GAUSS'S LAW

Coulomb's law can always be used to calculate the electric field intensity $\vec{E}$ for any discrete or continuous charge distribution of charges at rest. The sums or integrals might be complicated (and a computer might be needed to evaluate them numerically), but resulting electric field intensity $\vec{E}$ can always be found.

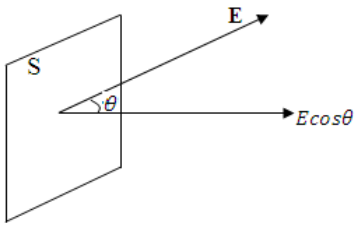
In this chapter, we discuss an alternative to Coulomb's law, called Gauss's law, that provides a more useful and instructive approach to calculating the electric field in the situations having certain symmetries.

The number of situations that can directly be analyzed using Gauss's law is small, but those cases can be done with extraordinary ease. Although Gauss's law and Coulomb's law gives identical results in the cases in which both can be used. Gauss's law is considered a more fundamental equation than Coulomb's law. It is fair to say that while Coulomb's law provides workhorse of electrostatics, Gauss's law provides the insight.

29.1 Electric Flux

The number of electric lines of force passing normally through a certain area is called the electric flux. It is measured by the product of area and the component of electric field intensity normal to the area. It is denoted by the symbol Φ_e.

Consider a surface 'S' placed in a uniform electric field of intensity '$\vec{E}$'. Let 'A' be the area of the surface. The component of $\vec{E}$normal to the area A is $E\cos\theta$ as shown in the figure below.



The electric flux through the surface S is given by

$$\Phi_e = A\,(E\cos\theta)$$

$$\Phi_e = EA\cos\theta$$

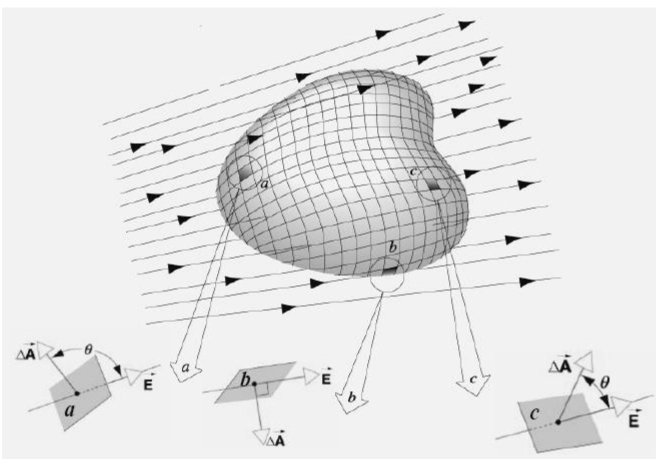
$$\Phi_e = \vec{E}.\vec{A}$$

Thus the electric flux is the scalar product of electric field intensity and the vector area. The SI unit of the electric flux is $\frac{Nm^2}{C}$.

29.2 Electric Flux through an Irregular Shaped Object

Consider an object of irregular shape placed in a non-uniform electric field. We want to find out the expression of electric flux through this irregular shaped object.

We divide the surface into n number of small patches having area $\Delta A_1, \Delta A_2, \Delta A_3, \dots\dots\dots, \Delta A_n$. Let$\vec{E}_1, \vec{E}_2, \vec{E}_3, \dots\dots\dots, \vec{E}_n$ are the electric field intensities which makes angle $\theta_1, \theta_2, \theta_3, \dots\dots\dots, \theta_n$ with thenormal to the area elements $\Delta A_1, \Delta A_2, \Delta A_3, \dots\dots\dots, \Delta A_n$, respectively. If $\Phi_1, \Phi_2, \Phi_3, \dots\dots\dots, \Phi_n$ be the electric flux through $\Delta A_1, \Delta A_2, \Delta A_3, \dots\dots\dots, \Delta A_n$, then the total electric flux $\Phi_e$will be:



$$\Phi_e = \Phi_1 + \Phi_2 + \Phi_3, + \dots\dots\dots + \Phi_n$$

$$\Rightarrow \Phi_e = E_1(\Delta A_1 \cos\theta_1) + E_2(\Delta A_2 \cos\theta_2) + E_3(\Delta A_3 \cos\theta_3) + \ \dots\dots\dots + E_n(\Delta A_n \cos\theta_n)$$

$$\Rightarrow \Phi_e = E_1\Delta A_1 \cos\theta_1 + E_2\Delta A_2 \cos\theta_2 + E_3\Delta A_3 \cos\theta_3 + \ \dots\dots\dots + E_n\Delta A_n \cos\theta_n$$

$$\Rightarrow \Phi_e = \vec{E}_1.\Delta\vec{A}_1 + \vec{E}_2.\Delta\vec{A}_2 + \vec{E}_3.\Delta\vec{A}_3 + \ \dots\dots\dots + \vec{E}_n.\Delta\vec{A}_n$$

Where $\Delta\vec{A}_1, \Delta\vec{A}_2, \Delta\vec{A}_3, \dots\dots\dots, \Delta\vec{A}_n$ are the vector areas corresponding to area elements $\Delta A_1, \Delta A_2, \Delta A_3, \dots\dots\dots, \Delta A_n$respectively.

$$\Phi_e = \sum_{i=1}^{n} \vec{E}_i.\Delta\vec{A}_i$$

When $n \to \infty$, or $\Delta A \to 0$, then the sigma is replaced by the surface integral i.e.,

$$\Phi_e = \int_S \vec{E}.d\vec{A}$$

By convention, the outward flux is taken as positive and inward flux is taken as negative.

29.3 Gauss's Law

Statement

The total electric flux through any close surface is $\frac{1}{\varepsilon_0}$ times the total charge enclosed by the surface.

Explanation

The Gauss's law gives the relation between total flux and total charge enclosed by the surface. Consider a collection of positive and negative charges in a certain region of space. According to Gauss's law:

$$\Phi_e = \frac{q}{\varepsilon_0} \quad ---- \quad (\mathbf{1})$$

where q is the net charge enclosed by the surface. Also,

$$\Phi_e = \oint \vec{E}.d\vec{A} \quad ---- \quad (\mathbf{2})$$

Comparing (1) and (2), we have:

$$\oint \vec{E}.d\vec{A} = \frac{q}{\varepsilon_0}$$

Thus we can describe the Gauss's law as

The surface normal integral of electric field intensity is equal to

$\frac{1}{\varepsilon_0}$*times the total charge enclosed by the surface.*

Problem 5. A point charge of $1.84\ \mu C$ is at center of cubical Gaussian surface of $55cm$edge. Find flux through the surface.

Solution:

$$q = 1.84\mu C = 1.84 \times 10^{-6} C$$

$\Phi_e = ?$

Applying Gauss law:

$$\Phi_e = \frac{q}{\varepsilon_0} = \frac{1.84 \times 10^{-6}}{8.85 \times 10^{-12}} = 2.07 \times 10^5 \frac{N - m^2}{C}$$

29.4Differential Form of Gauss's Law

If the charge is distributed into a volume having uniform volume charge density 'ρ', then charge enclosed q by Gaussian surface is described by expression:

$$q = \int_{\mathbf{v}} \rho\, dv$$

By Gauss's law:

$$\oint \vec{\mathrm{E}}.\, d\vec{\mathrm{A}} = \frac{q}{\varepsilon_0}$$

$$\Rightarrow \oint \vec{\mathrm{E}}.\, d\vec{\mathrm{A}} = \frac{1}{\varepsilon_0}\int_{\mathbf{v}} \rho\, dv \quad ---- \quad (\mathbf{1})$$

By Gauss's Divergence theorem,

$$\oint \vec{\mathrm{E}}.\, d\vec{\mathrm{A}} == \int_{\mathbf{v}} div\, \mathbf{E}\, dv \quad ---- \quad (\mathbf{2})$$

Comparing (1) and (2), we have:

$$\int_{\mathbf{v}} div\, \vec{\mathrm{E}}\, dv = \frac{1}{\varepsilon_0}\int_{\mathbf{v}} \rho\, dv$$

$$\Rightarrow \int_{\mathbf{v}} div\, \vec{\mathrm{E}}\, dv - \frac{1}{\varepsilon_0}\int_{\mathbf{v}} \rho\, dv = \mathbf{0}$$

$$\Rightarrow \int_{\mathbf{v}} \left(div\, \vec{\mathrm{E}} - \frac{1}{\varepsilon_0}\rho\right) dv = 0$$

As $dv \neq \mathbf{0}$

$$div\, \vec{\mathrm{E}} - \frac{1}{\varepsilon_0}\rho = 0$$

$$\Rightarrow div\, \vec{\mathrm{E}} = \frac{1}{\varepsilon_0}\rho$$

This is differential form of Gauss's law.

29.4 Integral Form of Gauss's Law

By Gauss's law:

$$\oint \vec{\mathrm{E}}.\, d\vec{\mathrm{A}} = \frac{q}{\varepsilon_0} \qquad \textit{where 'q' is the total charge enclosed}$$

If the charge is uniformly distributed into a volume having charge density 'ρ', then

$$\Phi_e = \oint \vec{\mathrm{E}}.\, d\vec{\mathrm{A}} = \frac{1}{\varepsilon_0}\int_{\mathbf{v}} \rho\, dv \quad ---- \quad (\mathbf{1})$$

If the charge is uniformly distributed over a surface having a surface charge density 'σ', then

$$\Phi_e = \oint \vec{\mathrm{E}}.\, d\vec{\mathrm{A}} = \frac{1}{\varepsilon_0}\int_{\mathbf{s}} \sigma\, dA \quad ---- \quad (\mathbf{2})$$

Equation (1) and (2) are the integral form of Gauss's law.

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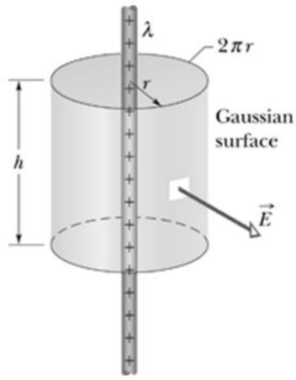
29.5 Applications of Gauss's law

Gauss's law can be used to calculate the electric field intensity due to certain charge distributions if the charge distribution has the greater symmetry.

29.5.1 Electric Field due to Infinite Line of Charge

Consider a section of infinite line of charge having uniform linear charge density 'λ' as shown in the figure below.



We want to find out electric field intensity at any point 'P' which is at distance 'r' from the wire. For this we consider cylindrical Gaussian surface which passes through point 'P'. The electric flux passing through the cylinder is given as

$$\Phi_e = \oint_{S} \vec{E}.d\vec{A}$$

The surface 'S' of the cylinder consist of three parts i.e., S_1, S_2 and S_3, where

S_1 = Area of top cross section of cylindrical Gaussian surface

S_2 = Area of bottom cross section of cylindrical Gaussian surface

S_3 = Area of curved part of Gaussian surface

Thus

$$\Phi_e = \int_{S_1} \vec{E}.d\vec{A} + \int_{S_2} \vec{E}.d\vec{A} + \int_{S_3} \vec{E}.d\vec{A}$$

Now

$$\int_{S_1} \vec{E}.d\vec{A} = \int_{S_2} \vec{E}.d\vec{A} = 0 \qquad \therefore \vec{E} \perp d\vec{A} \; for \; S_1 \; and \; S_2$$

$$\text{And} \int_{S_3} \vec{E}.d\vec{A} = \int_{S_3} E\,dA \cos 0^\circ = \int_{S_3} E\,dA \qquad \therefore \vec{E} \parallel d\vec{A} \; for \; S_3$$

Therefore

$$\Phi_e = \int_{S_3} E\,dA = E\int_{S_3} dA \quad ---- \quad (1) \qquad \therefore E \; is \; constant$$

For cylindrical symmetry

$$\int_{S_3} dA = 2\pi rh$$

So, equation (1) becomes:

$$\Phi_e = E(2\pi rh) = 2\pi rh\,E \quad ---- \quad (2)$$

By Gauss's law

$$\Phi_e = \frac{q}{\varepsilon_0} \quad ----- \quad (3)$$

As the line of charge has constant linear charge density λ, therefore:

$$\lambda = \frac{q}{h} \Longrightarrow q = \lambda h$$

So, the equation (3) becomes:

$$\Phi_e = \frac{\lambda\,h}{\varepsilon_0} \quad ----- \quad (4)$$

Comparing Eq. (2) and (4), we get:

$$2\pi rh\, E = \frac{\lambda h}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi r\varepsilon_0}$$

If '$\hat{r}$' gives the direction of electric field intensity, then

$$\vec{E} = \frac{\lambda}{2\pi r\varepsilon_0}\hat{r}$$

This expression gives the electric field intensity due to infinite line of charge.

Problem 20. An infinite line of charge produces a field of $4.52 \times 10^4 \frac{N}{C}$ at a distance of 1.96 m. calculate the linear charge density?

Solution:$E = 4.52 \times 10^4 \frac{N}{C}$

$r = 1.96\, m$

$\lambda = ?$

$$As\ E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

$$\Rightarrow \lambda = 2\pi\varepsilon_0 r.E$$

$$\Rightarrow \lambda = 2 \times 3.14 \times 8.85 \times 10^{-12} \times 1.96 \times 4.52 \times 10^4 = 4.92 \times 10^{-6}\frac{C}{m}$$

Sample problem 5. A plastic rod whose length is 220 cm and whose radius is 3.6 mm carries a negative charge q of magnitude $3.8 \times 10^{-7} C$ spread uniformly over its surface. What is the electric field near the midpoint of the rod at a point on its surface?

Solution:$L = 220cm = 2.2\, m$

$r = 3.6 \times 10^{-3} m$

$q = -3.8 \times 10^{-7} C$

$E = ?$

As Electric field intensity due to infinite line of charge is:

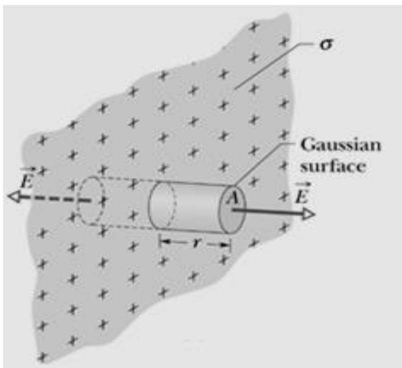
$$E = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r} - - - - - \quad (1)$$

$$\lambda = \frac{q}{L} = -\frac{3.8 \times 10^{-7}}{2.2} = -1.73 \times 10^{-7}\frac{C}{m}$$

$$Now E = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r} = \frac{1}{2 \times 3.14 \times 8.85 \times 10^{-12}}\frac{(-1.73 \times 10^{-7})}{3.6 \times 10^{-3}}$$

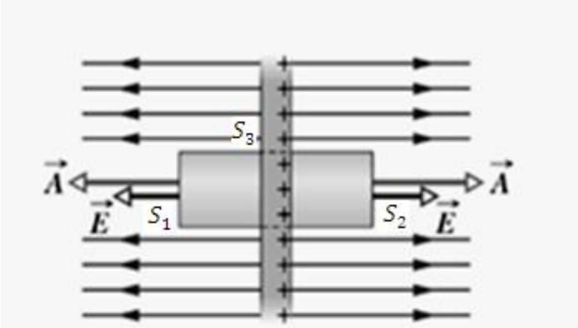
$$= -8.6 \times 10^5 \frac{N}{C}$$

29.5.2 Electric Field at a Point Due to Infinite Sheet of Charge



Consider an infinite sheet of charge having constant surface charge density 'σ'. The figure shows a small portion of such sheet.

We want to find electric field intensity at point 'P' which is at the distance 'r' from sheet. For this we consider a cylindrical Gaussian surface as shown in the figure below. The net electric flux passing through the cylinder is given as



$$\Phi_e = \int_S \vec{E}.d\vec{A}$$

We divide the cylindrical Gaussian surface into three parts i.e., S_1, S_2 and S_3, where

S_1 = Left cross sectional area of cylindrical Gaussian surface

S_2 = Right cross sectional area of cylindrical Gaussian surface

S_3 = Area of curved of cylindrical Gaussian surface

Thus

$$\Phi_e = \int_{S_1} \vec{E}.d\vec{A}_1 + \int_{S_2} \vec{E}.d\vec{A}_2 + \int_{S_3} \vec{E}.d\vec{A}_3 \quad --- \quad (1)$$

$$Now \int_{S_3} \vec{E}.d\vec{A}_3 = 0 \qquad \therefore \vec{E} \perp d\vec{A} \; for \; S_3$$

Therefore, the equation (1) becomes:

$$\Phi_e = \int_{S_1} E.dA_1 + \int_{S_2} E.dA_2$$

In case of surfaces S_1 and S_2, $\vec{E} \parallel d\vec{A}$ are parallel to each other i.e., θ = 0° and $|dA_1| = |dA_2| = dA$.

$$\Phi_e = \int_{S_1} E\,dA + \int_{S_2} E\,dA$$

$$\Phi_e = E\int_{S_1} dA + E\int_{S_2} dA \;\therefore E \; is \; constant$$

$$\Phi_e = E\,A + E\,A = 2\,E\,A \quad ----(2)$$

According to Gauss's law

$$\Phi_e = \frac{q}{\varepsilon_0} \qquad \therefore \sigma = \frac{q}{A} \Longrightarrow q = \sigma A$$

$$= \frac{\sigma A}{\varepsilon_0} \quad ---- \quad (3)$$

Comparing Eq. (2) and (3), we get

$$2\,E\,A = \frac{\sigma A}{\varepsilon_0}$$

$$\Longrightarrow E = \frac{\sigma}{2\varepsilon_0}$$

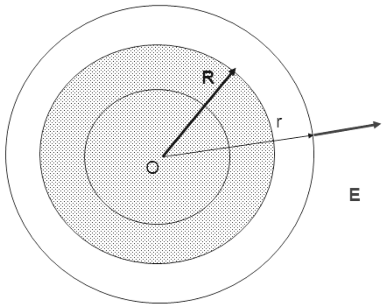
If '$\hat{r}$' gives the direction of electric field intensity, then

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{r}$$

This is the expression of electric field intensity due to infinite sheet of charge.

29.5.3 Electric Field due to Spherical Shell of Charge

Question: Show that the uniform spherical shell of charge behaves, for all external points, as if all its charge were concentrated at its center.



Proof: Consider a thin spherical shell of radius 'R' which have the charge 'q' with constant surface charge density 'σ'. The surface charge density

$$\sigma = \frac{q}{A} \Rightarrow \sigma = \frac{q}{4\pi r^2} \qquad \therefore A$$

$$= 4\pi r^2 \ (Surface\ area\ of\ sphere)$$

$$\Rightarrow q = 4\pi r^2 \sigma$$

Consider a point 'P' outside the shell. We want to find out electric field intensity due to this charge distribution. For this we consider a spherical Gaussian surface of radius $r > R$ which passes through point 'P' as shown in the figure below.

According to Gauss's law,

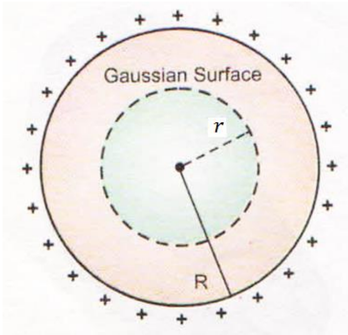
$$\oint \vec{E}.d\vec{A} = \frac{q}{\varepsilon_0}$$

$$\oint E\ dA \cos 0° = \frac{q}{\varepsilon_0} \therefore \vec{E} \parallel d\vec{A}$$

$$E \oint dA = \frac{q}{\varepsilon_0} \therefore E\ is\ constant$$

$$E(4\pi r^2) = \frac{q}{\varepsilon_0}$$

$$E = \frac{1}{4\pi\varepsilon_0}.\frac{q}{r^2}$$



Thus the uniform spherical shell of charge behaves like a point charge for all the points outside the shell.

Question: Show that the uniform spherical shell of charge exerts no electrostatic force on a charged particle placed inside the shell.

Consider a point 'P' inside the shell. We want to fine out electric field intensity 'E' at point 'P' due to this symmetrical charge distribution. For this we consider a spherical Gaussian surface of radius $r < R$ which passes through point 'P' as shown in the figure below.

According to Gauss's law,

$$\oint \vec{E}.d\vec{A} = \frac{q}{\varepsilon_0}$$

Because the Gaussian surface enclose no charge, therefore 'q = 0',

$$\oint \vec{E}.d\vec{A} = 0$$

$$\oint E\ dA\ \cos 0° = 0 \qquad \therefore \vec{E} \parallel d\vec{A}$$

$$E \oint dA = 0 \qquad \therefore E\ is\ constant$$

As $dA \neq 0$, therefore

$$E = 0$$

So the electric field does not exist inside a uniform shell of charge. So the test charge placed inside the charged shell would experience no force.

29.5.4 Electric Field due to Spherical Charge

Question. Find out the expression of electric field intensity outside solid sphere of charge.

Ans. Consider a spherical distribution of charge of radius 'R' with the uniform volume charge density 'ρ'. We want to find the electric field at point 'P' at a distance $r > R$ from the center of charged sphere. For we consider a spherical Gaussian surface which passes through point 'P' as shown in the figure below.

According to Gauss's law,

$$\oint \vec{E}.d\vec{A} = \frac{q}{\varepsilon_0}$$

$$\oint E\,dA\cos\theta = \frac{q}{\varepsilon_0}$$

As the electric field is radial, so the electric lines of force leave the Gaussian surface normally at all points. Therefore, the electric field intensity $\vec{E}$ and surface area element $d\vec{A}$ are in same direction i.e., $\theta = 0°$.

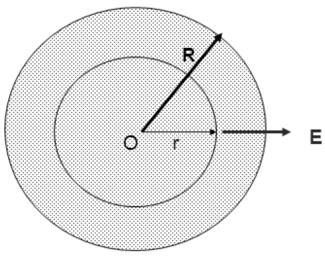
$$\oint E\,dA = \frac{q}{\varepsilon_0}$$

$$E\oint dA = \frac{q}{\varepsilon_0} \because E\ is\ constant$$

$$E(4\pi r^2) = \frac{q}{\varepsilon_0}$$

$$E = \frac{1}{4\pi\varepsilon_0}.\frac{q}{r^2}$$

Thus for all points outside the spherical charge distribution, the electric field has the same value as if the charge is concentrated at the center of sphere.



Q. Derive the expression of electric field intensity inside solid sphere of charge.

Ans. Consider a spherical distribution of charge of radius 'R' with the uniform volume charge density 'ρ'. The total charge in this uniform charge distribution is

$$\rho = \frac{q}{V} \Rightarrow q = \rho V$$

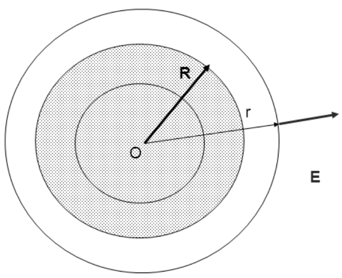
$$\Rightarrow q = \rho\left(\frac{4}{3}\pi R^3\right) \quad ---- \quad (1)$$

We want to find the electric field at point 'P' at a distance $r < R$ from the center of charged sphere. For this we consider a spherical Gaussian surface of which passes through point P as shown in the figure below.

Let the Gaussian surface encloses the charge $q' < q$ given by

$$q' = \rho\left(\frac{4}{3}\pi r^3\right) \quad ---- \quad (2)$$

Dividing eq. (1) and (2), we get



$$\frac{q'}{q} = \frac{\rho(\frac{4}{3}\pi r^3)}{\rho(\frac{4}{3}\pi R^3)}$$

$$\Rightarrow \frac{q'}{q} = \frac{r^3}{R^3}$$

$$\Rightarrow q' = \frac{r^3}{R^3}\, q \quad ----\quad (3)$$

According to Gauss's law

$$\oint \vec{E}.\, d\vec{A} = \frac{q'}{\varepsilon_0}$$

$$\Rightarrow \oint E\, dA \cos 0° = \frac{q'}{\varepsilon_0} \qquad \therefore E \textit{ is directed radially outward and } \vec{E} \parallel d\vec{A}$$

$$\Rightarrow E \oint dA = \frac{q'}{\varepsilon_0} \therefore E \textit{ is constant}$$

$$\Rightarrow E(4\pi r^2) = \frac{q'}{\varepsilon_0}$$

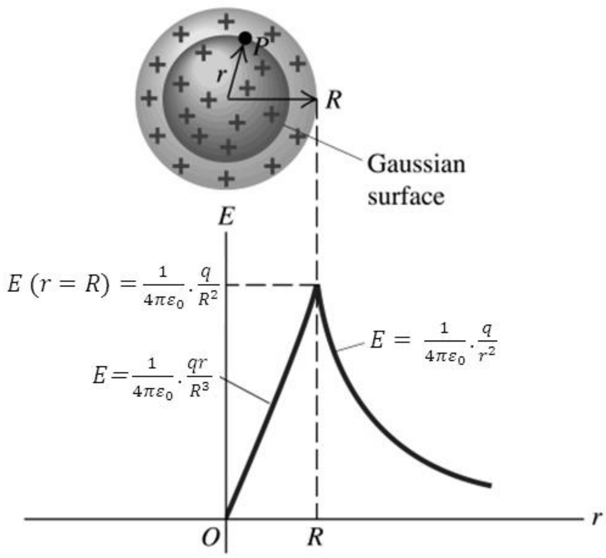
$$\Rightarrow E = \frac{1}{4\pi\varepsilon_0}.\frac{q'}{r^2}$$

Putting the value of q' from eq. (3), we get

$$E = \frac{1}{4\pi\varepsilon_0}.\frac{1}{r^2}.\frac{r^3}{R^3}\, q$$

$$\Rightarrow E = \frac{1}{4\pi\varepsilon_0}.\frac{r}{R^3}\, q$$

This is expression of electric field intensity inside solid sphere of charge.



Special case

To find out the expression of electric field intensity at the surface of solid sphere of charge, put $r = R$:

$$E\ (r = R) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}$$

Variation of Electric Field Intensity as a function of distance for Volume charge distribution

The graphical representation of the dependence of electric field strength on the radial distance 'r' from the center of this charge distribution is shown in the figure:

- Electric field intensity, inside solid sphere of charge, is directly proportional to distance as described by formula:

  $$E = \frac{1}{4\pi\varepsilon_0}\frac{r}{R^3}\, q$$

  So the graph between E and r is a straight line for the values of $r$from $0 \longrightarrow R$.
- The electric field intensity is maximum at the surface of sphere of charge:

  $$E\ (r = R) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}$$
- The solid sphere of charge behaves as a point charge for all the points outside the solid sphere of charge i.e., electric field intensity is inversely proportional to the square of the distance from center of sphere of charge.

Problem 21. The drum of a photocopying machine has the length of 42 cm and diameter of 12 cm having surface charge density equal to $2 \times 10^{-6} C/m^2$. What is the total charge on the drum? (b) The manufacturer wants to produce desktop version of machine. This requires reducing the length of drum to 28 cm and diameter of 8 cm. the electric field must remain unchanged. Calculate the charge of new drum.

Solution:

(a) $\sigma = 2 \times 10^{-6} C/m^2$

$h = 42\ cm = 0.42\ m$

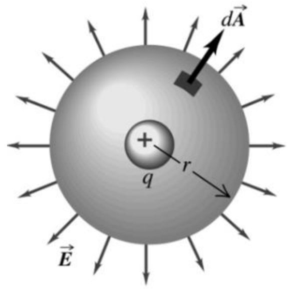
$d = 12\ cm = 0.12\ m$

$r = \frac{d}{2} = \frac{0.12}{2} = 0.06\ m$

$q = ?$

$\sigma = \frac{q}{A}$

$\Rightarrow q = \sigma A = \sigma.2\pi rh = 2 \times 10^{-6} \times 2 \times 3.14 \times 0.06 \times 0.42 = 322\ nC$



(b)

$\sigma = 2 \times 10^{-6}\ C/m^2$

$h' = 0.28\ m$

$d' = 0.08\ m$

$r' = 0.04 m$

As $\sigma = \frac{q'}{A'}$

$\Rightarrow q' = \sigma A' = \sigma.2\pi r'h' = 2 \times 10^{-6} \times 4 \times 3.14 \times 0.04 \times 0.028 = 141\ nC$

29.6 Deduction of Coulomb's Law from Gauss's Law

Coulomb's law can be deduced from Gauss's law under certain symmetry consideration. Consider positive point charge 'q'. In order to apply the Gauss's law, we assume a spherical Gaussian surface as shown in the figure below.

Considering the integral form of Gauss's law,

$$\oint \vec{E}.d\vec{A} = \frac{q}{\varepsilon_0}$$

Because the both vectors $\vec{E}$ and $d\vec{A}$are directed radially outward, so

$$\oint E\ dA \cos 0° = \frac{q}{\varepsilon_0} \qquad \therefore E \text{ is directed radially outward and } \vec{E} \parallel d\vec{A}$$

As E is constant for all the points on the spherical Gaussian surface,

$$E \oint dA = \frac{q}{\varepsilon_0}$$

$$E(4\pi r^2) = \frac{q}{\varepsilon_0} \therefore \text{ For spherical symmetry } \oint dA = 4\pi r^2$$

$$E = \frac{1}{4\pi\varepsilon_0}.\frac{q}{r^2}$$

This equation gives the magnitude of electric field intensity E at any point which is at the distance 'r' from an isolated point charge 'q'.

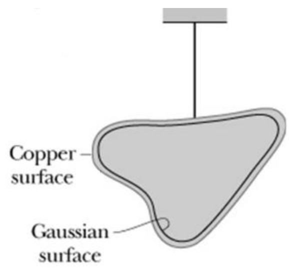
From the definition of electric field intensity, we know that

$$F = q_0 E$$

Where q_0 is the point charge placed at a point at which the value of electric field intensity has to be determined. Therefore

$$F = \frac{1}{4\pi\varepsilon_0} . \frac{qq_0}{r^2}$$

This is the mathematical form of Coulomb's la



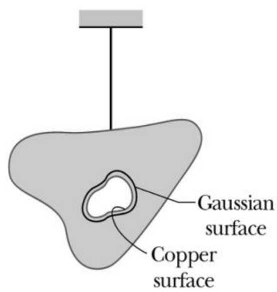
29.7 Prove that "An excess charge added to the isolated conductor moves entirely to its outer surface. None of the excess charge is found within the body of conductor".

Consider an isolated conductor (lump of copper) is hanging from a silk thread and carrying a net positive charge 'q' as shown in the figure below. The Gaussian surface lies inside the actual surface of the conductor.

Under the equilibrium conditions, the electric field inside the conductor must be zero. If it were not so, the field would exert the force on conduction electrons and the internal currents would be setup.

But there is no experimental evidence of such internal currents in isolated conductors. And if the extra charge is added to the surface, it redistribute itself on the surface in such a way that the electric field inside the conductor vanish.

If E is zero everywhere inside the conductor, it must be zero at all the points of Gaussian surface. This means that the flux through Gaussian surface must be zero. Gauss's law $(\Phi_e = \frac{q}{\varepsilon_0})$ then tells that the net charge inside the Gaussian surface must also be zero.



If the added charge is not inside the Gaussian surface, it can only be outside that surface. And the added charge must lie on the actual outer surface of the conductor.

29.8 Prove that the formation of cavity by cutting a natural material from the conductor does not change the distribution charge or pattern of electric field.

Consider an isolated conductor hanging from a silk thread carrying a net positive charge 'q'. Suppose a cavity is produced inside the conductor as shown in the figure below.

We draw a Gaussian surface around the cavity inside the conductor. Because E is zero inside the conductor, there is no flux though the Gaussian surface. So by Gauss's law, there is no net charge inside the Gaussian surface. So the total charge remains on the outer surface of conductor.

29.9 The External Electric Field

The electric field outside a charged isolated conductor can be find out by Gauss's law by considering the cylindrical Gaussian surface as shown in the figure below.

The flux through the interior end cap is zero, because E = 0 for all interior points of conductor. The flux through the cylindrical walls is also zero because the lines of E are parallel to the surface. But the flux through the outer cap will not be zero.

The total flux can then be calculated as:

$$\Phi_e = \int_{\substack{outer\\cap}} \vec{\mathrm{E}}.\,d\vec{\mathrm{A}} + \int_{\substack{inner\\cap}} \vec{\mathrm{E}}.\,d\vec{\mathrm{A}} + \int_{\substack{cylendrical\\walls}} \vec{\mathrm{E}}.\,d\vec{\mathrm{A}} \; ---- \; (1)$$

$$\int_{\substack{outer\\cap}} \vec{\mathrm{E}}.\,d\vec{\mathrm{A}} = \int_{\substack{outer\\cap}} E\,dA\cos 0° = \int_{\substack{outer\\cap}} E\,dA = E\int_{\substack{outer\\cap}} dA = EA \qquad \therefore \vec{\mathrm{E}} \parallel d\vec{\mathrm{A}} \text{for outer surface}$$

$$\int_{\substack{inner\\cap}} \vec{\mathrm{E}}.\,d\vec{\mathrm{A}} = 0 \qquad \therefore \text{As E is zero inside the body of conductor}$$

$$\int_{\substack{cylendrical\\walls}} \vec{\mathrm{E}}.\,d\vec{\mathrm{A}} = 0 \qquad \therefore \text{As}\vec{\mathrm{E}} \perp d\vec{\mathrm{A}}\text{for cylindrical walls}$$

Equation (1) will become:

$$\Phi_e = EA + 0 + 0 = EA \quad ---- \; (2)$$

According to Gauss's law

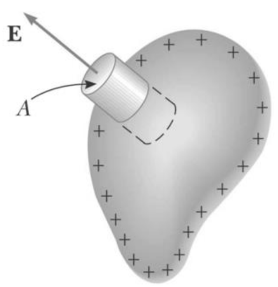
$$\Phi_e = \frac{q}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0} \quad ---- \; (3) \; \therefore q = \sigma A$$

Comparing Eq. (2) and (3), we get

$$E = \frac{\sigma}{\varepsilon_0}$$

This showed that the electric field intensity at any point is doubled than the value of E for an infinite sheet of charge.

Sample problem 3. The electric field just above the surface of the charged drum of a photo copying machine has a magnitude E of $2.3 \times 10^5 N/C$. What is the surface charge density on the drum if it is a conductor?



Solution: $E = 2.3 \times 10^5 N/C$

For conductors,

$$E = \frac{\sigma}{\varepsilon_0} \Longrightarrow \sigma = \varepsilon_0 E = 8.85 \times 10^{-12} \times 2.3 \times 10^5 = 2 \times 10^{-6}\,\frac{C}{m^2}$$

Sample problem 4. The magnitude of the average electric field normally present in the earth atmosphere just above the surface of the earth is $150\ N/C$ directed downward. What is the total net charge carried by the earth? Assume the earth to be conductor.

Solution: $E = -150\,\frac{N}{C}$

$q = ?$

We know

$$E = \frac{\sigma}{\varepsilon_0} \Longrightarrow \sigma = \varepsilon_0 E = 8.85 \times 10^{-12} \times (-150) = -1.33 \times 10^{-9}\,\frac{C}{m^2}$$

Now

$$\sigma = \frac{q}{A} = \frac{q}{4\pi r^2}$$

$$\Longrightarrow q = \sigma . 4\pi r^2 = -1.33 \times 10^{-9} \times 4 \times 3.14 \times (6.4 \times 10^6)^2 = -6.8 \times 10^5 C$$

ELECTRIC POTENTIAL

The energy approach in the study of dynamics of the particles can yield not only the simplification but also new insights.

One advantage of energy method is that, although force is a vector, energy is a scalar. In problems involving vector forces and fields, calculations involving sums and integrals are often complicated.

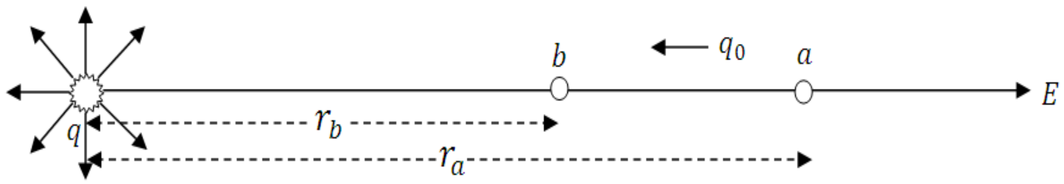
In this chapter, we introduce the energy method to the study of electrostatics.

30.1 Potential Difference

Potential difference 'ΔV' between two point is defined as "the amount of work done 'ΔW' per unit charge 'q_0' in moving it from one point to the other against the electric field and by keeping the system in equilibrium". Mathematically

$$\Delta V = \frac{\Delta W}{q_0}$$

Suppose a unit positive test charge 'q_0' is moved from one point 'a' to the point 'b' in the electric field '$\vec{E}$' of a large positive charge 'q' as shown in figure below:



The work done in moving 'q_0' from point 'a' to the point 'b' against the electric field '**E**' is

$$W_{a\to b} = \int_a^b \vec{F}.\overrightarrow{dr}$$

The electrical force of magnitude '$\vec{F} = -q_0\vec{E}$' must have to supplied in order to move 'q_0' against the electric field. Therefore

$$W_{a\to b} = \int_a^b (-q_0\vec{E}).\overrightarrow{dr}$$

$$W_{a\to b} = \int_a^b (-q_0\vec{E}).\overrightarrow{dr}$$

$$W_{a\to b} = -q_0\int_a^b \vec{E}.\overrightarrow{dr}$$

$$\frac{W_{a\to b}}{q_0} = -\int_a^b \vec{E}.\overrightarrow{dr}$$

$$\because \frac{W_{a\to b}}{q_0} = \Delta V = V_b - V_a$$

Therefore, the electrical potential difference between two points in an electrical field will be

$$V_b - V_a = -\int_a^b \vec{\mathrm{E}}.\overrightarrow{\mathrm{dr}}$$

30.2 Absolute Electrical Potential at a Point

Absolute electric potential at a point is defined as "the amount of work done per unit charge in moving it from infinity to a specific field point against the electric field and by keeping the system in equilibrium".

To find the absolute potential, the reference point is selected at which potential is zero. This point is situated at infinity i.e., out of the electric field. Thus, in equation (1)

$$V_a = V(\infty) = 0$$

$$\Rightarrow V_b - 0 = -\int_\infty^b \vec{\mathrm{E}}.\overrightarrow{\mathrm{dr}}$$

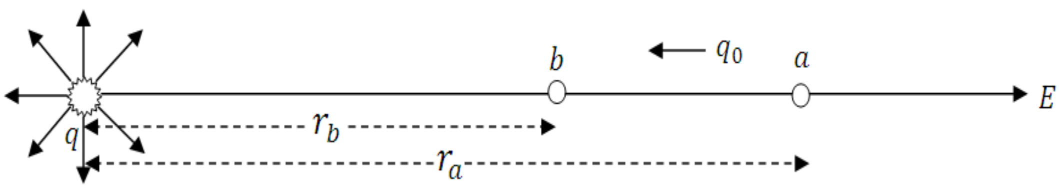
If the distance from the point 'b' to the charge 'q' is 'r', then in general

$$V(r) = -\int_\infty^r \vec{\mathrm{E}}.\overrightarrow{\mathrm{dr}}$$

30.3 Expression for the Electric Potential Difference due to a Point Charge

The potential difference between two points is the amount of work done per unit charge 'q_0' in moving it from one point to the other against the electric field 'E'. Mathematically, it is described as:

$$V_b - V_a = -\int_{r_a}^{r_b} \vec{\mathrm{E}}.\overrightarrow{\mathrm{dr}}$$



But the electric field intensity due to point charge: $\vec{\mathrm{E}} = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{q}}{\mathrm{r}^2}\hat{\mathrm{r}}$

So, $$V_b - V_a = -\int_{r_a}^{r_b} \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}.\overrightarrow{\mathrm{dr}}$$

$$\Rightarrow V_b - V_a = -\frac{q}{4\pi\varepsilon_0}\int_{r_a}^{r_b} \frac{\hat{r}.\overrightarrow{\mathrm{dr}}}{r^2}$$

$\therefore \vec{\mathrm{E}}$ *is directed radially outward, therefore* $\hat{r} \,//\, \overrightarrow{\mathrm{dr}}$

$\therefore \hat{r} \,.\, \overrightarrow{\mathrm{dr}} = |\hat{r}|\,|\overrightarrow{\mathrm{dr}}|\cos 0° = (1)(\mathrm{dr})(1) = \mathrm{dr}$

$$\Rightarrow V_b - V_a = -\frac{q}{4\pi\varepsilon_0}\int_{r_a}^{r_b}\frac{\mathrm{dr}}{r^2}$$

$$\Rightarrow V_b - V_a = -\frac{q}{4\pi\varepsilon_0}\left|-\frac{1}{r}\right|_{r_a}^{r_b} = \frac{q}{4\pi\varepsilon_0}\left|\frac{1}{r}\right|_{r_a}^{r_b}$$

$$\Rightarrow V_b - V_a = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_b} - \frac{1}{r_a}\right]$$

This is the expression for the potential difference between two points 'a' and 'b'.

30.4 Expression for the Absolute Electric Potential due to a Point Charge

The electric potential at any point is the amount of work done per unit charge in moving a unit positive charge (test charge) from infinity to that point, against the electric field.

If the point 'a' is at infinity then

$$V_a = V(\infty) = 0, \qquad r_a = \infty$$

Putting this value in equation in the expression of electric potential difference due to point charge, we get:

$$\Rightarrow V_b - 0 = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_b} - \frac{1}{\infty}\right] \therefore \frac{1}{\infty} = 0$$

$$\Rightarrow V_b = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_b}\right]$$

In general, the electric potential at point due to a point charge 'q' is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

Sample Problem 1. Two protons in the nucleus of U^{238} are 6 fm apart. What potential energy associated with the electric force that acts between them?

Solution:

$$r = 6\,fm = 6\times10^{-15}m$$

$\Delta U = ?$

$$q_1 = q_2 = 1e = 1.6\times10^{-19}C$$

As $$\Delta V = k\,\frac{q}{r} = 9\times10^9\times\frac{1.6\times10^{-19}}{6\times10^{-15}} = 2.34\times10^5$$

Now

$$\Delta U = q_1.\Delta V = 1.6\times10^{-19}\times2.34\times10^5 = 3.744\times10^{-14}\,J$$

$$\Rightarrow \Delta U = \frac{3.744\times10^{-14}}{1.6\times10^{-19}}eV = 2.34\times10^5\,eV$$

Sample problem 5. What must be the magnitude of an isolated positive point charge for the electric potential at 15 cm from the charge to be 120 V.

Solution:

$$r = 15\,cm = 15\times10^{-2}m$$

$$V = 120V$$

$$q = ?$$

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As $V = k\frac{q}{r}$

$$\Rightarrow q = \frac{V.r}{k} = \frac{120 \times 15 \times 10^{-2}}{9 \times 10^{9}} = 1.995 \times 10^{-9}C$$

Sample problem 5. What must be the magnitude of an isolated positive point charge for the electric potential at 15 cm from the charge to be 120 V.

Solution:

$r = 15\ cm = 15 \times 10^{-2}m$

$V = 120V$

$q = ?$

As $V = k\frac{q}{r} \Rightarrow q = \frac{V.r}{k} = \frac{120\times15\times10^{-2}}{9\times10^{9}} = 1.995 \times 10^{-9}C$

Sample problem 6. What is the electric potential at the surface of the gold nucleus. The radius of the gold nucleus is $\mathbf{7 \times 10^{-15}m}$ and atomic number is 79.

Solution:

$r = 7 \times 10^{-15}m$

$Z = 79$

$q = 79e = 79 \times 1.6 \times 10^{-19}C$

$V = ?$

As $V = k\frac{q}{r} = 9 \times 10^{9} \times \frac{79\times1.6\times10^{-19}}{7\times10^{-15}} = 1.6 \times 10^{7}\ V$

Sample Problem 3. An alpha particle $\boldsymbol{q = 2e}$ in a nuclear accelerator move from one terminal at a potential of $\boldsymbol{V_a = 6.5 \times 10^{6}V}$ to $\boldsymbol{V_b = 0}$. What is the corresponding change in potential energy. (b) Assuming the terminals and their charges do not move and that no external force act on this system, what is the change in kinetic energy of the particle?

Solution:

$V_a = 6.5 \times 10^{6}V$

$V_b = 0$

$\Delta U = ?$

$q = 2e$

(a) $\Delta U = q.\Delta V = q.(V_b - V_a) = 2e(0 - 6.5 \times 10^{6}) = -13 \times 10^{6}\ eV$

(b) $T.E = P.E + K.E$

$\Rightarrow 0 = P.E + K.E$

$\Rightarrow K.E = -P.E = -(-13 \times 10^{6}\ eV) = 13 \times 10^{6}\ eV$

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Problem 6. Two parallel flat conducting surfaces spacing $d = 1\ cm$ have a potential difference of 10.3 kV. An electron is projected from one plate directly towards second. What is the initial velocity of electron if it will come to rest just at the surface of the 2nd plate?

Solution: $q = 1.6 \times 10^{-19} C$

$d = 1\ cm = 0.01\ m$

$\Delta V = 10.3 \times 10^3\ V$

$velocity\ v = ?$

$\Delta U = q.\Delta V = 1.6 \times 10^{-19} \times 10.3 \times 10^3$

$Now \quad Total\ Loss\ of\ P.E = Gain\ in\ K.E$

$$\Rightarrow \Delta U = \frac{1}{2} m v^2$$

$$\Rightarrow v = \sqrt{\frac{2.\Delta U}{m}} = \sqrt{\frac{2 \times 1.6 \times 10^{-19} \times 10.3 \times 10^3}{9.1 \times 10^{-31}}} = \_\_\_\_\_\_\_\_ m/s$$

Problem 7. In a typical lighting flash, the potential difference between discharge points is about $1 \times 10^9\ V$ and the quantity of charge transfer is about 30 C. (a) How much energy is released? (b) If all energy released could be used to accelerate a 1200 kg automobile from rest, what would its final speed? (c) If it would be used to melt ice, how much ice it would melt at $0^0 C$?

Solution: (a) $\Delta V = 10^9\ V$

$q = 30C$

$\Delta U = q.\Delta V = 30 \times 10^9 J$

(b) Now

$Total\ Loss\ of\ P.E = Gain\ in\ K.E$

$$\Delta U = \frac{1}{2} m v^2 \Rightarrow v = \sqrt{\frac{2.\Delta U}{m}} = \sqrt{\frac{2 \times 30 \times 10^9}{1200}} = \_\_\_\_\_\_\_\_ m/s$$

(c) $\Delta U = mH_f \Rightarrow m = \frac{\Delta U}{H_f} = \frac{30 \times 10^9}{336000} = \_\_\_\_\_\_\_\_ kg$

Problem 56. Find (a) charge and (b) charge density on the surface of conducting sphere of radius 15.2 cm whose potential is 215 V?

Solution: $q = ?$

$\sigma = ?$

$r = 0.15\ m$

$V = 215\ V$

(a) As $V = k.\frac{q}{r}$

$$\Rightarrow q = \frac{V.r}{k} = \frac{215 \times 0.15}{9 \times 10^9} C = 3.5 \times 10^{-9} C$$

(b) $\sigma = \frac{q}{A} = \frac{q}{4\pi R^2} = \frac{3.5 \times 10^{-9}}{4 \times 3.14 \times (0.15)^2} = 12.3 \times\times 10^{-9} C$

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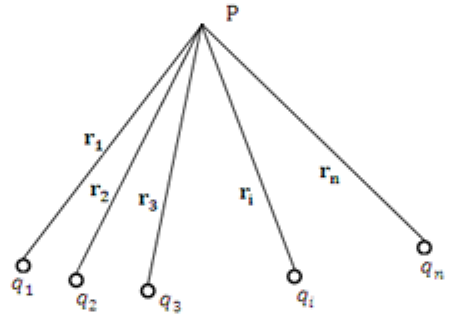
30.5 Electric Potential Due to a Collection of Point Charges

Let "$q_1, q_2, q_3, \ldots\ldots\ldots, q_n$" are the 'n' point charges which are at distances "$r_1, r_2, r_3, \ldots\ldots\ldots, r_n$" from a point '$P$', as shown in the figure:



Now if $V_1, V_2, V_3, \ldots\ldots, V_n$ be the electric potential at a field point 'P' due to the point charges $q_1, q_2, q_3, \ldots\ldots\ldots, q_n$ respectively. Then, the total electric potential at a field point 'P' due to assembly of 'n' point charges will be;

$$V = V_1 + V_2 + V_3 + \ldots\ldots + V_n \quad ---- \ (1)$$

where

V_1 = Electric Potential at a Field Point 'P' due to Point Charge 'q_1' $= \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1}$

V_2 = Electric Potential at a Field Point 'P' due to Point Charge 'q_2' $= \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2}$

V_3 = Electric Potential at a Field Point 'P' due to Point Charge 'q_3' $= \frac{1}{4\pi\varepsilon_0}\frac{q_3}{r_3}$

⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮

⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮

V_n = Electric Potential at a Field Point 'P' due to Point Charge 'q_n' $= \frac{1}{4\pi\varepsilon_0}\frac{q_n}{r_n}$

Putting values in equation (1), we get

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1} + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2} + \frac{1}{4\pi\varepsilon_0}\frac{q_3}{r_3} + \ldots\ldots + \frac{1}{4\pi\varepsilon_0}\frac{q_n}{r_n}$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \ldots\ldots + \frac{q_n}{r_n}\right)$$

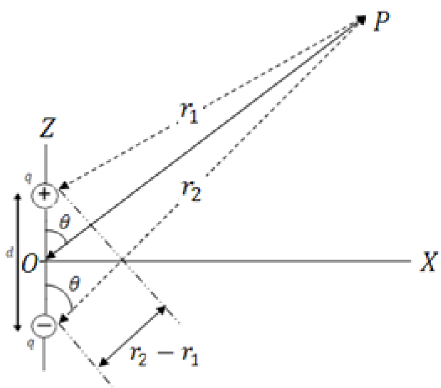
$$= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}$$

This is the net potential at any point 'P' due to collection of 'n' point charges.

30.6 Electric Potential due to a Dipole

Consider two point charges $+q$ and $-q$ of equal magnitude lying distance 'd' apart as shown in the figure.

The electric potential at point 'P' is the sum of potential due to '$+q$' and '$-q$'. If V_+ and V_- are the electric potential due to the charges $+q$ and $-q$. Therefore,



$$V = V_+ + V_-$$

$$\Rightarrow V = \frac{1}{4\pi\varepsilon_0}\frac{+q}{r_1} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{r_2}$$

$$\Rightarrow V = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

$$\Rightarrow V = \frac{q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right]$$

Normally for a dipole, $r \gg d$

Therefore, $r_1 r_2 \approx r^2$

And from figure, $r_2 - r_1 = d\,cos\theta$

$$\Rightarrow V = \frac{q}{4\pi\varepsilon_0}\left[\frac{d\,cos\theta}{r^2}\right]$$

$$\Rightarrow V = \frac{1}{4\pi\varepsilon_0}\frac{qd\,cos\theta}{r^2}$$

$$\Rightarrow V = \frac{1}{4\pi\varepsilon_0}\frac{p\,cos\theta}{r^2} \qquad \therefore p = qd = Dipole\ Moment$$

$$\Rightarrow V = \frac{1}{4\pi\varepsilon_0}\frac{p\,r\,cos\theta}{r^3}$$ ∴ Multiplying and dividing by r

$$\Rightarrow V = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}.\vec{r}}{r^3}$$

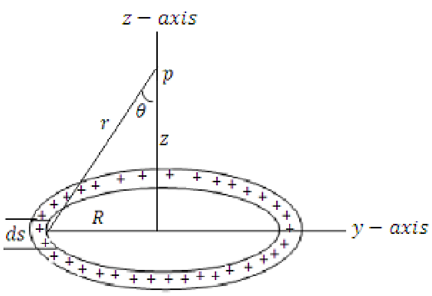
This is the expression of electric potential at any point 'P' due to a dipole.

30.7 Electric Potential of Continuous Charge Distribution

30.7.1 Electric Potential due to a Ring of Charge

Consider a ring of positive charge of radius 'R' as shown in the figure. We want to find out the electric potential at point 'P' which is at the distance 'z' from the plane of ring.

As the charge is distributed uniformly over it, so it has constant linear charge density λ. For an infinitesimal length element 'ds' of ring,



$$\lambda = \frac{dq}{ds}$$

$$dq = \lambda\, ds$$

The electric potential due to the charge dq at point 'P' is given by:

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, ds}{r}$$

$$\Rightarrow dV = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, ds}{\sqrt{z^2 + R^2}} \qquad \therefore From\ Figure, r = \sqrt{z^2 + R^2}$$

The total value of electric potential is

$$V = \int dV = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda\, ds}{\sqrt{z^2 + R^2}}$$

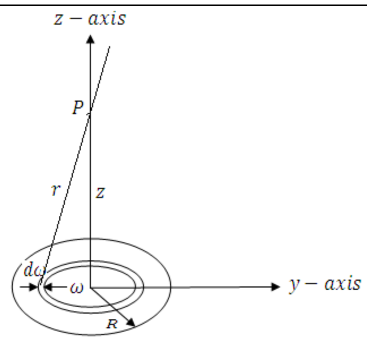
$$\Rightarrow V = \frac{1}{4\pi\varepsilon_0}\frac{1}{\sqrt{z^2 + R^2}}\int \lambda\, ds \qquad \therefore \int \lambda\, ds = q$$

Therefore,

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{z^2 + R^2}}$$

This is the expression of electric potential at any point 'P' due to a ring of charge.

30.7.2 Electric Potential due to a Disk of Charge



Consider a circular disk of uniform surface charge density as shown in the figure below. We want to find out the electric potential at point 'P' which is at the distance 'z' from the plane of disk.

Consider a small element of the disk in the ring shape of radius 'ω' and width '$d\omega$'. If 'dq' is the charge on this element of ring, then

$$\sigma = \frac{dq}{dA} \qquad \therefore \textit{where } dA \textit{ is the is the area of length element}$$

$$\Rightarrow dq = \sigma\, dA$$

$$\Rightarrow dq = \sigma\,(2\pi\omega\, d\omega) \qquad \therefore dA = 2\pi\omega\, d\omega$$

We know the electric potential due to the ring of charge which has the radius ω is given by;

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r}$$

$$\Rightarrow dV = \frac{1}{4\pi\varepsilon_0}\frac{\sigma\,(2\pi\omega\, d\omega)}{z^2 + \omega^2}$$

The electrical potential at point 'P' due to whole disk is:

$$V = \frac{1}{4\pi\varepsilon_0}\int_{\omega=0}^{\omega=R}\frac{\sigma\,(2\pi\omega\, d\omega)}{z^2 + \omega^2}$$

$$\Rightarrow V = \frac{\sigma}{4\varepsilon_0}\int_{\omega=0}^{\omega=R}(z^2 + \omega^2)^{-\frac{1}{2}}(2\omega)\, d\omega$$

$$\Rightarrow V = \frac{\sigma}{4\varepsilon_0}\left|\frac{(z^2 + \omega^2)^{\frac{-1}{2}+1}}{\frac{1}{2}}\right|_0^R$$

$$\Rightarrow V = \frac{\sigma}{2\varepsilon_0}\left[(z^2 + R^2)^{\frac{1}{2}} - (z^2)^{\frac{1}{2}}\right]$$

$$\Rightarrow V = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{z^2 + R^2} - z\right]$$

This is the expression of electric potential at point 'P' due to disk of charge.

Special Case:

If $z \gg R$, then $\sqrt{z^2 + R^2} = (z^2 + R^2)^{\frac{1}{2}} = z\left[1 + \frac{R^2}{z^2}\right]^{\frac{1}{2}} = z\left[1 + \frac{1}{2}\frac{R^2}{z^2} + \ldots\ldots\ldots\right]$

$$\Rightarrow \sqrt{z^2 + R^2} = \left[z + \frac{R^2}{2z}\right] \qquad \therefore \textit{Neglecting Higher Terms}$$

$$So, V = \frac{\sigma}{2\varepsilon_0}\left[z + \frac{R^2}{2z} - z\right] = \frac{\sigma R^2}{4\varepsilon_0\, z} = \frac{\sigma\, \pi R^2}{4\pi\varepsilon_0\, z} = \frac{\sigma\,(\pi R^2)}{4\pi\varepsilon_0\, z} \qquad \therefore \sigma\,(\pi R^2) = q\ (\textit{total charge on disk})$$

$$\Rightarrow V = \frac{1}{4\pi\varepsilon_0}\frac{q}{z}$$

Thus, the disk of charge behave like a point charge for the case $z \gg R$.

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30.8 Equipotential Surfaces

If all the points on a surface have the same value of electric potential, then, it is known as equipotential surface. The examples of some equipotential surfaces are given below

- In case of uniform electric field, the equipotential surfaces are the planes as shown by the dashed line in fig. (a). The potential at points B_1, B_2 and B_3 is same. So, no work will be done in moving a test charge from B_1 to B_2 or B_3.
- The concentric spheres about a point charge '$+q$' are the equipotential surfaces as shown in the fig. (b) by dashed lines.
- The Equipotential surfaces of a dipole are shown by the dashed lines in fig. (c)

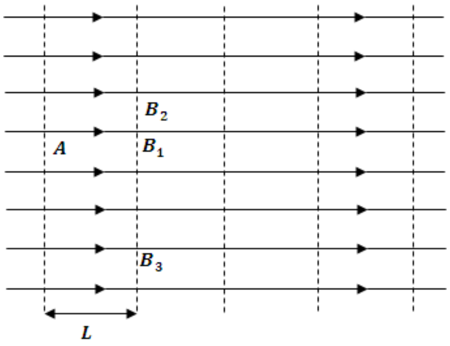


Fig. (a)

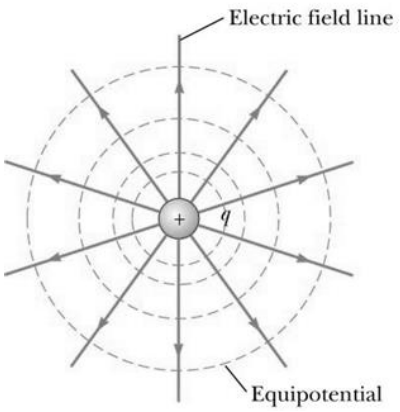


Fig. (b)

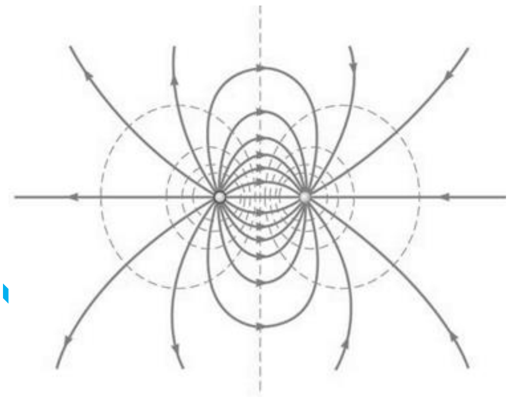
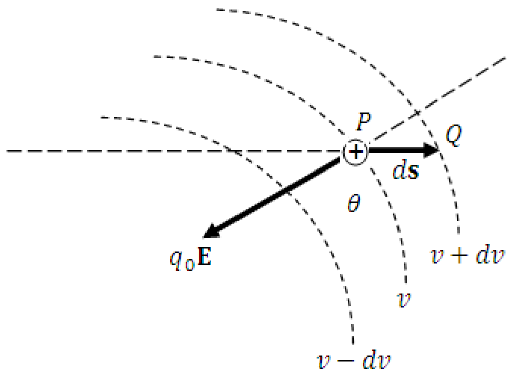


Fig. (c)

30.9 Calculating Electric Field from the Electric Potential



Consider a test charge 'q_0' is displaced in an electric field form point 'P' to point 'Q'. If the potential difference between the points 'P' and 'Q' is dV, then the work done 'dW' by the electric field is

$$dW = -q_0 dV \quad ---- \quad (1)$$

If the test charge is placed in an electric field, then the electrical force on the test charge is

$$\vec{F} = q_0\vec{E}$$

So, the corresponding work done in diplacing it through displacemnet dx:

$$dW = \vec{F}.\vec{ds}$$

$$= q_0\vec{E}.\vec{ds} \quad ---- \quad (2) \qquad \therefore \vec{F} = q_0\vec{E}$$

Comparing eq. (1) and (2)

$$q_0\vec{E}.\vec{ds} = -q_0 dV$$

$$q_0 E\, ds \cos\theta = -q_0 dV$$

$$E\cos\theta = -\frac{dV}{ds}$$

Consider '$E\cos\theta = E_s$' which is the component of '**E**' along '$d\mathbf{s}$'

$$E_s = -\frac{dV}{ds}$$

There is only one direction in which the rate of change of electric potential w r t position is maximum, which is direction of electric field. Thus

$$E = -\left(\frac{dV}{ds}\right)_{max}$$

The maximum value of $\frac{dV}{ds}$ at a given point is called the 'potential gradient. Therefore,

$$E = -\vec{\nabla}V$$

$$\Longrightarrow E = -\text{grad } V$$

Hence proved that the electric field is negative gradient of electric potential.

Question: Find the expression of electric field from electric potential due to a point charge.

Solution: the electric potential due to a point charge is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{x^2 + y^2 + z^2}} ----(1) \qquad \because r = \sqrt{x^2 + y^2 + z^2}$$

Now $\quad \text{grad}V = (\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k})\, V$

$$= (\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k})$$

Now consider $\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{x^2 + y^2 + z^2}}\right]$

$$= \frac{q}{4\pi\varepsilon_0}\frac{-\frac{1}{2}\,(2x)}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{q}{4\pi\varepsilon_0}\frac{x}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{q}{4\pi\varepsilon_0}\frac{x}{r^3}$$

Similarly

$$\frac{\partial V}{\partial y} = -\frac{q}{4\pi\varepsilon_0}\frac{y}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{q}{4\pi\varepsilon_0}\frac{y}{r^3}$$

$$\frac{\partial V}{\partial z} = -\frac{q}{4\pi\varepsilon_0}\frac{z}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{q}{4\pi\varepsilon_0}\frac{z}{r^3}$$

Putting these values in eq. (3)

$$\text{grad}V = -\frac{q}{4\pi\varepsilon_0}\left[\frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{r^3}\right]$$

$$\Longrightarrow \text{grad}V = -\frac{q}{4\pi\varepsilon_0}\left[\frac{\vec{r}}{r^3}\right] \qquad \because \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\Longrightarrow \text{grad}V = -\frac{q}{4\pi\varepsilon_0}\left[\frac{r\,\hat{r}}{r^3}\right] \qquad \because \vec{r} = r\,\hat{r}$$

,

$$\Rightarrow \text{grad}V = -\frac{q}{4\pi\varepsilon_0}\left[\frac{\hat{r}}{r^2}\right]$$

$$\Rightarrow -\text{grad}V = \frac{q}{4\pi\varepsilon_0}\left[\frac{\hat{r}}{r^2}\right]$$

$$\vec{\text{E}} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\hat{r}}{r^2}\right] \qquad \therefore \vec{\text{E}} = -\text{grad}V$$

which is the expression of electric field intensity due to a point charge.

Problem. Using $V = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+Z^2} - Z\right)$, find the values of electric field of uniformly charged disk.

Solution:

From statement

$$V = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+Z^2} - Z\right)$$

From symmetry, E must lie along axis of the disk i.e., along z-axis.

$$E_z = -\frac{\partial V}{\partial Z} = -\frac{\partial}{\partial Z}\left[\frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+Z^2} - Z\right)\right] = -\frac{\sigma}{2\varepsilon_0}\frac{\partial}{\partial Z}\left(\sqrt{R^2+Z^2} - Z\right)$$

$$\Rightarrow E_z = -\frac{\sigma}{2\varepsilon_0}\left(\frac{1}{2}(R^2+Z^2)^{-1/2}.2Z - 1\right)$$

$$\Rightarrow E_z = -\frac{\sigma}{2\varepsilon_0}\left(\frac{Z}{\sqrt{R^2+Z^2}} - 1\right) = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{Z}{\sqrt{R^2+Z^2}}\right)$$

Problem 54. If the earth had a net charge equivalent to $1\frac{e}{m^2}$ of the surface area. (a) What would be the earth potential? (b) What would be the electric field earth just outside?

Solution:

$$\sigma = 1\frac{e}{m^2} = 1.6\times10^{-19}\frac{C}{m^2}$$

Radius of the earth $R = 6.4\times10^6 m$

$$V = ?$$

$$E = ?$$

As

$$\sigma = \frac{q}{A} \Rightarrow q = \sigma A = \sigma.4\pi R^2 = 1.6\times10^{-19}\times4\times3.14\times(6.4\times10^6)^2$$

Now

$$V = k.\frac{q}{r} = 9\times10^9\times\frac{1.6\times10^{-19}\times4\times3.14\times(6.4\times10^6)^2}{6.4\times10^6} = 1201\times10^{-4}\,V$$

And $E = \frac{V}{d} = \frac{1201\times10^{-4}}{6.4\times10^6} = 1.80\times10^{-8}\frac{N}{C}$

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30.10 Electric Potential and Electric Field Inside and Outside an Isolated Conductor

Consider an isolated conductor in the form of a spherical shell having uniform surface charge density.

Electric Field Inside and Outside an Isolated Conductor

- If the excess of charge is placed on an isolated conductor, then it moves entirely on to the outer surface of the conductor. In equilibrium, none of the charge is inside the body of conductor or on any interior surface (even the conductor has internal cavities). Thus the electric field at any point inside sphere is zero.
- The electric field has maximum on the surface of the spherical shell.
- The electric field decreases as we move away from the sphere because

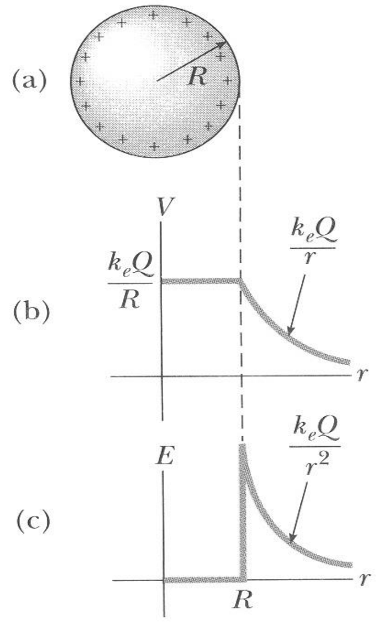
$$E \propto \frac{1}{r^2}$$

Electric Potential Inside and Outside an Isolated Conductor

- As the excess of charge placed on an isolated conductor distributes itself on the surface so that all points of conductor come to the same potential. So the electric potential remains constant at every point inside the sphere. Also, if a test charge is pushed inside the sphere through the hole, then it experiences no force and no work will be done.
- The value of electric potential is maximum at the surface of sphere.
- The electric potential decreases as we move away from the sphere, because

$$V \propto \frac{1}{r}$$

The variation of electric potential and electric field with respect to 'r' is shown in the figure:



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