Propositional Logic

[Propositions, Logical Connectives, Negation, Converse, Contrapositive, Logical Equivalence]

Introduction to Logic

Propositional calculus (or logic) is the study of the logical relationship between objects called propositions and forms the basis of all mathematical reasoning.

Definition

A *proposition* is a statement that is either *true* or *false*, but not both (we usually denote a proposition by letters; $p, q, r, s, \ldots$).

Propositions - Examples

- **p : 2+2 = 4**

statement p is a proposition, because answer is TRUE

- **q: 3+3 > 8 …**

Statement q is proposition, because answer is FALSE

- r: **How many students are in the class? ----**

Statement r is NOT a proposition … because answer can not be TRUE or FALSE

Definition

The value of a proposition is called its *truth value*; denoted by T or 1 if it is true and F or 0 if it is false.

Opinions, interrogative and imperative sentences are not propositions.

Truth table:

p
0
1

Examples

Example (Propositions)

- Today is Monday.
- The derivative of $\sin x$ is $\cos x$.
- Every even number has at least two factors.

Example (Not Propositions)

- `C++` is the best language.
- When is the pretest?
- Do your homework.

Which of the following statements are propositions?

- Washington DC is the capital of USA
- Jakarta is the capital of Indonesia
- $2 + 3 = 5$
- $5 + 7 = 10$
- $x + 2 = 11$
- $x + 2 = 11$, $x = 5$;
- Answer this question
- $2^n \geq 100$

Which of the following statements are propositions?

- Washington DC is the capital of USA
- Jakarta is the capital of Indonesia
- $2 + 3 = 5$
- $5 + 7 = 10$
- $x + 2 = 11$
- Answer this question
- $2^n \geq 100$

Compound Proposition: Logical Connectives

Connectives are used to create a *compound* proposition from two or more other propositions.

- Negation (denoted $\neg$ or !)
- And (denoted $\wedge$) or Logical Conjunction
- Or (denoted $\vee$) or Logical Disjunction
- Exclusive Or (XOR, denoted $\oplus$)
- Implication (denoted $\rightarrow$)
- Biconditional; "if and only if" (denoted $\leftrightarrow$)

Negation: $\neg P$, $\tilde{P}$, $\bar{P}$, $!P$

e.g. P: Today is Monday

$\neg P$: Today is NOT Monday. (It is not the case that today is Monday)

Q: $x > 10$

$\neg Q$: $x \ngtr 10$ or $x \leq 10$

Negation

A proposition can be negated. This is also a proposition. We usually denote the negation of a proposition p by $\neg p$.

Example (Negated Propositions)

- Today is *not* Monday.
- *It is not the case that* today is Monday.
- *It is not the case that* the derivative of $\sin x$ is $\cos x$.

Truth table:

p	$\neg$p
0	1
1	0

Negation

- P: Today is Monoday
- $\neg P$: Todays in NOT Monday

Truth Table:

P: Today is Monday	$\neg P$: Todays in NOT Monday
0 (False)	1 (True)
1 (True)	0 (False)

Negation

- Negation of TRUE is FASLE ---- Negation of 1 is 0 --- $\neg 1 = 0$
- Negation of FALSE is TRUE --- Negation of 0 is 1 --- $\neg 0 = 1$

Truth Table:

P: Today is Monday	$\neg P$: Todays in NOT Monday
0 (False)	1 (True)
1 (True)	0 (False)

Logical AND

The logical connective AND is true only if *both* of the propositions are true. It is also referred to as a *conjunction*.

Example (Logical Connective: AND)

- It is raining and it is warm.
- $(2+3=5) \wedge (\sqrt{2} < 2)$
- Schrödinger's cat is dead and Schrödinger's cat is not dead.

Truth table:

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Logical AND

- P: $2 + 3 \;=\; 5$
- Q: $\sqrt{2} < 2$
- P AND Q: $P \wedge Q$: $(2 + 3 = 5) \wedge \left(\sqrt{2} < 2\right)$: TRUE AND TRUE : $1 \wedge 1 = 1$

Logical AND

- $P \wedge \neg P = 0\ (FALSE)$

- If we have number of propositions $p_1, p_2, p_3, \cdots, p_n$ …. If any of these statements is **FASLE**, then whole **CONJUNCTION** is **FASLE**

$$p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n = FASLE$$

AND Truth Table

Inputs		Output
A	B	Y = A.B
0	0	0
0	1	0
1	0	0
1	1	1

Logical OR

The logical disjunction (or logical or) is true if one or both of the propositions are true.

Example (Logical Connective: OR)

- It is raining or it is the second day of lecture.
- $(2+2=5) \vee (\sqrt{2} < 2)$
- You may have cake or ice cream.[1]

Truth table:

p	q	$p \wedge q$	$p \vee q$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

Logical OR

- $P \vee \neg P = 1\ (TRUE)$

- If we have number of propositions $p_1, p_2, p_3, \cdots, p_n$ …. If any of these statements is **TRUE**, then whole **DISJUNCTION** is **TRUE**

$$p_1 \vee p_2 \vee p_3 \vee \cdots \vee p_n = TRUE$$

AND Truth Table

Inputs		Output
A	B	Y = A.B
0	0	0
0	1	0
1	0	0
1	1	1

OR Truth Table

Inputs		Output
A	B	Y = A+B
0	0	0
0	1	1
1	0	1
1	1	1

Logical XOR

The exclusive or of two propositions is true when exactly *one* of its propositions is true and the other one is false.

Example (Logical Connective: Exclusive Or)

- The circuit is either is on or off.
- Let $ab < 0$, then either $a < 0$ or $b < 0$ but not both.
- You may have cake or ice cream, but not both.

Truth table:

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

AND Truth Table

Inputs		Output
A	B	Y = A.B
0	0	0
0	1	0
1	0	0
1	1	1

OR Truth Table

Inputs		Output
A	B	Y = A+B
0	0	0
0	1	1
1	0	1
1	1	1

XOR Truth Table

Inputs		Output
A	B	$Y = A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

Implications

Definition

Let p and q be propositions. The implication

$$p \to q$$

is the proposition that is false when p is true and q is false and true otherwise.

Here, p is called the "hypothesis" (or "antecedent" or "premise") and q is called the "conclusion" or "consequence".

Implications

Definition

Let p and q be propositions. The implication

$$p \to q$$

is the proposition that is false when p is true and q is false and true otherwise.

Here, p is called the "hypothesis" (or "antecedent" or "premise") and q is called the "conclusion" or "consequence".

Truth table:

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

Implication: *an example*

- If you get 100% marks in the final, **then** you will get an A
- You will get an A, if you get 100% marks in the final.

$$p \rightarrow q$$

Hypothesis → Conclusion

- When this implication is **TRUE**, and when is it **FALSE**?

Implication: *an example*

- If you get 100% marks in the final, then you will get an A
    - When this implication is TRUE, and when is it FALSE?

- Think of an implication as the promise that **if P happen, then Q will definitely happen.**
    - It is FALSE if P happens and Q does not happen **(the promise has been broken)**
    - **In any other case, the promise is not broken,** and the implication is TRUE.

Implication: *an example*

- If you get 100% marks in the final, then you will get an A
  - When this implication is TRUE, and when is it FALSE?
- Think of an implication as the promise that if P happen, then Q will definitely happen.
  - It is FALSE if P happens and Q does not happen (the promise has been broken)
  - In any other case, the promise is not broken, and the implication is TRUE.
- The implication $\boldsymbol{Q \rightarrow P}$ is called the **CONVERSE**

Implication: *an example*

- If you get 100% marks in the final, then you will get an A
    - When this implication is TRUE, and when is it FALSE?
- Think of an implication as the promise that if P happen, then Q will definitely happen.
    - It is FALSE if P happens and Q does not happen (the promise has been broken)
    - In any other case, the promise is not broken, and the implication is TRUE.
- The implication $\boldsymbol{Q \rightarrow P}$ is called the **CONVERSE**

  If you get an A, then you have got 100% marks in the final

Not same as $\boldsymbol{P \rightarrow Q}$

Implications

The implication $p \to q$ can be equivalently read as

- if p then q
- p implies q
- if p, q
- p only if q
- q if p
- q when p
- q whenever p
- p is a sufficient condition for q (p is sufficient for q)
- q is a necessary condition for p (q is necessary for p)
- q follows from p

Examples

Example

- If you buy your air ticket in advance, it is cheaper.
- If $2+2=5$ then all unicorns are pink.
- If x is a real number, then $x^2 \geq 0$.
- If it rains, the grass gets wet.
- If the sprinklers operate, the grass gets wet.

Exercise

Which of the following implications is true?

- If -1 is a positive number, then $2+2=5$.
- If -1 is a positive number, then $2+2=4$.
- If $\sin x = 0$ then $x = 0$.

Exercise

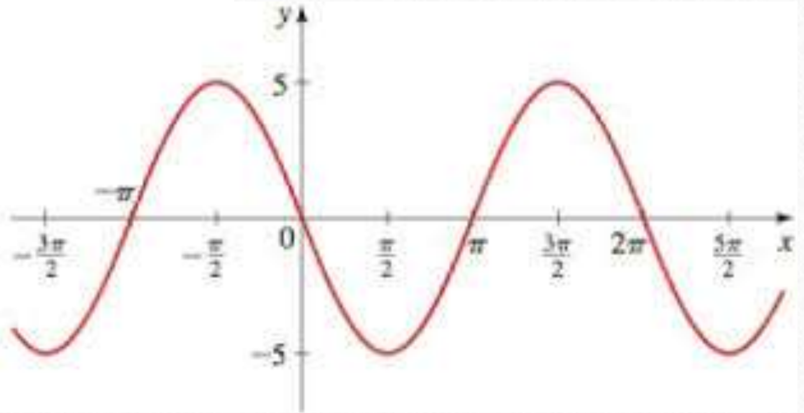
Which of the following implications is true?

- If -1 is a positive number, then $2+2=5$.
  true: the hypothesis is obviously false, thus no matter what the conclusion, the implication holds.
- If -1 is a positive number, then $2+2=4$.
- If $\sin x = 0$ then $x = 0$.

Exercise

Which of the following implications is true?

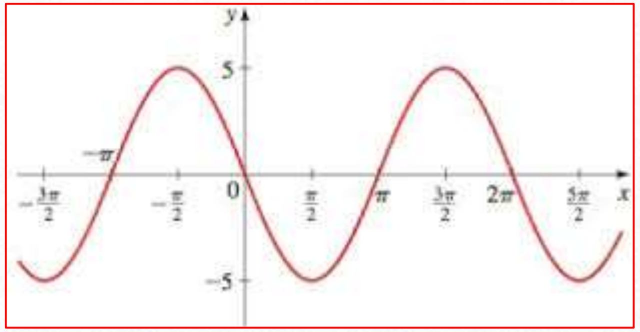
- If -1 is a positive number, then $2+2=5$.
  true: the hypothesis is obviously false, thus no matter what the conclusion, the implication holds.
- If -1 is a positive number, then $2+2=4$.
  true: for the same reason as above
- If $\sin x = 0$ then $x = 0$.



Exercise

Which of the following implications is true?

- If -1 is a positive number, then $2 + 2 = 5$.
  true: the hypothesis is obviously false, thus no matter what the conclusion, the implication holds.
- If -1 is a positive number, then $2 + 2 = 4$.
  true: for the same reason as above
- If $\sin x = 0$ then $x = 0$.
  false: x can be any multiple of π; i.e. if we let $x = 2\pi$ then clearly $\sin x = 0$, but $x \neq 0$. The implication "if $\sin x = 0$ then $x = k\pi$ for some integer k" is true.



Exercise

If $\sin x = 0$, then $x = 0$ ---- FALSE

Because $\sin x$ is ZERO at many other places as well.

- **Let's look at its CONVERSE**

If $x = 0$, then $\sin x = 0$. --- TRUE

If $\sin x = 0$, then $x = 0$ ---- FALSE

Because $\sin x$ is ZERO at many other places as well.

- Let's look at its CONVERSE

If $x = 0$, then $\sin x = 0$. --- TRUE

Exercise

If $\sin x = 0$, then $x = 0$ ---- FALSE

Because $\sin x$ is ZERO at many other places as well.

- **Let's look at its CONVERSE**

If $x = 0$, then $\sin x = 0$. --- TRUE

p	q	$p \rightarrow q$	Converse: $q \rightarrow p$
0	0	1	1
0	1	1	0
1	0	0	1
1	1	1	1

If $\sin x = 0$, then $x = 0$ ---- FALSE
Because $\sin x$ is ZERO at many other places as well.
- Let's look at its CONVERSE
If $x = 0$, then $\sin x = 0$. --- TRUE

Quick – Do it by yourself - Be ready for class activity

- Let p and q be the propositions: "The election is decided", and "The votes have been counted". Express the following compound propositions in English:
  - $\neg p$ $\quad$ $p \vee q$ $\quad$ $\neg p \wedge q$ $\quad$ $q \rightarrow p$ $\quad$ $\neg q \rightarrow \neg p$ $\quad$ $\neg p \rightarrow \neg q$

Quick – Do it by yourself - Be ready for class activity in next class

- Let p and q be the propositions:
  - p: "It is below freezing", q: "It is snowing".
- Express the following using p and q, and logical connectives:
  - It is below freezing and snowing
  - It is below freezing but not snowing
  - It is not below freezing and it is not snowing
  - It is either snowing or below freezing (but not both)
  - If it is below freezing, it is also snowing
  - Either it is below freezing or it is snowing, but it is not snowing if it is below freezing.

Quick – Do it by yourself - Be ready for class activity in next class

- Let p and q be the propositions:
  - p: "It is below freezing", q: "It is snowing".
- Express the following using p and q, and logical connectives:
  - It is below freeing and snowing ---- $p \wedge q$
  - It is below freezing but not snowing $p \wedge \sim q$
  - It is not below freezing and it is not snowing --- $\neg p \wedge \neg q$
  - It is either snowing or below freezing (but not both) $\boldsymbol{p \; XOR \; q}$
  - If it is below freezing, it is also snowing --- $p \rightarrow q$
  - Either it is below freezing or it is snowing, but it is not snowing if it is below freezing

$\boldsymbol{(p \; XOR \; q) \wedge (p \rightarrow \neg q)}$.

Converse, Contrapositive, Inverse

Consider the proposition $p \to q$:

- Its *converse* is the proposistion $q \to p$.
- Its *inverse* is the proposistion $\neg p \to \neg q$.
- Its *contrapositive* is the proposistion $\neg q \to \neg p$.

$p \to q$: If it is below freezing, then it is snowing

- Converse $q \to p$: If it is snowing, then it is below freezing
- Inverse $\neg p \to \neg q$: If it is NOT below freezing, then it is NOT snowing.
- Contrapositive: $\neg q \to \neg p$: If it is NOT snowing, then it is NOT below freezing

Exercise: *Correct or Incorrect*

If x is divisible by 4, then x is even.

$p \to q$, where p is 'x is divisible by 4' and q is 'x is even'.

Exercise: *Correct or Incorrect*

If x is divisible by 4, then x is even.

$p \rightarrow q$, where p is 'x is divisible by 4' and q is 'x is even'.

- $q \rightarrow p$: If x is even, then x is divisible by 4

Exercise: *Correct or Incorrect*

If x is divisible by 4, then x is even.

$p \to q$, where p is 'x is divisible by 4' and q is 'x is even'.

- $q \to p$: If x is even, then x is divisible by 4 — Incorrect

Exercise: *Correct or Incorrect*

If x is divisible by 4, then x is even.

$p \rightarrow q$, where p is 'x is divisible by 4' and q is 'x is even'.

- $q \rightarrow p$: If x is even, then x is divisible by 4 Incorrect
- $\neg p \rightarrow \neg q$: If x is not divisible by 4, then x is not even

Exercise: *Correct or Incorrect*

If x is divisible by 4, then x is even.

$p \rightarrow q$, where p is 'x is divisible by 4' and q is 'x is even'.

- $q \rightarrow p$: If x is even, then x is divisible by 4 Incorrect
- $\neg p \rightarrow \neg q$: If x is not divisible by 4, then x is not even Incorrect
- $\neg q \rightarrow \neg p$: If x is not even, then x is not divisible by 4.

Exercise: *Correct or Incorrect*

If x is divisible by 4, then x is even.

$p \to q$, where p is 'x is divisible by 4' and q is 'x is even'.

- $q \to p$: If x is even, then x is divisible by 4 Incorrect
- $\neg p \to \neg q$: If x is not divisible by 4, then x is not even Incorrect
- $\neg q \to \neg p$: If x is not even, then x is not divisible by 4. Correct

Exercise: *Correct or Incorrect*

If x is divisible by 4, then x is even.

$p \rightarrow q$, where p is 'x is divisible by 4' and q is 'x is even'.

- $q \rightarrow p$: If x is even, then x is divisible by 4 Incorrect
- $\neg p \rightarrow \neg q$: If x is not divisible by 4, then x is not even Incorrect
- $\neg q \rightarrow \neg p$: If x is not even, then x is not divisible by 4. Correct

- IMPLICATION is EQUIVALENT to its CONTRAPOSITIVE

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Exercise: *Correct or Incorrect*

- ~~IMPLICATION is EQUIVALENT to its CONTRAPOSITIVE~~

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

p	q	$p \rightarrow q$	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
0	0	1	1	1	1
0	1	1	0	1	1
1	**0**	**0**	**1**	**0**	**0**
1	1	1	0	0	1

Biconditional Proposition

Definition

The *biconditional*

$$p \leftrightarrow q$$

is the proposition that is true when p and q have the same truth values. It is false otherwise.

Note that it is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$

Biconditional

Definition

The *biconditional*

$$p \leftrightarrow q$$

is the proposition that is true when p and q have the same truth values. It is false otherwise.

Note that it is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$

- What does the following line mean?
  - You can take the flight if and only if you buy a ticket

  If you take the flight, then you have bought ticket, AND, if you buy ticket then you take the flight
- Can you draw the truth table of a bi-condition $p \leftrightarrow q$?

Biconditional

Definition

The *biconditional*

$$p \leftrightarrow q$$

is the proposition that is true when p and q have the same truth values. It is false otherwise.

Note that it is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$

Truth table:

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

- What does the following line mean?
  - You can take the flight if and only if you buy a ticket
- Can you draw the truth table of a bi-condition $p \leftrightarrow q$?

Examples

$p \leftrightarrow q$ can be equivalently read as

- p if and only if q
- p is necessary and sufficient for q
- if p then q, and conversely
- p iff q (Note typo in textbook, page 9, line 3.)

Example

- $x > 0$ if and only if x^2 is positive.
- The alarm goes off iff a burglar breaks in.
- You may have pudding if and only if you eat your meat.[1]

Exercise

Which of the following biconditionals is true?

- $x^2 + y^2 = 0$ if and only if $x = 0$ and $y = 0$
- $2 + 2 = 4$ if and only if $\sqrt{2} < 2$
- $x^2 \geq 0$ if and only if $x \geq 0$.

Exercise

Which of the following biconditionals is true?

- $x^2 + y^2 = 0$ if and only if $x = 0$ and $y = 0$
  true: both implications hold.
- $2 + 2 = 4$ if and only if $\sqrt{2} < 2$
- $x^2 \geq 0$ if and only if $x \geq 0$.

Exercise

Which of the following biconditionals is true?

- $x^2 + y^2 = 0$ if and only if $x = 0$ and $y = 0$
  true: both implications hold.
- $2 + 2 = 4$ if and only if $\sqrt{2} < 2$
  true: for the same reason above.
- $x^2 \geq 0$ if and only if $x \geq 0$.

Exercise

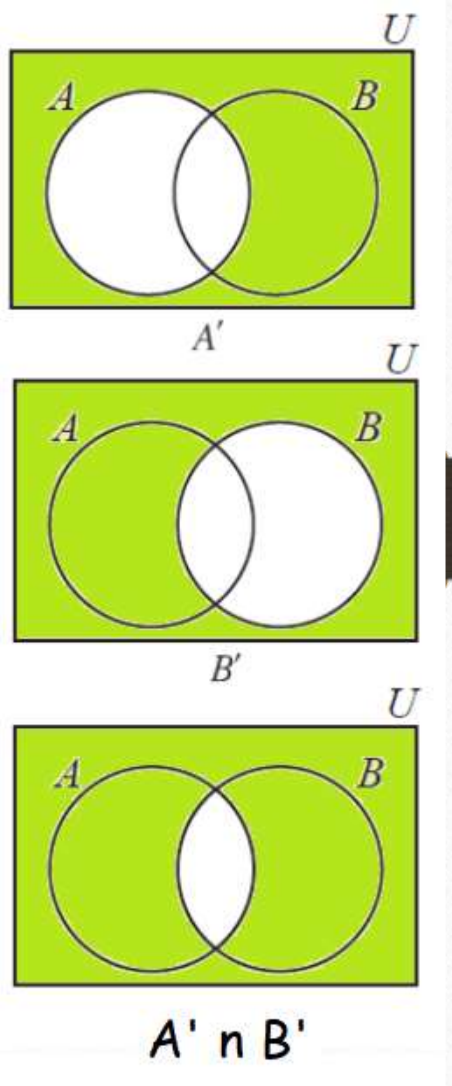
Which of the following biconditionals is true?

- $x^2 + y^2 = 0$ if and only if $x = 0$ and $y = 0$
  true: both implications hold.
- $2 + 2 = 4$ if and only if $\sqrt{2} < 2$
  true: for the same reason above.
- $x^2 \geq 0$ if and only if $x \geq 0$.
  false: The converse holds. That is, "if $x \geq 0$ then $x^2 \geq 0$". However, the implication is false; consider $x = -1$. Then the hypothesis is true, $1^2 \geq 0$ but the conclusion fails.

De Morgan's Law

$\overline{(A \cup B)} \equiv \bar{A} \cap \bar{B}$

$\overline{(A \cap B)} \equiv \bar{A} \cup \bar{B}$



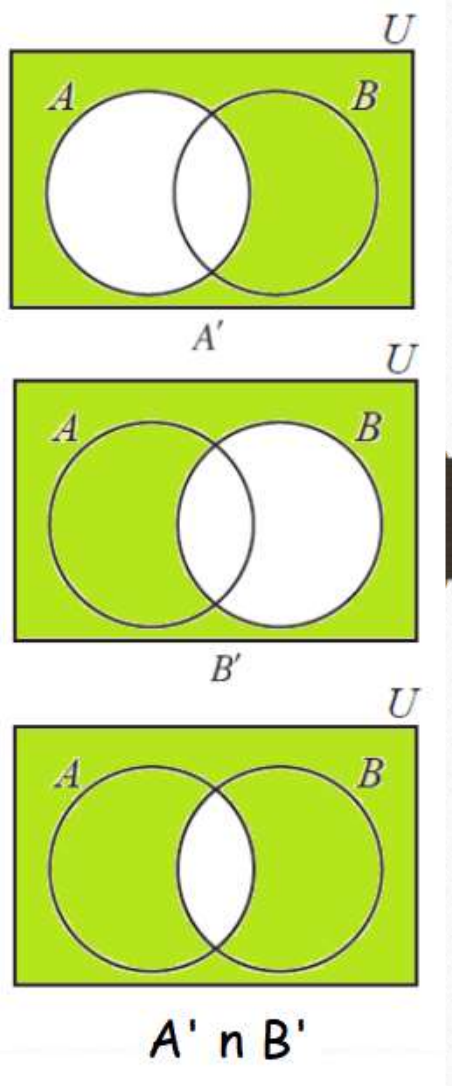
De Morgan's Law

- **Union is similar to OR operation** $\overline{(A \cup B)} \equiv \bar{A} \cap \bar{B}$
- **Intersection is similar to AND operation** $\overline{(A \cap B)} \equiv \bar{A} \cup \bar{B}$

$$\neg(P \wedge Q) \iff (\neg P) \vee (\neg Q)$$

and

$$\neg(P \vee Q) \iff (\neg P) \wedge (\neg Q),$$



De Morgan's Law

$$\neg(P \wedge Q) \iff (\neg P) \vee (\neg Q)$$

and

$$\neg(P \vee Q) \iff (\neg P) \wedge (\neg Q),$$

Example:

P: Votes have been counted

Q: Election is decided

$\neg(P \wedge Q)$: *NOT(Votes have been counted and Election is decided) = Votes have NOT been counted OR Election is NOT decided*

Negation of Implication

- $p \to q$

- Negation: $\sim(\boldsymbol{p} \to \boldsymbol{q})$ **(is not equal to inverse)**

$$\neg(\boldsymbol{p} \to \boldsymbol{q}) \equiv \overline{(\boldsymbol{p} \to \boldsymbol{q})} \equiv \overline{(\neg \boldsymbol{p} \vee \boldsymbol{q})} \equiv$$

$(\boldsymbol{use\ de\ Morgan's\ Law})$

$$\neg\neg \boldsymbol{p} \wedge \neg \boldsymbol{q} \equiv \boldsymbol{p} \wedge \neg \boldsymbol{q}$$

$p \Rightarrow q \equiv \neg p \vee q$

Truth table check:

Same - so equiv.

p	q	$\neg p$	$p \Rightarrow q$	$\neg p \vee q$
T	T	F	T	T
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F

Negation of Implication

- $p \to q$

- Negation:

$$\neg(\boldsymbol{p \to q}) \equiv \overline{(\boldsymbol{p \to q})} \equiv \overline{(\neg \boldsymbol{p} \vee \boldsymbol{q})} \equiv$$

$(\boldsymbol{use\ de\ Margan's\ Law})$

$$\neg\neg \boldsymbol{p} \wedge \neg \boldsymbol{q} \equiv \boldsymbol{p} \wedge \neg \boldsymbol{q}$$

Be Careful: Negation is NOT $\neg \boldsymbol{p} \to \neg\ \boldsymbol{q}$ ---

this is INVERSE

Negation is $\boldsymbol{p} \wedge \neg \boldsymbol{q}$.

The statement "p implies q" is equivalent to "p or not q".

$$p \Rightarrow q \equiv \neg p \vee q$$

Truth table check:

Same - so equiv.

p	q	$\neg p$	$p \Rightarrow q$	$\neg p \vee q$
T	T	F	T	T
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F

Negation of Implication – Truth Table

- $p \to q$

Negation:

$$\neg(p \to q) \equiv \overline{(p \to q)} \equiv \overline{(\neg p \vee q)} \equiv p \wedge \neg q$$

p	q	$\neg q$	$p \to q$	$\overline{(p \to q)}$	$p \wedge \neg q$
0	0	1	1	0	0
0	1	0	1	0	0
1	0	1	0	1	1
1	1	0	1	0	0

- $p \to q$

Negation:

$$\neg(p \to q) \equiv \overline{(p \to q)} \equiv \overline{(\neg p \vee q)} \equiv p \wedge \neg q$$

Negation of Implication – Example

- $p \rightarrow q$
- Negation:

$\neg(p \rightarrow q) \equiv \overline{(p \rightarrow q)} \equiv \overline{(\neg p \vee q)} \equiv p \wedge \neg q$

$p \rightarrow q$: If it is below freezing, then it is snowing

NEGATION?

$p \rightarrow q \equiv \neg p \vee q$: It is NOT below freezing OR it is Snowing

$\boldsymbol{\neg(p \rightarrow q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q}$: **It is below freezing AND it is NOT snowing**

Negation of Bicondition

$\neg(p \leftrightarrow q) \equiv \neg(p \rightarrow q \wedge q \rightarrow p) \equiv (using\ De\ Morgan's\ Law)$

$= \neg(p \rightarrow q) \vee \neg(q \rightarrow p)$

$= \neg(\neg p \vee q) \vee \neg(\neg q \vee p)$ *Use De Morgan's Law*

$= (p \wedge \neg q) \vee (q \wedge \neg p)$ *Same as XOR operation*

p	q	$\neg p$	$\neg q$	$(p \wedge \neg q) \vee (q \wedge \neg p)$	$\neg(p \leftrightarrow q)$
0	0	1	1	0 V 0 = **0**	$\neg 1 =$ **0**
0	1	1	0	0 V 1 = **1**	$\neg 0 =$ **1**
1	0	0	1	1 V 0 = **1**	$\neg 0 =$ **1**
1	1	0	0	0 V 0 = **0**	$\neg 1 =$ **0**

Express the following in mathematical notation:

- Consider the following propositions:
  - p: Robbers have been seen in the city
  - q: It is safe to drive on the roads
  - r: Police stations are opened everywhere
- Express the following compound propositions in mathematical notation:
  - Robbers have not been seen in the city and driving on the road is safe but police stations are opened everywhere
  - Driving on the roads is not safe whenever (robbers have been seen in the city and police stations are not open everywhere)
  - For it to be safe to drive on the roads, it is necessary that police stations are everywhere and robbers have not been seen in the city.

Express the following in mathematical notation:

- Consider the following propositions:
  - p: Robbers have been seen in the city
  - q: It is safe to drive on the roads
  - r: Police stations are opened everywhere
- Express the following compound propositions in mathematical notation:
  - Robbers have not been seen in the city and driving on the road is safe but police stations are opened everywhere
    $\neg p \land q \land r$
  - Driving on the roads is not safe whenever (robbers have been seen in the city and police stations are not open everywhere)
  - For it to be safe to drive on the roads, it is necessary that police stations are everywhere and robbers have not been seen in the city.

Express the following in mathematical notation:

- Consider the following propositions:
  - p: Robbers have been seen in the city
  - q: It is safe to drive on the roads
  - r: Police stations are opened everywhere
- Express the following compound propositions in mathematical notation:
  - Robbers have not been seen in the city and driving on the road is safe but police stations are opened everywhere
    $\neg p \wedge q \wedge r$
  - Driving on the roads is not safe whenever (robbers have been seen in the city and police stations are not open everywhere) $(p \wedge \neg r) \rightarrow \neg q$
  - For it to be safe to drive on the roads, it is necessary that police stations are everywhere and robbers have not been seen in the city.

Express the following in mathematical notation:

- Consider the following propositions:
  - p: Robbers have been seen in the city
  - q: It is safe to drive on the roads
  - r: Police stations are opened everywhere
- Express the following compound propositions in mathematical notation:
  - Robbers have not been seen in the city and driving on the road is safe but police stations are opened everywhere
    $\neg p \land q \land r$
  - Driving on the roads is not safe whenever (robbers have been seen in the city and police stations are not open everywhere) $(p \land \neg r) \rightarrow \neg q$
  - For it to be safe to drive on the roads, it is necessary that police stations are everywhere and robbers have not been seen in the city. $q \rightarrow (r \land \neg p)$

Contrapositive

- Consider these proposition:

p: Robbers have been seen in the city

r: Police stations are open everywhere

Let us say that we know that $p \rightarrow r$ (if robbers have been seen in the city then police stations are open everywhere)

Then what does it mean if you saw that police stations are not open everywhere?

Contrapositive

- Implication $\boldsymbol{p \rightarrow q}$ has CONTRAPOSITIVE $\boldsymbol{\neg q \rightarrow \neg p}$
- An implication and its contrapositive are logically equivalent. (they have identical truth tables)
  - ***Can you quickly draw the truth tables and see this?***
- Proving one of them to be TRUE will prove the other to be TRUE and vice versa.

Tautologies and Contradictions

- A compound proposition which is always TRUE, no matter what the truth values of the propositional variables that occur in it, are, is called a **tautology**

  e.g. $p \vee \neg p$ = true

- A compound proposition which is always FALSE, no matter what the truth values of the propositional variables that occur in it, are, is called a **contradiction**

  e.g. $p \wedge \neg p$ = false

- **Question:** If you start from a proposition p, and a series of valid implications leads to a contradiction, what does that mean about p?

Truth Tables

Truth Tables are used to show the relationship between the truth values of individual propositions and the compound propositions based on them.

p	q	$p \wedge q$	$p \vee q$	p $\oplus$ q	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

Table: Truth Table for Logical Conjunction, Disjunction, Exclusive Or, and Implication

Exercise

Construct the Truth Table for the following compound proposition.

$$((p \wedge q) \vee \neg q)$$

p	q	$p \wedge q$	$\neg q$	$((p \wedge q) \vee \neg q)$
0	0			
0	1			
1	0			
1	1			

Exercise

Construct the Truth Table for the following compound proposition.

$$((p \wedge q) \vee \neg q)$$

p	q	$p \wedge q$	$\neg q$	$((p \wedge q) \vee \neg q)$
0	0	0	1	1
0	1	0	0	0
1	0	0	1	1
1	1	1	0	1