Discrete Structures

Today's Lecture

- Properties of relations
- Reflexive, Symmetric and Transitive Relations
- Properties of "Less than" relations
- Properties of Congruence Modulo 3
- Transitive closure of a relations

Representing Relations Using Digraphs

Definition

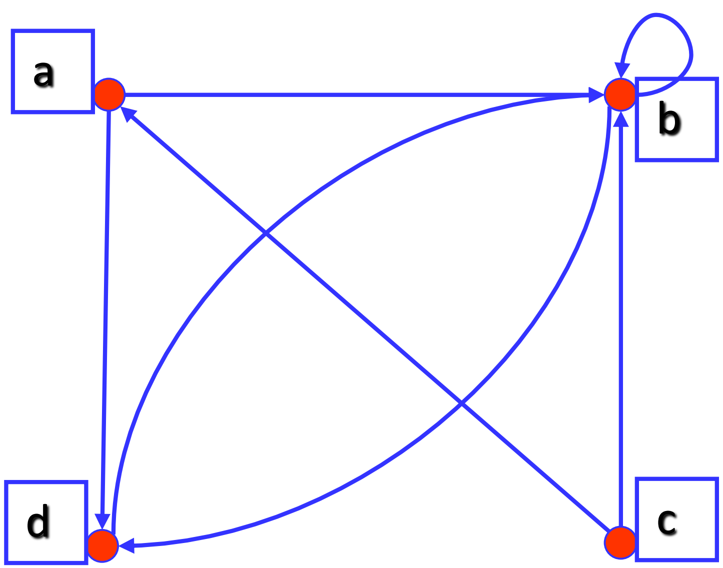
A **directed graph**, or **digraph**, consists of a set V of **vertices** (or **nodes**) together with a set E of ordered pairs of elements of V called **edges** (or **arcs**).

The vertex a is called the **initial vertex** of the edge (a, b), and the vertex b is called the **terminal vertex** of this edge.

We can use arrows to display graphs.

Representing Relations Using Digraphs

Example: Display the digraph with V = {a, b, c, d} and E = {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)}.



An edge of the form (b, b) is called a **loop.**

Representing Relations Using Digraphs

Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows:

$$\text{For all } x, y \in A,\ \ x\, R\, y \Leftrightarrow 2 \mid (x - y).$$

Draw the directed graph of R.

Note that $3\ R\ 3$ because $3 - 3 = 0$ and $2 \mid 0$ since $0 = 2 \cdot 0$. Thus there is a loop from 3 to itself. Similarly, there is a loop from 4 to itself, from 5 to itself, and so forth, since the difference of each integer with itself is 0, and $2 \mid 0$.

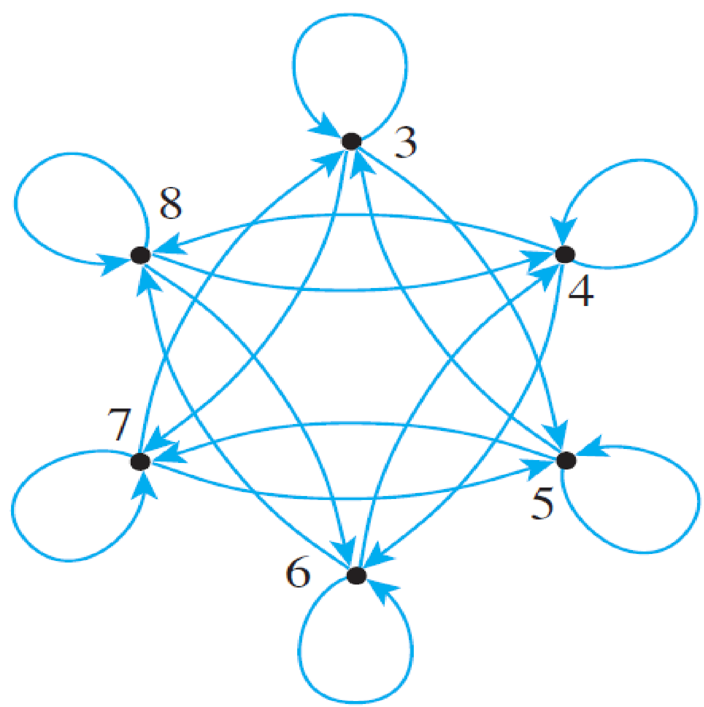
Note also that $3\ R\ 5$ because $3 - 5 = -2 = 2 \cdot (-1)$. And $5\ R\ 3$ because $5 - 3 = 2 = 2 \cdot 1$. Hence there is an arrow from 3 to 5 and also an arrow from 5 to 3. The other arrows in the directed graph, are obtained by similar reasoning.

Directed Graph of a Relation

Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows:

$$\text{For all } x, y \in A,\ x\,R\,y \Leftrightarrow 2 \mid (x - y).$$

Draw the directed graph of R.



Properties of Relations

We will now look at some useful ways to classify relations.

Definition: A relation R on a set A is called **reflexive,** if $(a, a) \in R$ for every element $a \in A$.

Are the following relations on {1, 2, 3, 4} reflexive?

- $R = \{(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)\}$ No
- $R = \{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$ Yes
- $R = \{(1, 1), (2, 2), (3, 3)\}$ No

A relation on a set A is not **reflexive** if there exist some element $a \in A$ such that $(a, a) \notin R$.

Properties of Relations

Definitions

- A relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.
- A relation R on a set A is called **antisymmetric** if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$.
- A relation R on a set A is called **asymmetric** if $(a, b) \in R$ implies that $(b, a) \notin R$ for all $a, b \in A$.

Properties of Relations

Are the following relations on {1, 2, 3, 4} symmetric, antisymmetric, or asymmetric?

- R = {(1, 1), (1, 2), (2, 1), (3, 3), (4, 4)} Symmetric
- R = {(1, 1)} Antisymmetric.
- R = {(1, 3), (3, 2), (2, 1)} Asymmetric.
- R = {(4, 4), (3, 3), (1, 4)} Antisymmetric.

Properties of Relations

Definition: A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

Are the following relations on $\{1, 2, 3, 4\}$ transitive?

- $R = \{(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)\}$ Yes
- $R = \{(1, 3), (3, 2), (2, 1)\}$ No
- $R = \{(2, 4), (4, 3), (2, 3), (4, 1)\}$ No

Properties of Relations

Let $A = \{0, 1, 2, 3\}$ and define relations R, S, and T on A as follows:

- $R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$,
- $S = \{(0, 0), (0, 2), (0, 3), (2, 3)\}$,
- $T = \{(0, 1), (2, 3)\}$.

a. Is R reflexive? symmetric? transitive?

b. Is S reflexive? symmetric? transitive?

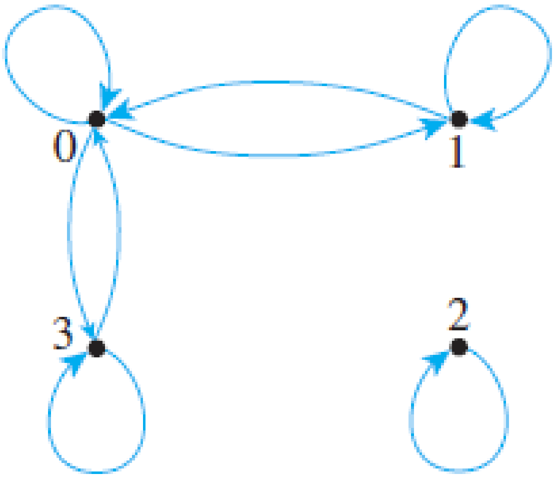
c. Is T reflexive? symmetric? transitive?

Properties of Relations

Let $A = \{0, 1, 2, 3\}$ and define relations R, S, and T on A as follows:

- $R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\};$

The directed graph of R has the appearance shown below

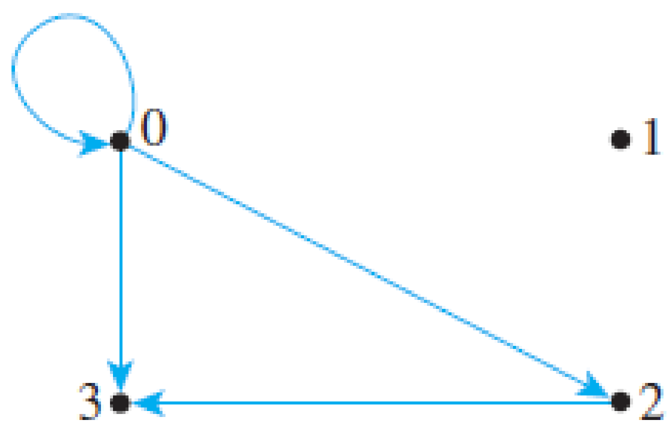


Properties of Relations

Let $A = \{0, 1, 2, 3\}$ and define relations R, S, and T on A as follows:

- $S = \{(0, 0), (0, 2), (0, 3), (2, 3)\}$;

The directed graph of S has the appearance shown below

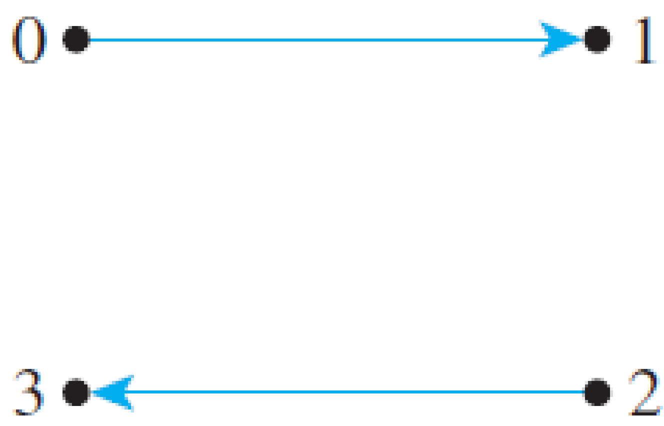


Properties of Relations

Let $A = \{0, 1, 2, 3\}$ and define relations R, S, and T on A as follows:

- $T = \{(0, 1), (2, 3)\}$.

The directed graph of T has the appearance shown below



Properties of Relations

How many different reflexive relations can be defined on a set A containing n elements?

Solution: Relations on R are subsets of $A \times A$, which contains n^2 elements.

- Therefore, different relations on A can be generated by choosing different subsets out of these n^2 elements, so there are 2^{n^2} relations.
- A **reflexive** relation, however, **must** contain the n elements (a, a) for every $a \in A$.
- Consequently, we can only choose among $n^2 - n = n(n - 1)$ elements to generate reflexive relations, so there are $2^{n(n-1)}$ of them.

Properties of "Less Than" relation

Define a relation R on **R** (the set of all real numbers) as follows:

For all $x, y \in R$, $x\ R\ y \Leftrightarrow x < y$.

a. Is R reflexive? b. Is R symmetric? c. Is R transitive?

***R is not reflexive*:**

R is reflexive if, and only if, $\forall x \in \mathbf{R}, x\ R\ x$. By definition of R, this means that $\forall x \in \mathbf{R}$, $x < x$. But this is false: $\exists x \in \mathbf{R}$ such that $x \not< x$.

As a counterexample, let $x = 0$ and note that $0 \not< 0$. Hence R is not reflexive.

Properties of "Less Than" relation

Define a relation R on **R** (the set of all real numbers) as follows:

For all $x, y \in R$, $x \, R \, y \Leftrightarrow x < y$.

a. Is R reflexive? b. Is R symmetric? c. Is R transitive?

***R is not symmetric*:**

R is symmetric if, and only if, $\forall x, y \in \mathbf{R}$, if $x \, R \, y$ then $y \, R \, x$.

By definition of R, this means that $\forall x, y \in \mathbf{R}$, if $x < y$ then $y < x$. But this is false: $\exists x, y \in \mathbf{R}$ such that $x < y$ and $y \not< x$. As a counterexample, let $x = 0$ and $y = 1$ and note that $0 < 1$ but $1 \not< 0$. Hence R is not symmetric.

Properties of "Less Than" relation

Define a relation R on **R** (the set of all real numbers) as follows:

For all $x, y \in R$, $x \, R \, y \Leftrightarrow x < y$.

a. Is R reflexive? b. Is R symmetric? c. Is R transitive?

***R is not transitive*:**

R is transitive if, and only if, for all $x, y, z \in$ **R**, if $x \, R \, y$ and $y \, R \, z$ then $x \, R \, z$.

By definition of R, this means that for all $x, y, z \in$ **R**, if $x < y$ and $y < z$, then $x < z$. But this statement is true by the transitive law of order for real numbers. Hence R is transitive.

Properties of Congruence Modulo 3

Define a relation T on **Z** (the set of all integers) as follows:

For all integers m and n, $m \, T \, n \Leftrightarrow 3 \mid (m - n)$.

This relation is called **congruence modulo 3.**

a. Is T reflexive? b. Is T symmetric? c. Is T transitive?

T is Reflexive

Suppose m is a particular but arbitrarily chosen integer.

[We must show that m T m.] Now $m - m = 0$. But $3 \mid 0$ since $0 = 3 \cdot 0$.

Hence $3 \mid (m - m)$. Thus, by definition of T, $m \, T \, m$.

Hence T is reflexive.

Properties of Congruence Modulo 3

Define a relation T on **Z** (the set of all integers) as follows:

For all integers m and n, $m\ T\ n \Leftrightarrow 3 \mid (m - n)$.

This relation is called **congruence modulo 3.**

a. Is T reflexive? b. Is T symmetric? c. Is T transitive?

T is Symmetric

Suppose m and n are particular but arbitrarily chosen integers that satisfy the condition $m\ T\ n$. *[We must show that $n\ T\ m$.]* By definition of T , since $m\ T\ n$ then $3 \mid (m - n)$. By definition of "divides," this means that $m - n = 3k$, for some integer k. Multiplying both sides by -1 gives $n - m = 3(-k)$. Since $-k$ is an integer, this equation shows that $3 \mid (n - m)$. Hence, by definition of T, $n\ T\ m$.

Properties of Congruence Modulo 3

Define a relation T on **Z** (the set of all integers) as follows:

For all integers m and n, $m\ T\ n \Leftrightarrow 3 \mid (m - n)$.

This relation is called **congruence modulo 3.**

a. Is T reflexive? b. Is T symmetric? c. Is T transitive?

T is Transitive

Suppose m, n, and p are particular but arbitrarily chosen integers that satisfy the condition $m\ T\ n$ and $n\ T\ p$. *[We must show that $m\ T\ p$.]* By definition of T , since $m\ T\ n$ and $n\ T\ p$, then $3 \mid (m - n)$ and $3 \mid (n - p)$. By definition of "divides," this means that $m - n = 3r$ and $n - p = 3s$, for some integers r and s. Adding the two equations gives $(m - n) + (n - p) = 3r + 3s$, and simplifying gives that $m - p = 3(r + s)$. Since $r + s$ is an integer, this equation shows that $3 \mid (m - p)$. Hence, by definition of T , $m\ T\ p$.

Combining Relations

- Relations are sets, and therefore, we can apply the usual **set operations** to them.
- If we have two relations R_1 and R_2, and both of them are from a set A to a set B, then we can combine them to $R_1 \cup R_2$, $R_1 \cap R_2$, or $R_1 - R_2$.
- In each case, the result will be **another relation from A to B**.

Combining Relations

There is another important way to combine relations.

Definition: Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite** of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $\mathbf{S \circ R}$.

In other words, if relation R contains a pair (a, b) and relation S contains a pair (b, c), then $S \circ R$ contains a pair (a, c).

Combining Relations

Example: Let D and S be relations on A = {1, 2, 3, 4}.

- D = {(a, b) | b = 5 - a}   "b equals (5 – a)"
- S = {(a, b) | a < b}   "a is smaller than b"

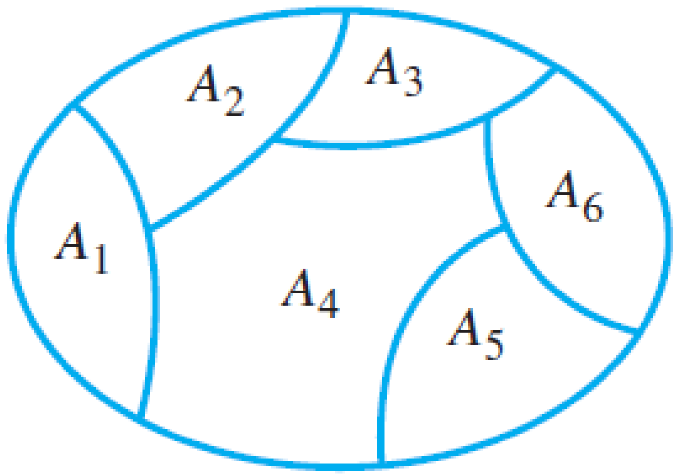
- D = {(1, 4), (2, 3), (3, 2), (4, 1)}
- S = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}
- S∘D = { (2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)}

- D maps an element a to the element (5 – a), and afterwards S maps (5 – a) to all elements larger than (5 – a), resulting in

  S∘D = {(a,b) | 5 – a < b} or **S∘D = {(a,b) | a + b > 5}.**

The Relation Induced by a Partition

A **partition** of a set A is a finite or infinite collection of nonempty, mutually disjoint subsets whose union is A. A partition of a set A by subsets $A_1, A_2, \ldots, A_6$.



$A_i \cap A_j = \emptyset$, whenever $i \neq j$

$A_i \cup A_2 \cup \cdots \cup A_6 = A$

The Relation Induced by a Partition

Definition

Given a partition of a set A, the **relation induced by the partition,** R, is defined on A as follows:

For all $x, y \in A$, $x\ R\ y \Leftrightarrow$ there is a subset A_i of the partition such that both x and y are in A_i .

Example

Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A:

$$\{0, 3, 4\}, \{1\}, \{2\}.$$

Find the relation R induced by this partition.

Solution Since $\{0, 3, 4\}$ is a subset of the partition,

$0 \, R \, 3$ because both 0 and 3 are in $\{0, 3, 4\}$,
$3 \, R \, 0$ because both 3 and 0 are in $\{0, 3, 4\}$,
$0 \, R \, 4$ because both 0 and 4 are in $\{0, 3, 4\}$,
$4 \, R \, 0$ because both 4 and 0 are in $\{0, 3, 4\}$,
$3 \, R \, 4$ because both 3 and 4 are in $\{0, 3, 4\}$,
$4 \, R \, 3$ because both 4 and 3 are in $\{0, 3, 4\}$,

Also, $0 \, R \, 0$ because both 0 and 0 are in $\{0, 3, 4\}$,
$3 \, R \, 3$ because both 3 and 3 are in $\{0, 3, 4\}$,
$4 \, R \, 4$ because both 4 and 4 are in $\{0, 3, 4\}$.

Example (Contd.)

Since {1} is a subset of the partition,

1 *R* 1 because both 1 and 1 are in {1},

and since {2} is a subset of the partition,

2 *R* 2 because both 2 and 2 are in {2}.

Hence

$R = \{ (0, 0), (0, 3), (0, 4), (1, 1), (2, 2), (3, 0), (3, 3), (3, 4), (4, 0), (4, 3), (4, 4) \}.$

Lecture Summery

- Properties of relations
- Reflexive, Symmetric and Transitive Relations
- Properties of “Less than” relations
- Properties of Congruence Modulo 3
- Transitive closure of a relations