Discrete Structures

Previous Lecture Summery

- Sum/Difference of Two Functions
- Equality of Two Functions
- One-to-One Function
- Onto Function
- Bijective Function (One-to-One correspondence)

Today's Lecture

- Inverse Functions
- Finding an Inverse Function
- Composition of Functions
- Composition of Functions
- Composition of Functions defined on finite sets
- Plotting Functions

Inverse Functions

Theorem

Suppose $F: X \to Y$ is a one-to-one correspondence; that is, suppose F is one-to-one and onto. Then there is a function $F^{-1}: Y \to X$ that is defined as follows:

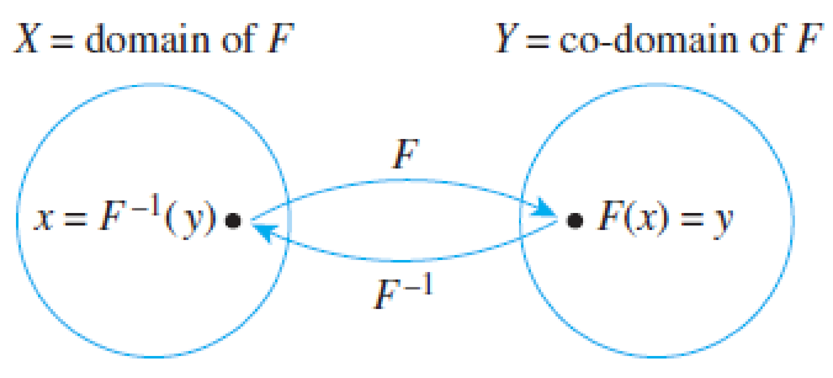
Given any element y in Y,

$$F^{-1}(y) = \text{that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y.$$

In other words,

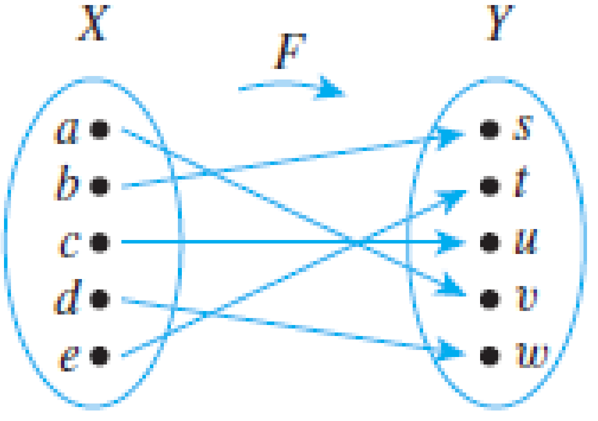
$$F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).$$

The function F^{-1} is called inverse function.



Inverse Functions

Given an arrow diagram for a function. Draw the arrow diagram for the inverse of this function



Finding an Inverse Function

The function $f : \mathbf{R} \to \mathbf{R}$ defined by the formula

$$f(x) = 4x - 1, \quad \text{for all real numbers } x$$

Solution For any *[particular but arbitrarily chosen]* y in $\mathbf{R}$, by definition of f^{-1},

$$f^{-1}(y) = \text{that unique real number } x \text{ such that } f(x) = y.$$

But

$$\begin{aligned} & f(x) = y \\ \Leftrightarrow \quad & 4x - 1 = y && \text{by definition of } f \\ \Leftrightarrow \quad & x = \frac{y+1}{4} && \text{by algebra.} \end{aligned}$$

Hence $f^{-1}(y) = \dfrac{y+1}{4}$.

Theorem

If X and Y are sets and $F: X \to Y$ is one-to-one and onto, then $F^{-1}: Y \to X$ is also one-to-one and onto.

Proof:

F^{-1} is one-to-one: Suppose y_1 and y_2 are elements of Y such that $F^{-1}(y_1) = F^{-1}(y_2)$. *[We must show that $y_1 = y_2$.]* Let $x = F^{-1}(y_1) = F^{-1}(y_2)$. Then $x \in X$, and by definition of F^{-1},

$$F(x) = y_1 \quad \text{since } x = F^{-1}(y_1)$$

and

$$F(x) = y_2 \quad \text{since } x = F^{-1}(y_2).$$

Consequently, $y_1 = y_2$ since each is equal to $F(x)$. This is what was to be shown.

F^{-1} is onto: Suppose $x \in X$. *[We must show that there exists an element y in Y such that $F^{-1}(y) = x$.]* Let $y = F(x)$. Then $y \in Y$, and by definition of F^{-1}, $F^{-1}(y) = x$. This is what was to be shown.

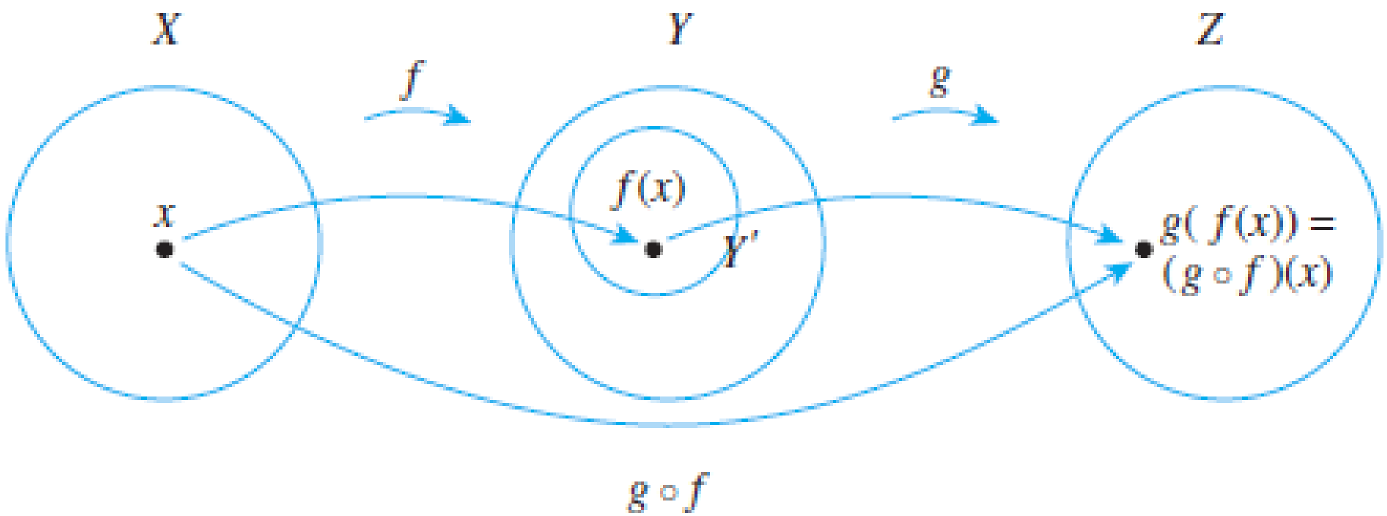
Composition of Functions

• Definition

Let $f: X \to Y'$ and $g: Y \to Z$ be functions with the property that the range of f is a subset of the domain of g. Define a new function $g \circ f: X \to Z$ as follows:

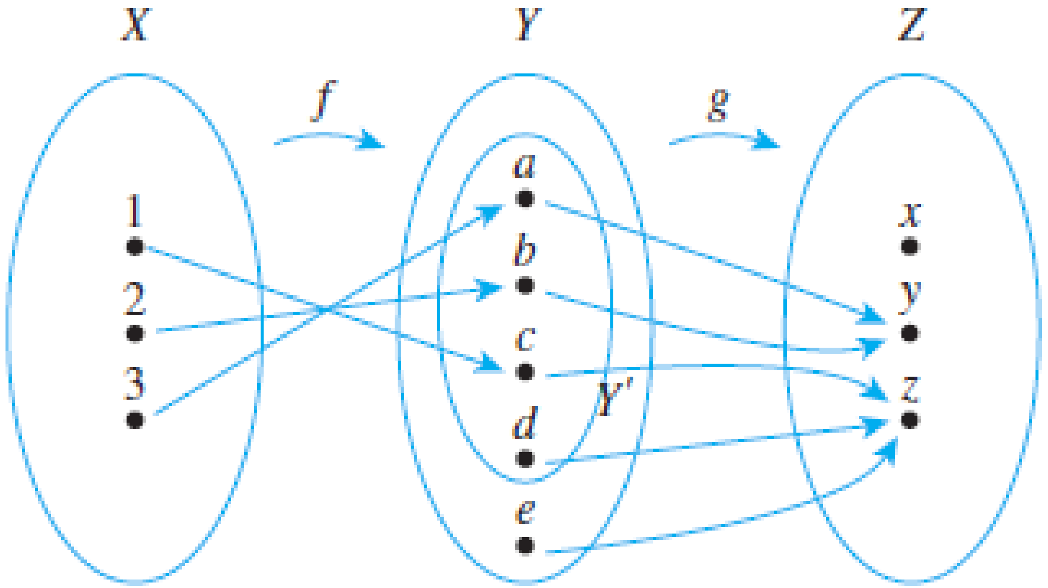
$$(g \circ f)(x) = g(f(x)) \quad \text{for all } x \in X,$$

where $g \circ f$ is read "g circle f" and $g(f(x))$ is read "g of f of x." The function $g \circ f$ is called the **composition of f and g.**



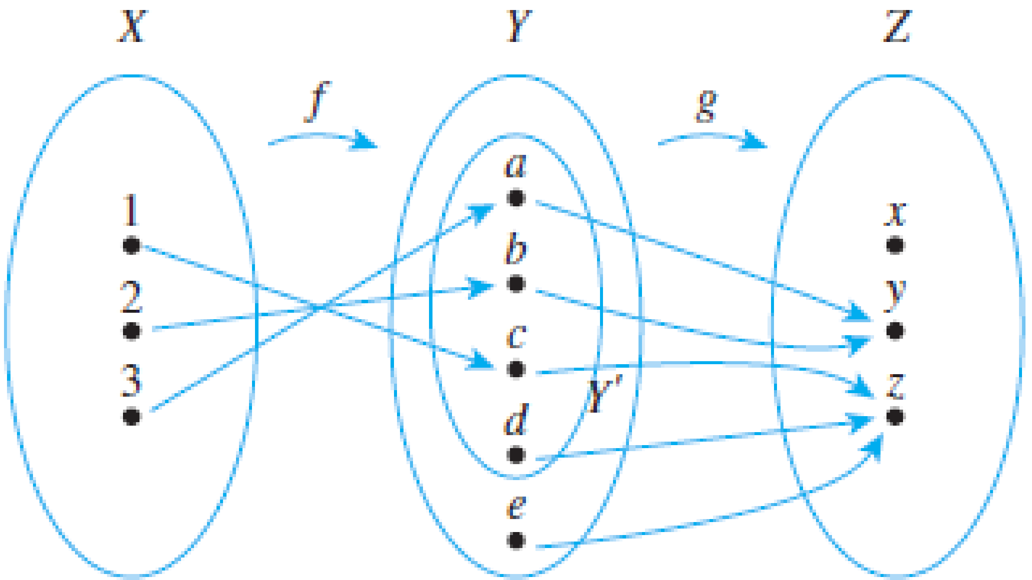
Composition of Functions defined on finite sets

Let $X = \{1, 2, 3\}$, $Y' = \{a, b, c, d\}$, $Y = \{a, b, c, d, e\}$, and $Z = \{x, y, z\}$. Define functions $f : X \to Y'$ and $g: Y \to Z$ by the arrow diagrams below.



Draw the arrow diagram for $g \circ f$. What is the range of $g \circ f$?

Composition of Functions defined on finite sets



$$(g \circ f)(1) = g(f(1)) = g(c) = z$$

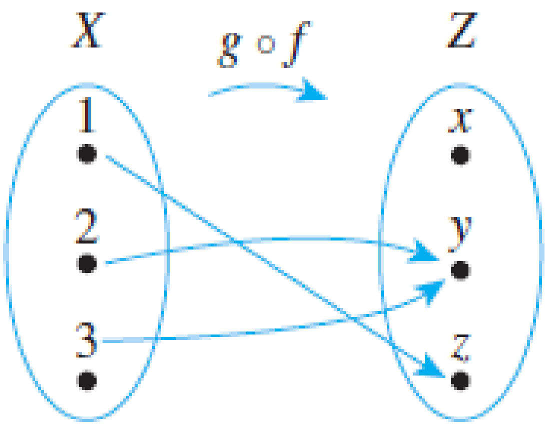
$$(g \circ f)(2) = g(f(2)) = g(b) = y$$

$$(g \circ f)(3) = g(f(3)) = g(a) = y$$

Composition of Functions defined on finite sets

Let $X = \{1, 2, 3\}$, $Y' = \{a, b, c, d\}$, $Y = \{a, b, c, d, e\}$, and $Z = \{x, y, z\}$.

To find the arrow diagram for $g \circ f$, just trace the arrows all the way across from X to Z through Y . The result is shown below.



$$(g \circ f)(1) = g(f(1)) = g(c) = z$$
$$(g \circ f)(2) = g(f(2)) = g(b) = y$$
$$(g \circ f)(3) = g(f(3)) = g(a) = y$$

The range of $g \circ f$ is $\{y, z\}$.

Composition of Functions defined on Infinite Sets

Let $f : \mathbf{Z} \rightarrow \mathbf{Z}$, and $g: \mathbf{Z} \rightarrow \mathbf{Z}$ be two functions. s.t.,

$f(n) = n + 1$ for all $n \in \mathbf{Z}$ and $g(n) = n^2$ for all $n \in \mathbf{Z}$.

a. Find the compositions $g \circ f$ and $f \circ g$.

b. Is $g \circ f = f \circ g$? Explain.

a. The functions $g \circ f$ and $f \circ g$ are defined as follows:

$(g \circ f)(n) = g(f(n)) = g(n + 1) = (n + 1)^2$ for all $n \in \mathbf{Z}$,

$(f \circ g)(n) = f(g(n)) = f(n^2) = n^2 + 1$ for all $n \in \mathbf{Z}$.

Composition of Functions defined on Infinite Sets

Let $f : \mathbf{Z} \to \mathbf{Z}$, and $g: \mathbf{Z} \to \mathbf{Z}$ be two functions. s.t., $f(n) = n + 1$ for all $n \in \mathbf{Z}$ and $g(n) = n^2$ for all $n \in \mathbf{Z}$.

b. Is $g \circ f = f \circ g$? Explain.

b. Two functions from one set to another are equal if, and only if, they always take the same values. In this case,

$$(g \circ f)(1) = (1+1)^2 = 4, \text{ whereas } (f \circ g)(1) = 1^2 + 1 = 2.$$

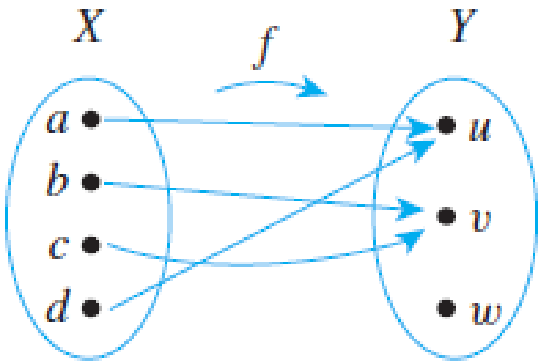
Thus the two functions $g \circ f$ and $f \circ g$ are not equal:

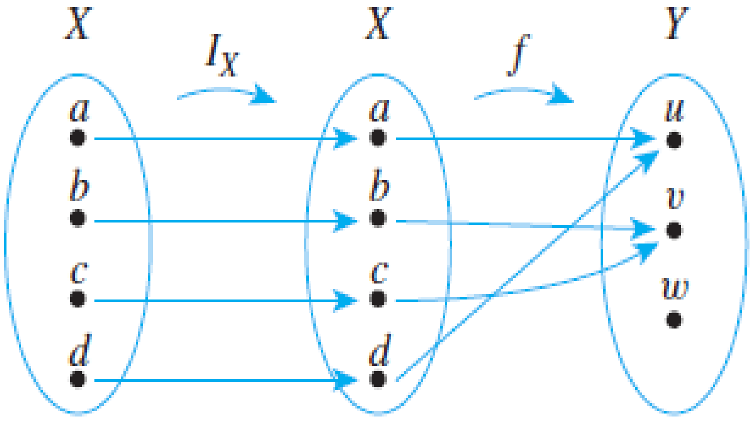
$$g \circ f \neq f \circ g.$$

Composition with Identity Function

Let $X = \{a, b, c, d\}$ and $Y = \{u, v, w\}$, and suppose $f : X \to Y$ is given by the arrow diagram

Find $f \circ I_X$ and $I_Y \circ f$.





$$(f \circ I_X)(a) = f(I_X(a)) = f(a) = u$$
$$(f \circ I_X)(b) = f(I_X(b)) = f(b) = v$$
$$(f \circ I_X)(c) = f(I_X(c)) = f(c) = v$$
$$(f \circ I_X)(d) = f(I_X(d)) = f(d) = u$$

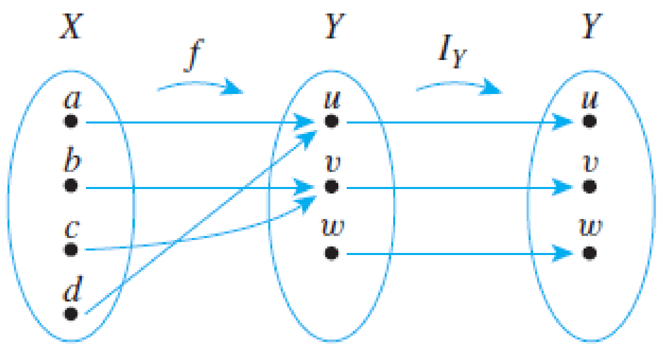
Composition with Identity Function

Note that for all elements x in X,

$$(f \circ I_X)(x) = f(x).$$

By definition of equality of functions, this means that $f \circ I_X = f$.

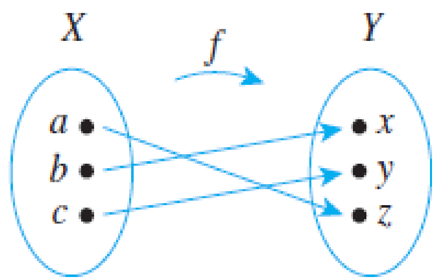
Similarly, the equality $I_Y \circ f = f$ can be verified by tracing through the arrow diagram below for each x in X and noting that in each case, $(I_Y \circ f)(x) = f(x)$.



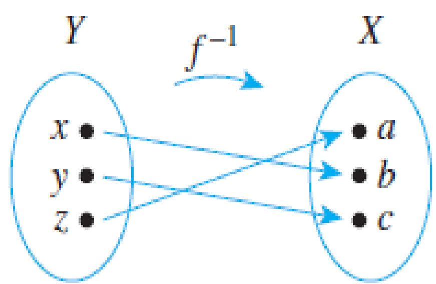
More generally, the composition of any function with an identity function equals the function.

Composing a Function with its Inverse

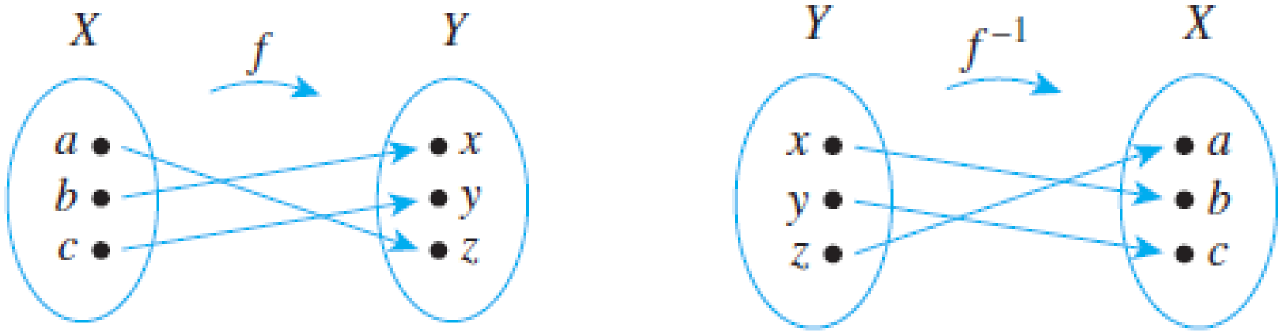
Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Define $f: X \to Y$ by the following arrow diagram.



Then f is one-to-one and onto. Thus f^{-1} exists and is found by tracing the arrows backwards, as shown below.



Composing a Function with its Inverse



$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(z) = a$$
$$(f^{-1} \circ f)(b) = f^{-1}(f(b)) = f^{-1}(x) = b$$
$$(f^{-1} \circ f)(c) = f^{-1}(f(c)) = f^{-1}(y) = c.$$

$$f^{-1} \circ f = I_X.$$

In a similar way, we can show that

$$f \circ f^{-1} = I_Y.$$

Composition of One-to-One Functions

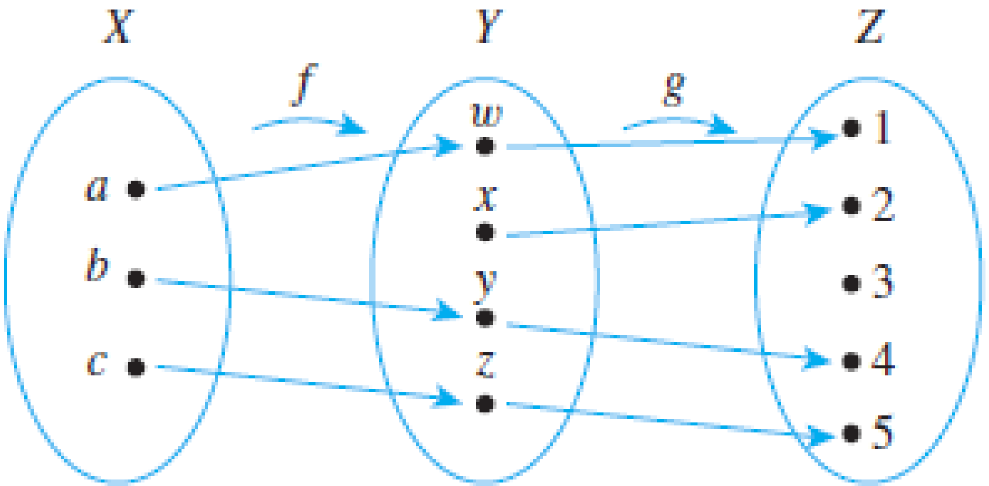


TABLE 1.1 Types of intervals

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \leq x \leq b\}$	Closed	
	$[a, b)$	$\{x \mid a \leq x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \leq b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \geq a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \leq b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

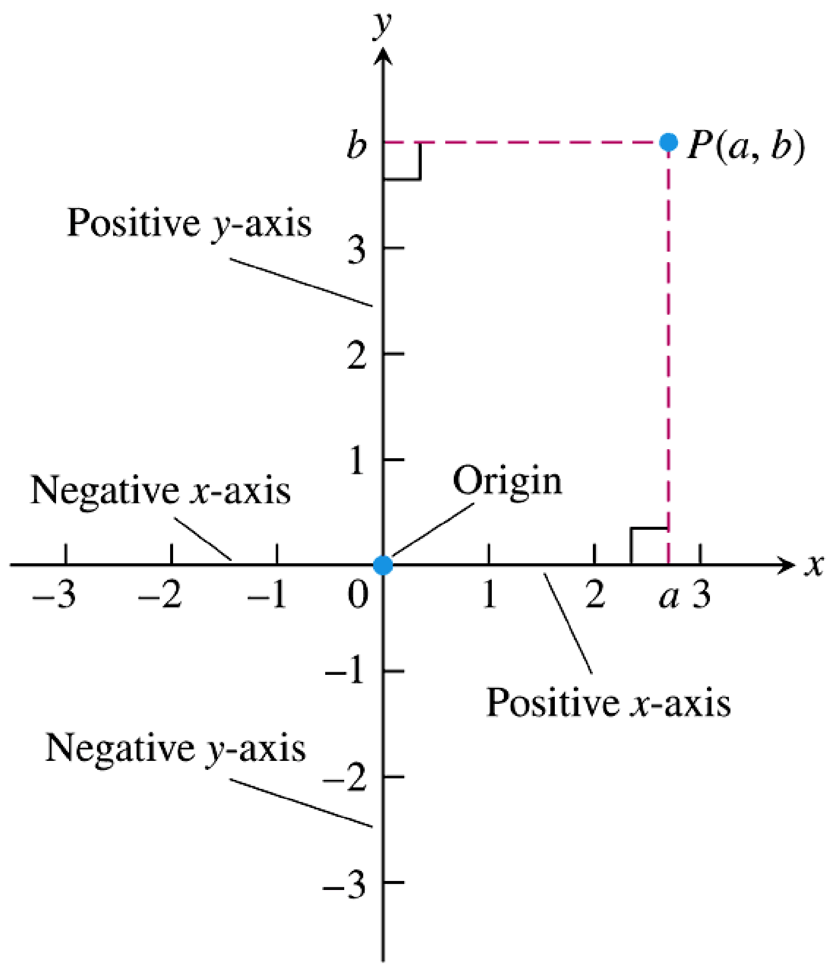
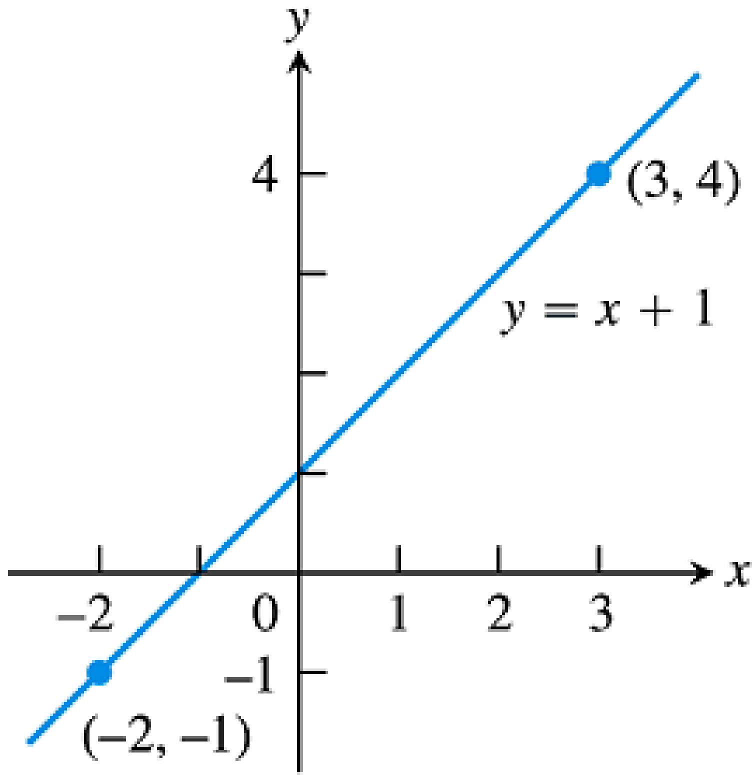
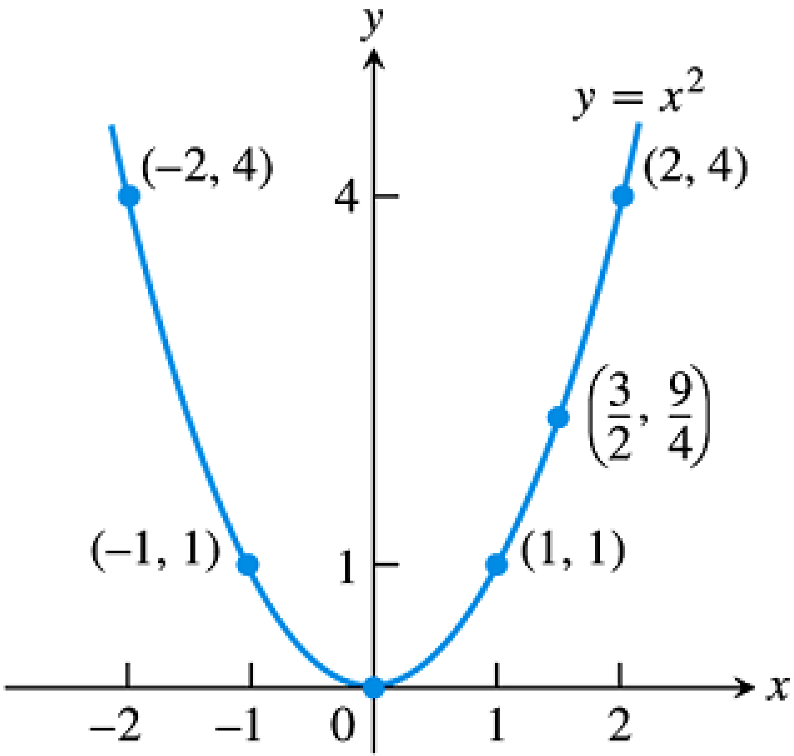


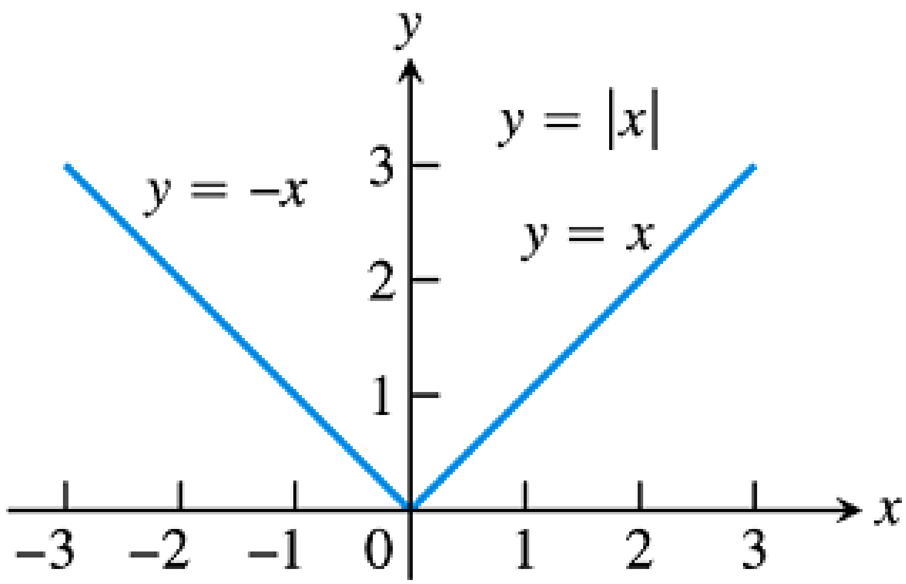
FIGURE 1.5 Cartesian coordinates in the plane are based on two perpendicular axes intersecting at the origin.


y
4
(3, 4)
y = x + 1
x
–2
0
1
2
3
–1
(–2, –1)

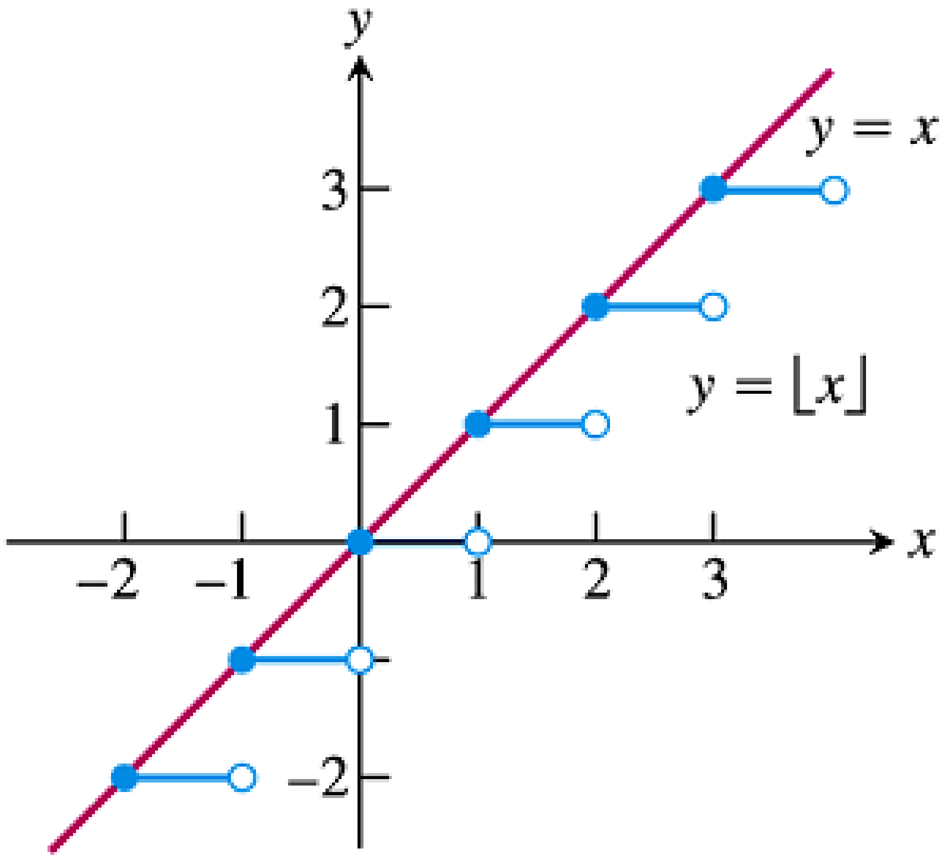


y
$y = x^2$
(−2, 4)
(2, 4)
4
$\left(\frac{3}{2}, \frac{9}{4}\right)$
(−1, 1)
(1, 1)
1
x
−2
−1
0
1
2

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$



Domain = $(-\infty, \infty)$ Range = $[0, \infty)$



y
x
y = x
y = ⌊x⌋
3
2
1
−2
−2
−1
1
2
3

Lecture Summery

- One-to-One Function
- Onto Function
- Bijective Function (One-to-One correspondence)
- Inverse Functions