Discrete Structures

Mathematical Induction

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- Principle of Mathematical Induction
- Method of proof
- Finding Terms of Sequences
- Sum of Geometric Series

Principal Of Mathematical Induction

Principle of Mathematical Induction

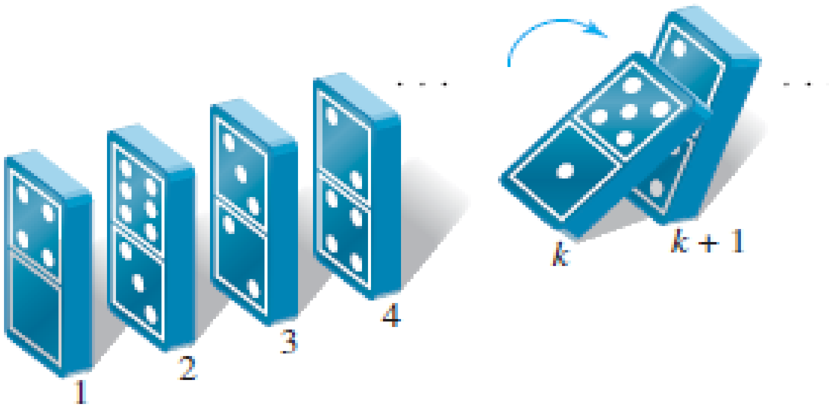
Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. $P(a)$ is true.
2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

Then the statement

$$\text{for all integers } n \geq a,\ P(n)$$

is true.



If the kth domino falls backward, it pushes the $(k+1)$st domino backward also.

Method Of Proof

Method of Proof by Mathematical Induction

Consider a statement of the form, "For all integers $n \geq a$, a property $P(n)$ is true." To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true. To perform this step,

suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

[This supposition is called the **inductive hypothesis.** *]*

Then

show that $P(k+1)$ is true.

Finding Terms of Sequences

Proposition: For all integers $n \geq 8$, n ¢ can be obtained using 3¢ and 5¢ coins

Proof: Let the property *P(n)* be the sentence n ¢ can be obtained using 3¢ and 5¢ coins. ← *P(n)*

Show that P(8) is true:

P(8) is true because 8¢ can be obtained using one 3¢ coin and one 5¢ coin.

Show that for all integers k ≥ 8, if P(k) is true then P(k+1) is also true:

Suppose that k is any integer with $k \geq 8$ such that k¢ can be obtained using 3¢ and 5¢ coins. ← *P(k)* inductive hypothesis

Propositions

Proposition: For all integers $n \geq 1$,

$$1 + 2 + \cdots + n = n(n + 1)/2$$

Proof:

Show that P(1) is true:

To establish $P(1)$, we must show that

$$1 = \frac{1(1+1)}{2} \qquad \leftarrow \quad P(1)$$

But the left-hand side of this equation is 1 and the right-hand side is

$$\frac{1(1+1)}{2} = \frac{2}{2} = 1$$

also. Hence $P(1)$ is true.

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is also true:

[Suppose that $P(k)$ is true for a particular but arbitrarily chosen integer $k \geq 1$. That is:] Suppose that k is any integer with $k \geq 1$ such that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \qquad \leftarrow P(k)$$

inductive hypothesis

Cont…

[We must show that $P(k+1)$ is true. That is:] We must show that

$$1+2+3+\cdots+(k+1)=\frac{(k+1)[(k+1)+1]}{2},$$

or, equivalently, that

$$1+2+3+\cdots+(k+1)=\frac{(k+1)(k+2)}{2}. \quad \leftarrow P(k+1)$$

[We will show that the left-hand side and the right-hand side of $P(k+1)$ are equal to the same quantity and thus are equal to each other.]

The left-hand side of $P(k+1)$ is

$$\begin{aligned}
&1+2+3+\cdots+(k+1) \\
&\quad = 1+2+3+\cdots+k+(k+1) && \text{by making the next-to-last term explicit} \\
&\quad = \frac{k(k+1)}{2}+(k+1) && \text{by substitution from the inductive hypothesis} \\
&\quad = \frac{k(k+1)}{2}+\frac{2(k+1)}{2}
\end{aligned}$$

Cont…

$$= \frac{k^2 + k}{2} + \frac{2k + 2}{2}$$

$$= \frac{k^2 + 3k + 1}{2} \qquad \text{by algebra.}$$

And the right-hand side of $P(k+1)$ is

$$\frac{(k+1)(k+2)}{2} = \frac{k^2 + 3k + 1}{2}.$$

Thus the two sides of $P(k+1)$ are equal to the same quantity and so they are equal to each other. Therefore the equation $P(k+1)$ is true *[as was to be shown]*.

[Since we have proved both the basis step and the inductive step, we conclude that the theorem is true.]

Sum of Geometric Series

For any real number r except 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

Proof (by mathematical induction):

Suppose r is a particular but arbitrarily chosen real number that is not equal to 1, and let the property $P(n)$ be the equation

$$\sum_{i=0}^{n} r^i = \frac{r^{i+1} - 1}{r - 1} \quad \leftarrow P(n)$$

We must show that $P(n)$ is true for all integers $n \geq 0$. We do this by mathematical induction on n.

Show that P(0) is true:

To establish $P(0)$, we must show that

$$\sum_{i=0}^{0} r^i = \frac{r^{0+1} - 1}{r - 1} \quad \leftarrow P(0)$$

Cont….

The left-hand side of this equation is $r^0 = 1$ and the right-hand side is

$$\frac{r^{0+1}-1}{r-1} = \frac{r-1}{r-1} = 1$$

also because $r^1 = r$ and $r \neq 1$. Hence $P(0)$ is true.

Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is also true:

[Suppose that $P(k)$ is true for a particular but arbitrarily chosen integer $k \geq 0$. That is:]
Let k be any integer with $k \geq 0$, and suppose that

$$\sum_{i=0}^{k} r^i = \frac{r^{k+1}-1}{r-1} \quad \leftarrow P(k) \text{ inductive hypothesis}$$

[We must show that $P(k+1)$ is true. That is:] We must show that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{(k+1)+1}-1}{r-1},$$

or, equivalently, that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{k+2}-1}{r-1}. \quad \leftarrow P(k+1)$$

Cont…

$$\begin{aligned}
\sum_{i=0}^{k+1} r^i &= \sum_{i=0}^{k} r^i + r^{k+1} && \text{by writing the } (k+1)\text{st term separately from the first } k \text{ terms} \\
&= \frac{r^{k+1}-1}{r-1} + r^{k+1} && \text{by substitution from the inductive hypothesis} \\
&= \frac{r^{k+1}-1}{r-1} + \frac{r^{k+1}(r-1)}{r-1} && \text{by multiplying the numerator and denominator of the second term by } (r-1) \text{ to obtain a common denominator} \\
&= \frac{(r^{k+1}-1) + r^{k+1}(r-1)}{r-1} && \text{by adding fractions} \\
&= \frac{r^{k+1}-1+r^{k+2}-r^{k+1}}{r-1} && \text{by multiplying out and using the fact that } r^{k+1}\cdot r = r^{k+1}\cdot r^1 = r^{k+2} \\
&= \frac{r^{k+2}-1}{r-1} && \text{by canceling the } r^{k+1}\text{'s.}
\end{aligned}$$

which is the right-hand side of $P(k+1)$ *[as was to be shown.]*

Deducing Additional Formula

Applying the Formula for the Sum of a Geometric Sequence

In each of (a) and (b) below, assume that m is an integer that is greater than or equal to 3. Write each of the sums in closed form.

a. $1 + 3 + 3^2 + \cdots + 3^{m-2}$

b. $3^2 + 3^3 + 3^4 + \cdots + 3^m$

Solution

a. $$1 + 3 + 3^2 + \cdots + 3^{m-2} = \frac{3^{(m-2)+1} - 1}{3 - 1}$$ by applying the formula for the sum of a geometric sequence with $r = 3$ and $n = m - 2$

$$= \frac{3^{m-1} - 1}{2}.$$

b. $$3^2 + 3^3 + 3^4 + \cdots + 3^m = 3^2 \cdot (1 + 3 + 3^2 + \cdots + 3^{m-2})$$ by factoring out 3^2

$$= 9 \cdot \left(\frac{3^{m-1} - 1}{2}\right)$$ by part (a). ■

Cont…

As with the formula for the sum of the first *n* integers, there is a way to think of the formula for the sum of the terms of a geometric sequence that makes it seem simple and intuitive. Let

$$S_n = 1 + r + r^2 + \cdots + r^n.$$

Then

$$rS_n = r + r^2 + r^3 + \cdots + r^{n+1},$$

and so

$$\begin{aligned} rS_n - S_n &= (r + r^2 + r^3 + \cdots + r^{n+1}) - (1 + r + r^2 + \cdots + r^n) \\ &= r^{n+1} - 1. \end{aligned}$$

But

$$rS_n - S_n = (r-1)S_n.$$

Equating the right-hand sides of equations and dividing by $r - 1$ gives

$$S_n = \frac{r^{n+1} - 1}{r - 1}.$$

Proving a Divisibility Property

Proposition: For all integers $n \geq 0, 2^{2n} - 1$ is divisible by 3.

Proof:

Let the property $P(n)$ be the sentence "$2^{2n} - 1$ is divisible by 3."

$$2^{2n} - 1 \text{ is divisible by 3.} \qquad \leftarrow P(n)$$

Show that P(0) is true:

To establish $P(0)$, we must show that

$$2^{2 \cdot 0} - 1 \text{ is divisible by 3.} \qquad \leftarrow P(0)$$

But

$$2^{2 \cdot 0} - 1 = 2^0 - 1 = 1 - 1 = 0$$

and 0 is divisible by 3 because $0 = 3 \cdot 0$. Hence $P(0)$ is true.

Show that for all integers $k \geq 0$, if P(k) is true then P(k + 1) is also true:

[Suppose that P(k) is true for a particular but arbitrarily chosen integer $k \geq 0$. That is:]

Let k be any integer with $k \geq 0$, and suppose that

Cont….

$$2^{2k} - 1 \text{ is divisible by } 3. \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

By definition of divisibility, this means that

$$2^{2k} - 1 = 3r \quad \text{for some integer } r.$$

[We must show that $P(k+1)$ is true. That is:] We must show that

$$2^{2(k+1)} - 1 \text{ is divisible bv } 3. \qquad \leftarrow P(k+1)$$

But

$$\begin{aligned}
2^{2(k+1)} - 1 &= 2^{2k+2} - 1 \\
&= 2^{2k} \cdot 2^2 - 1 && \text{by the laws of exponents} \\
&= 2^{2k} \cdot 4 - 1 \\
&= 2^{2k}(3+1) - 1 \\
&= 2^{2k} \cdot 3 + (2^{2k} - 1) && \text{by the laws of algebra} \\
&= 2^{2k} \cdot 3 + 3r && \text{by inductive hypothesis} \\
&= 3(2^{2k} + r) && \text{by factoring out the 3.}
\end{aligned}$$

Cont…

$$2k + 1 < 2^k. \quad \leftarrow P(k) \text{ inductive hypothesis}$$

[We must show that $P(k+1)$ is true. That is:] We must show that

$$2(k+1) + 1 < 2^{(k+1)},$$

or, equivalently,

$$2k + 3 < 2^{(k+1)}. \quad \leftarrow P(k+1)$$

But

$$\begin{aligned} 2k + 3 &= (2k+1) + 2 && \text{by algebra} \\ &< 2^k + 2^k && \text{because } 2k+1 < 2^k \text{ by the inductive hypothesis and because } 2 < 2^k \text{ for all integers } k \geq 2 \\ \therefore\ 2k + 3 &< 2 \cdot 2^k = 2^{k+1} && \text{by the laws of exponents.} \end{aligned}$$

[This is what we needed to show.]

[Since we have proved the basis step and the inductive step, we conclude that the proposition is true.]

Cont…

Solution

a. $a_1 = 2.$

$a_2 = 5a_{2-1} = 5a_1 = 5 \cdot 2 = 10$

$a_3 = 5a_{3-1} = 5a_2 = 5 \cdot 10 = 50$

$a4 = 5a_{4-1} = 5a_3 = 5 \cdot 50 = 250.$

b. To use mathematical induction to show that every term of the sequence satisfies the equation, begin by showing that the first term of the sequence satisfies the equation. Then suppose that an arbitrarily chosen term a_k satisfies the equation and prove that the next term a_{k+1} also satisfies the equation

Cont…

Proof:

Let $a_1, a_2, a_3, \ldots$ be the sequence defined by specifying that $a_1 = 2$ and $a_k = 5a_{k-1}$ for all integers $k \geq 2$, and let the property $P(n)$ be the equation

$$a_n = 2 \cdot 5^{n-1}. \qquad \leftarrow P(n)$$

We will use mathematical induction to prove that for all integers $n \geq 1$, $P(n)$ is true.

Show that P(1) is true:
To establish $P(1)$, we must show that

$$a_1 = 2 \cdot 5^{1-1}. \qquad \leftarrow P(1)$$

But the left-hand side of $P(1)$ is

$$a_1 = 2 \qquad \text{by definition of } a_1, a_2, a_3, \ldots,$$

and the right-hand side of $P(1)$ is

$$2 \cdot 5^{1-1} = 2 \cdot 5^0 = 2 \cdot 1 = 2.$$

Cont…

But the left-hand side of $P(1)$ is

$$a_1 = 2 \qquad \text{by definition of } a_1, a_2, a_3, \ldots,$$

and the right-hand side of $P(1)$ is

$$2 \cdot 5^{1-1} = 2 \cdot 5^0 = 2 \cdot 1 = 2.$$

Thus the two sides of $P(1)$ are equal to the same quantity, and hence $P(1)$ is true.

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is also true:
[Suppose that $P(k)$ is true for a particular but arbitrarily chosen integer $k \geq 1$. That is:] Let k be any integer with $k \geq 0$, and suppose that

$$a_k = 2 \cdot 5^{k-1}. \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

By definition of divisibility, this means that

$$a_k = 2 \cdot 5^{k-1}.$$

Cont….

[We must show that P (k + 1) is true. That is:] We must show that

$$a_{k+1} = 2 \cdot 5^{(k+1)-1},$$

or, equivalently,

$$a_{k+1} = 2 \cdot 5^k. \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
a_{k+1} &= 5a_{(k+1)-1} && \text{by definition of } a_1, a_2, a_3, \ldots \\
&= 5a_k && \text{since } (k+1)-1 = k \\
&= 5 \cdot (2 \cdot 5^{k-1}) && \text{by inductive hypothesis} \\
&= 2 \cdot (5 \cdot 5^{k-1}) && \text{by regrouping} \\
&= 2 \cdot 5^k && \text{by the laws of exponents}
\end{aligned}$$

which is the right-hand side of the equation *[as was to be shown.]*

[Since we have proved the basis step and the inductive step, we conclude that the formula holds for all terms of the sequence.] ■

Examples

Problem:

Proof (by mathematical induction): Let the property $P(n)$ be the equation

$$(1+\frac{1}{1})(1+\frac{1}{2})\cdots(1+\frac{1}{n})=n+1$$

Show that the property is true for $\boldsymbol{n=1}$: When $n=1$, the left-hand side of the equation is $1+\frac{1}{1}=2$ and the right hand side is $1+1=2$ also. So the property is true for $n=1$.

Show that for all integers $\boldsymbol{k\geq 1}$***, if the property is true for*** $\boldsymbol{n=k}$ ***then it is true for*** $\boldsymbol{n=k+1}$: Suppose $(1+\frac{1}{1})(1+\frac{1}{2})\cdots(1+\frac{1}{k})=k+1$, for some integer $k\geq 1$. *[This is the inductive hypothesis.]* We must show that $(1+\frac{1}{1})(1+\frac{1}{2})\cdots(1+\frac{1}{k+1})=(k+1)+1$. But the left-hand side of this equation is

Cont….

$$
\begin{aligned}
&(1+\tfrac{1}{1})(1+\tfrac{1}{2})\cdots(1+\tfrac{1}{k+1}) \\
&= (1+\tfrac{1}{1})(1+\tfrac{1}{2})\cdots(1+\tfrac{1}{k})(1+\tfrac{1}{k+1}) && \text{by making the next-to-last factor explicit} \\
&= (k+1)(1+\tfrac{1}{k+1}) && \text{by inductive hypothesis} \\
&= (k+1)+1 && \text{by algebra,}
\end{aligned}
$$

and this is the right-hand side of the equation *[as was to be shown]*.

Problem:

For each positive integer n, let $P(n)$ be the property

$$2^n < (n+1)!.$$

a. Write $P(2)$. Is $P(2)$ true?
b. Write $P(k)$.
c. Write $P(k+1)$.
d. In a proof by mathematical induction that this inequality holds for all integers $n \geq 2$, what must be shown in the inductive step?

Cont…

Solution:

$a.$ $P(n)$: $2^n < (n+1)!$

$P(2)$ is true because $2^2 = 4 < 6 = (2+1)!$.

$b.$ $P(k)$: $2^k < (k+1)!$

$c.$ $P(k+1)$: $2^{k+1} < ((k+1)+1)!$

$d.$ *Must show:* If k is any integer with $k \geq 2$ and $2^k < (k+1)!$, then $2^{k+1} < ((k+1)+1)!$

Example

Proposition: $7^n - 1$ is divisible by 6, for each integer $n \geq 0$.

Solution:

Proof (by mathematical induction): Let the property $P(n)$ be the sentence "$7^n - 1$ is divisible by 6."

Show that the property is true for $n = 0$: The property is true for $n = 0$ because $7^0 - 1 = 1 - 1 = 0$ and 0 is divisible by 6 (since $0 = 0 \cdot 6$).

Show that for all integers $k \geq 0$***, if the property is true for*** $n = k$ ***then it is true for*** $n = k + 1$: Suppose $7^k - 1$ is divisible by 6 for some integer $k \geq 0$. *[This is the inductive hypothesis.]* We must show that $7^{k+1} - 1$ is divisible by 6. By definition of divisibility, the inductive hypothesis is equivalent to the statement $7^k - 1 = 6r$ for some integer r. Then by the laws of algebra, $7^{k+1} - 1 = 7 \cdot 7^k - 1 = (6+1)7^k - 1 = 6 \cdot 7^k + (7^k - 1) = 6 \cdot 7^k + 6r$, where the last equality holds by inductive hypothesis. Thus, by factoring out the 6 from the extreme right-hand side and by equating the extreme left-hand and extreme right-hand sides, , we have $7^{k+1} - 1 = 6(7^k + r)$, which is divisible by 6 because $7^k + r$ is an integer (since products and sums of integers are integers). Therefore, $7^{k+1} - 1$ is divisible by 6 *[as was to be shown]*.