Discrete Structures

Previous Lecture Summery

- Relations and Functions
- Definition of Function
- Examples of Functions

Functions

Today’s Lecture

- One-to-One Function
- Onto Function
- Bijective Function (One-to-One correspondence)
- Inverse Functions

Sum/difference of Functions

Let F: $\mathbf{R} \to \mathbf{R}$ and G: $\mathbf{R} \to \mathbf{R}$ be functions. Define new functions $F + G$: $\mathbf{R} \to \mathbf{R}$: For all $x \in \mathbf{R}$,

$$(F + G)(x) = F(x) + G(x)$$

F and G must have same Domains and Codomains.

Equality of Functions

Theorem: If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

Example

Let $J_3 = \{0, 1, 2\}$, and define functions f and g from J_3 to J_3 as follows: For all x in J_3,

$$f(x) = (x^2 + x + 1)\ mod\ 3 \quad \text{and} \quad g(x) = (x + 2)^2\ mod\ 3.$$

Does $f = g$?

x	x^2+x+1	$f(x)=(x^2+x+1)\ mod\ 3$	$(x+2)^2$	$g(x)=(x+2)^2\ mod\ 3$
0	1	$1\ mod\ 3=1$	4	$4\ mod\ 3=1$
1	3	$3\ mod\ 3=0$	9	$9\ mod\ 3=0$
2	7	$7\ mod\ 3=1$	16	$16\ mod\ 3=1$

One-to-One Functions

- **Definition**

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

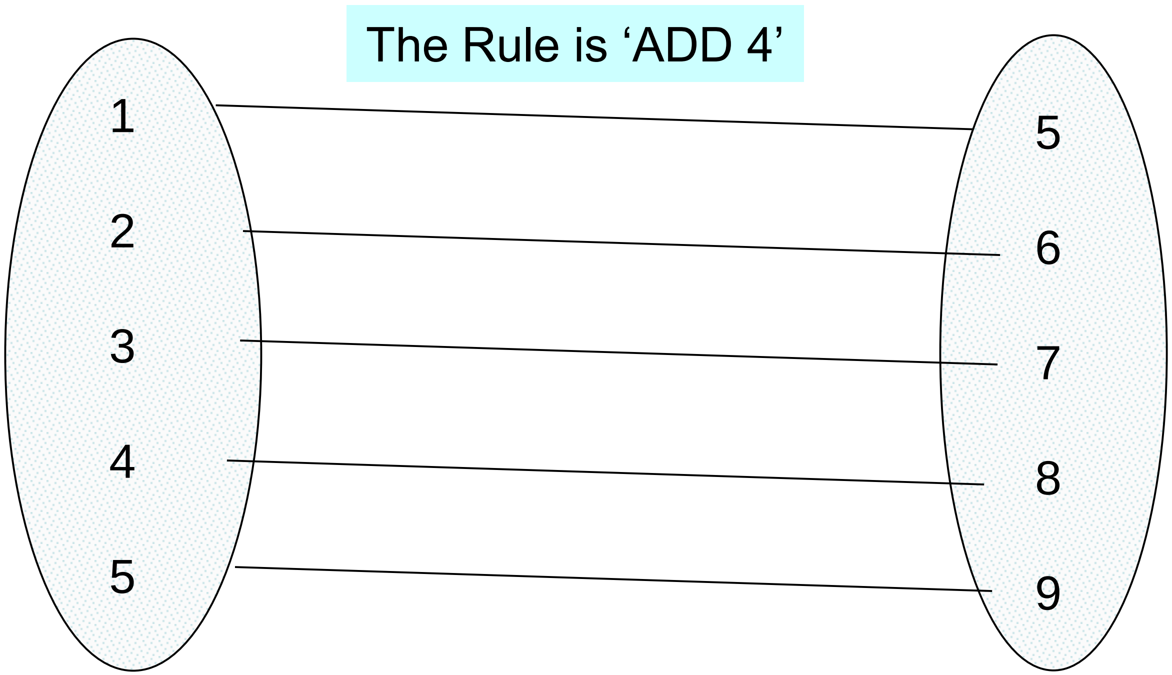
or, equivalently,

$$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

Symbolically,

$$F\colon X \to Y \text{ is one-to-one} \quad \Leftrightarrow \quad \forall x_1, x_2 \in X, \text{ if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

One-to-One Functions

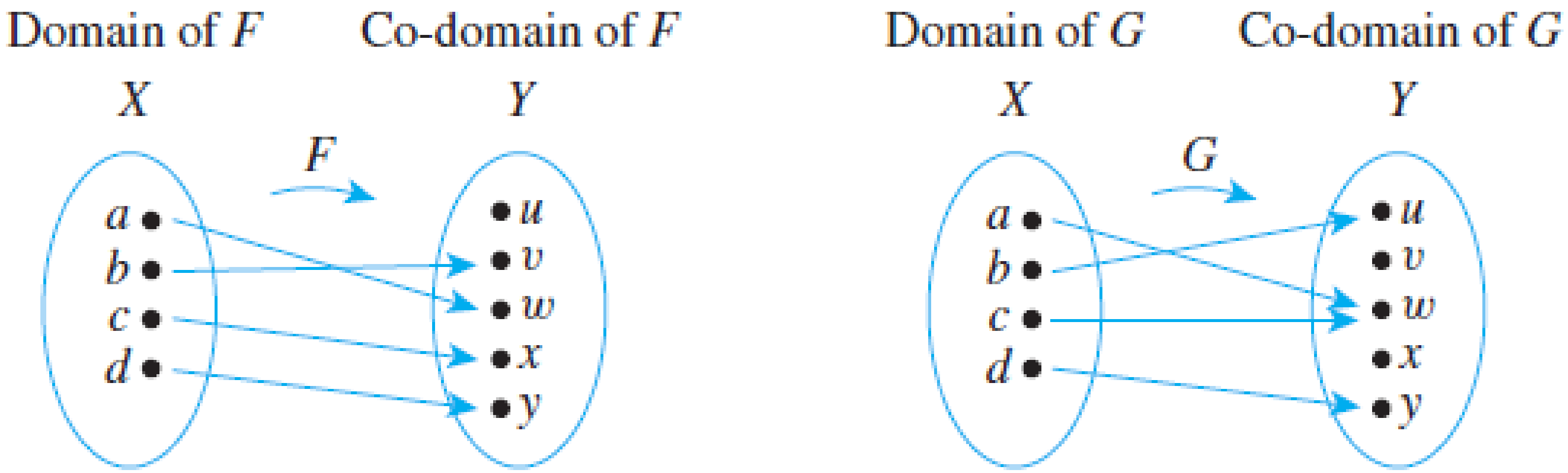


Dom (R) = {1, 2, 3, 4, 5}

Codomain(R)={5, 6, 7, 8, 9,10}

One-to-One Functions

Identifying One-to-One functions defined on sets



One-to-One Functions on Infinite Sets

Now suppose f is a function defined on an infinite set X. By definition, f is one-to-one if, and only if, the following universal statement is true:

$$\forall x_1, x_2 \in X, \text{if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

Thus, to prove f is one-to-one, you will generally use the method of direct proof:

suppose x_1 and x_2 are elements of X such that $f(x_1) = f(x_2)$

and **show** that $x_1 = x_2$.

To show that f is *not* one-to-one, you will ordinarily

find elements x_1 and x_2 in X so that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

One-to-One Functions on Infinite Sets

Define $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{Z} \to \mathbf{Z}$ by the rules

$$f(x) = 4x - 1 \quad \text{for all} \quad x \in \mathbf{R}$$

and

$$g(n) = n^2 \quad \text{for all} \quad n \in \mathbf{Z}.$$

a. Is f one-to-one? Prove or give a counterexample.

b. Is g one-to-one? Prove or give a counterexample.

One-to-One Functions on Infinite Sets

If the function $f: \mathbf{R} \to \mathbf{R}$ is defined by the rule $f(x) = 4x - 1$, for all real numbers x, then f is one-to-one.

Proof:

Suppose x_1 and x_2 are real numbers such that $f(x_1) = f(x_2)$. *[We must show that $x_1 = x_2$.]* By definition of f,

$$4x_1 - 1 = 4x_2 - 1.$$

Adding 1 to both sides gives

$$4x_1 = 4x_2,$$

and dividing both sides by 4 gives

$$x_1 = x_2,$$

which is what was to be shown.

One-to-One Functions on Infinite Sets

If the function $g: \mathbf{Z} \to \mathbf{Z}$ is defined by the rule $g(n) = n^2$, for all $n \in \mathbf{Z}$, then g is not one-to-one.

Counterexample:

Let $n_1 = 2$ and $n_2 = -2$. Then by definition of g,

$$g(n_1) = g(2) = 2^2 = 4 \quad \text{and also}$$
$$g(n_2) = g(-2) = (-2)^2 = 4.$$

Hence $$g(n_1) = g(n_2) \quad \text{but} \quad n_1 \neq n_2,$$

and so g is not one-to-one.

Onto Functions on Sets

- **Definition**

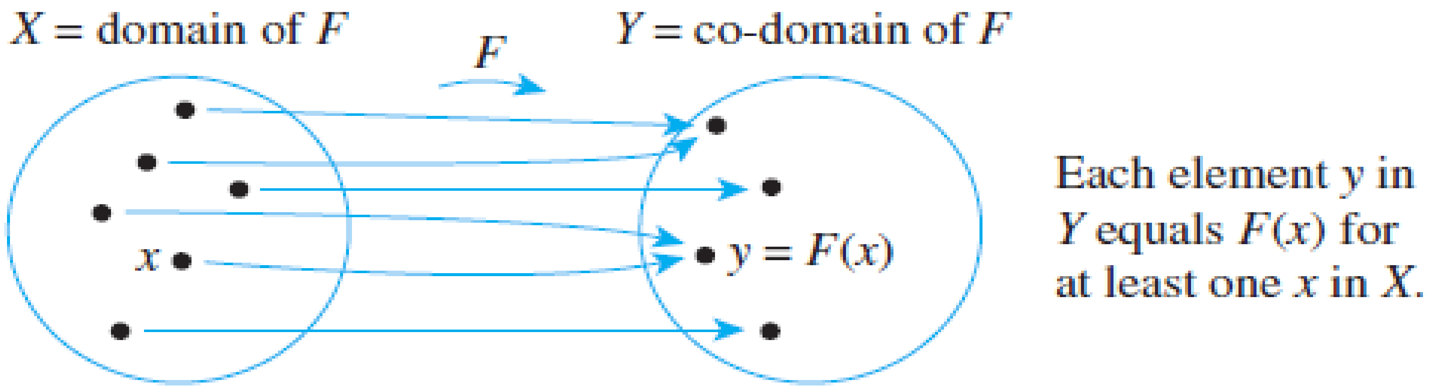
Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.

Symbolically:

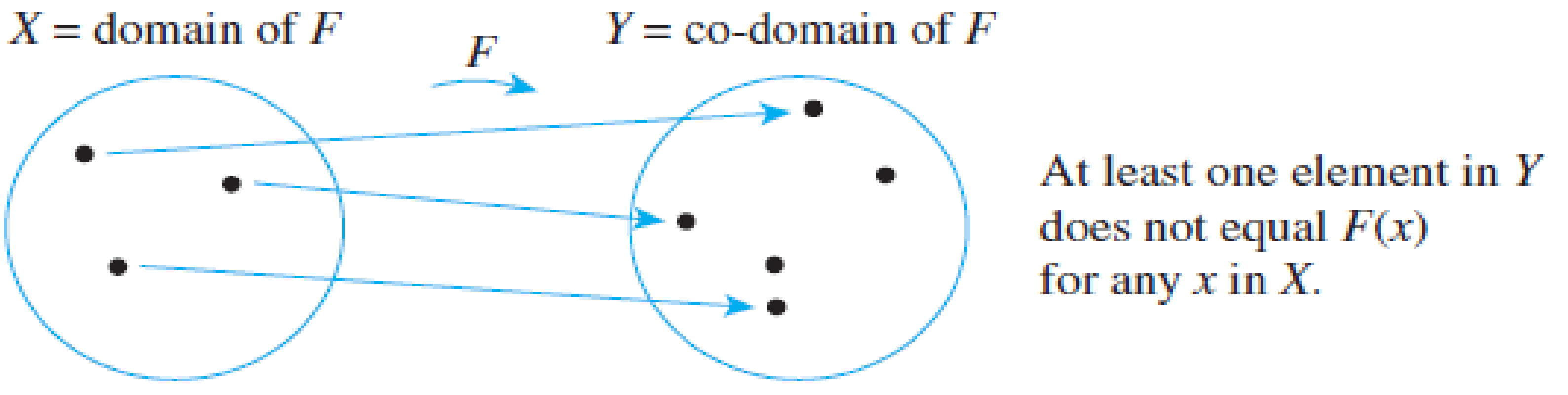
$$F: X \to Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

$$F: X \to Y \text{ is } \textit{not} \text{ onto} \quad \Leftrightarrow \quad \exists y \text{ in } Y \text{ such that } \forall x \in X,\ F(x) \neq y.$$

Onto Functions on Sets



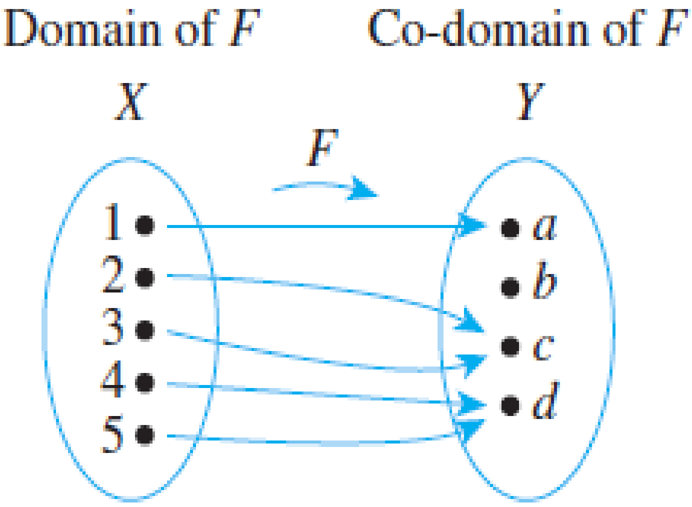
A Function That Is Onto

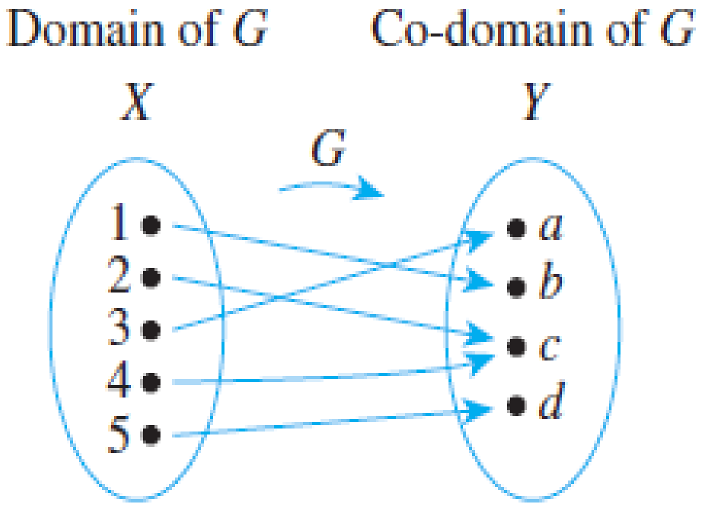


A Function That Is Not Onto

Onto Functions on Sets

Identifying Onto Functions





Onto Functions on Infinite Sets

Now suppose F is a function from a set X to a set Y, and suppose Y is infinite. By definition, F is onto if, and only if, the following universal statement is true:

$$\forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

Thus to prove F is onto, you will ordinarily use the method of generalizing from the generic particular:

suppose that y is any element of Y

and **show** that there is an element X of X with $F(x) = y$.

To prove F is *not* onto, you will usually

find an element y of Y such that $y \neq F(x)$ for *any* x in X.

Onto Functions on Infinite Sets

Define $f: \mathbf{R} \to \mathbf{R}$ and $h: \mathbf{Z} \to \mathbf{Z}$ by the rules

$$f(x) = 4x - 1 \quad \text{for all } x \in \mathbf{R}$$

and

$$h(n) = 4n - 1 \quad \text{for all } n \in \mathbf{Z}.$$

a. Is f onto? Prove or give a counterexample.

b. Is h onto? Prove or give a counterexample.

Onto Functions on Infinite Sets

To prove that *f* is onto, you must prove

$$\forall y \in Y, \exists x \in X \text{ such that } f(x) = y.$$

1. There exists real number x such that y = f(x)?
2. Does f really send x to y?

Onto Functions on Infinite Sets

If $f : \mathbf{R} \to \mathbf{R}$ is the function defined by the rule $f(x) = 4x - 1$ for all real numbers x, then f is onto.

Proof:

Let $y \in \mathbf{R}$. *[We must show that $\exists x$ in* $\mathbf{R}$ *such that $f(x) = y$.]* Let $x = (y+1)/4$. Then x is a real number since sums and quotients (other than by 0) of real numbers are real numbers. It follows that

$$
\begin{aligned}
f(x) &= f\left(\frac{y+1}{4}\right) && \text{by substitution} \\
&= 4 \cdot \left(\frac{y+1}{4}\right) - 1 && \text{by definition of } f \\
&= (y+1) - 1 = y && \text{by basic algebra.}
\end{aligned}
$$

[This is what was to be shown.]

Onto Functions on Infinite Sets

If the function $h: \mathbf{Z} \to \mathbf{Z}$ is defined by the rule $h(n) = 4n - 1$ for all integers n, then h is not onto.

Onto Functions on Infinite Sets

If the function $h: \mathbf{Z} \to \mathbf{Z}$ is defined by the rule $h(n) = 4n - 1$ for all integers n, then h is not onto.

Counterexample:

The co-domain of h is $\mathbf{Z}$ and $0 \in \mathbf{Z}$. But $h(n) \neq 0$ for any integer n. For if $h(n) = 0$, then

$$4n - 1 = 0 \qquad \text{by definition of } h$$

which implies that

$$4n = 1 \qquad \text{by adding 1 to both sides}$$

and so

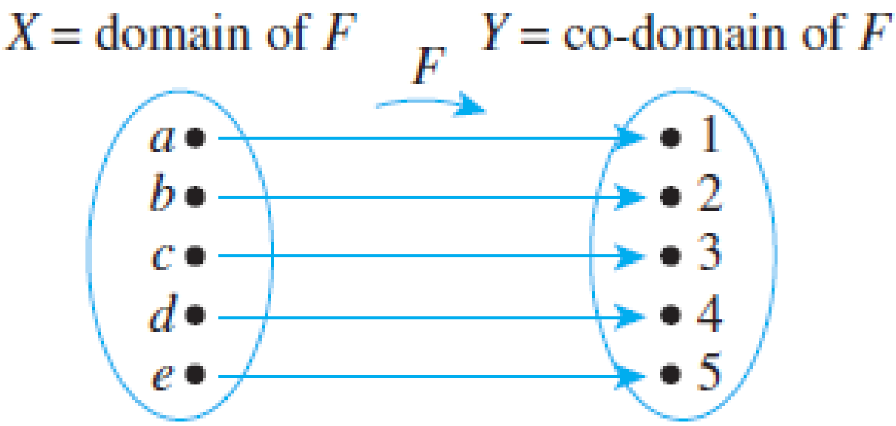
$$n = \frac{1}{4} \qquad \text{by dividing both sides by 4.}$$

But $1/4$ is not an integer. Hence there is no integer n for which $f(n) = 0$, and thus f is not onto.

One-to-One Correspondence (Bijection)

• Definition

A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function $F: X \to Y$ that is both one-to-one and onto.



An Arrow Diagram for a One-to-One Correspondence

One-to-One Correspondence (Bijection)

Inverse Functions

Theorem

Suppose $F: X \to Y$ is a one-to-one correspondence; that is, suppose F is one-to-one and onto. Then there is a function $\boldsymbol{F^{-1}: Y \to X}$ that is defined as follows:

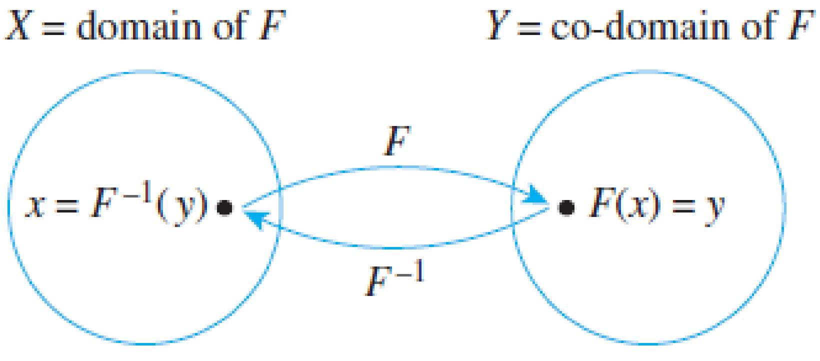
Given any element y in Y,

$$F^{-1}(y) = \text{that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y.$$

In other words,

$$F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).$$

The function F^{-1} is called inverse function.



Finding an Inverse Function

The function f : **R** → **R** defined by the formula

$f(x) = 4x - 1$, for all real numbers x

Solution For any *[particular but arbitrarily chosen]* y in **R**, by definition of f^{-1},

$$f^{-1}(y) = \text{that unique real number } x \text{ such that } f(x) = y.$$

But

$$\begin{aligned} & f(x) = y \\ \Leftrightarrow\quad & 4x - 1 = y && \text{by definition of } f \\ \Leftrightarrow\quad & x = \frac{y+1}{4} && \text{by algebra.} \end{aligned}$$

Hence $f^{-1}(y) = \dfrac{y+1}{4}$.

Theorem

If X and Y are sets and $F: X \to Y$ is one-to-one and onto, then $F^{-1}: Y \to X$ is also one-to-one and onto.

Proof:

F^{-1} is one-to-one: Suppose y_1 and y_2 are elements of Y such that $F^{-1}(y_1) = F^{-1}(y_2)$. *[We must show that $y_1 = y_2$.]* Let $x = F^{-1}(y_1) = F^{-1}(y_2)$. Then $x \in X$, and by definition of F^{-1},

$$F(x) = y_1 \quad \text{since } x = F^{-1}(y_1)$$

and

$$F(x) = y_2 \quad \text{since } x = F^{-1}(y_2).$$

Consequently, $y_1 = y_2$ since each is equal to $F(x)$. This is what was to be shown.

F^{-1} is onto: Suppose $x \in X$. *[We must show that there exists an element y in Y such that $F^{-1}(y) = x$.]* Let $y = F(x)$. Then $y \in Y$, and by definition of F^{-1}, $F^{-1}(y) = x$. This is what was to be shown.

Lecture Summery

- One-to-One Function
- Onto Function
- Bijective Function (One-to-One correspondence)
- Inverse Functions