Discrete Structures

Functions

Today’s Lecture

- ❖ Relations and Functions
- ❖ Definition of Function
- ❖ Examples of Functions
- ❖ One-to-One Function
- ❖ Onto Function
- ❖ Bijective Function (One-to-One correspondence)
- ❖ Inverse Functions

Relations

If we want to describe a relationship between elements of two sets A and B, we can use **ordered pairs** with their first element taken from A and their second element taken from B.
Since this is a relation between **two sets**, it is called a **binary relation**.

Definition: Let A and B be sets. A binary relation R from A to B is a subset of $A \times B$.

In other words, for a binary relation R we have $R \subseteq A \times B$. We use the notation aRb to denote that $(a, b) \in R$ and $a\underline{R}b$ to denote that $(a, b) \notin R$.

Relations

If we have two sets $A = \{1,2,3,4,5\}$ and $B = \{5,6,7,8,9\}$

The cartesian product of A and B is

$$
\begin{aligned}
A \times B = \{ &(1,5), (1,6), (1,7), (1,8), (1,9),\\
&(2,5), (2,6), (2,7), (2,8), (2,9),\\
&(3,5), (3,6), (3,7), (3,8), (3,9),\\
&(4,5), (4,6), (4,7), (4,8), (4,9),\\
&(5,5), (5,6), (5,7), (5,8), (5,9) \}.
\end{aligned}
$$

The rule is to add 4:

$$R = \{ (1,5), (2,6), (3,7), (4,8), (5,9) \}.$$

The **domain** is the set of all values which are first members of the ordered pairs in the relation, i.e.,

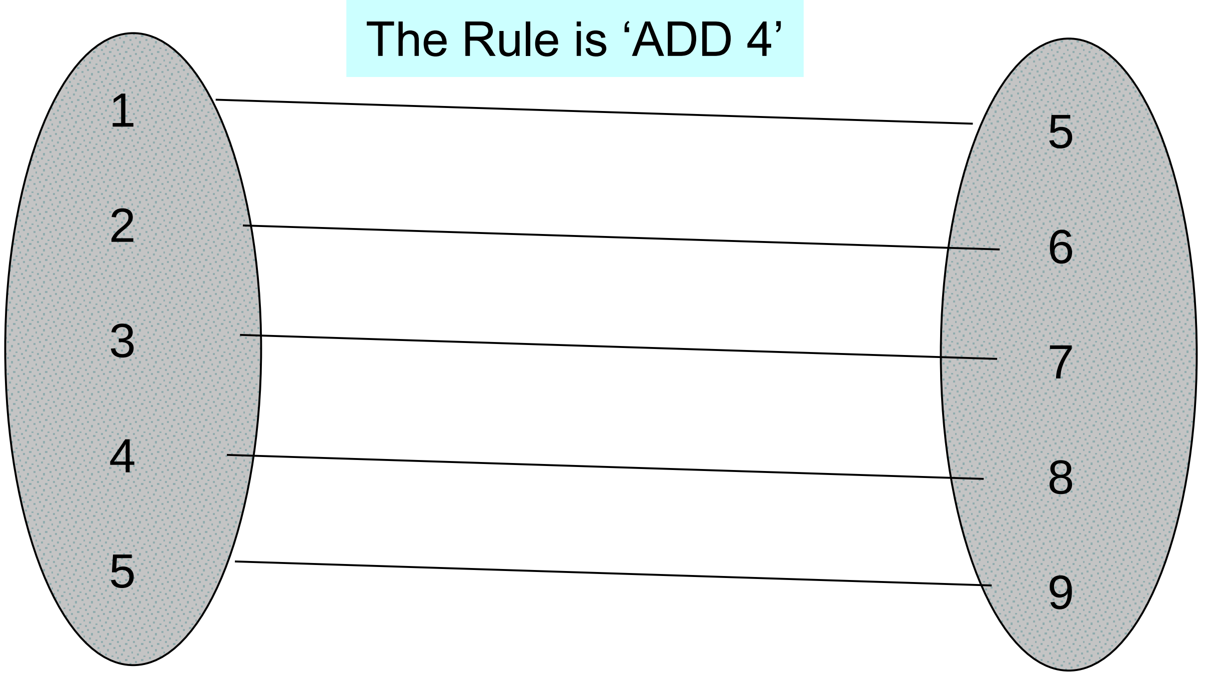
$$\text{Dom}(R) = \{1, 2, 3, 4, 5\}$$

The **range** is the set of all values which are second members of the ordered pairs in the relation, i.e.,

$$\text{Range}(R) = \{5, 6, 7, 8, 9\}$$

Relations

One to One Relations

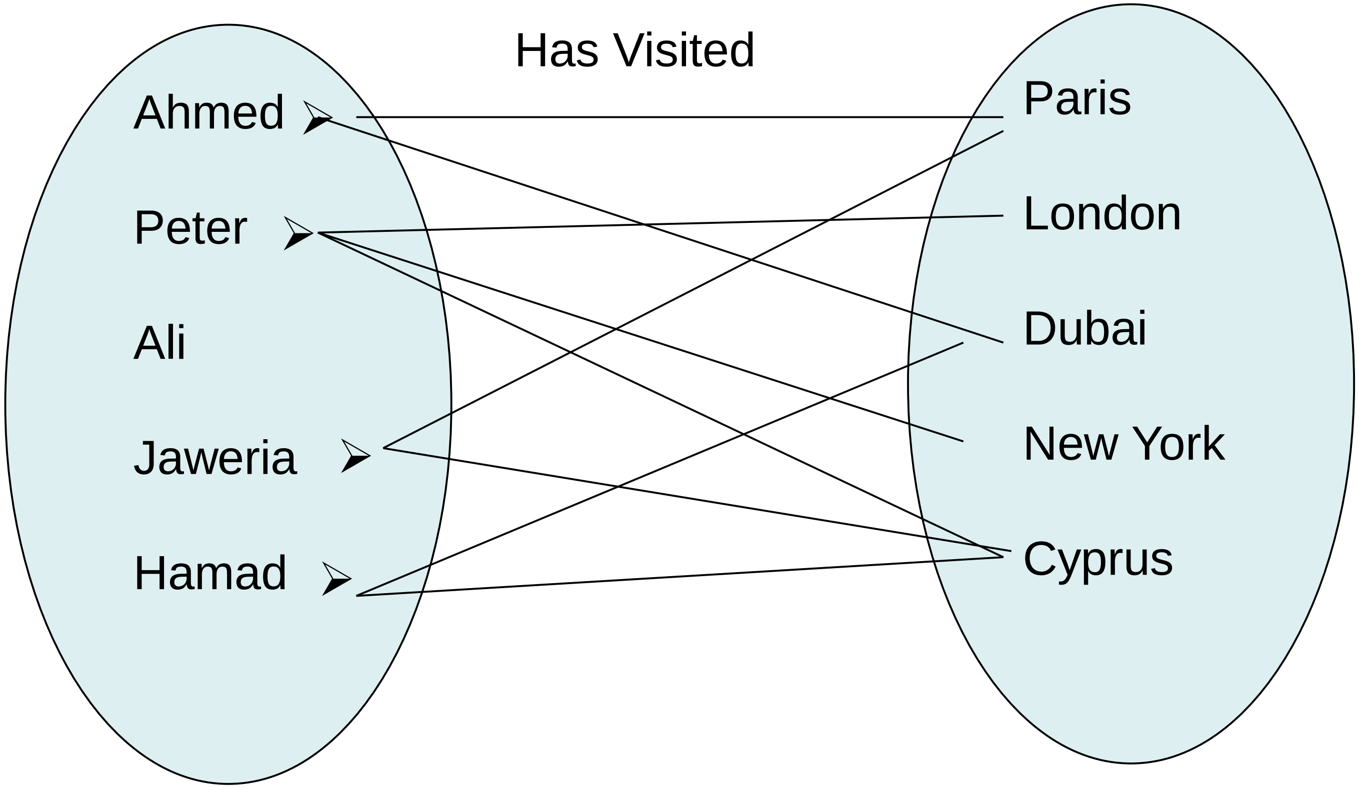


Dom (R) = {1, 2, 3, 4, 5}

Range (R) = {5, 6, 7, 8, 9}

Relations

Many to Many relation

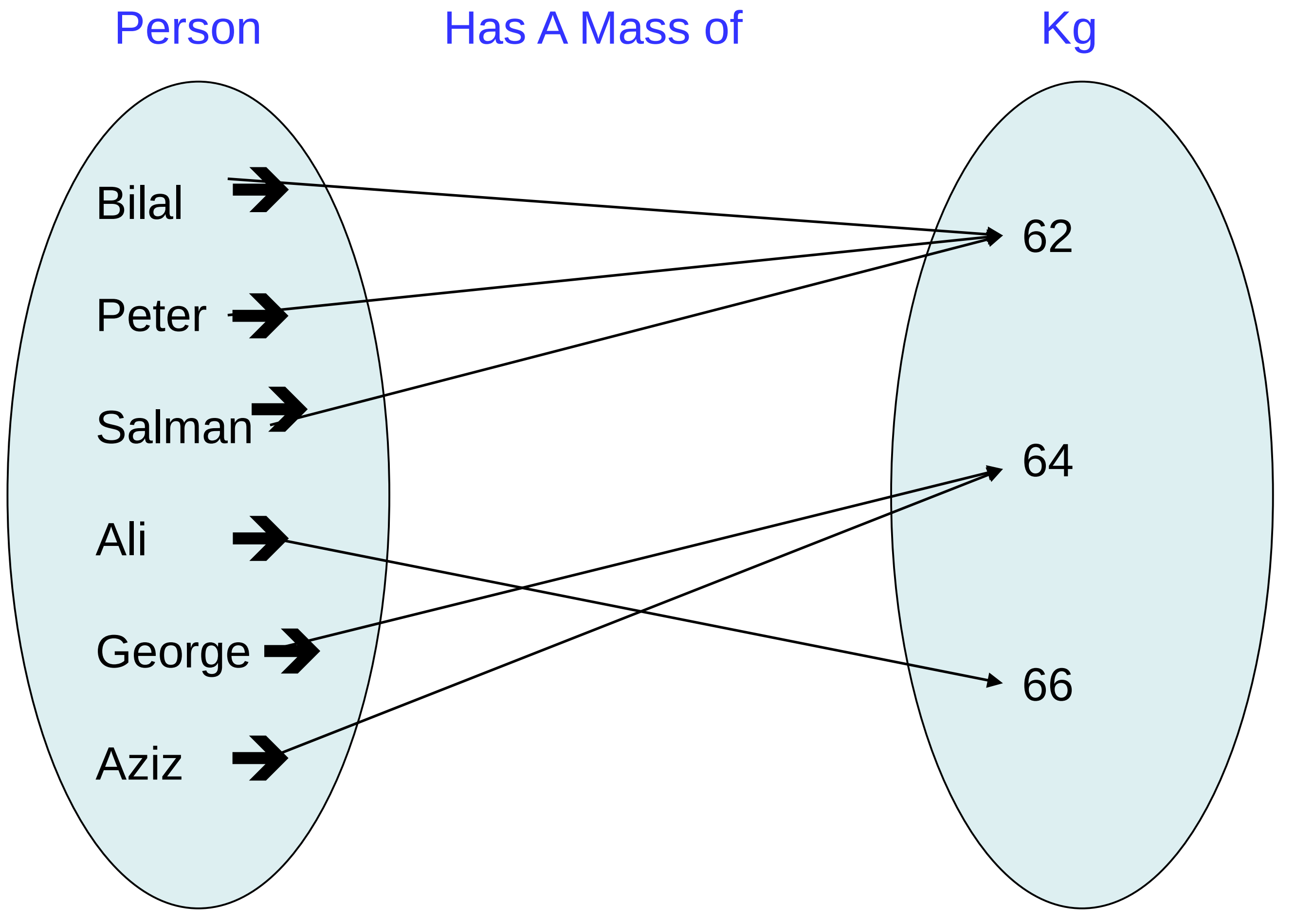


Dom (R) = {Ahmad, Peter, Jaweria, Hamad}

Range (R) = {Paris, London, Dubai, New York, Cyprus}

Note That: Ali is not in the Domain

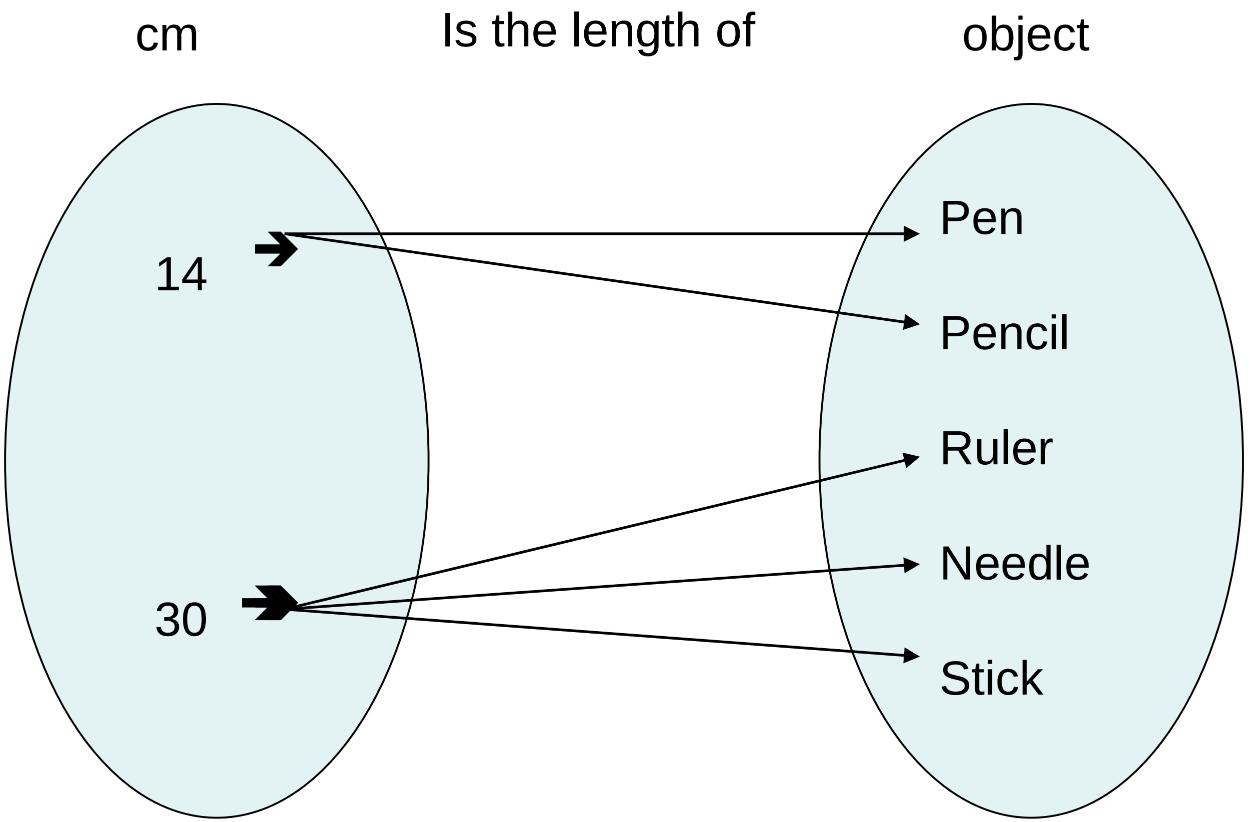
Relations (Many to One relation)



Dom (R) = {Bilal, Peter, Salman, Ali, George,Aziz}

Range (R) = {62, 64, 66}

Relations (One to Many relation)

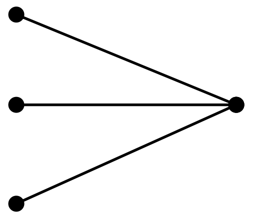


Range (R) = {14, 30}

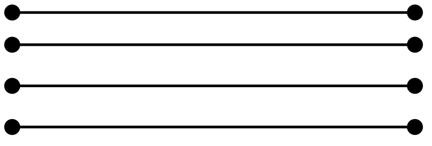
Range (R) = {Pen, Pencil, Ruler, Needle, Stick}

Relations

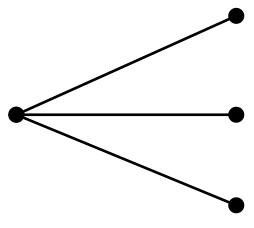
Many to One Relationship



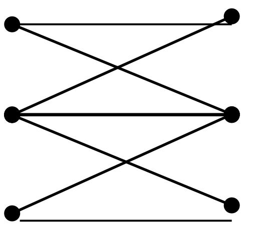
One to One Relationship



One to Many Relationship

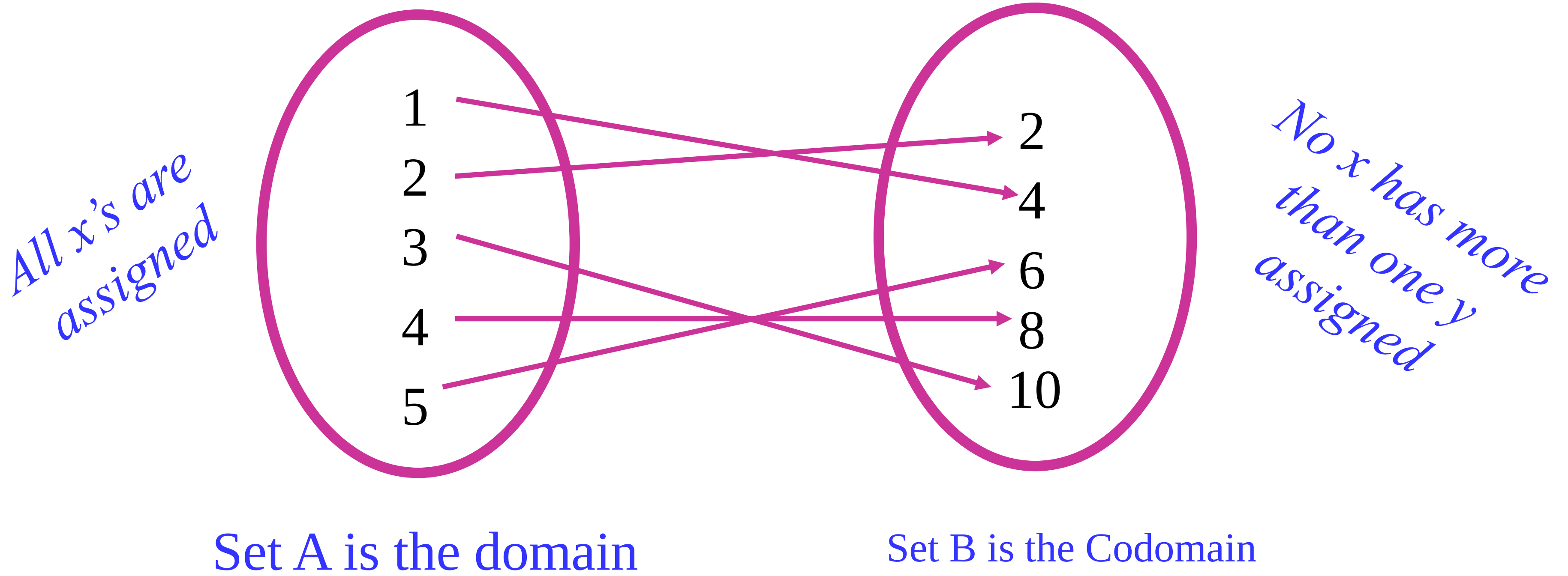


Many to Many Relationship



Function

A function *f* from set A to set B is a relation (rule of correspondence) that assigns each element *x* in the set A to exactly one element *y* in the set B.



1). Must use all the x's in A.

2). The x value can only be assigned to one y in B.

Function (Definition)

• Definition

A function f **from a set** X **to a set** Y, denoted $f\colon X \to Y$, is a relation from X, the **domain**, to Y, the **co-domain**, that satisfies two properties: (1) every element in X is related to some element in Y, and (2) no element in X is related to more than one element in Y. Thus, given any element x in X, there is a unique element in Y that is related to x by f. If we call this element y, then we say that "f sends x to y" or "f maps x to y" and write $x \xrightarrow{f} y$ or $f\colon x \to y$. The unique element to which f sends x is denoted

$f(x)$ and is called **f of x**, or
the output of f for the input x, or
the value of f at x, or
the image of x under f.

The set of all values of f taken together is called the *range of f* or the *image of X under f*. Symbolically,

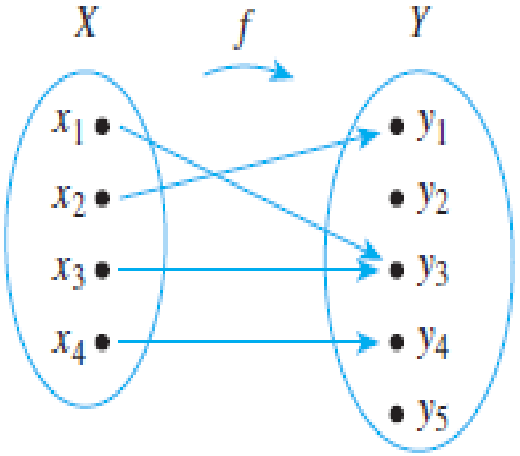
$$\textbf{range of } f = \textbf{image of } X \textbf{ under } f = \{y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}.$$

Function (Arrow diagrams)

If *X* and *Y* are finite sets, you can define a function *f* from *X* to *Y* by drawing an arrow diagram. You make a list of elements in *X* and a list of elements in *Y*, and draw an arrow from each element in *X* to the corresponding element in *Y.*

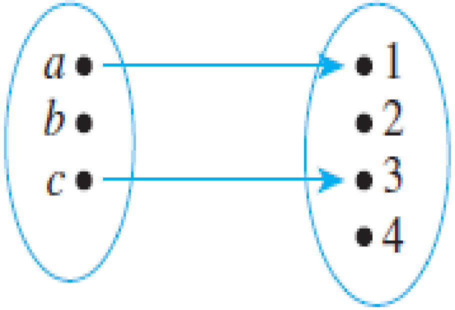
This arrow diagram does define a function because

1. Every element of *X* has an arrow coming out of it.
2. No element of *X* has two arrows coming out of it that point to two different elements of *Y*.

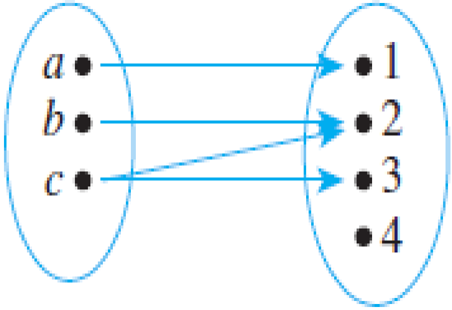


Functions and non functions

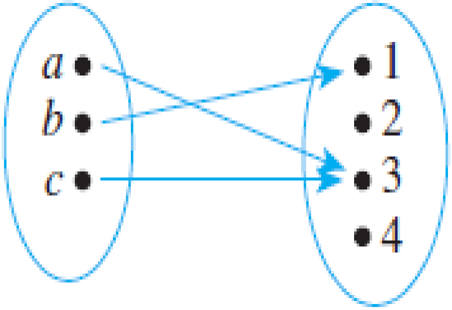
Which of the arrow diagrams define functions from $X = \{a, b, c\}$ to $Y = \{1, 2, 3, 4\}$?



(a)



(b)



(c)

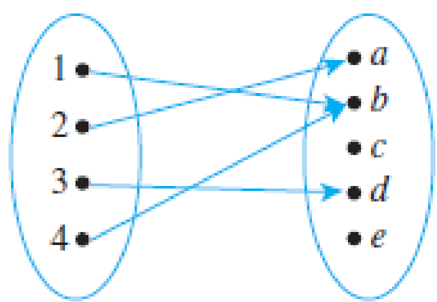
Function (Inverse Image)

Definition

Given an element y in Y, there may exist elements in X with y as their image. If $f(x) = y$, then x is called **a preimage of** y or **an inverse image of** y. The set of all inverse images of y is called *the inverse image of* y. Symbolically,

$$\textbf{the inverse image of } y = \{x \in X \mid f(x) = y\}.$$

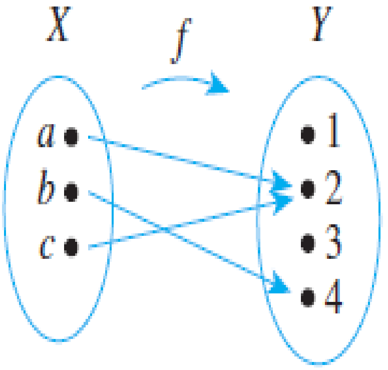
Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e\}$, and define $F\colon\ X \to Y$ by the following arrow diagram:



Let $A = \{1, 4\}$, $C = \{a, b\}$, and $D = \{c, e\}$. Find $F(A)$, $F(X)$, $F^{-1}(C)$, and $F^{-1}(D)$.

Functions defined by Arrow diagrams

Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Define a function f from X to Y by the arrow diagram in Figure 7.1.3.



a. Write the domain and co-domain of f.

b. Find $f(a)$, $f(b)$, and $f(c)$.

c. What is the range of f?

d. Is c an inverse image of 2? Is b an inverse image of 3?

e. Find the inverse images of 2, 4, and 1.

f. Represent f as a set of ordered pairs.

Examples of Functions

The Identity Function on a Set

Given a set X, define a function I_X from X to X by

$$I_X(x) = x, \quad \text{for all } x \text{ in } X.$$

The function I_X is called the **identity function on *X*** because it sends each element of X to the element that is identical to it. Thus the identity function can be pictured as a machine that sends each piece of input directly to the output chute without changing it in any way.

Sum/difference of Functions

Let F: $\mathbf{R} \rightarrow \mathbf{R}$ and G: $\mathbf{R} \rightarrow \mathbf{R}$ be functions. Define new functions $F + G$: $\mathbf{R} \rightarrow \mathbf{R}$: For all $x \in \mathbf{R}$,

$$(F + G)(x) = F(x) + G(x)$$

F and G must have same Domains and Codomains.

Equality of Functions

Theorem: If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

Example

Let $J_3 = \{0, 1, 2\}$, and define functions f and g from J_3 to J_3 as follows: For all x in J_3,

$$f(x) = (x^2 + x + 1)\ mod\ 3 \quad \text{and} \quad g(x) = (x+2)^2\ mod\ 3.$$

Does $f = g$?

x	x^2+x+1	$f(x) = (x^2+x+1) \bmod 3$	$(x+2)^2$	$g(x) = (x+2)^2 \bmod 3$
0	1	$1 \bmod 3 = 1$	4	$4 \bmod 3 = 1$
1	3	$3 \bmod 3 = 0$	9	$9 \bmod 3 = 0$
2	7	$7 \bmod 3 = 1$	16	$16 \bmod 3 = 1$

One-to-One Functions

- **Definition**

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,

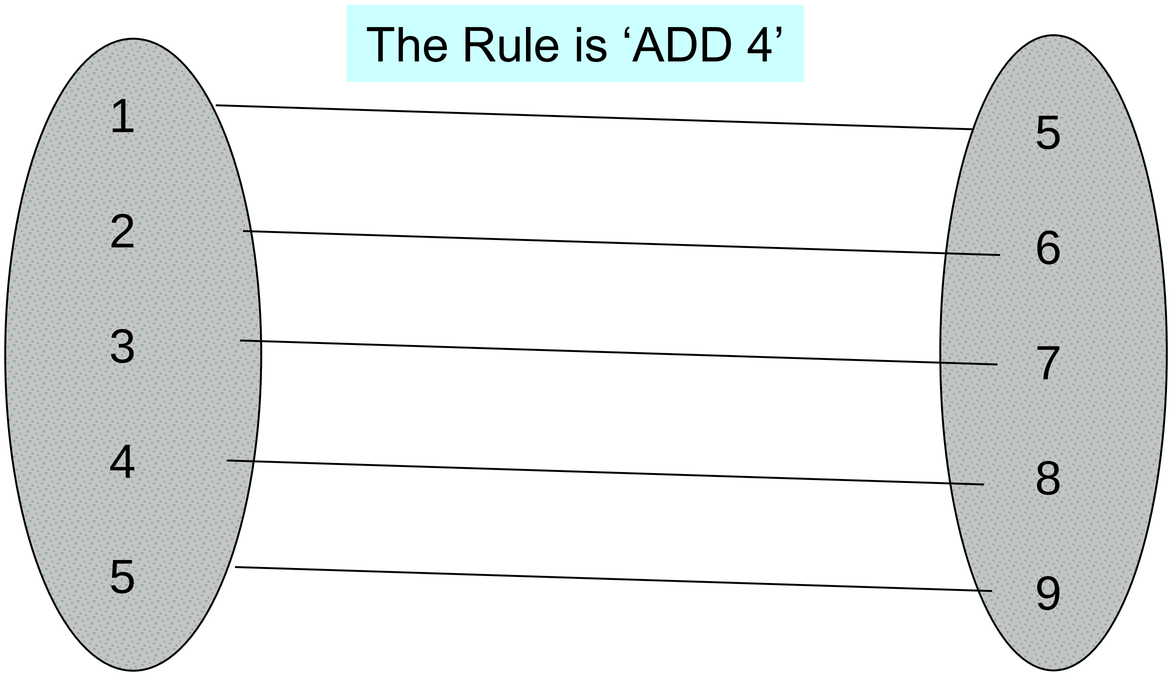
$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently, $\quad \text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$

Symbolically,

$$F\colon X \to Y \text{ is one-to-one} \quad \Leftrightarrow \quad \forall x_1, x_2 \in X, \text{ if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

One-to-One Functions

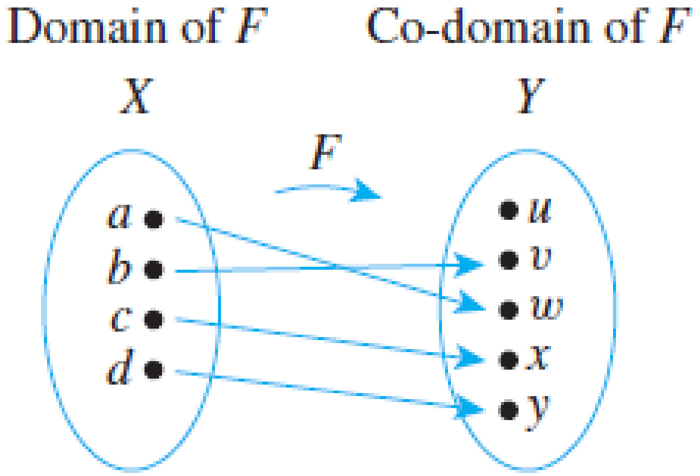


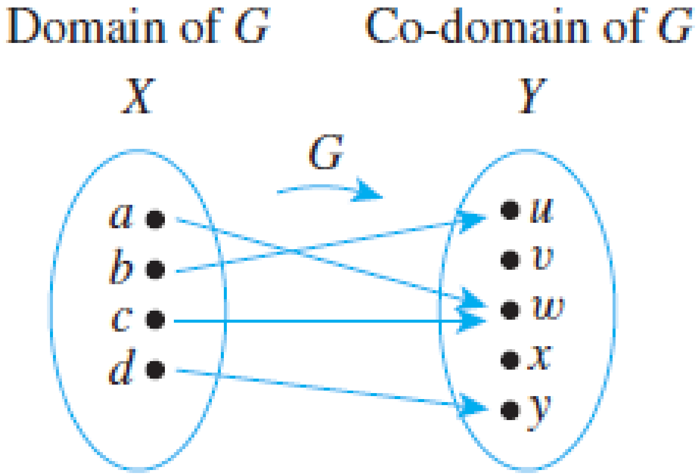
Dom (R) = {1, 2, 3, 4, 5}

Codomain(R)={5, 6, 7, 8, 9,10}

One-to-One Functions

Identifying One-to-One functions defined on sets





One-to-One Functions on Infinite Sets

Now suppose f is a function defined on an infinite set X. By definition, f is one-to-one if, and only if, the following universal statement is true:

$$\forall x_1, x_2 \in X, \text{if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

Thus, to prove f is one-to-one, you will generally use the method of direct proof:

suppose x_1 and x_2 are elements of X such that $f(x_1) = f(x_2)$

and **show** that $x_1 = x_2$.

To show that f is *not* one-to-one, you will ordinarily

find elements x_1 and x_2 in X so that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

One-to-One Functions on Infinite Sets

Define $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{Z} \to \mathbf{Z}$ by the rules

$$f(x) = 4x - 1 \quad \text{for all} \quad x \in \mathbf{R}$$

and

$$g(n) = n^2 \quad \text{for all} \quad n \in \mathbf{Z}.$$

a. Is f one-to-one? Prove or give a counterexample.

b. Is g one-to-one? Prove or give a counterexample.

One-to-One Functions on Infinite Sets

If the function $f: \mathbf{R} \to \mathbf{R}$ is defined by the rule $f(x) = 4x - 1$, for all real numbers x, then f is one-to-one.

Proof:

Suppose x_1 and x_2 are real numbers such that $f(x_1) = f(x_2)$. *[We must show that $x_1 = x_2$.]* By definition of f,

$$4x_1 - 1 = 4x_2 - 1.$$

Adding 1 to both sides gives

$$4x_1 = 4x_2,$$

and dividing both sides by 4 gives

$$x_1 = x_2,$$

which is what was to be shown.

One-to-One Functions on Infinite Sets

If the function $g: \mathbf{Z} \to \mathbf{Z}$ is defined by the rule $g(n) = n^2$, for all $n \in \mathbf{Z}$, then g is not one-to-one.

Counterexample:

Let $n_1 = 2$ and $n_2 = -2$. Then by definition of g,

$$g(n_1) = g(2) = 2^2 = 4 \quad \text{and also}$$

$$g(n_2) = g(-2) = (-2)^2 = 4.$$

Hence $$g(n_1) = g(n_2) \quad \text{but} \quad n_1 \neq n_2,$$

and so g is not one-to-one.

Onto Functions on Sets

- **Definition**

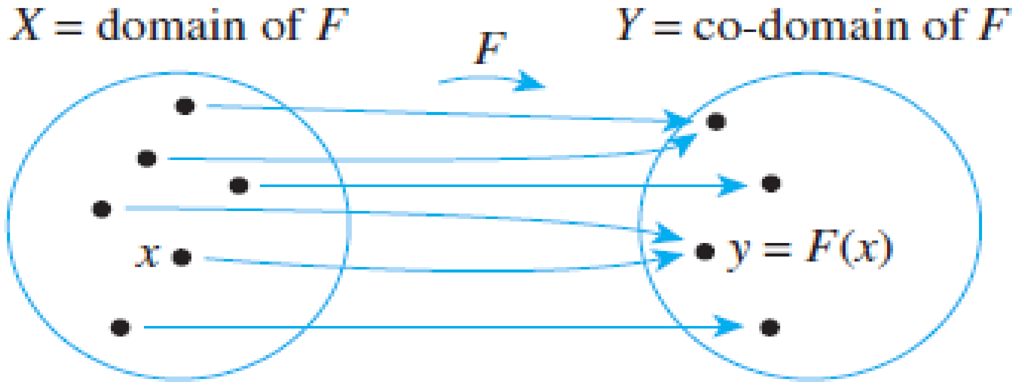
Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.

Symbolically:

$$F: X \to Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

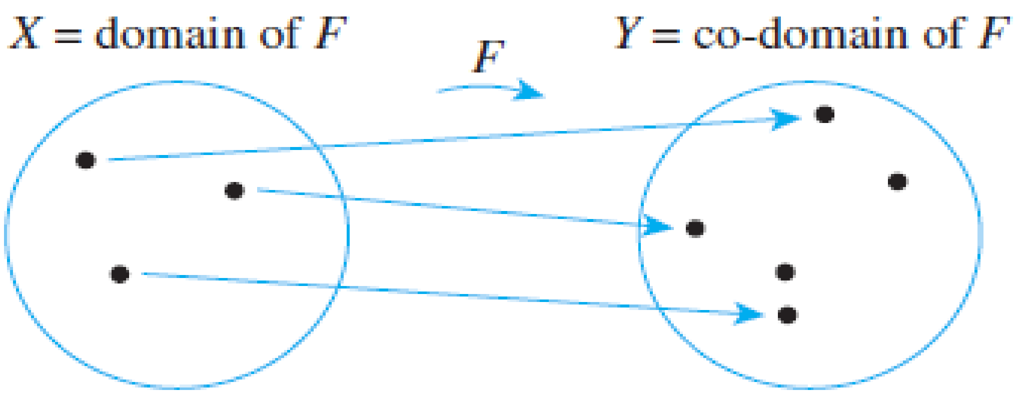
$$F: X \to Y \text{ is } \textit{not} \text{ onto} \quad \Leftrightarrow \quad \exists y \text{ in } Y \text{ such that } \forall x \in X, F(x) \neq y.$$

Onto Functions on Sets



Each element y in Y equals $F(x)$ for at least one x in X.

A Function That Is Onto

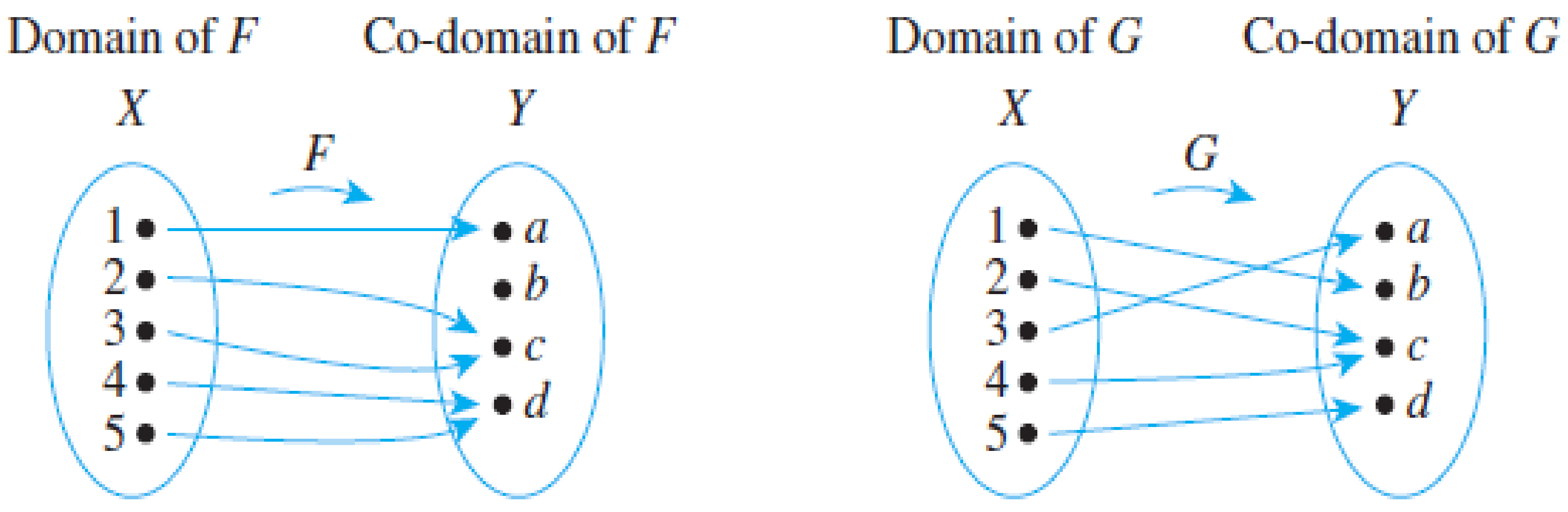


At least one element in Y does not equal $F(x)$ for any x in X.

A Function That Is Not Onto

Onto Functions on Sets

Identifying Onto Functions



Onto Functions on Infinite Sets

Now suppose F is a function from a set X to a set Y, and suppose Y is infinite. By definition, F is onto if, and only if, the following universal statement is true:

$$\forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

Thus to prove F is onto, you will ordinarily use the method of generalizing from the generic particular:

suppose that y is any element of Y

and **show** that there is an element X of X with $F(x) = y$.

To prove F is *not* onto, you will usually

find an element y of Y such that $y \neq F(x)$ for *any* x in X.

Onto Functions on Infinite Sets

Define $f\colon \mathbf{R} \to \mathbf{R}$ and $h\colon \mathbf{Z} \to \mathbf{Z}$ by the rules

$$f(x) = 4x - 1 \quad \text{for all } x \in \mathbf{R}$$

and

$$h(n) = 4n - 1 \quad \text{for all } n \in \mathbf{Z}.$$

a. Is f onto? Prove or give a counterexample.

b. Is h onto? Prove or give a counterexample.

Onto Functions on Infinite Sets

If $f: \mathbf{R} \to \mathbf{R}$ is the function defined by the rule $f(x) = 4x - 1$ for all real numbers x, then f is onto.

Proof:

Let $y \in \mathbf{R}$. *[We must show that $\exists x$ in $\mathbf{R}$ such that $f(x) = y$.]* Let $x = (y+1)/4$. Then x is a real number since sums and quotients (other than by 0) of real numbers are real numbers. It follows that

$$\begin{aligned} f(x) &= f\left(\frac{y+1}{4}\right) && \text{by substitution} \\ &= 4 \cdot \left(\frac{y+1}{4}\right) - 1 && \text{by definition of } f \\ &= (y+1) - 1 = y && \text{by basic algebra.} \end{aligned}$$

[This is what was to be shown.]

Onto Functions on Infinite Sets

If the function $h: \mathbf{Z} \to \mathbf{Z}$ is defined by the rule $h(n) = 4n - 1$ for all integers n, then h is not onto.

Onto Functions on Infinite Sets

If the function $h: \mathbf{Z} \to \mathbf{Z}$ is defined by the rule $h(n) = 4n - 1$ for all integers n, then h is not onto.

Counterexample:

The co-domain of h is $\mathbf{Z}$ and $0 \in \mathbf{Z}$. But $h(n) \neq 0$ for any integer n. For if $h(n) = 0$, then

$$4n - 1 = 0 \qquad \text{by definition of } h$$

which implies that

$$4n = 1 \qquad \text{by adding 1 to both sides}$$

and so

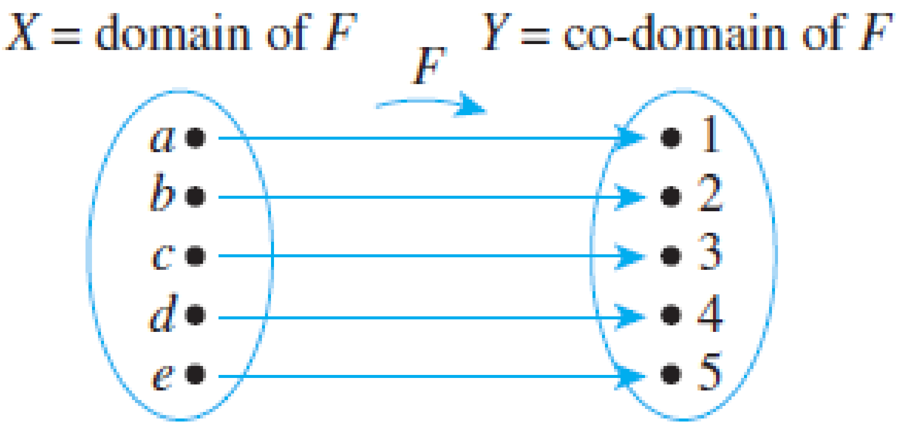
$$n = \frac{1}{4} \qquad \text{by dividing both sides by 4.}$$

But $1/4$ is not an integer. Hence there is no integer n for which $f(n) = 0$, and thus f is not onto.

One-to-One Correspondence (Bijection)

• **Definition**

A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function $F: X \to Y$ that is both one-to-one and onto.



An Arrow Diagram for a One-to-One Correspondence

One-to-One Correspondence (Bijection)

Inverse Functions

Theorem

Suppose $F: X \to Y$ is a one-to-one correspondence; that is, suppose F is one-to-one and onto. Then there is a function $\boldsymbol{F^{-1}: Y \to X}$ that is defined as follows:

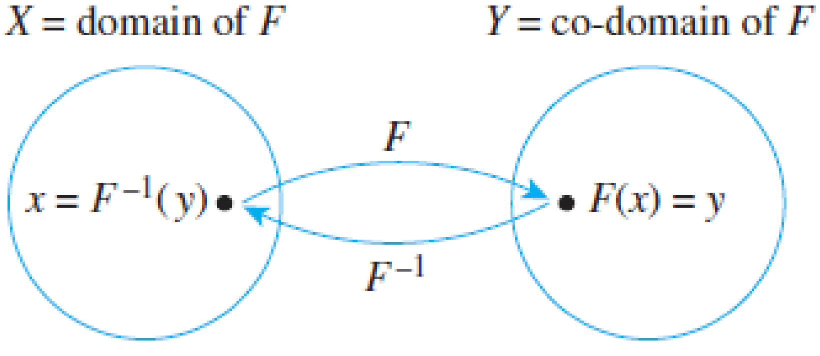
Given any element y in Y,

$$F^{-1}(y) = \text{that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y.$$

In other words,

$$F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).$$

The function F^{-1} is called inverse function.



Finding an Inverse Function

The function $f : \mathbf{R} \to \mathbf{R}$ defined by the formula

$$f(x) = 4x - 1, \quad \text{for all real numbers } x$$

Solution For any *[particular but arbitrarily chosen]* y in $\mathbf{R}$, by definition of f^{-1},

$$f^{-1}(y) = \text{that unique real number } x \text{ such that } f(x) = y.$$

But

$$\begin{aligned} & f(x) = y \\ \Leftrightarrow \quad & 4x - 1 = y && \text{by definition of } f \\ \Leftrightarrow \quad & x = \frac{y+1}{4} && \text{by algebra.} \end{aligned}$$

Hence $f^{-1}(y) = \dfrac{y+1}{4}$.

Theorem

If X and Y are sets and $F: X \to Y$ is one-to-one and onto, then $F^{-1}: Y \to X$ is also one-to-one and onto.

Proof:

F^{-1} is one-to-one: Suppose y_1 and y_2 are elements of Y such that $F^{-1}(y_1) = F^{-1}(y_2)$. *[We must show that $y_1 = y_2$.]* Let $x = F^{-1}(y_1) = F^{-1}(y_2)$. Then $x \in X$, and by definition of F^{-1},

$$F(x) = y_1 \quad \text{since } x = F^{-1}(y_1)$$

and

$$F(x) = y_2 \quad \text{since } x = F^{-1}(y_2).$$

Consequently, $y_1 = y_2$ since each is equal to $F(x)$. This is what was to be shown.

F^{-1} is onto: Suppose $x \in X$. *[We must show that there exists an element y in Y such that $F^{-1}(y) = x$.]* Let $y = F(x)$. Then $y \in Y$, and by definition of F^{-1}, $F^{-1}(y) = x$. This is what was to be shown.

Lecture Summery

- Properties of relations
- Reflexive, Symmetric and Transitive Relations
- Equivalence Relations
- Properties of Congruence Modulo n
- Transitive closure of a relations
- Combining Relations
- Partial Order Relations
- Hasse Diagrams