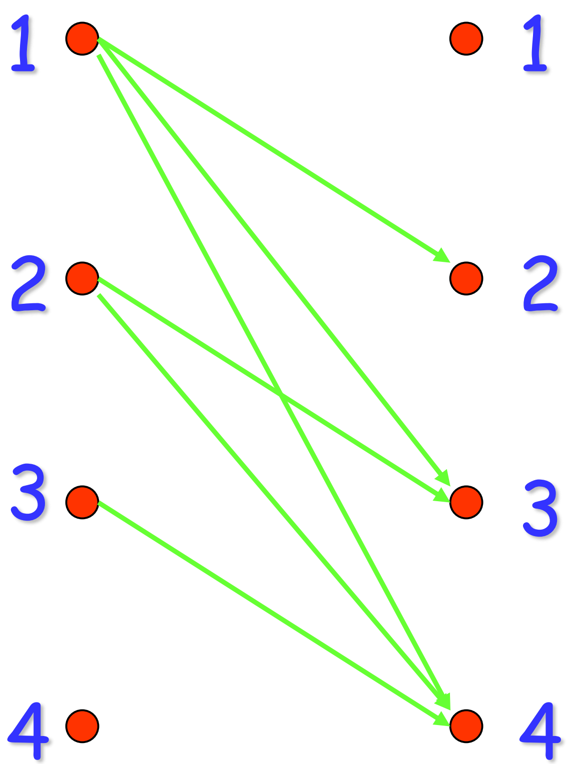
Definition: A relation on the set A is a relation from A to A.

In other words, a relation on the set A is a subset of $A \times A$.

Example: Let A = {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a < b} ?

Relation on a Set

R = { (1, 2), (1, 3),(1, 4),(2, 3),(2, 4),(3, 4) }



R	1	2	3	4
1		X	X	X
2			X	X
3				X
4				

Relation on a Set

How many different relations can we define on a set A with n elements?

- A relation on a set A is a subset of $A \times A$.
- How many elements are in $A \times A$?
- There are n^2 elements in $A \times A$, so how many subsets (= relations on A) does $A \times A$ have?
- The number of subsets that we can form out of a set with m elements is 2^m. Therefore, 2^{n^2} subsets can be formed out of $A \times A$.

Answer: We can define 2^{n^2} different relations on A.

Relation on an infinite Set

The Less-than Relation for Real Numbers

Define a relation L from **R** to **R** as follows: For all real numbers x and y,

$$x \; L \; y \Leftrightarrow x < y.$$

a. Is $57 \; L \; 53$?

b. Is $(-17) \; L \; (-14)$?

c. Is $143 \; L \; 143$?

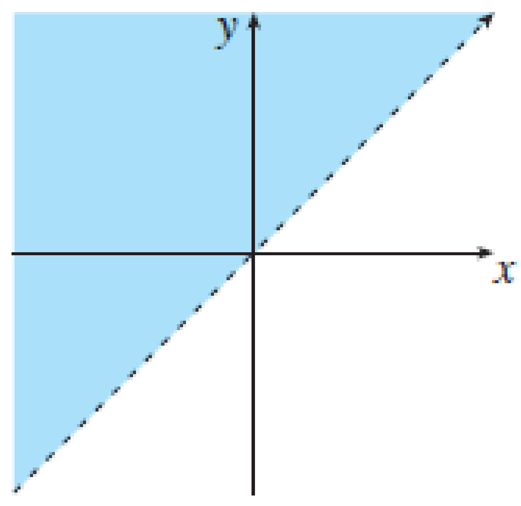
d. Is $(-35) \; L \; 1$?

e. Draw the graph of L as a subset of the Cartesian plane $\mathbf{R} \times \mathbf{R}$

Relation on an infinite Set

The Less-than Relation for Real Numbers

For each value of *x*, all the points *(x, y)* with *y* > *x* are on the graph. So the graph consists of all the points above the line *x* = *y*.



Relation on an infinite Set

The Congruence Modulo 2 Relation

Define a relation E from $\mathbf{Z}$ to $\mathbf{Z}$ as follows:

For all $(m, n) \in \mathbf{Z} \times \mathbf{Z}$, $\quad m\ E\ n \Leftrightarrow m - n$ is even.

a. Is $4\ E\ 0$?

b. Is $2\ E\ 6$?

c. Is $3\ E\ (-3)$?

d. Is $5\ E\ 2$?

b. List five integers that are related by E to 1.

c. Prove that if n is any odd integer, then $n\ E\ 1$.

Relation on an infinite Set

The Congruence Modulo 2 Relation

Define a relation E from **Z** to **Z** as follows: For all $(m, n) \in \mathbf{Z} \times \mathbf{Z}$,
$m \; E \; n \Leftrightarrow m - n$ is even.

List five integers that are related by E to 1.

There are many such lists. One is

1 because $1 - 1 = 0$ is even,
3 because $3 - 1 = 2$ is even,
5 because $5 - 1 = 4$ is even,
-1 because $-1 - 1 = -2$ is even,
-3 because $-3 - 1 = -4$ is even.

Relation on an infinite Set

The Congruence Modulo 2 Relation

Define a relation E from $\mathbf{Z}$ to $\mathbf{Z}$ as follows:
For all $(m, n) \in \mathbf{Z} \times \mathbf{Z}$, $m\ E\ n \Leftrightarrow m - n$ is even.

Prove that if n is any odd integer, then $n\ E\ 1$.

Proof: Suppose n is any odd integer. Then $n = 2k + 1$ for some integer k. Now by definition of E, $n\ E\ 1$ if, and only if, $n - 1$ is even. But by substitution,

$n - 1 = (2k + 1) - 1 = 2k,$

and since k is an integer, $2k$ is even. Hence $n\ E\ 1$

A Relation on a Power Set

Let $X = \{a, b, c\}$. Then $P(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define a relation **S** from $P(X)$ to **Z** as follows:

For all sets A and B in $P(X)$, A **S** $B \Leftrightarrow A$ has at least as many elements as B.

a. Is $\{a, b\}$ **S** $\{b, c\}$?

b. Is $\{a\}$ **S** $\emptyset$?

c. Is $\{b, c\}$ **S** $\{a, b, c\}$?

d. Is $\{c\}$ **S** $\{a\}$?

The Inverse of a Relation

If R is a relation from A to B, then a relation $R-1$ from B to A can be defined by interchanging the elements of all the ordered pairs of R.

Definition

Let R be a relation from A to B. Define the inverse Relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.$$

Equivalently For all $x \in A$ and $y \in B$, $(y, x) \in R-1 \Leftrightarrow (x, y) \in R$.

The Inverse of a Finite Relation

Let $A = \{2, 3, 4\}$ and $B = \{2, 6, 8\}$ and let R be the "divides" relation from A to B:

For all $(x, y) \in A \times B, \quad x \, R \, y \Leftrightarrow x \mid y$

State explicitly which ordered pairs are in R and R^{-1}, and draw arrow diagrams for R and R^{-1}.

$R = \{(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)\}$

$R^{-1} = \{(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)\}$

For all $(y, x) \in B \times A, \quad y \, R^{-1} \, x \Leftrightarrow y$ is a multiple of x.

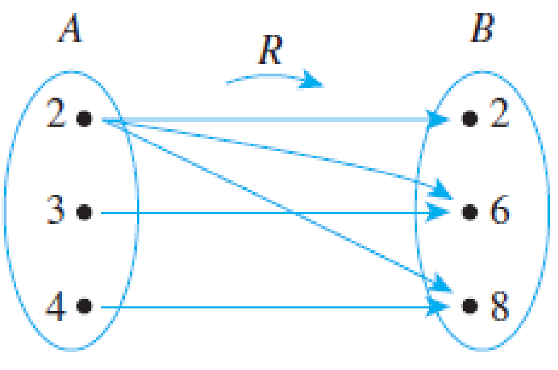
The Inverse of a Finite Relation

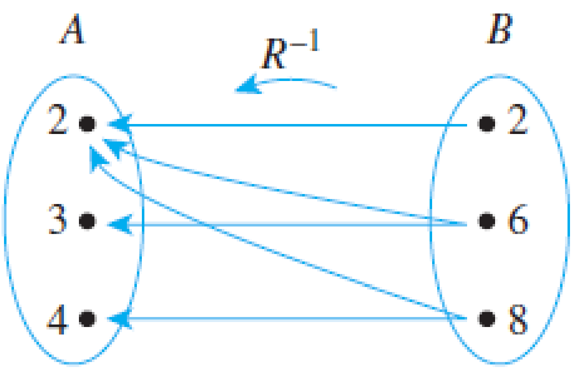
Let $A = \{2, 3, 4\}$ and $B = \{2, 6, 8\}$ and let R be the "divides" relation from A to B:

For all $(x, y) \in A \times B, \quad x \, R \, y \Leftrightarrow x \mid y$

$R = \{(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)\}$

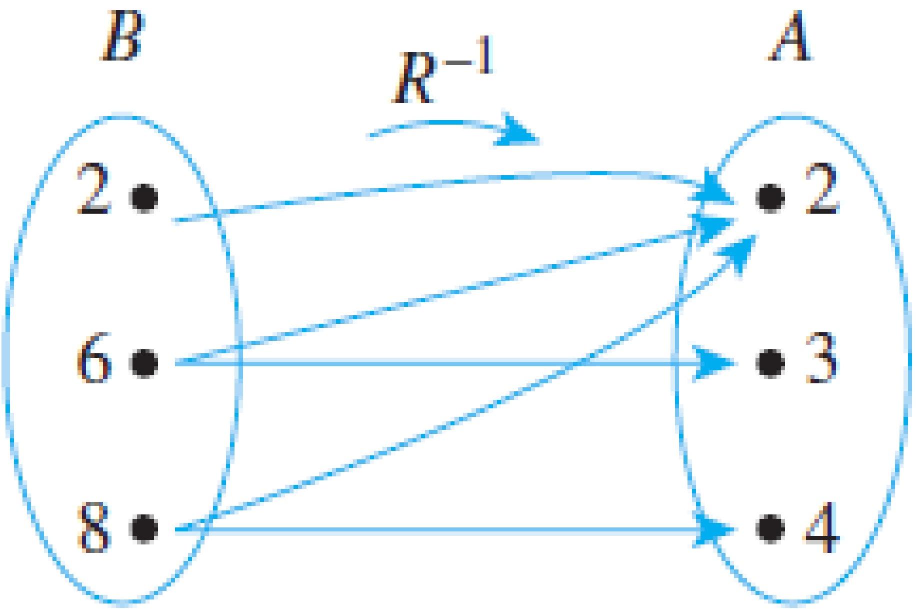
$R^{-1} = \{(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)\}$





The Inverse of a Finite Relation

$$R^{-1} = \{(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)\}$$



The Inverse of an infinite Relation

Define a relation R from **R** to **R** as follows:

For all $(x, y) \in \mathbf{R} \times \mathbf{R}$, $x \, R \, y \Leftrightarrow y = 2|x|$.

Draw the graphs of R and R^{-1} in the Cartesian plane.

$$R = \{(x, y) \mid y = 2|x|\}$$

x	y
0	0
1	2
−1	2
2	4
−2	4

1st coordinate 2nd coordinate

$$R^{-1} = \{(y, x) \mid y = 2|x|\}$$

y	x
0	0
2	1
2	−1
4	2
4	−2

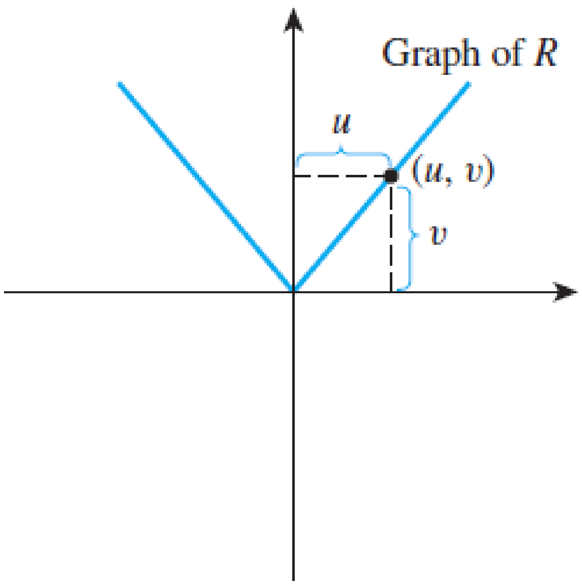
1st coordinate 2nd coordinate

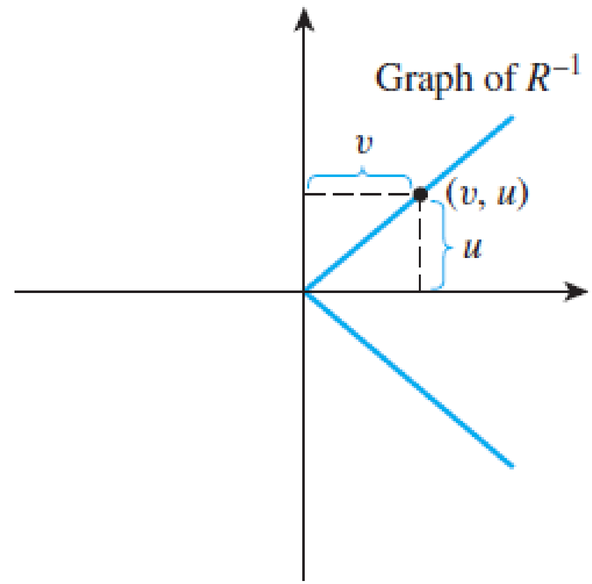
The Inverse of an infinite Relation

Define a relation *R* from **R** to **R** as follows:

For all $(x, y) \in \mathbf{R} \times \mathbf{R}$, $x \; R \; y \Leftrightarrow y = 2|x|$.

Draw the graphs of *R* and R^{-1} in the Cartesian plane.





Representing Relations Using Digraphs

Definition

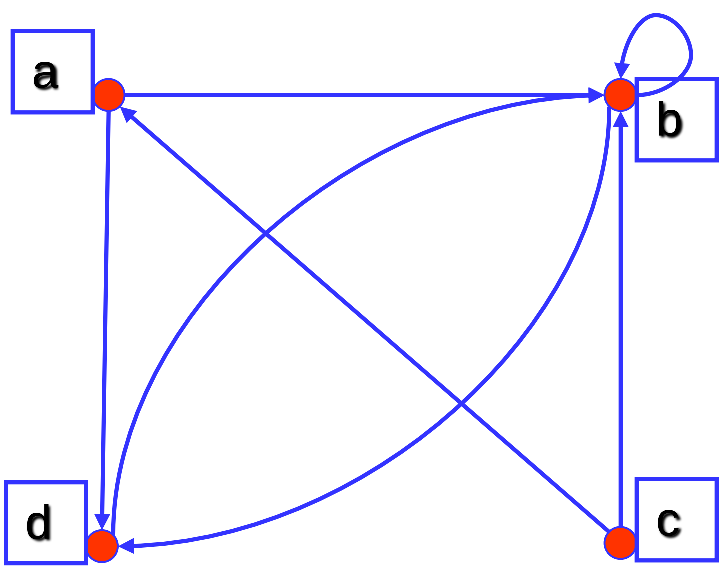
A **directed graph**, or **digraph**, consists of a set V of **vertices** (or **nodes**) together with a set E of ordered pairs of elements of V called **edges** (or **arcs**).

The vertex a is called the **initial vertex** of the edge (a, b), and the vertex b is called the **terminal vertex** of this edge.

We can use arrows to display graphs.

Representing Relations Using Digraphs

Example: Display the digraph with V = {a, b, c, d} and
E = {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)}.



An edge of the form (b, b) is called a **loop.**

Representing Relations Using Digraphs

Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows:

$$\text{For all } x, y \in A, \ \ x\,R\,y \Leftrightarrow 2 \mid (x - y).$$

Draw the directed graph of R.

Note that $3\,R\,3$ because $3 - 3 = 0$ and $2 \mid 0$ since $0 = 2 \cdot 0$. Thus there is a loop from 3 to itself. Similarly, there is a loop from 4 to itself, from 5 to itself, and so forth, since the difference of each integer with itself is 0, and $2 \mid 0$.

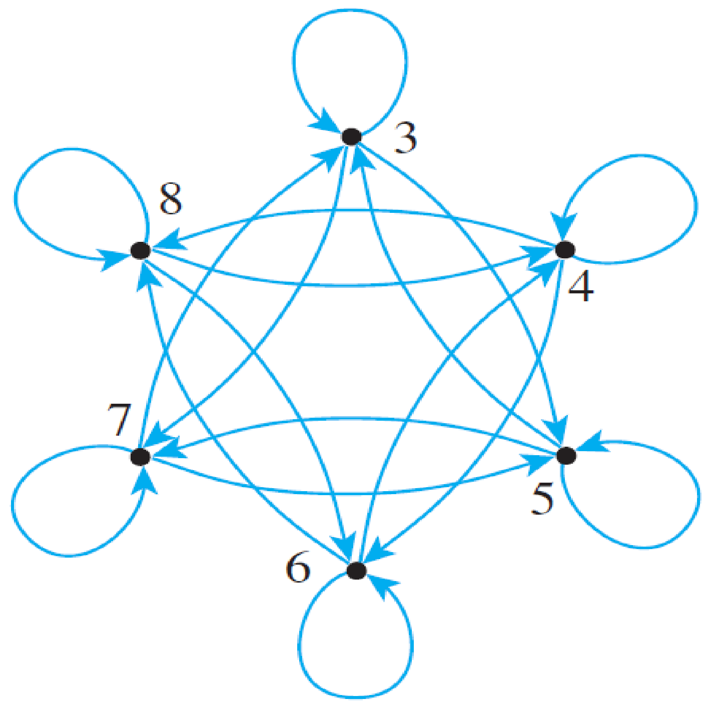
Note also that $3\,R\,5$ because $3 - 5 = -2 = 2 \cdot (-1)$. And $5\,R\,3$ because $5 - 3 = 2 = 2 \cdot 1$. Hence there is an arrow from 3 to 5 and also an arrow from 5 to 3. The other arrows in the directed graph, are obtained by similar reasoning.

Directed Graph of a Relation

Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows:

$$\text{For all } x, y \in A,\ \ x\, R\, y \Leftrightarrow 2 \mid (x - y).$$

Draw the directed graph of R.



N-*ary Relations*

A binary relation is a subset of a Cartesian product of two sets, similarly, an *n-ary* relation is a subset of a Cartesian product of *n* sets.

Definition

Given sets $A_1, A_2, . . . , A_n$, an ***n*-ary relation** R on $A_1 \times A_2 \times \cdot\cdot\cdot \times A_n$ is a subset of $A_1 \times A_2 \times \cdot\cdot\cdot \times A_n$. The special cases of 2-ary, 3-ary, and 4-ary relations are called **binary, ternary,** and **quaternary relations,** respectively.

The sets A_1, A_2, …, A_n are called the **domains** of the relation, and n is called its **degree**.

N-ary Relations

Example

Let $R = \{(a, b, c) \mid a = 2b \wedge b = 2c \text{ with } a, b, c \in \mathbf{N}\}$

What is the degree of R?

The degree of R is 3, so its elements are triples.

What are its domains?

Its domains are all equal to the set of integers.

Is (2, 4, 8) in R?

No.

Is (4, 2, 1) in R?

Yes.

A Simple Database (Application)

Example: Consider a database of students, whose records are represented as 4-tuples with the fields **Student Name**, **ID Number**, **Major**, and **GPA**:

R = {(Ackermann, 231455, CS, 3.88),
(Adams, 888323, Physics, 3.45),
(John, 102147, CS, 3.79),
(Mac, 453876, Math, 3.45),
(Rao, 678543, Math, 3.90),
(Stevens, 786576, Psych, 2.99)}

Relations that represent databases are also called **tables**, since they are often displayed as tables.

Lecture Summery

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- The inverse of a relation
- Representing relations using Digraphs
- N-*ary* Relations