Discrete Structures

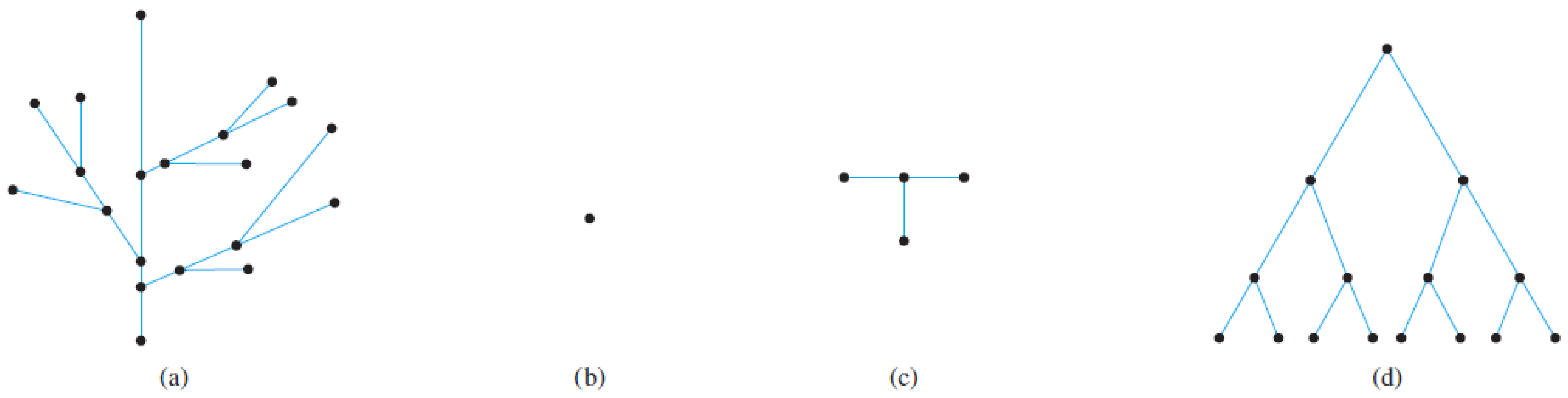
Trees

Trees

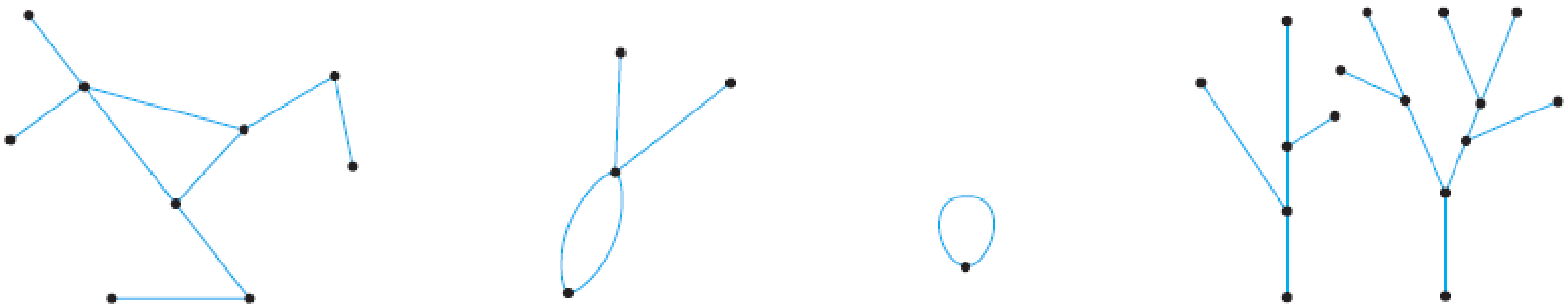
• Definition

A graph is said to be **circuit-free** if, and only if, it has no circuits. A graph is called a **tree** if, and only if, it is circuit-free and connected. A **trivial tree** is a graph that consists of a single vertex. A graph is called a **forest** if, and only if, it is circuit-free and not connected.

Trees and Non Trees



Trees and Non Trees

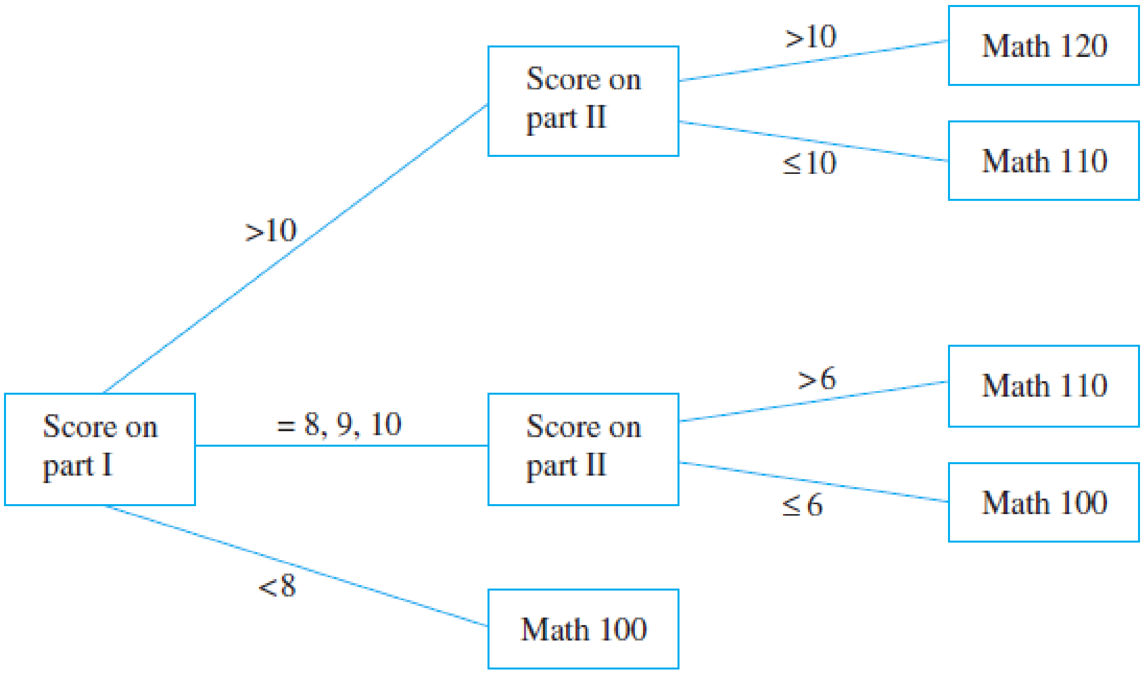


(a) (b) (c) (d)

Examples of Trees

A Decision Tree

During orientation week, a college administers an exam to all entering students to determine placement in the mathematics curriculum. The exam consists of two parts, and placement recommendations are made as indicated by the tree shown in figure. Read the tree from left to right to decide what course should be recommended for a student who scored 9 on part I and 7 on part II.



Characterizing Trees

Theorems

- Any tree that has more than one vertex has at least one vertex of degree 1.
- For any positive integer n, any tree with n vertices has $n - 1$ edges.
- For any positive integer n, if G is a connected graph with n vertices and $n - 1$ edges, then G is a tree.

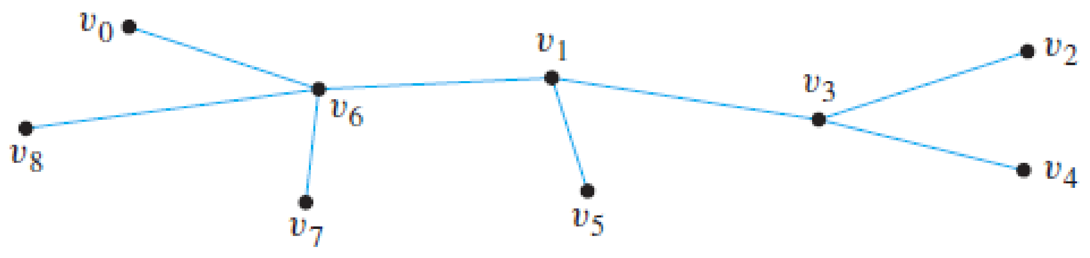
Characterizing Trees

• Definition

Let T be a tree. If T has only one or two vertices, then each is called a **terminal vertex.** If T has at least three vertices, then a vertex of degree 1 in T is called a **terminal vertex** (or a **leaf**), and a vertex of degree greater than 1 in T is called an **internal vertex** (or a **branch vertex**).

Example

Find all terminal vertices and all internal vertices in the following tree:



The terminal vertices are v_0,v_2,v_4,v_5,v_7, and v_8. The internal vertices are v_6,v_1, and v_3.

Characterizing Trees

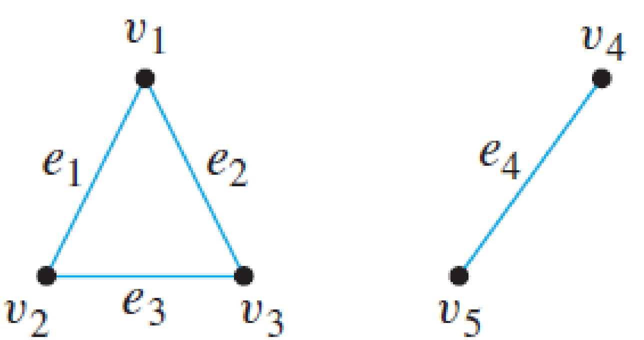
Example

A graph G has ten vertices and twelve edges. Is it a tree?

No. By Theorem, any tree with ten vertices has nine edges, not twelve.

Example

A Graph with n Vertices and n − 1 Edges That Is Not a Tree.
By Theorem, such a graph cannot be connected. One example of such an unconnected graph is shown below.



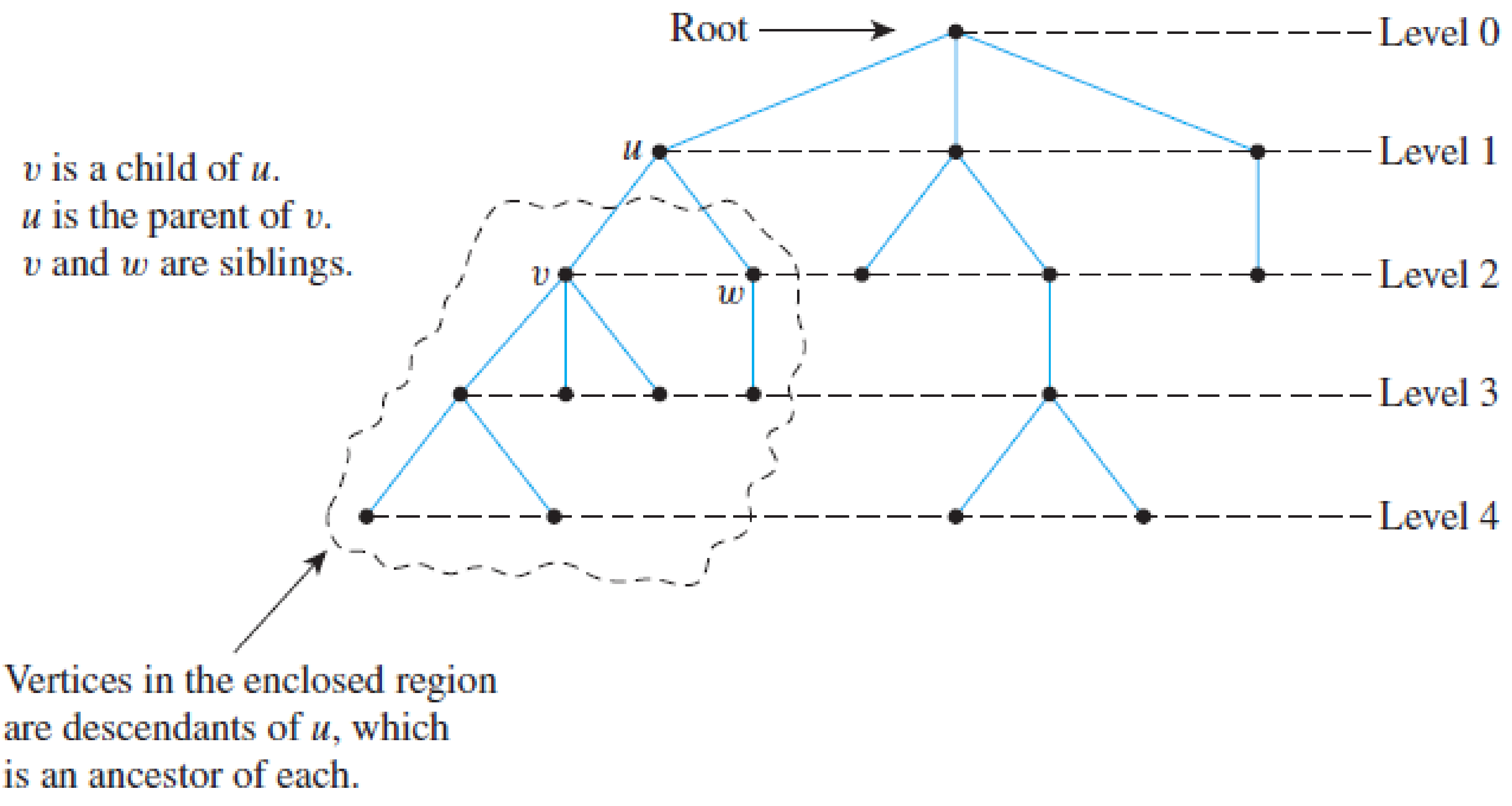
Rooted Trees

- **Definition**

A **rooted tree** is a tree in which there is one vertex that is distinguished from the others and is called the **root.** The **level** of a vertex is the number of edges along the unique path between it and the root. The **height** of a rooted tree is the maximum level of any vertex of the tree. Given the root or any internal vertex v of a rooted tree, the **children** of v are all those vertices that are adjacent to v and are one level farther away from the root than v. If w is a child of v, then v is called the **parent** of w, and two distinct vertices that are both children of the same parent are called **siblings.** Given two distinct vertices v and w, if v lies on the unique path between w and the root, then v is an **ancestor** of w and w is a **descendant** of v.

Rooted Trees

These terms are illustrated in the following Figure of Rooted Tree



Rooted Trees

Example

Consider the tree with root v_0 shown below.

a. What is the level of v_5?

b. What is the level of v_0?

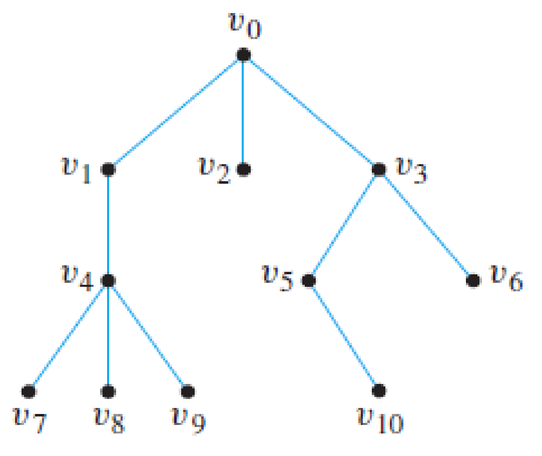
c. What is the height of this rooted tree?

d. What are the children of v_3?

e. What is the parent of v_2?

f. What are the siblings of v_8?

g. What are the descendants of v_3?



a. 2 b. 0 c. 3 d. v_5 and v_6 e. v_0 f. v_7 and v_9 g. v_5, v_6, v_{10}

Binary Trees

• Definition

A **binary tree** is a rooted tree in which every parent has at most two children. Each child in a binary tree is designated either a **left child** or a **right child** (but not both), and every parent has at most one left child and one right child. A **full binary tree** is a binary tree in which each parent has exactly two children.

Given any parent v in a binary tree T, if v has a left child, then the **left subtree** of v is the binary tree whose root is the left child of v, whose vertices consist of the left child of v and all its descendants, and whose edges consist of all those edges of T that connect the vertices of the left subtree. The **right subtree** of v is defined analogously.