Discrete Structures (Mathematics)

Topic: Logic of Compoud Statements

What are discrete Structures?

- Discrete mathematics is the part of mathematics devoted to the study of discrete objects (Kenneth H. Rosen, 6th edition).

- Discrete mathematics is the study of mathematical structures that are fundamentally discrete rather than continuous (wikipedia).

Discrete vs Continuous

- Examples of discrete Data
  - Number of boys in the class.
  - Number of candies in a packet.
  - Number of suitcases lost by an airline.

- Examples of continuous Data
  - Height of a person.
  - Time in a race.
  - Distance traveled by a car.

Applications

- How can a circuit that adds two integers be designed?
- How many ways are there to choose a valid password on a computer?
- What is the shortest path between two cities using transportation system?
- How can I encrypt a message so that no unintended recipient can read it?
- How many valid internet addresses are there?
- How can a list of integers be sorted so that the integers are in increasing order?

Problems from Ex 2.1

Todays Lecture

- Propositional Logic
- Logic of Compound Statements
- Conditional Statements
- Logical Equivalences
- Valid and Invalid Arguments

Problems from Ex 2.1

Propositional Logic

Proposition: A proposition (or Statement) is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

Examples

1. Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

Almaty is the capital of Kazakhstan.

This makes a declarative statement, and hence is a proposition. The proposition is False (F).

Propositions Cont….

2. Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

Can Alinur come with you?.

This is a question not the declarative sentence and hence not a proposition.

3. Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

Take two aspirins.

This is an imperative sentence not the declarative sentence and therefore not a proposition.

Propositions Cont…

4. Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

$$x + 4 > 9.$$

Because this is true for certain values of x (such as $x = 6$) and false for other values of x (such as $x = 5$), it is not a proposition.

5. Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

He is a University student.

Because truth or falsity of this proposition depend on the reference for the pronoun *he*. it is not a proposition.

Notations

- The small letters are commonly used to denote the propositional variables, that is, variables that represent propositions, such as, $\underbrace{p, q, r, s, \ldots.}_{\text{represents some statement}}$

- The truth value of a proposition is true, denoted by T or 1, if it is a true proposition and false, denoted by F or 0, if it is a false proposition.

Compound Propositions

Producing new propositions from existing propositions.

Logical Operators or Connectives

1. Not $\neg$
2. And $\wedge$
3. OR $\vee$
4. Exclusive OR $\oplus$
5. Implication $\rightarrow$
6. Biconditional $\leftrightarrow$

Operator (NOT, $\neg$)

Let p be a proposition. The negation of p, denoted by $\neg p$ ***(also denoted by ~p)***, is the statement

"It is not the case that p".

The proposition $\neg p$ is read as "not p". The truth values of the negation of p, $\neg p$, is the opposite of the truth value of p.

p	¬p
true	false
false	true

Problems from Ex 2.1 **Truth Table (NOT)**

Examples

1. Find the negation of the following proposition

p : Today is Friday.

The negation is

$\neg\ p$: It is not the case that today is Friday.

This negation can be more simply expressed by

$\neg p$: Today is not Friday.

Cont…

2. Write the negation of

“6 is negative”.

The negation is

“It is not the case that 6 is negative”.

or “6 is nonnegative”.

Conjunction (AND, $\wedge$)

Let p and q be propositions. The conjunction of p and q, **denoted by** $\boldsymbol{p \wedge q}$, is the proposition "*p and q*".

The conjunction $p \wedge q$ is true when *p and q* are both true and is false otherwise.

p	q	$p \wedge q$
true	true	true
true	false	false
false	true	false
false	false	false

Truth Table (AND)

Problems from Ex 2.1

Examples

1. Find the conjunction of the propositions p and q, where

$$p : \text{Today is Friday.}$$

$$q : \text{It is raining today.}$$

The conjunction is

$$p \wedge q : \text{Today is Friday and it is raining today.}$$

Disjunction (OR, ∨)

Let p and q be propositions. The disjunction of p and q, **denoted by** $\boldsymbol{p \vee q}$, is the proposition "*p or q*".

The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

p	q	$p \vee q$
true	true	true
true	false	true
false	true	true
false	false	false

Truth Table (OR)

Problems from Ex 2.1

Examples

1. Find the disjunction of the propositions p and q, where

p : Today is Friday.

q : It is raining today.

The disjunction is

$p \vee q$: Today is Friday or it is raining today.

Translating English to Logic

I did not buy a lottery ticket this week or I bought a lottery ticket and won the million dollar on Friday.

Let p and q be two propositions

p: I bought a lottery ticket this week.

q: I won the million dollar on Friday.

In logic form

$$\neg p \lor (p \land q)$$

Conditional Statements (Implication)

Let p and q be propositions. The *conditional statement* $p \to q$, is the proposition *"If p, then q"*.

The *conditional statement* $p \to q$ is false when p is true and q is false and is true otherwise.

where p is called hypothesis, antecedent or premise.

q is called conclusion or consequence

Example: If $\underbrace{\text{Alinur lives in Almaty,}}_{\text{hypothesis}}$ then $\overbrace{\text{he lives in Kazakhstan}}^{\text{conclusion}}$.

Problems from Ex 2.1

Truth Table (if – then, Symbol: →)

P	Q	P → Q
true	true	true
true	false	false
false	true	true
false	false	true

Alternatively: *p* is a necessary condition for *q* also means "if *p* then *q*."

Biconditional Statements

Let *p* and *q* be propositions. The *biconditional statement p↔q*, is the proposition *"p if and only if q"*.

The *biconditional (bi-implication) statement* *p* ↔ *q* is true when *p* and *q* have same truth values and is false otherwise.

P	Q	P ↔ Q
true	true	true
true	false	false
false	true	false
false	false	true

Examples

1. $\overbrace{\text{Tomorrow is Wednesday}}^{p}$ if and only if $\overbrace{\text{today is Tuesday.}}^{q}$
2. $x^2 - 4 = 0$ if and only if $x = \pm 2$.
3. An angle is right if and only if it measures 90^o
4. I will pass if and only if I study hard.

Alternatively: *p* is a necessary and sufficient condition for *q* means "*p* if, and only if, *q*."

Interpreting Necessary and sufficient conditions

Example: Consider the proposition

‘if John is eligible to vote then he is at least 18 year old’.

The truth of the condition ‘John is eligible to vote’ is sufficient to ensure the truth of the condition ‘John is at least 18 year old’.

In addition, the condition ‘John is at least 18 year old’ is necessary for the condition ‘John is eligible to vote’ to be true. If John were younger than 18, then he would not eligible to vote.

Composite Statements

Statements and operators can be combined in any way to form new statements.

P	Q	$\neg P$	$\neg Q$	$(\neg P)\vee(\neg Q)$
true	true	false	false	false
true	false	false	true	true
false	true	true	false	true
false	false	true	true	true

Equivalent Statements, Symbol $\equiv$

Two statements are called **logically equivalent** if and only if they have identical truth tables

The statements $\neg(P\wedge Q)$ and $(\neg P)\vee(\neg Q)$ are logically equivalent, because $\neg(P\wedge Q)\leftrightarrow(\neg P)\vee(\neg Q)$ is always **true.**

P	Q	$\neg(P\wedge Q)$	$(\neg P)\vee(\neg Q)$	$\neg(P\wedge Q)\leftrightarrow(\neg P)\vee(\neg Q)$
true	true	false	false	true
true	false	true	true	true
false	true	true	true	true
false	false	true	true	true

Activity

- Show that

$$p \rightarrow q \equiv \neg p \vee q$$

This shows that a conditional proposition is simple a proposition form that uses **a not and an or**.

- Show that

$$\neg (p \rightarrow q) \equiv p \wedge \neg q$$

This means that negation of ‘**if p then q**’ is logically equivalent to ‘**p and not q**’.

Solution

p	q	$p \to q$	$\neg p \vee q$	$\neg (p \to q)$	$p \wedge \neg q$
T	T	T	T	T	T
T	F	F	F	F	F
F	T	T	T	T	T
F	F	T	T	T	T

From the above table it is obvious that conditional proposition is equivalent to a "not or proposition" and that its negation is not of the form 'if then'.

Valid and Invalid Argument

An argument is a sequence of statements such that

- all statements but the last are called *hypotheses*
- the final statement is called the *conclusion*
- the symbol ∴ read "therefore" is usually placed just before the conclusion.

$$\left.\begin{array}{l} p \wedge \sim q \to r \\ p \vee q \\ q \to p \\ \therefore r \end{array}\right\} \text{Example of argument}$$

An argument is said to be* valid *if whenever all hypotheses are true, the conclusion must be true

Valid Argument

$p \wedge (q \vee r)$

$\sim q$

$\therefore p \wedge r$

p	q	r	$p \wedge (q \vee r)$	$\sim q$	$p \wedge r$
T	T	T	T	F	T
T	T	F	T	F	F
T	F	T	T	T	T
T	F	F	F	T	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	T	F
F	F	F	F	T	F

Invalid Argument

$p \to q \vee \sim r$

$q \to p \vee r$

$\therefore p \to r$

	p	q	r	$p \to q \vee \sim r$	$q \to p \vee r$	$p \to r$
	T	T	T	T	T	T
	T	T	F	T	F	
	T	F	T	F	T	
Invalid row	T	F	F	T	T	F
	F	T	T	T	F	
	F	T	F	T	F	
	F	F	T	T	T	T
	F	F	F	T	T	T

Practice Exercises of the Text Book

Discrete Mathematics with Applications, 4th edition (International Edition)

Ex 2.1,2.2, and 2.3

What we Discussed

- Propositions
- Logical Connectives
- Truth Tables
- Compound propositions
- Translating English to logic and logic to English.
- Valid and Invalid Statements

Laws of Logic

1. Commutative laws

   $p \wedge q \equiv q \wedge p$; $p \vee q \equiv q \vee p$

2. Associative laws

   $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$; $p \vee (q \vee r) \equiv (p \vee q) \vee r$

3. Distributive laws

   $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

4. Identity laws

   $p \wedge t \equiv p$; $p \vee c \equiv p$

5. Negation laws

   $p \vee \neg p \equiv t$; $p \wedge \neg p \equiv c$

6. Double negation law

   $\neg(\neg p) \equiv p$

Laws of Logic

7. Idempotent laws

   $p \wedge p \equiv p$; $p \vee p \equiv p$

8. Universal bound laws

   $p \vee t \equiv t$; $p \wedge c \equiv c$

9. Absorption laws

   $p \wedge (p \vee q) \equiv p$; $p \vee (p \wedge q) \equiv p$

10. Negation of *t* and *c*

   $\neg t \equiv c$; $\neg c \equiv t$

Discrete Mathematics

Topic: Logic of Compoud Statements

Write the statements in symbolic form using the symbols , $\neg$, $\vee$ and $\wedge$ ` and the indicated letters to represent component statements.

1. Let s = "stocks are increasing" and i = "interest rates are steady."
   a. Stocks are increasing but interest rates are steady.
   b. Neither are stocks increasing nor are interest rates steady.

2. Let p = "$x > 5$," q ="$x = 5$," and r = "$10 > x$."
   a. $x \geq 5$
   b. $10 > x > 5$
   c. $10 > x \geq 5$

Problems from Ex 2.1

3. Write truth tables for the statement , $\neg (p \wedge q) \vee (p \vee q)$.

4. Verify the logical equivalence, $p \wedge (\neg q \vee p) \equiv p$.

5. Write each of the following three statements in symbolic form and determine which pairs are logically equivalent. Include truth tables and a few words of explanation.

a) If it walks like a duck and it talks like a duck, then it is a duck.

b) Either it does not walk like a duck or it does not talk like a duck, or it is a duck.

c) If it does not walk like a duck and it does not talk like a duck, then it is not a duck.

Solution: Let p represent "It walks like a duck," q represent "It talks like a duck," and *r* represent "It is a duck."

Truth Table

p	q	r	$\sim p$	$\sim q$	$p \wedge q$	$\sim p \vee \sim q$	$p \wedge q \to r$	$(\sim p \vee \sim q) \vee r$
T	T	T	F	F	T	F	T	T
T	T	F	F	F	T	F	F	F
T	F	T	F	T	F	T	T	T
T	F	F	F	T	F	T	T	T
F	T	T	T	F	F	T	T	T
F	T	F	T	F	F	T	T	T
F	F	T	T	T	F	T	T	T
F	F	F	T	T	F	T	T	T

same truth values

p	q	r	$\sim p$	$\sim q$	$\sim r$	$p \wedge q$	$\sim p \wedge \sim q$	$p \wedge q \to r$	$(\sim p \wedge \sim q) \to \sim r$	
T	T	T	F	F	F	T	F	T	T	
T	T	F	F	F	T	T	F	F	T	$\leftarrow$
T	F	T	F	T	F	F	F	T	T	
T	F	F	F	T	T	F	F	T	T	
F	T	T	T	F	F	F	F	T	T	
F	T	F	T	F	T	F	F	T	T	
F	F	T	T	T	F	F	T	T	F	$\leftarrow$
F	F	F	T	T	T	F	T	T	T	

different truth values

Problems from Ex 2.1

Rewrite the statements in if-then form.

1. Payment will be made on fifth unless a new hearing is granted.

   If a new hearing is not granted, payment will be made on the fifth.

2. Ann will go unless it rains.

   If it doesn't rain, then Ann will go.

3. This door will not open unless a security code is entered.

   If a security code is not entered, then the door will not open.

Problems from Ex 2.2

Use truth tables to determine whether the argument forms are valid

$p \wedge q \rightarrow \sim r$

$p \vee \sim q$

$\sim p \rightarrow p$

$\therefore \quad \sim r$

						premises			conclusion	
p	q	r	$\sim q$	$\sim r$	$p \wedge q$	$p \wedge q \rightarrow \sim r$	$p \vee \sim q$	$\sim q \rightarrow p$	$\sim r$	
T	T	T	F	F	T	T	T	T	F	critical row
T	T	F	F	T	T	F	T	T	T	
T	F	T	T	F	F	T	T	T	F	critical row
T	F	F	T	T	F	T	T	T	T	
F	T	T	F	F	F	T	F	T	F	
F	T	F	F	T	F	T	F	T	T	
F	F	T	T	F	F	T	T	F	F	
F	F	F	T	T	F	T	T	F	T	

Problems from Ex 2.3