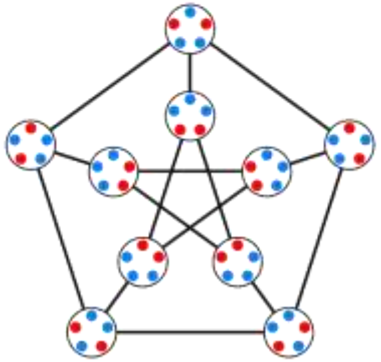
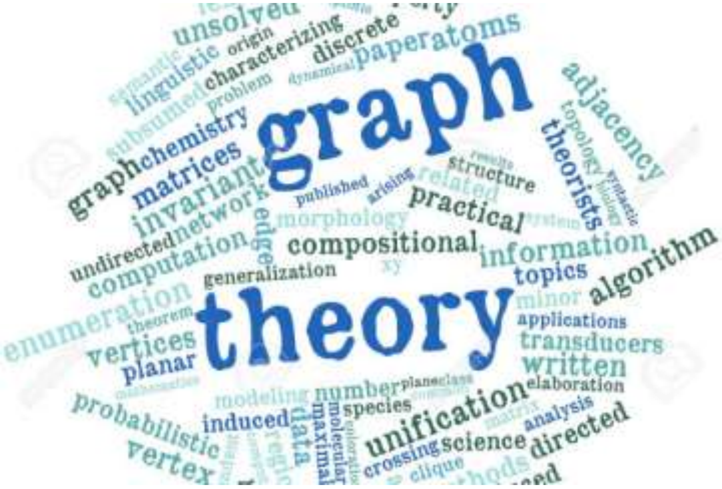


Graph Theory
Basics

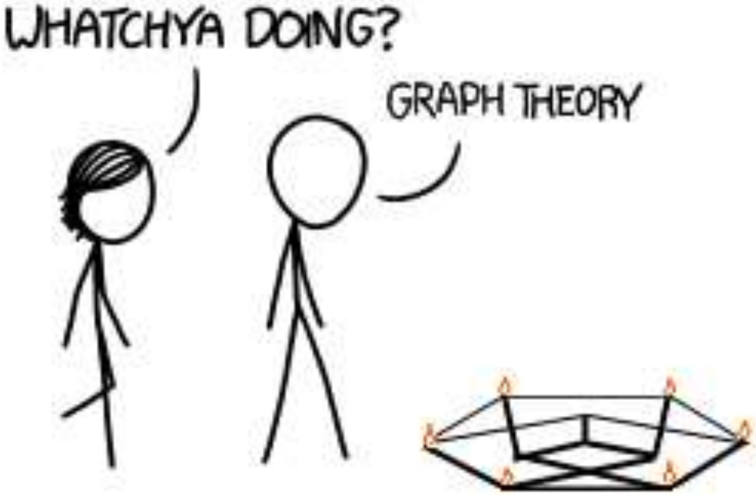
Mr. Asim Raza

Email: asim.raza@ucp.edu.pk



Graphs

$$G(V, E)$$



Basics of Graph Theory

Even and Odd Degrees

Even: $2k$

Odd: $2k + 1$

Prove that at a party with 51 people, there is always a person who knows an even number of others.

Basics of Graph Theory

Even and Odd Degrees

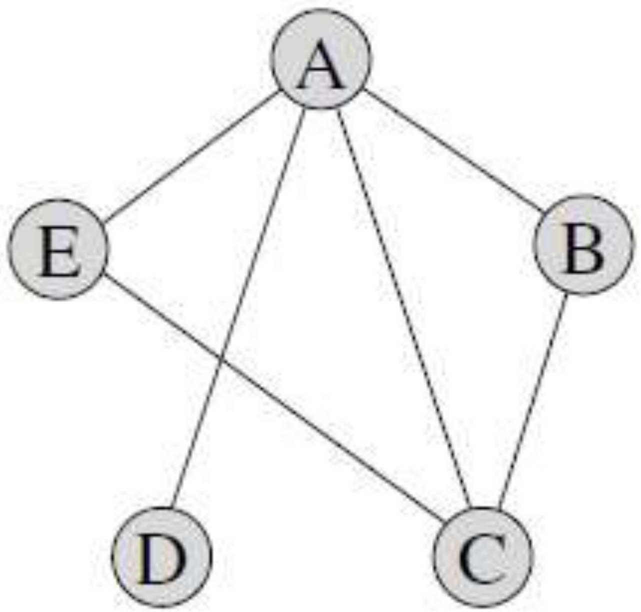
at a party with an odd number of people, there is always a person who knows an even number of others.

Basics of Graph Theory

Even and Odd Degrees

Suppose we have 5 friends:

Alice, Bob, Carl, Diane, Eve



. The graph depicting acquaintance between our friends

What is a Graph ?

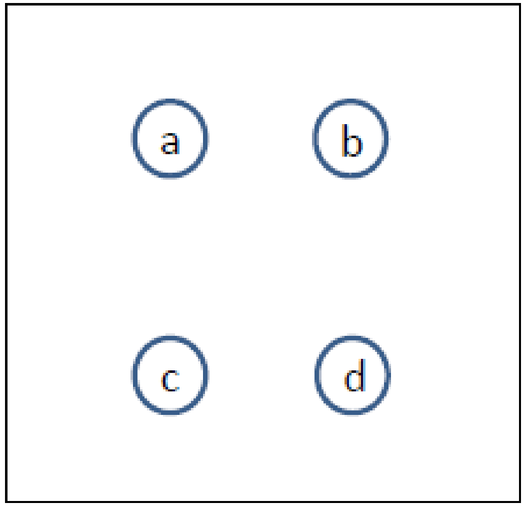
A graph consists of a nonempty set V of vertices and a set E of edges, where each edge in E connects two (may be the same) vertices in V.

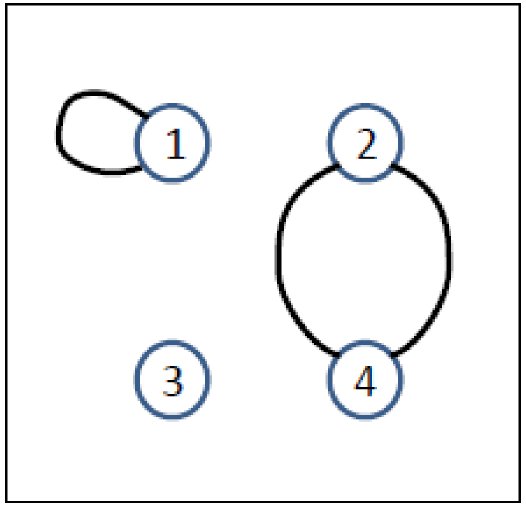
- Let G be a graph associated with a vertex set V and an edge set E

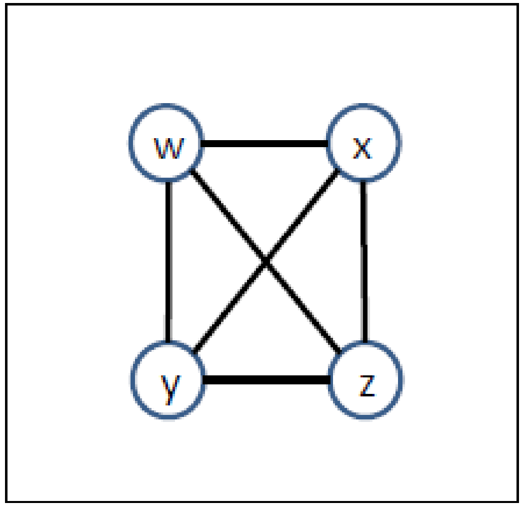
  We usually write G = (V, E) to indicate the above relationship

Basics of Graph Theory

Examples







- Furthermore, if each edge connects two different vertices, and no two edges connect the same pair of vertices, then the graph is a simple graph
- Which of the above is a simple graph ?

Terminology (Directed Graph)

- Sometimes we want to specify a direction on each edge
- Example: Vertices may represent cities, and edges may represent roads (can be one-way)
- This gives directed graph as follows:

> A directed graph G consists of a non empty set V of vertices and a set E of directed edges, where each edge is associated with an ordered pair of vertices. We write G=(V,E) to denote graph.

Basics of Graph Theory

Terminology (Undirected Graph)

- Let e be an edge that connects vertices u and v

  We say (i) e is incident with u and v

  (ii) u and v are the endpoints of e ;

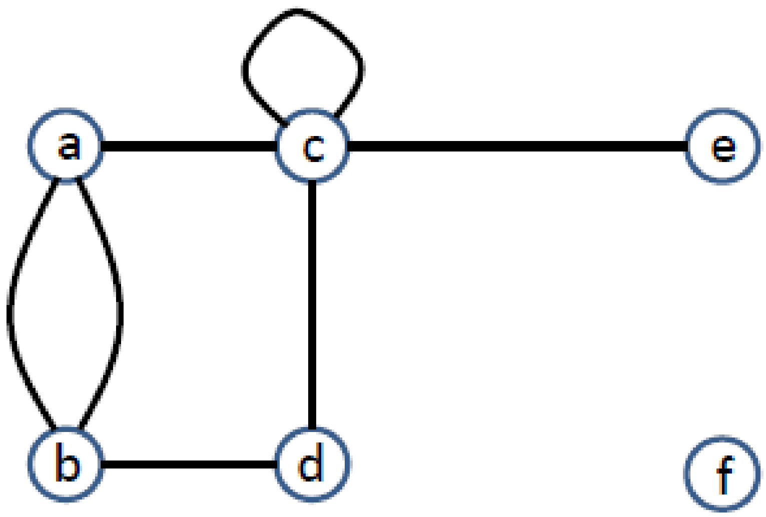
  (iii) u and v are adjacent (or neighbors)

  (iv) if u = v, the edge e is called a loop

- The degree of a vertex v, denoted by deg(v), is the number of edges incident with v, except that a loop at v contributes twice to the degree of v

Example

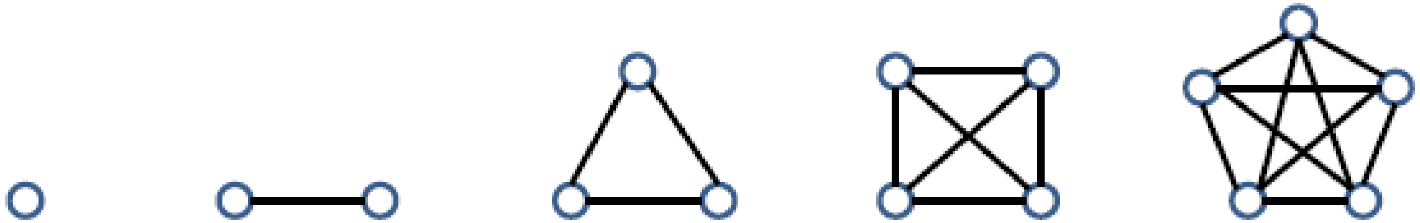
- What are the degrees and neighbors of each vertex in the following graph ?



Some Special Simple Graphs

Definition: A complete graph on n vertices, denoted by K_n, is a simple graph that contains one edge between each pair of distinct vertices

Examples :

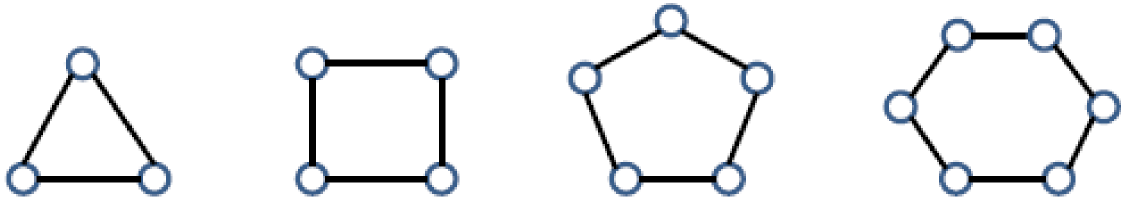


K_1 K_2 K_3 K_4 K_5

Some Special Simple Graphs

Definition: A cycle C_n, $n \geq 3$, is a graph that consists of n vertices $v_1, v_2, \ldots, v_n$ and n edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, …, $\{v_{n-1}, v_n\}$, $\{v_n, v_1\}$

Examples :

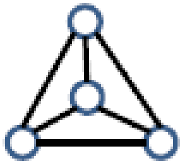


C_3 C_4 C_5 C_6

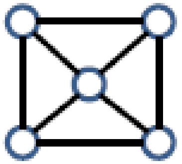
Some Special Simple Graphs

Definition: A wheel W_n, $n \geq 3$, is a graph that consists of a cycle C_n with an extra vertex that connects to each vertex in C_n

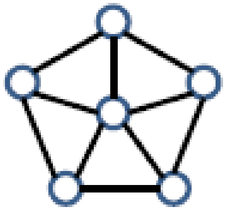
Examples :



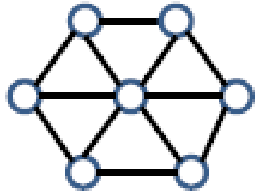
W_3



W_4



W_5

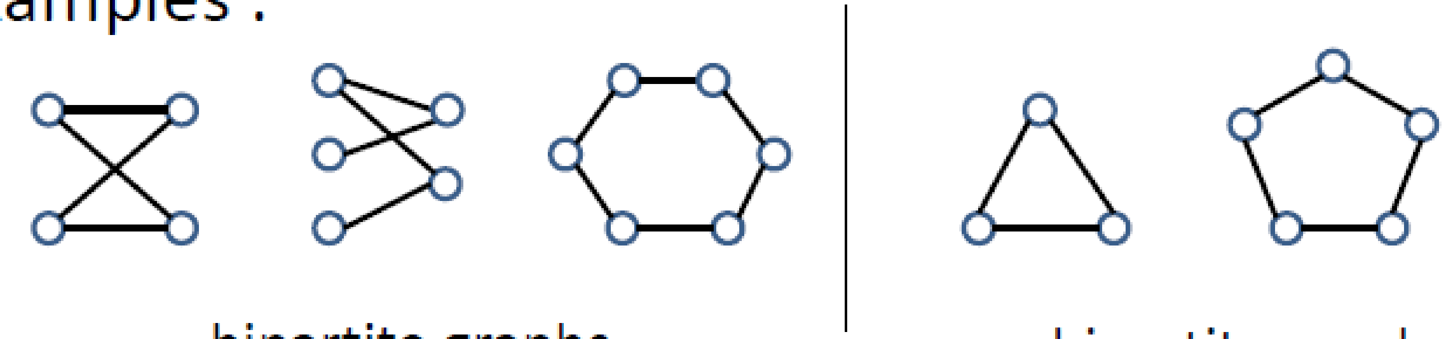


W_6

Some Special Simple Graphs

Definition: A bipartite graph is a graph such that the vertices can be partitioned into two sets V and W, so that each edge has exactly one endpoint from V, and one endpoint from W

Examples :



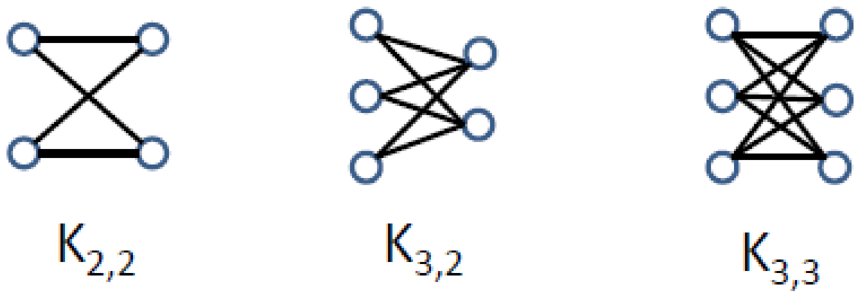
bipartite graphs

non-bipartite graphs

Some Special Simple Graphs

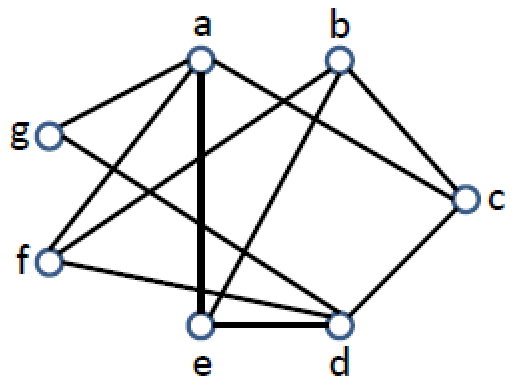
Definition: A complete bipartite graph $K_{m,n}$ is a bipartite graph with vertices partitioned into two subsets V and W of size m and n, respectively, such that there is an edge between each vertex in V and each vertex in W

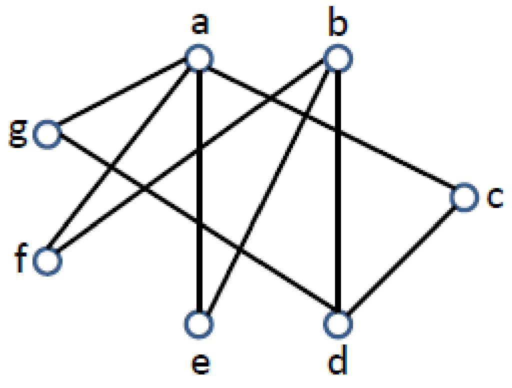
Examples :



Some Special Simple Graphs

- Which of the following is a bipartite graph?





Basics of Graph Theory

Even and Odd Degrees

at a party with an odd number of people, there is always a person who knows an even number of others.

If a graph has an odd number of nodes, then it has a node with even degree.

Basics of Graph Theory

Even and Odd Degrees

If a graph has an odd number of nodes, then it has a node with even degree.

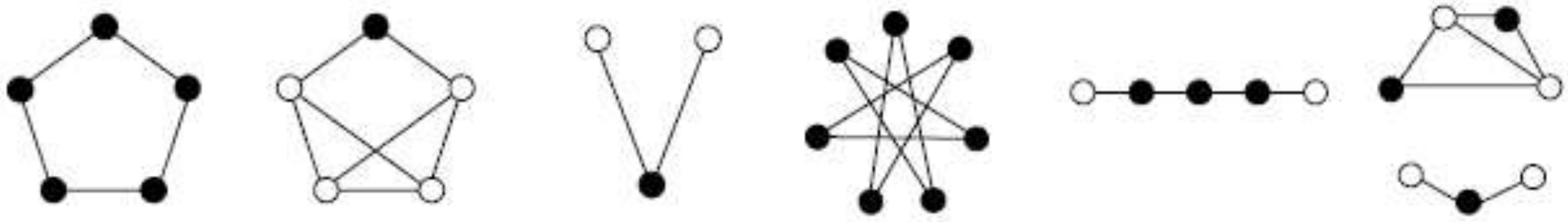


FIGURE 7.2. Some graphs with an odd number of nodes. Black circles mark nodes of even degree.

Basics of Graph Theory

Even and Odd Degrees

Even: $2k$

Odd: $2k + 1$

If a graph has an odd number of nodes, then it has a node with even degree.

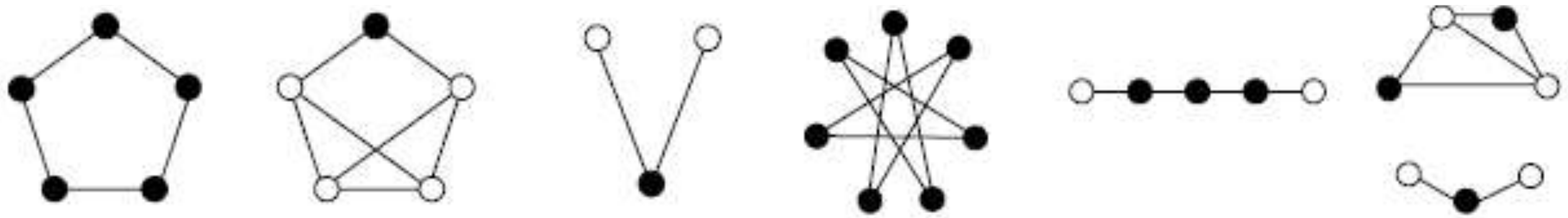


FIGURE 7.2. Some graphs with an odd number of nodes. Black circles mark nodes of even degree.

If a graph has an odd number of nodes, then the number of nodes with even degree is odd,

Basics of Graph Theory

Even and Odd Degrees

If a graph has an odd number of nodes, then the number of nodes with even degree is odd,

This implies

If a graph has an odd number of nodes,

then the number of nodes with odd degree is even

Because

Odd + Even = Odd

|Even degree nodes| + |Odd degree nodes| = |Total nodes|

Basics of Graph Theory

Even and Odd Degrees

if a graph has an even number of nodes, then the number of nodes with even degree is even.

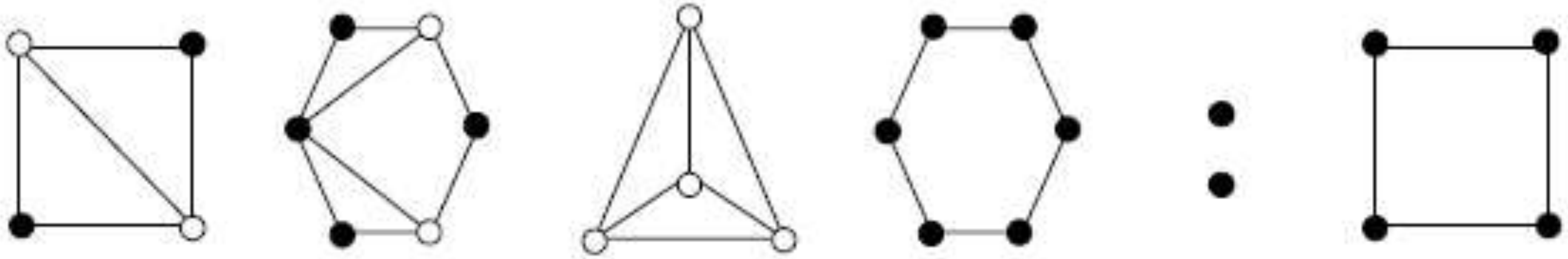
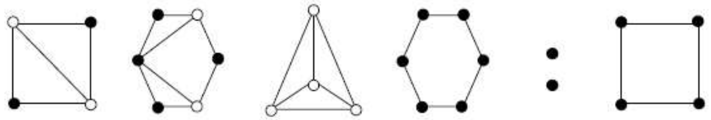


FIGURE 7.3. Some graphs with an even number of nodes. Black circles mark nodes of even degree.

Basics of Graph Theory

Even and Odd Degrees



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This implies

If a graph has an even number of nodes, then the number of nodes with odd degree is even

Because

Even + Even = Even

|Even degree nodes| + |Odd degree nodes| = |Total nodes|

Basics of Graph Theory

Even and Odd Degrees

Theorem 7.1.1 *In every graph, the number of nodes with odd degree is even.*

Basics of Graph Theory

Proof

Proof. One way of proving the theorem is to build up the graph one edge at a time, and observe how the parities of the degrees change. An example is shown in Figure 7.4. We start with a graph with no edge, in which every degree is 0, and so the number of nodes with odd degree is 0, which is an even number.

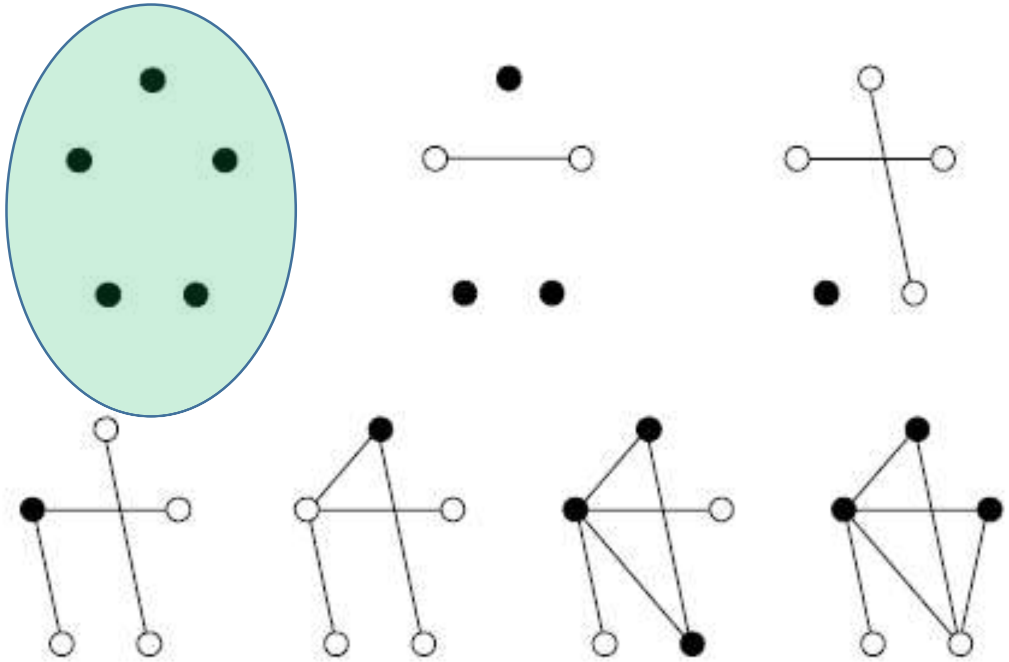

FIGURE 7.4. Building up a graph one edge at a time. Black circles mark nodes of even degree.

Basics of Graph Theory

Proof

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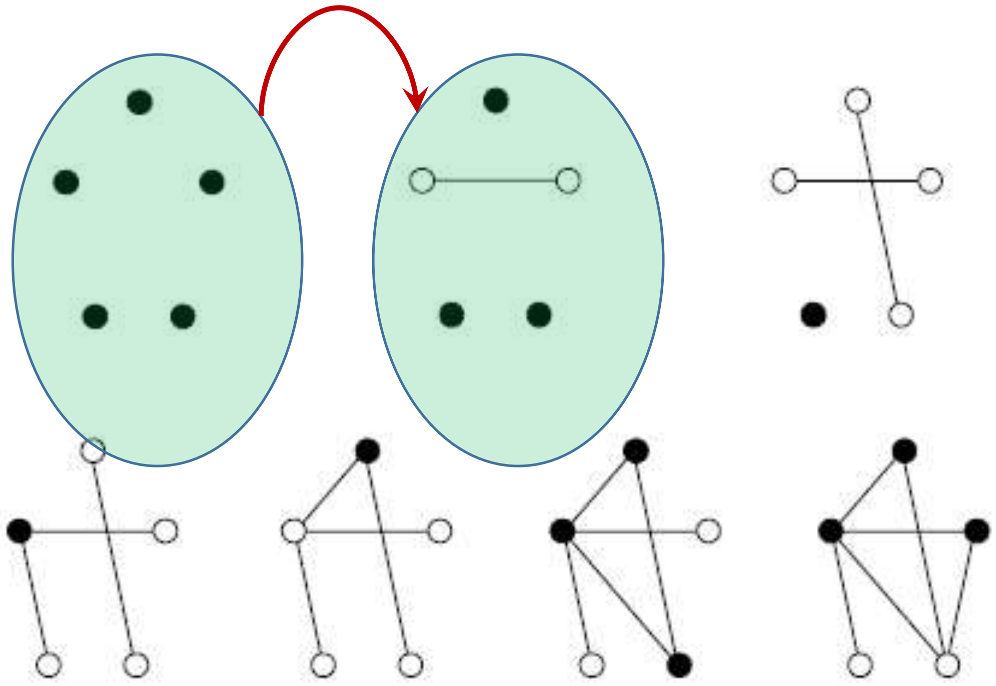


FIGURE 7.4. Building up a graph one edge at a time. Black circles mark nodes of even degree.

Basics of Graph Theory

Proof

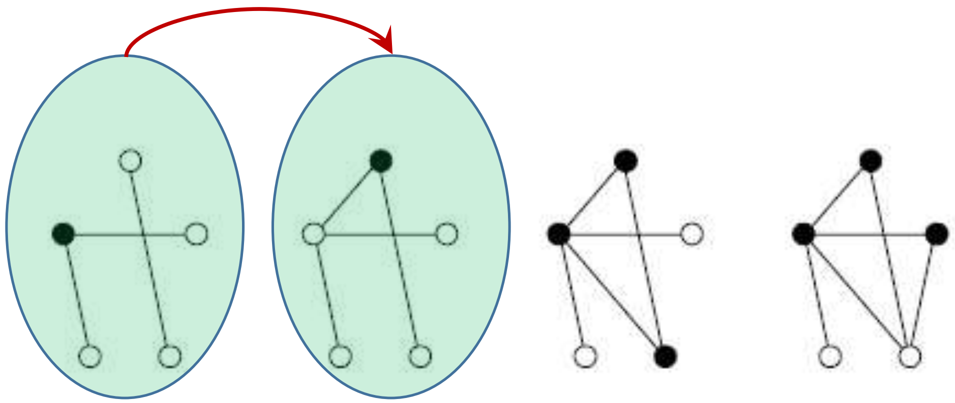


FIGURE 7.4. Building up a graph one edge at a time. Black circles mark nodes of even degree.

Now if we connect two nodes by a new edge, we change the parity of the degrees at these nodes. In particular,

|Odd-Degree Nodes| = Even

Even + 2 = Even

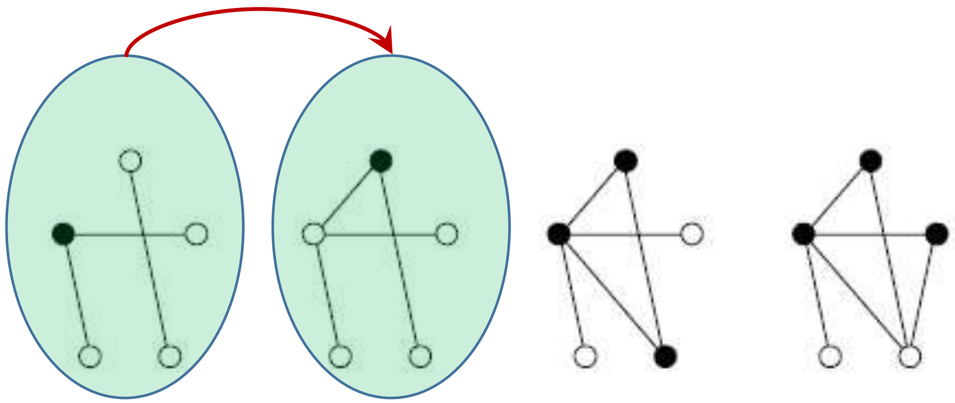
— if both endpoints of the new edge had even degree, we increase the number of nodes with odd degree by 2;

— if both endpoints of the new edge had odd degree, we decrease the number of nodes with odd degree by 2;

— if one endpoint of the new edge had even degree and the other had odd degree, then we don't change the number of nodes with odd degree.

Basics of Graph Theory

Proof



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— if both endpoints of the new edge had odd degree, we decrease the number of nodes with odd degree by 2;

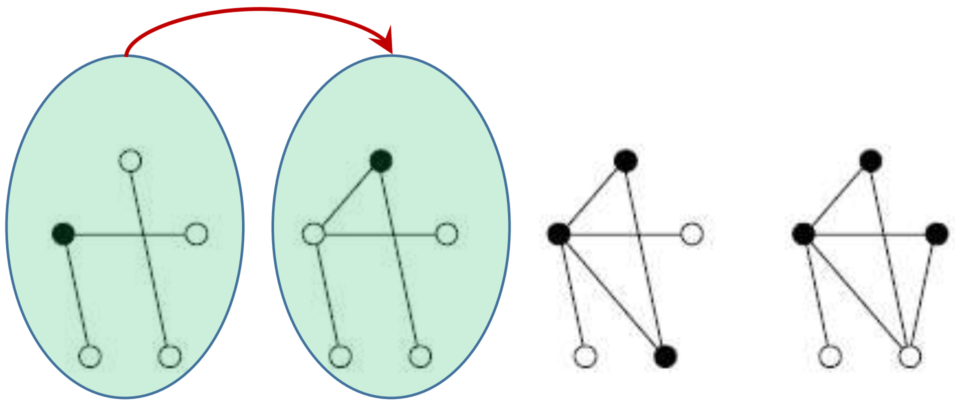
— if one endpoint of the new edge had even degree and the other had odd degree, then we don't change the number of nodes with odd degree.

|Odd-Degree Nodes| = Even

Even – 2 = Even

Basics of Graph Theory

Proof



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— if one endpoint of the new edge had even degree and the other had odd degree, then we don't change the number of nodes with odd degree.

|Odd-Degree Nodes| = Even

Odd + 1 = Even

Even + 1 = Odd

Basics of Graph Theory

Even and Odd Degrees

Theorem 7.1.1 *In every graph, the number of nodes with odd degree is even.*

Theorem 7.1.2 *The sum of degrees of all nodes in a graph is twice the number of edges.*

Handshaking Theorem

- Let G = (V, E) be an undirected graph with m edges

Theorem:

$$\sum_{v \in V} \deg(v) = 2m$$

- Proof : Each edge e contributes exactly twice to the sum on the left side (one to each endpoint).

Corollary : An undirected graph has an even number of vertices of odd degree.

Practice Problems

Try to solve at your own

7.1.1 Find all graphs with 2, 3, and 4 nodes.

7.1.2 (a) Is there a graph on 6 nodes with degrees 2, 3, 3, 3, 3, 3?

(b) Is there a graph on 6 nodes with degrees 0, 1, 2, 3, 4, 5?

(c) How many graphs are there on 4 nodes with degrees 1, 1, 2, 2?

(d) How many graphs are there on 10 nodes with degrees 1, 1, 1, 1, 1, 1, 1, 1, 1, 1?

7.1.3 At the end of the party with n people, everybody knows everybody else. Draw the graph representing this situation. How many edges does it have?

Practice Problems

Try to solve at your own

7.1.4 (a) Draw a graph with nodes representing the numbers $1, 2, \ldots, 10$, in which two nodes are connected by an edge if and only if one is a divisor of the other.

(b) Draw a graph with nodes representing the numbers $1, 2, \ldots, 10$, in which two nodes are connected by an edge if and only if they have no common divisor larger than 1.

(c) Find the number of edges and the degrees in these graphs, and check that Theorem 7.1.1 holds.

7.1.5 What is the largest number of edges a graph with 10 nodes can have?

Practice Problems

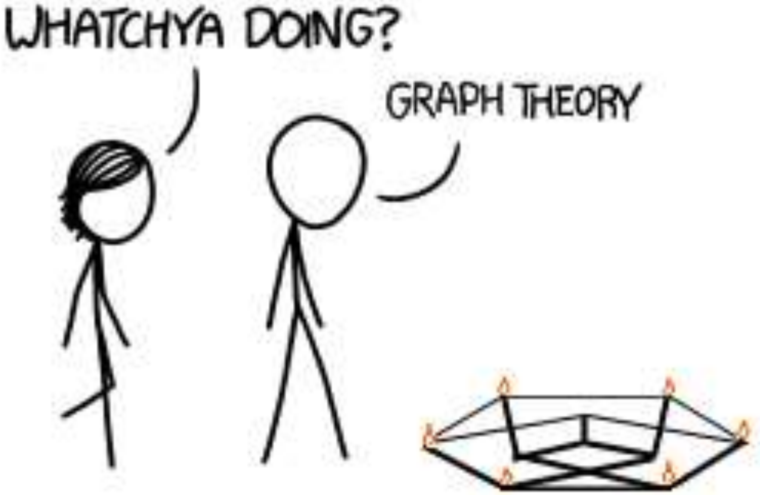
Try to solve at your own

7.1.6 How many graphs are there on 20 nodes? (To make this question precise, we have to make sure we know what it means that two graphs are the same. For the purpose of this exercise, we consider the nodes given, and labeled, say, as Alice, Bob, ... The graph consisting of a single edge connecting Alice and Bob is different from the graph consisting of a single edge connecting Eve and Frank.)

7.1.7 Formulate the following assertion as a theorem about graphs, and prove it: At every party one can find two people who know the same number of other people (like Bob and Eve in our first example).

Graphs

Paths, Cycles and Connectivity



Paths and Cycles

(Section 7.2 of textbook)

- Paths

- Cycles

Recap

- Graph: $G(V, E)$ – V: Set of vertices $\{v_0, v_1, v_2, \cdots, v_n\}$,

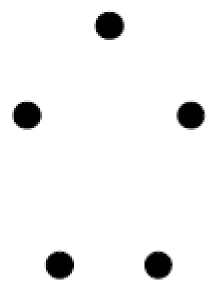
  E: Set of edges $\{e_0, e_1, e_2, \cdots, e_m\}$

Recap

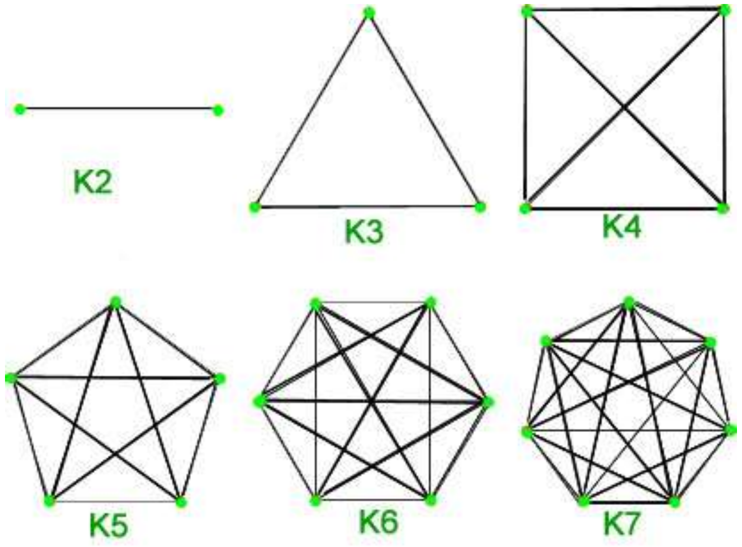
- **Graph:** $G(V, E)$ – V: Set of vertices $\{v_0, v_1, v_2, \cdots, v_n\}$,

  E: Set of edges $\{e_0, e_1, e_2, \cdots, e_m\}$

- **Empty Graph:** n vertices and Zero edges



- **Complete Graph:** K_n: n vertices and All Possible edges



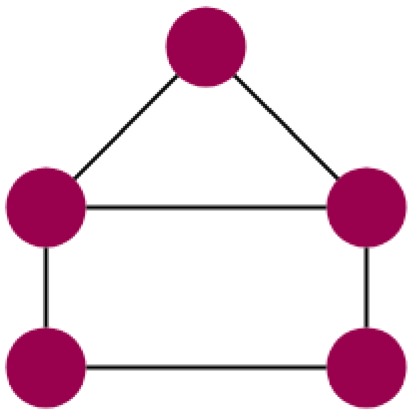
Number of Edges = $\binom{n}{2} = \frac{n(n-1)}{2}$

Recap

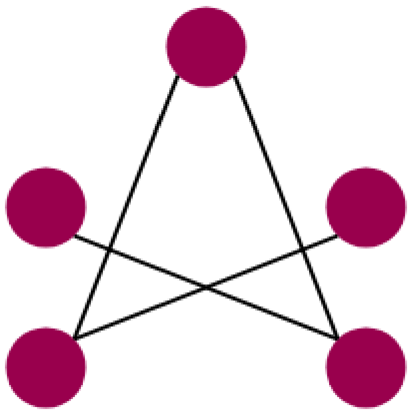
- Graph: $G(V,E)$ – V: Set of vertices $\{v_0, v_1, v_2, \cdots, v_n\}$,
  E: Set of edges $\{e_0, e_1, e_2, \cdots, e_m\}$

- Complement of a Graph: $\bar{G}(V,E)$:

Draw the missing edges, and delete the existing edges (Just take the complement of Edge set E)



Graph G



Complement Graph $\overline{G}$

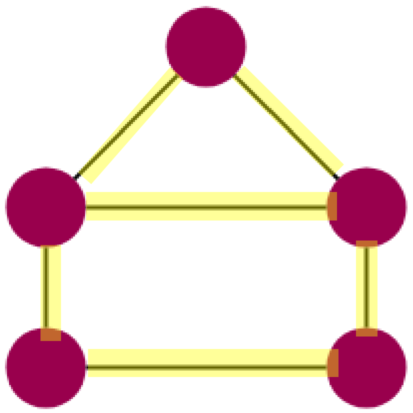
Recap

- **Graph:** $G(V, E)$ – V: Set of vertices $\{v_0, v_1, v_2, \cdots, v_n\}$,

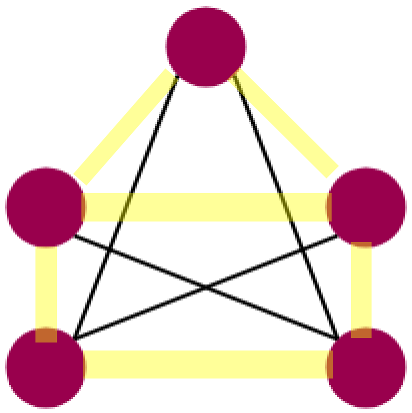
  E: Set of edges $\{e_0, e_1, e_2, \cdots, e_m\}$

- **Complement of a Graph:** $\bar{G}(V, E)$:

Draw the missing edges, and delete the existing edges (Just take the complement of Edge set E)



Graph G



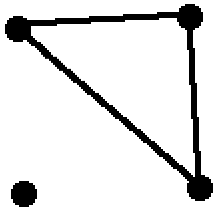
Complement Graph $\overline{G}$

Recap

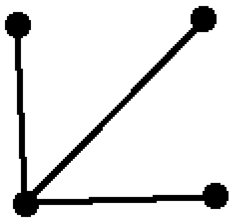
- **Examples:** Verify these at your own

Complement of a Graph: $\bar{G}(V, E)$:

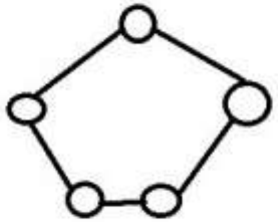
Draw the missing edges, and delete the existing edges (Just take the complement of Edge set E)



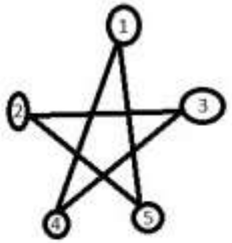
G



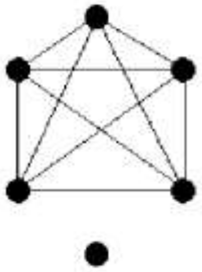
$\bar{G}$
(Complement of G)



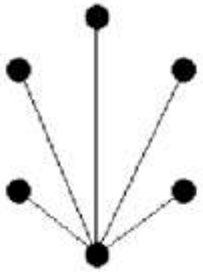
C5



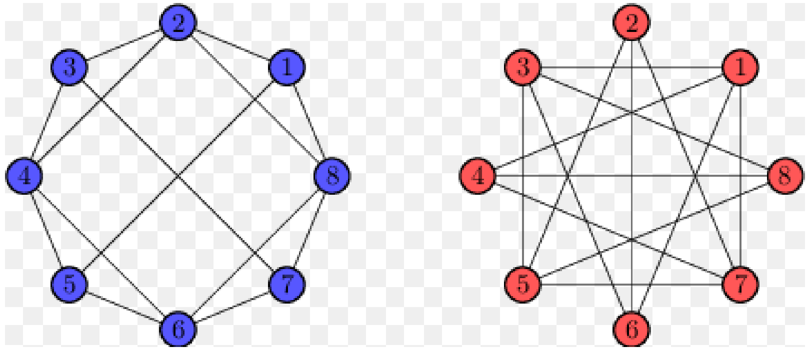
complement of c5

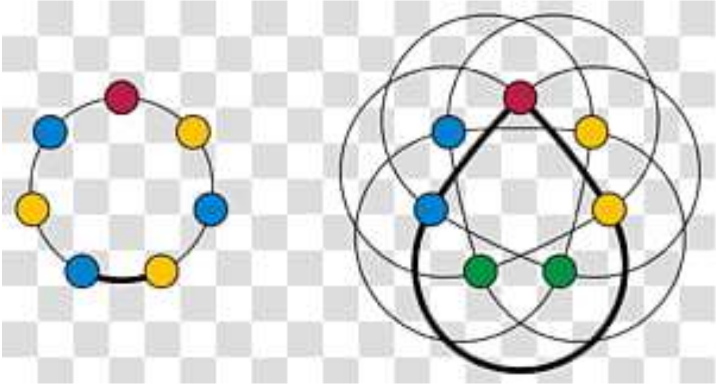


G



G_c



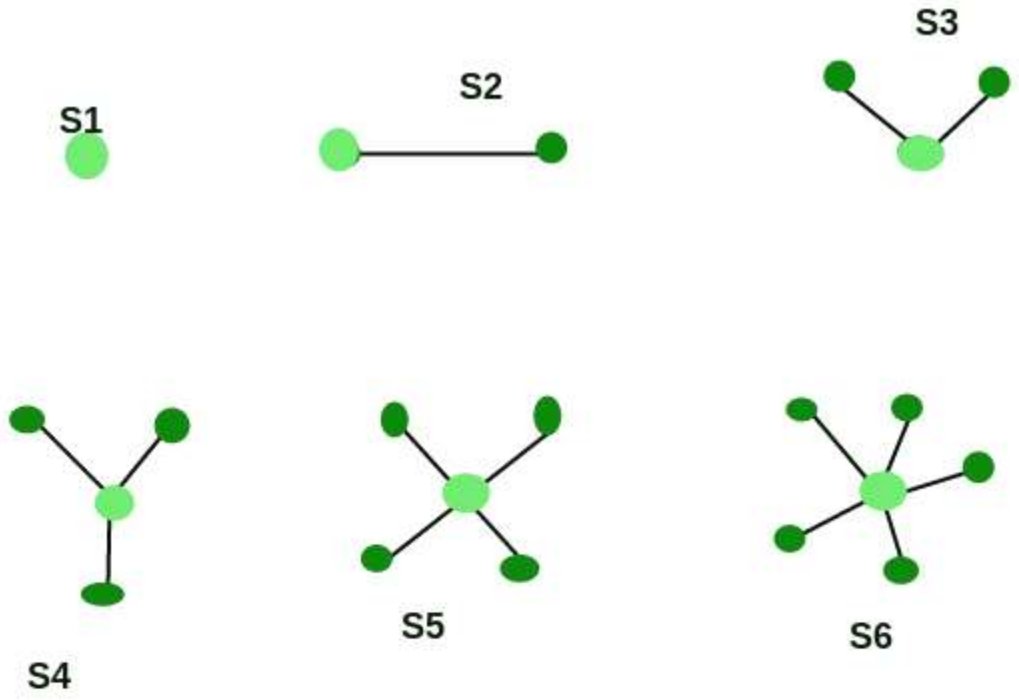


Recap

- **Star Graphs:**

Take a particular node and connect it to all other $n - 1$ nodes.

This graph has $n - 1$ edges.

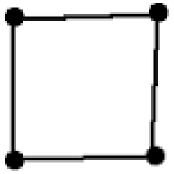


Star Graph of order-n (Sn)

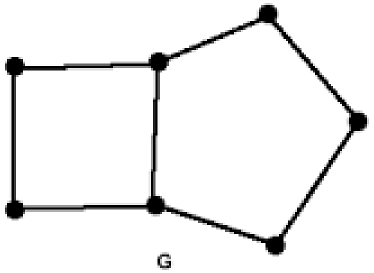
Recap

- ## SubGraphs:

Take **<u>any part</u>** of the graph – it will be Subgraph of the Graph (***just like subset of a set***)

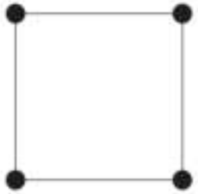


S
(Subgraph of G)



G

G



G'

G''

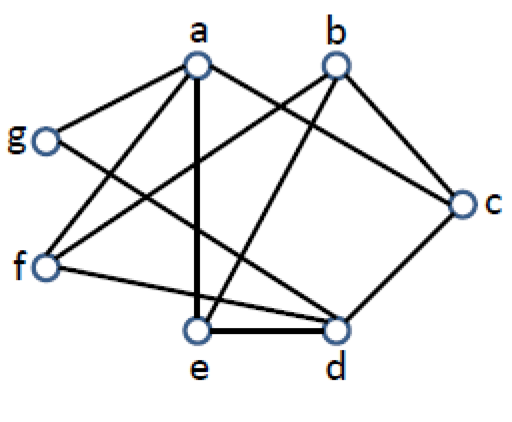
K_4

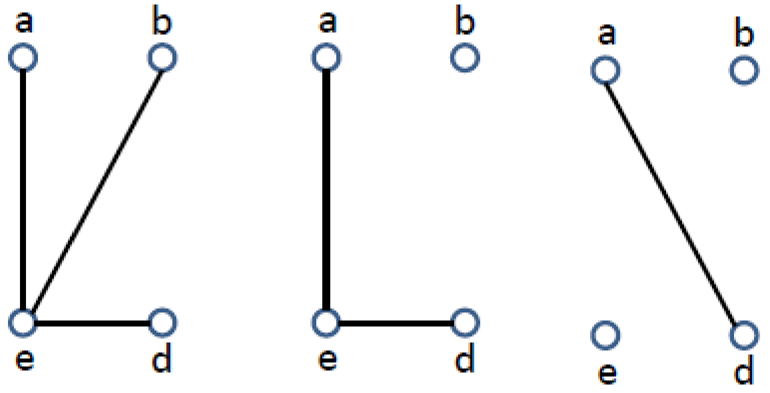
C_4

Subgraphs and Complements

If G = (V, E) is a graph, then G' = (V', E') is called a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$.

- Which one is a subgraph of the leftmost graph G ?



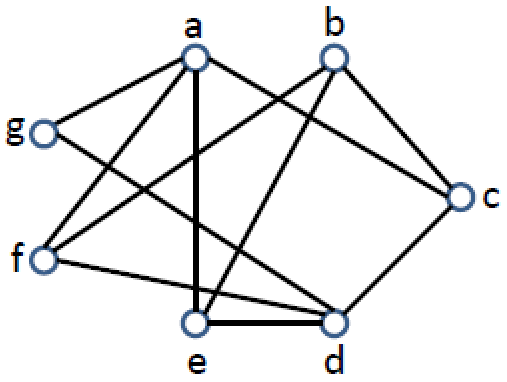


Subgraphs and Complements

If G = (V, E) is a graph, then the subgraph of G induced by $U \subseteq V$ is a graph with the vertex set U and contains exactly those edges from G with both endpoints from U

Ex : Consider the graph on the right side

What is its subgraph induced by the vertex set { a, b, c, g } ?



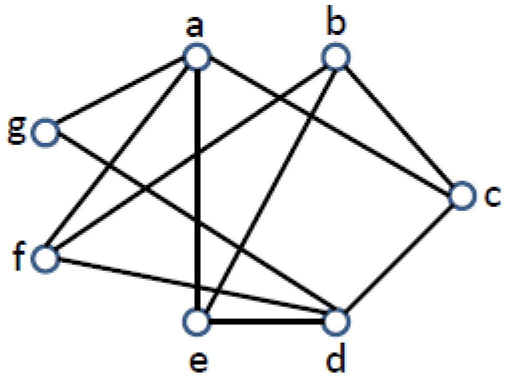
Subgraphs and Complements

If G = (V, E) is a graph, then the complement of G, denoted by $\overline{G}$, is a graph with the same vertex set, such that

an edge e exists in $\overline{G}$ ⇔ e does not exist in G

Ex : Consider the graph on the right side

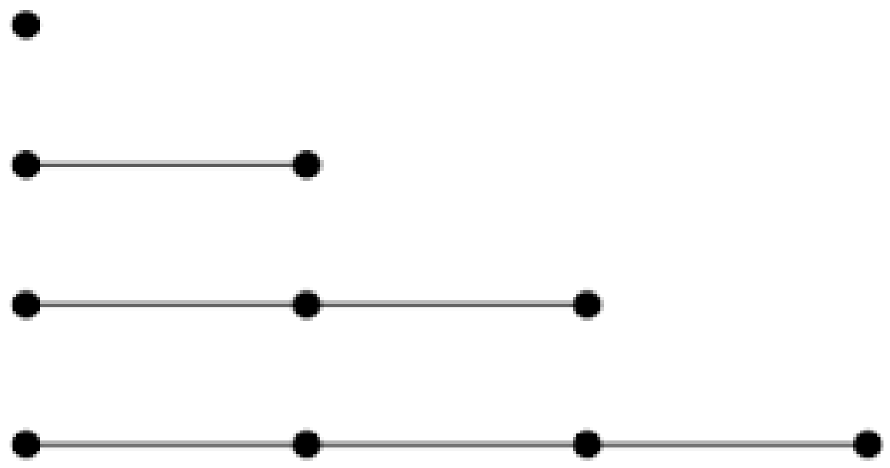
What is its complement ?



Paths and Cycles

(Section 7.2 of textbook)

- **Path:** Place all the nodes in a row and connect the consecutive nodes – we shall get a chain (or ***path***) with $n - 1$ edges.



The first and last nodes are called the end points.

End points have degree = 1

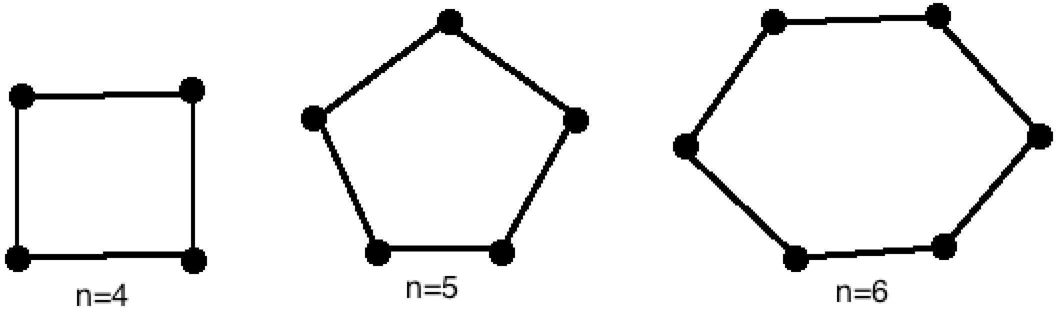
Intermediate nodes have degree = 2

Paths and Cycles

(Section 7.2 of textbook)

- **Path:** Place all the nodes in a row and connect the consecutive nodes – we shall get a chain (or ***path***) with $n - 1$ edges.

- If we also connect the first and last node – we complete a circular path – it is called **CYCLE.**



Paths and Cycles

(Section 7.2 of textbook)

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We may draw same graph in many ways

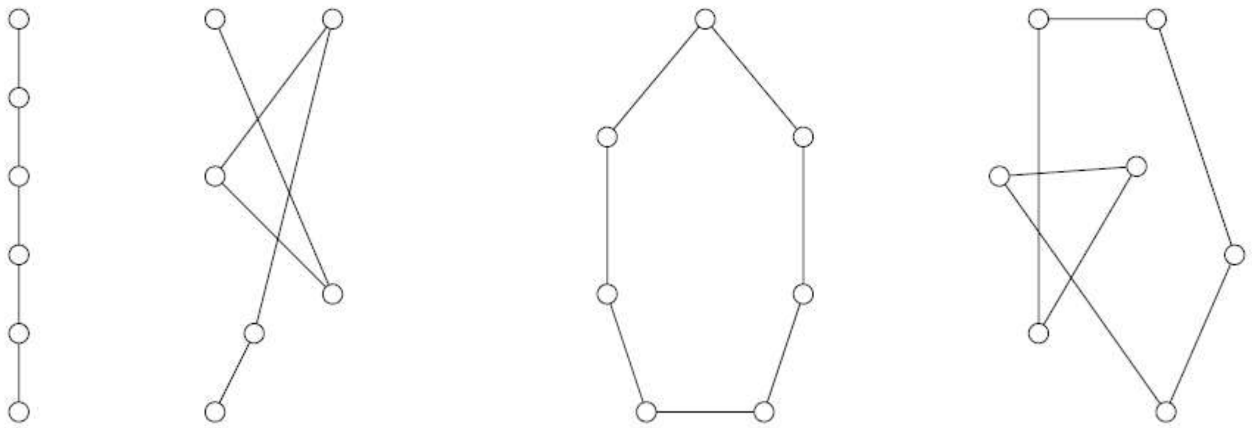

FIGURE 7.5. Two paths and two cycles

Paths and Cycles

(Section 7.2 of textbook)

- **Connected Graph:** If every two nodes in the graph are connected by a path.

We can reach from any node a to a node b, following a path.

Paths and Cycles

(Section 7.2 of textbook)

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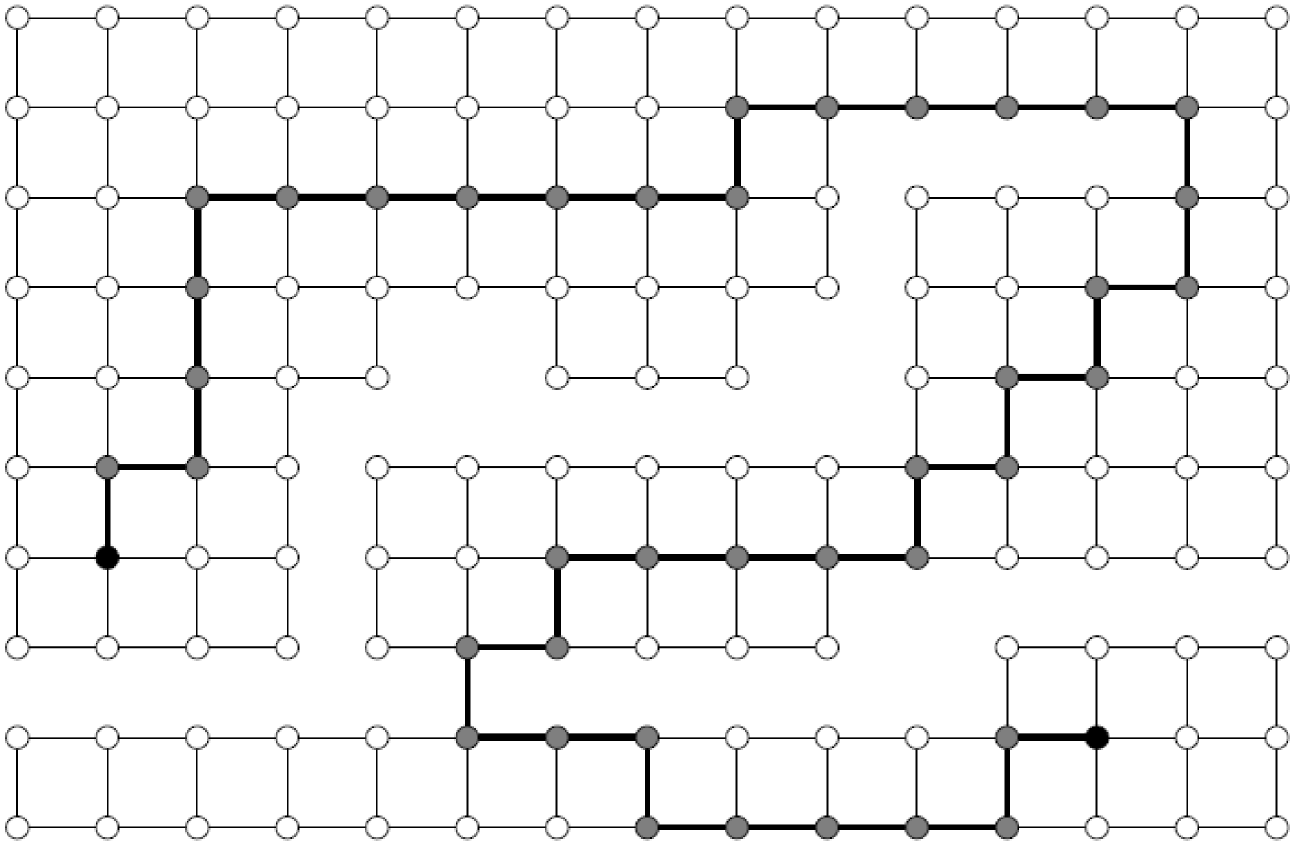

FIGURE 7.6. A path in a graph connecting two nodes

Paths and Cycles

(Section 7.2 of textbook)

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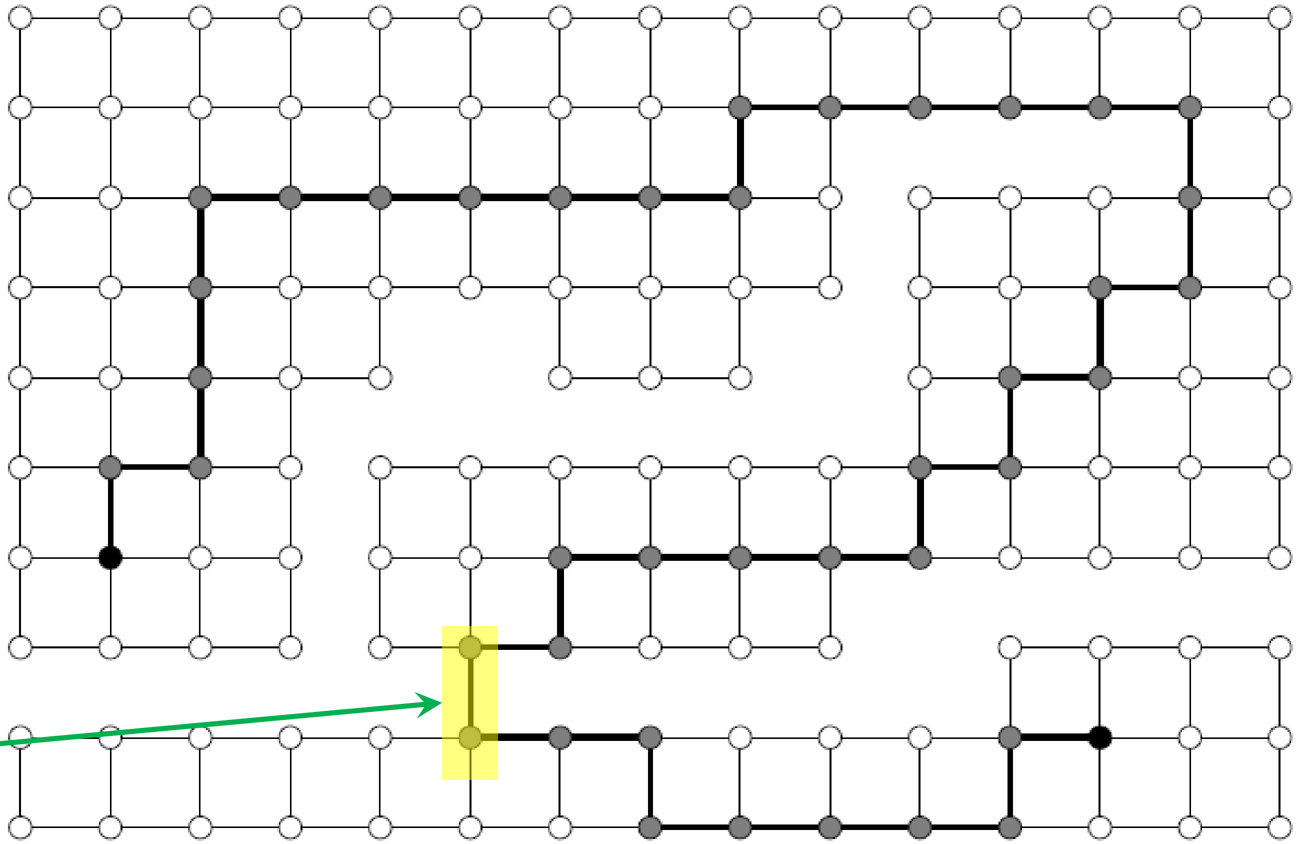


FIGURE 7.6. A path in a graph connecting two nodes

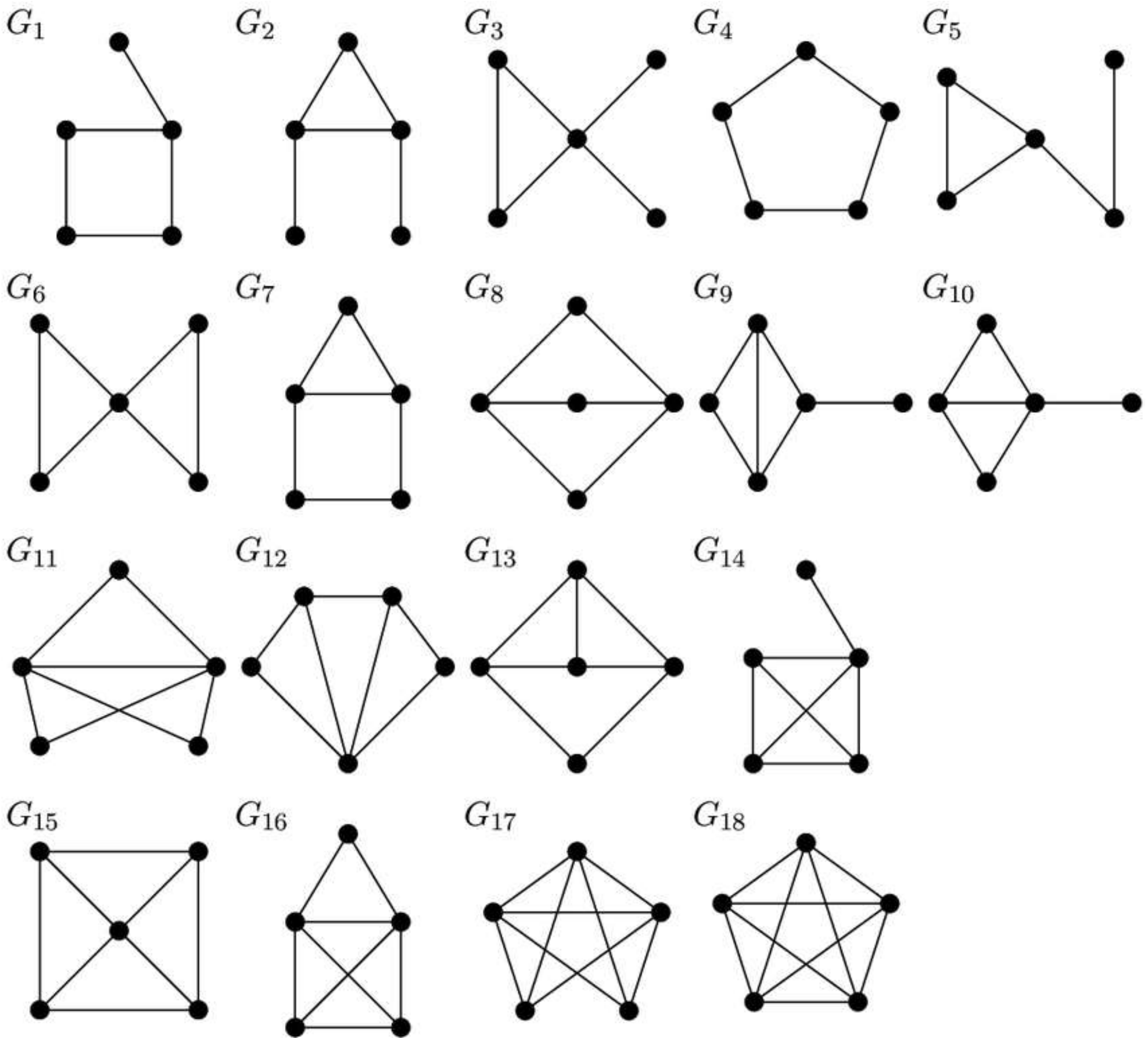
Bridge: *if we remove this edge, the graph becomes disconnected*

Paths and Cycles

(Section 7.2 of textbook)

Example:

Connected Graphs on 5 nodes



Paths and Cycles

(Section 7.2 of textbook)

- **Walk:** A walk in a graph G is a sequence of nodes $v_0, v_1, \cdots, v_n$ which are connected by edges in the given sequence.

It is just like a PATH where nodes may be **REPEATED**.

Paths and Cycles

(Section 7.2 of textbook)

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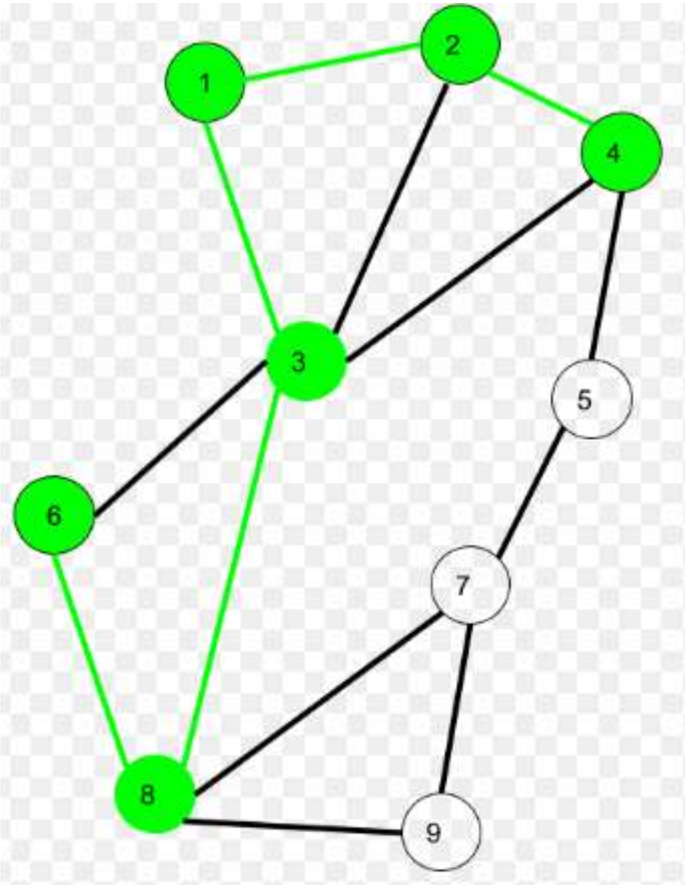

Paths and Cycles

(Section 7.2 of textbook)

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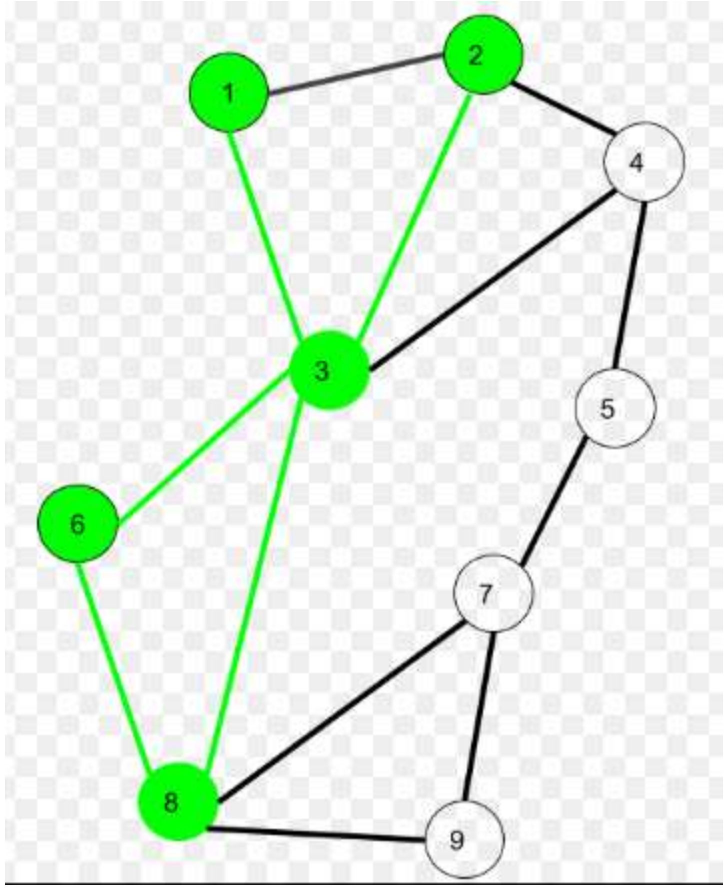

Paths and Cycles

(Section 7.2 of textbook)

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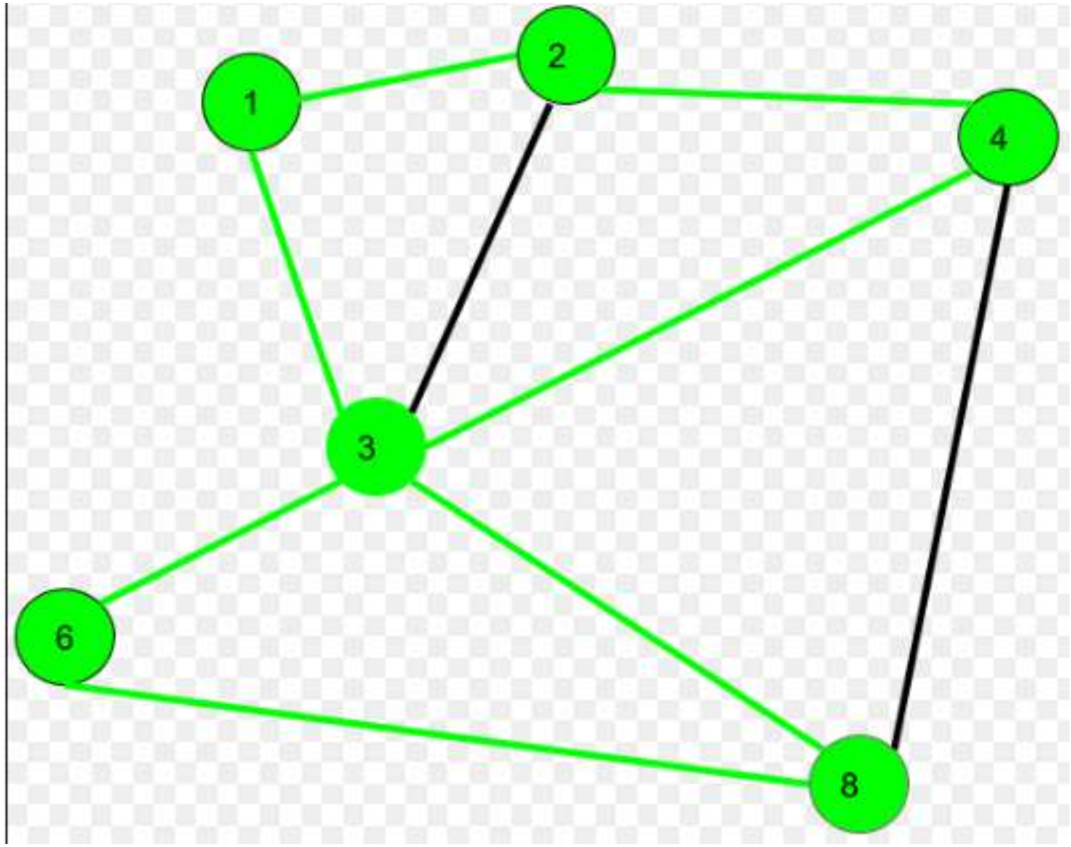


Eulerian Walk and Hamiltonian Cycles

(Section 7.3 of textbook)

- **Eulerian Walk:** A walk in a graph G such that each edge is visited **ONLY ONCE**.

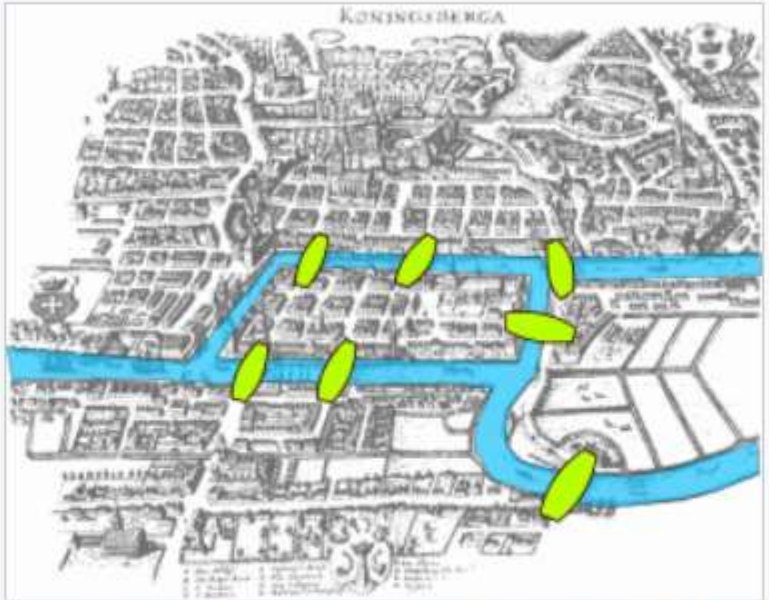
FIGURE 7.8. Leonhard Euler 1707–1783

Eulerian Walk and Hamiltonian Cycles

(Section 7.3 of textbook)

Seven Bridges of Königsberg

- Cross each of seven bridges once and return back to starting point.



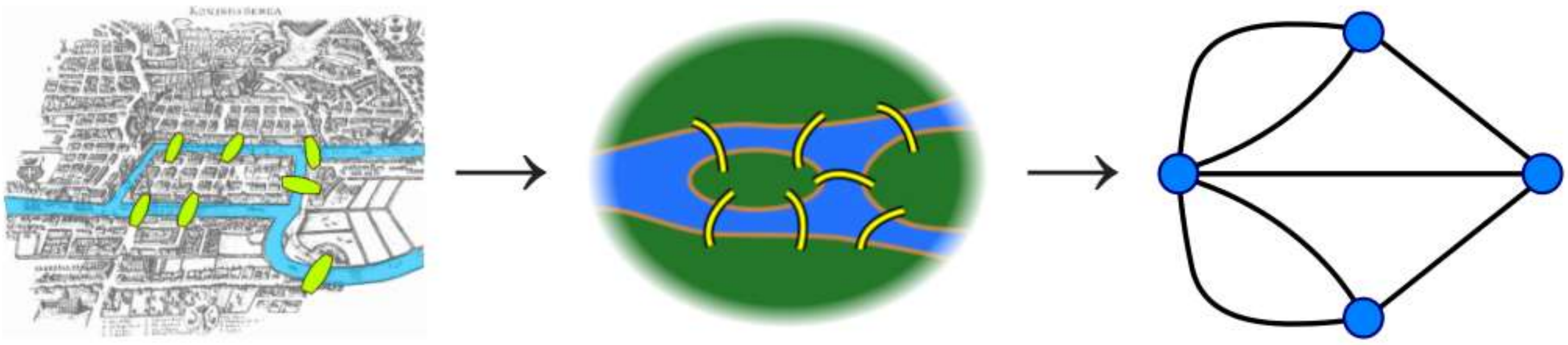
Map of Königsberg in Euler's time showing the actual layout of the seven bridges, highlighting the river Pregel and the bridges

Eulerian Walk and Hamiltonian Cycles

(Section 7.3 of textbook)

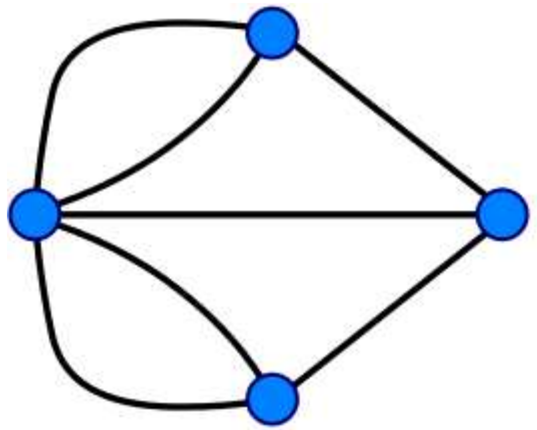
Seven Bridges of Königsberg

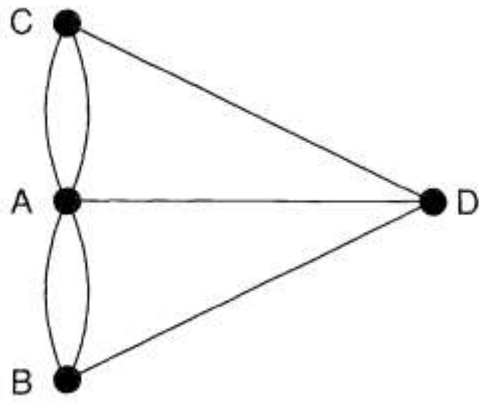
- ***Cross each of seven bridges once and return back to starting point.***
- **Let's make graph of the problem. Replace LAND by <u>NODES</u> and BRIDGES by <u>EDGES</u>, connecting the lands.**



Let’s Re-draw it: … and the Eulerian Trail

- ***Cross each of seven bridges once and return back to starting point.***
- **Eulerian Trail/Path: In a graph G, there is a closed path which traverse each edge ONLY ONCE and returns back to the starting point. ➔ such graph is called Eulerian Graph.**

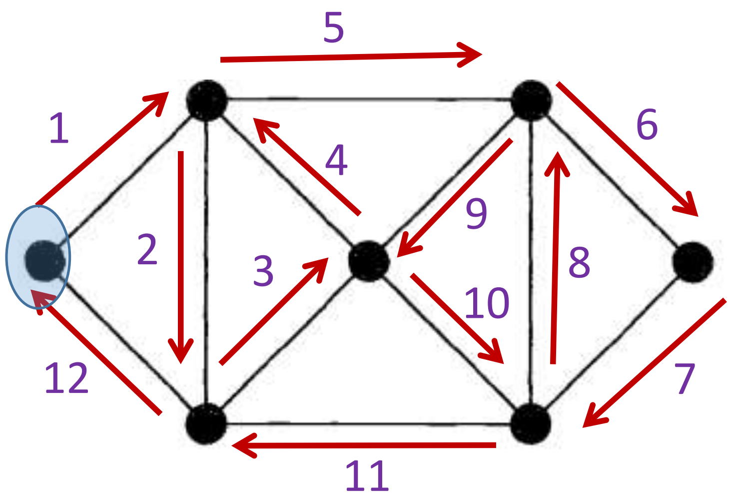


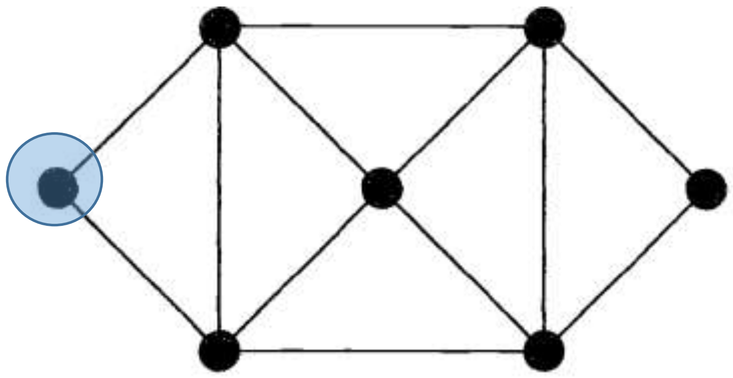


Eulerian Graph

(Section 7.3 of textbook)

- **Eulerian Graph:** A graph containing Eulerian Path



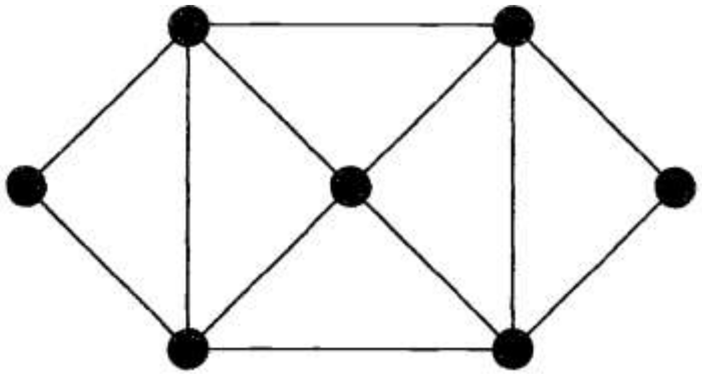


Eulerian Graph

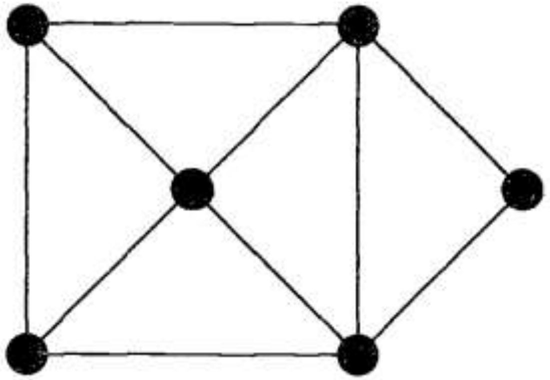
(Section 7.3 of textbook)

- **Eulerian Graph:** A graph containing Eulerian Path

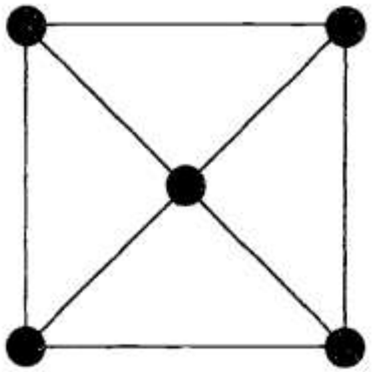
Eulerian Graph



Semi-Eulerian Graph



Non-Eulerian Graph

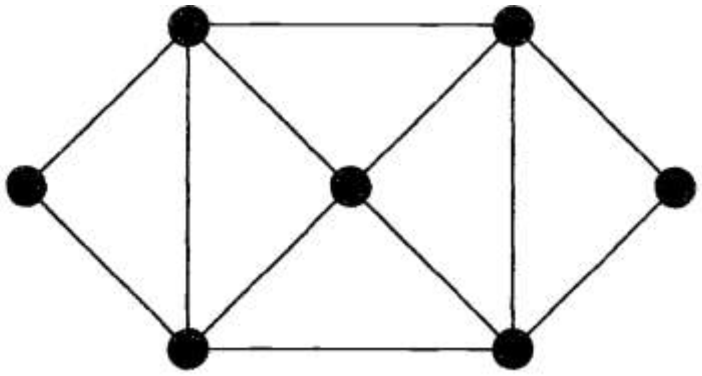


Eulerian Graph

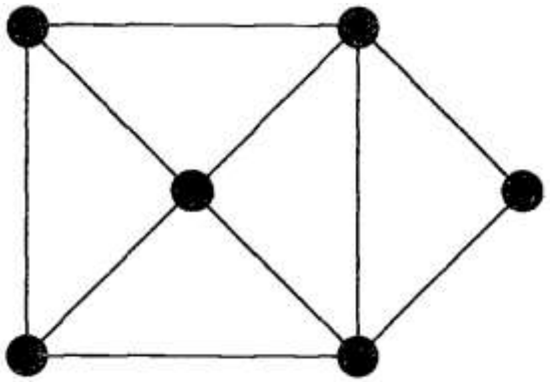
(Section 7.3 of textbook)

- Which graph is an **Eulerian Graph** ?

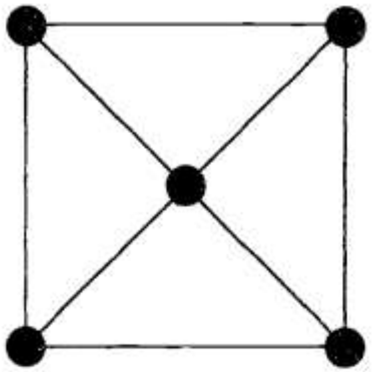
Eulerian Graph



Semi-Eulerian Graph



Non-Eulerian Graph

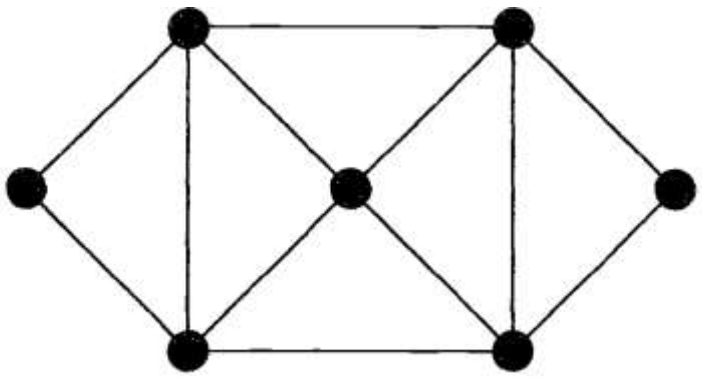


Eulerian Graph

(Section 7.3 of textbook)

- Which graph is an **Eulerian Graph** ?
- *Lemma:* If in a graph G, every vertex has a degree at least 2, then G contains a cycle.

Eulerian Graph

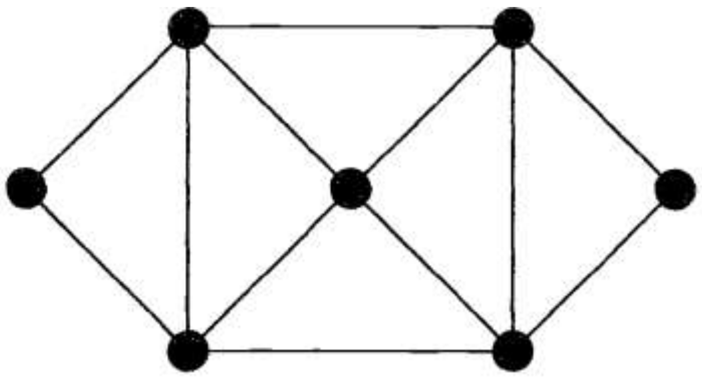


Eulerian Graph

(Section 7.3 of textbook)

- Which graph is an **Eulerian Graph** ?
- *Lemma:* If in a graph G, every vertex has a degree at least 2, then G contains a cycle.
- *Theorem (Euler, 1736):* A connected graph G is Eulerian if and only if the degree of each vertex is even.
  - *Proof:* ☺

Eulerian Graph



Eulerian Graph

(Section 7.3 of textbook)

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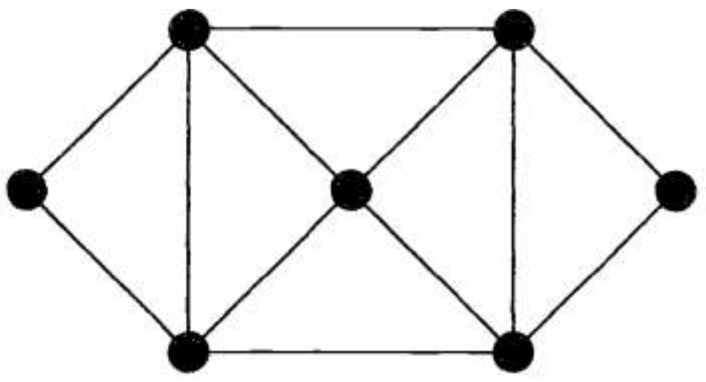
- *Theorem (Euler, 1736):* A connected graph G is Eulerian if and only if the degree of each vertex is even.
  - *Proof:* ☺
- Semi-Eulerian: if and only if it has exactly two vertices of odd degree.

Eulerian Graph

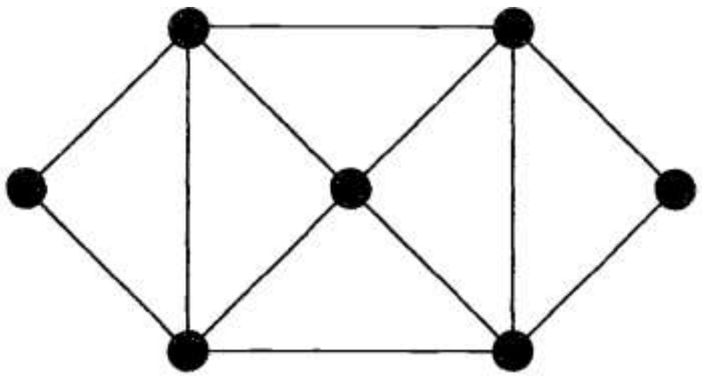


Eulerian Graph

(Section 7.3 of textbook)

- *Theorem (Euler, 1736):* A connected graph G is Eulerian if and only if the degree of each vertex is even.
  - *Proof:* ☺
- How to construct: *Fleury's Algorithm*:

Eulerian Graph

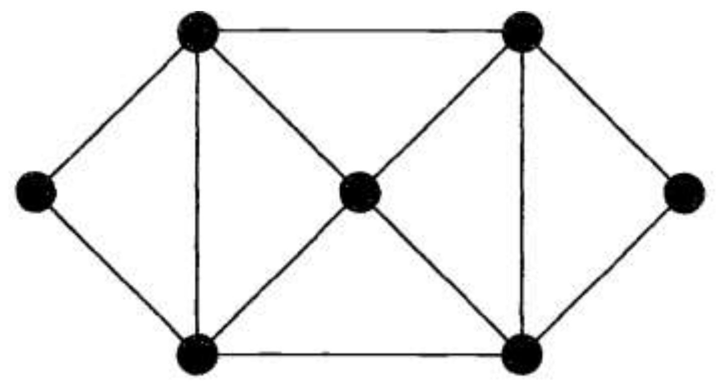


Eulerian Graph

(Section 7.3 of textbook)

Eulerian Graph: *Fleury's Algorithm*

- How to construct:



Start at any vertex *u* and traverse the edges in an arbitrary manner, subject only to the following rules:

(i) erase the edges as they are traversed, and if any isolated vertices result, erase them too;

(ii) at each stage, use a bridge only if there is no alternative.

Eulerian Graph

(Section 7.3 of textbook)

Eulerian Graph: *Fleury's Algorithm*

- Try with these:

Start at any vertex u and traverse the edges in an arbitrary manner, subject only to the following rules:

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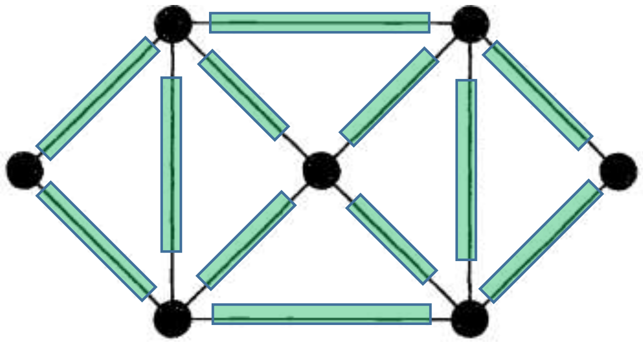
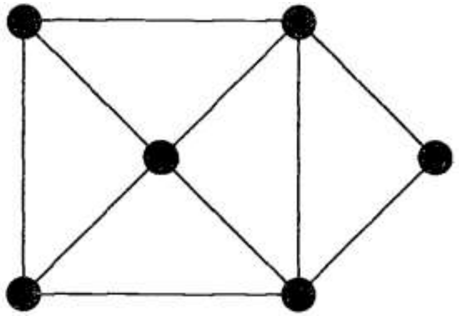
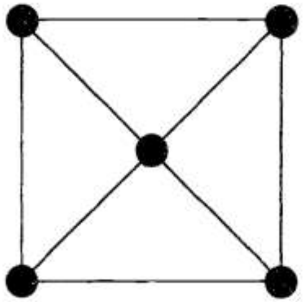
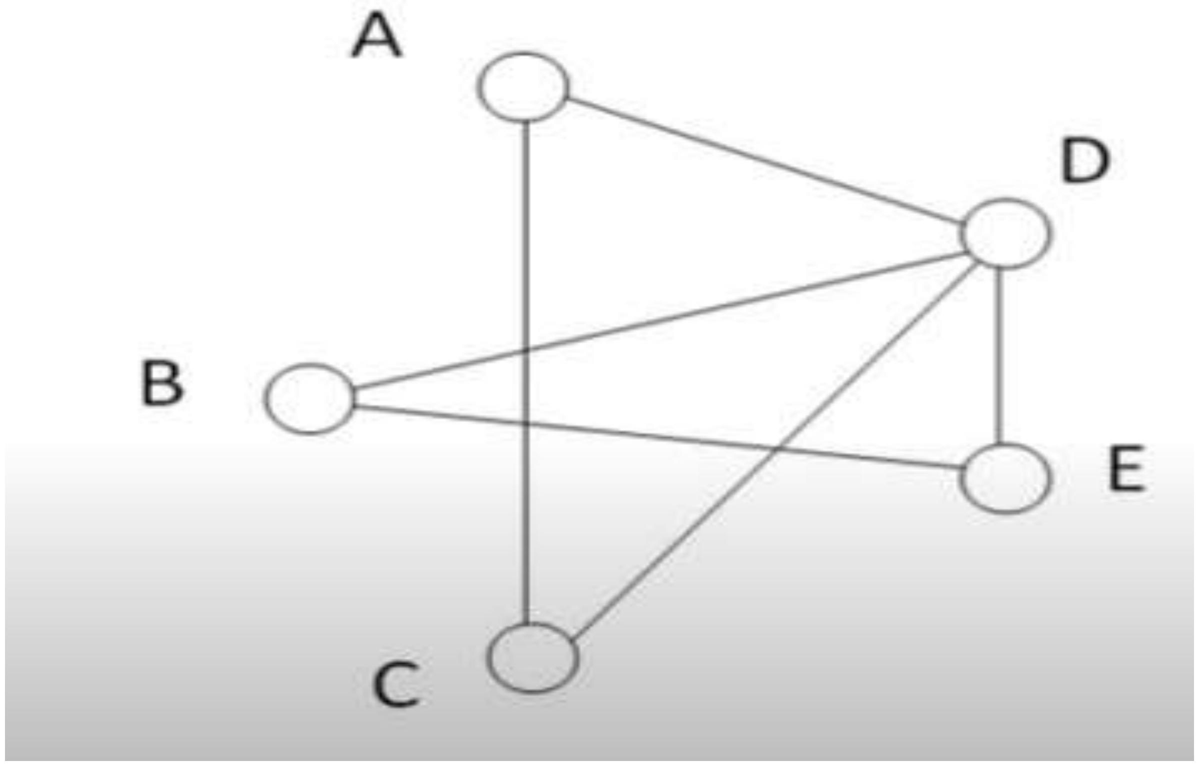






Try this



Eulerian Graph

(Section 7.3 of textbook)

Eulerian Graph: *Fleury's Algorithm*

- Try with these: **Seven Bridges of Königsberg**

Start at any vertex u and traverse the edges in an arbitrary manner, subject only to the following rules:

(i) *erase the edges as they are traversed, and if any isolated vertices result, erase them too;*

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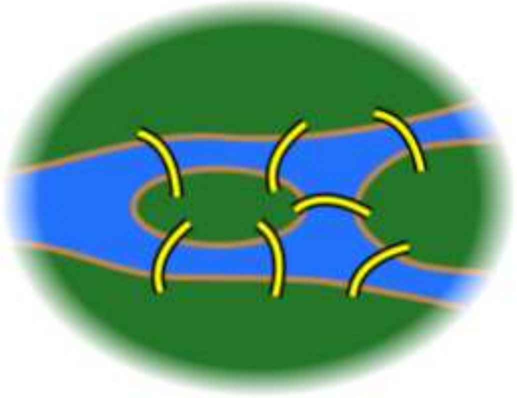

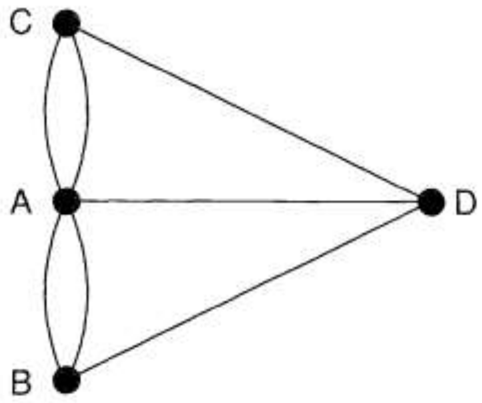

Eulerian Graph

(Section 7.3 of textbook)

Eulerian Graph: *Fleury's Algorithm*

- Try with these: **Seven Bridges of Königsberg**

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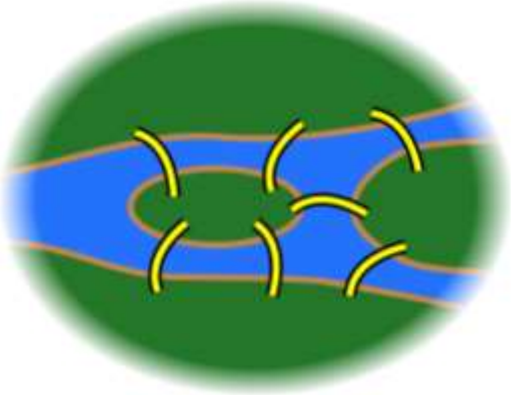
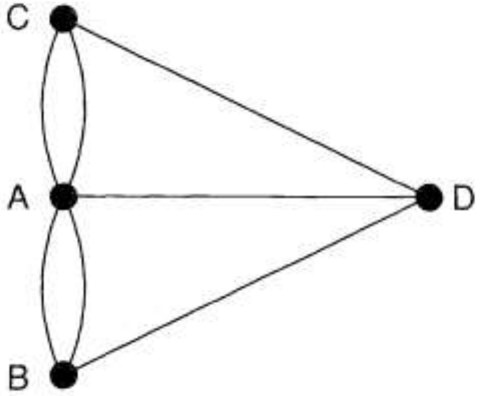




NOT SOLVABLE

Eulerian Graph

(Section 7.3 of textbook)

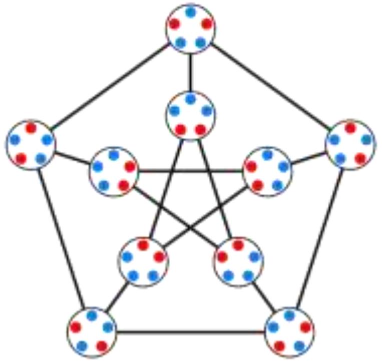
Exercises

Which of the following graphs are Eulerian? semi-Eulerian?

the complete graph K_5;

the complete bipartite graph $K_{2,3}$;

the Petersen graph.



For which values of n is K_n Eulerian?

Which complete bipartite graphs are Eulerian?

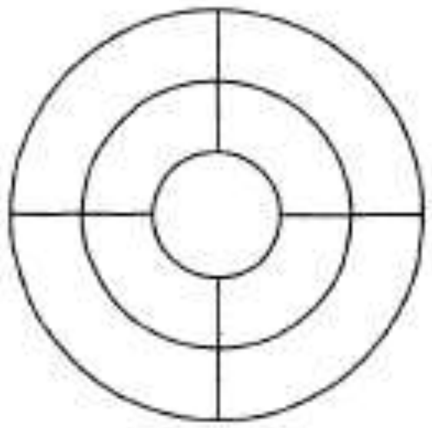
For which values of n is the wheel W_n Eulerian?

Eulerian Graph

(Section 7.3 of textbook)

Exercises

How many continuous pen-strokes are needed to draw the diagram without repeating any line?

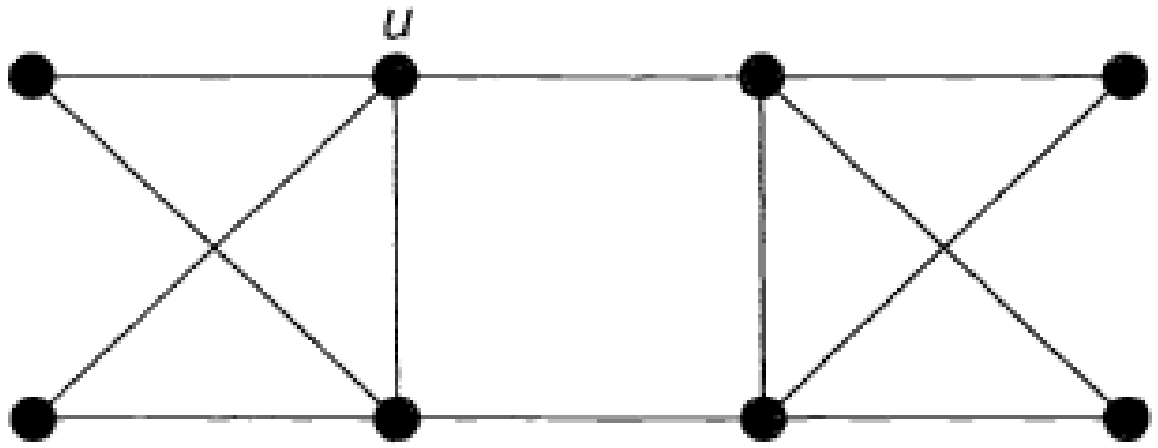


Eulerian Graph

(Section 7.3 of textbook)

Exercises

Use Fleury's algorithm to produce an Eulerian trail for the graph



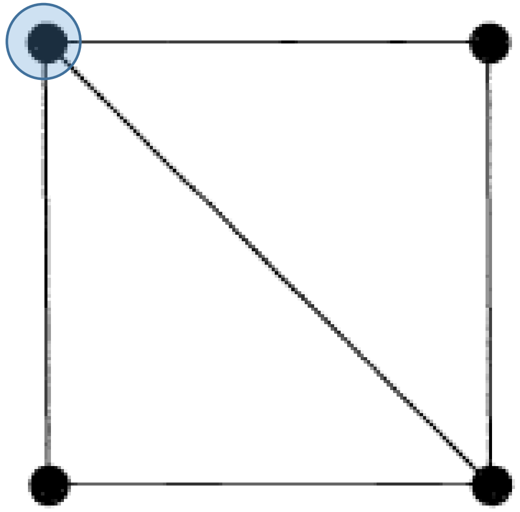
Eulerian Walk and Hamiltonian Cycles

(Section 7.3 of textbook)

- Hamiltonian Cycle: A cycle in a graph G such that each **NODE** is visited **ONLY ONCE**.

Hamiltonian Graph

- **Hamiltonian Cycle:** **In a graph G, there is a closed path which traverse each *vertex* ONLY ONCE and returns back to the starting point. ➔ such graph is called Hamiltonian Graph.**



Hamiltonian Graph

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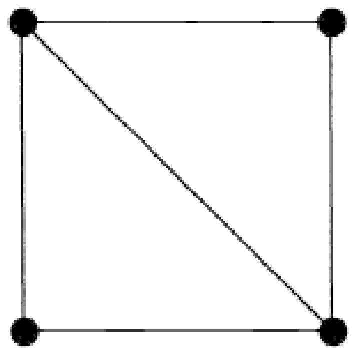
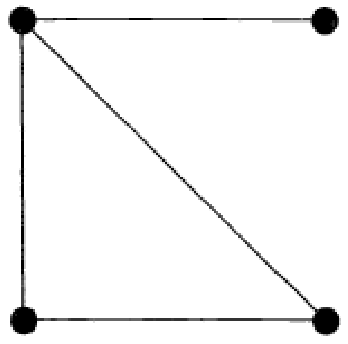
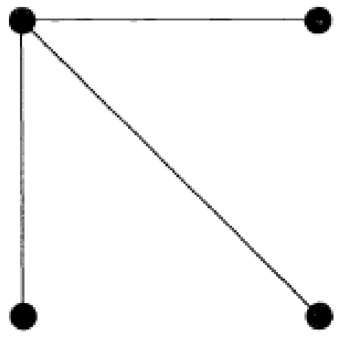
Hamiltonian Graph



Semi-Hamiltonian Graph

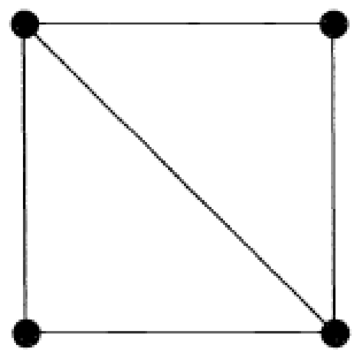


Non-Hamiltonian Graph



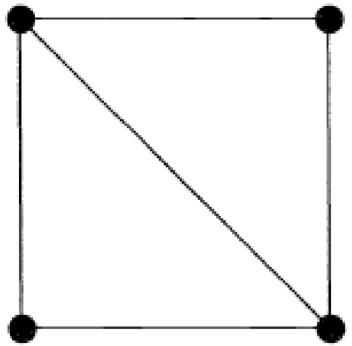
Hamiltonian Graph

- **Which graphs are Hamiltonian?. ☹**



Hamiltonian Graph

- **Which graphs are <u>Hamiltonian?.</u> ☹**
- <u>*Theorem* (Dirac 1952)</u>**:If G is a simple graph with $n \geq 3$, and if $deg(v) \geq n/2$ for each vertex v, then G is Hamiltonian.**



Hamiltonian Graph

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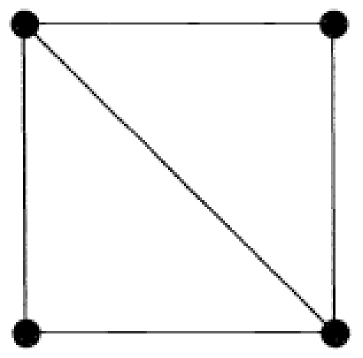
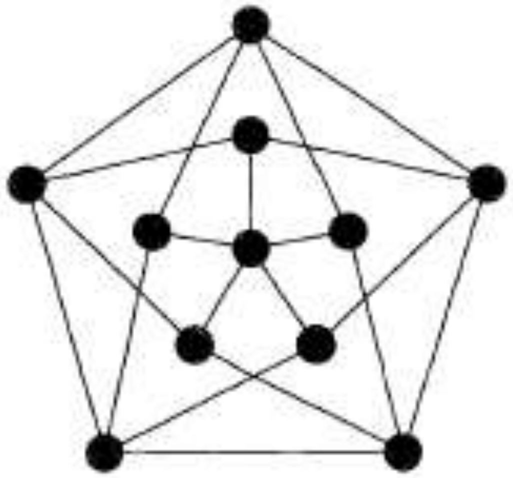
- <u>*Theorem* (Ore 1960)</u>**:If G is a simple graph with $n \geq 3$, and if $deg(v) + deg(w) \geq n$, for each pair of non-adjacent vertices v and w, then G is Hamiltonian.**



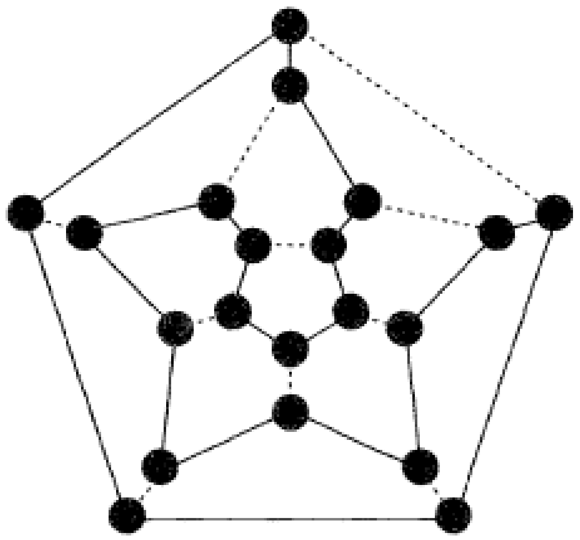
Exercises

Show that the Grötzsch graph is Hamiltonian.



Exercises

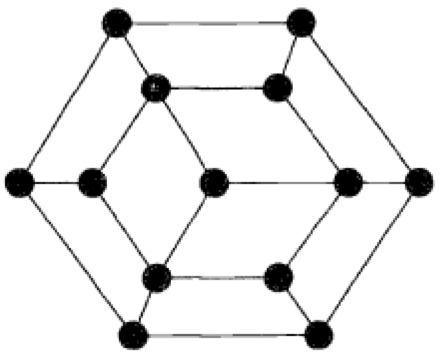
Dodecahedron is Hamiltonian.



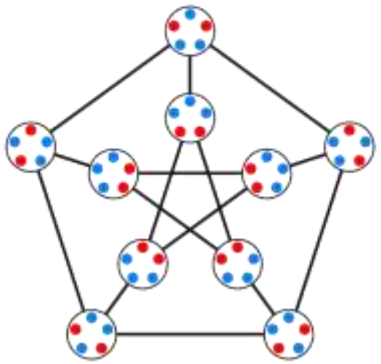
Exercises

Prove that, if G is a bipartite graph with an odd number of vertices, then G is non-Hamiltonian.

Deduce that the graph in Fig. is non-Hamiltonian.



Prove that the Petersen graph is non-Hamiltonian.



Show that, if n is odd, it is not possible for a knight to visit all the squares of an $n \times n$ chessboard exactly once by knight's moves and return to its starting point.

Exercises

In a complete graph on n nodes, K_n, how many Hamiltonian Cycles should be there?

e.g. K_5

Hamiltonian Cycles:

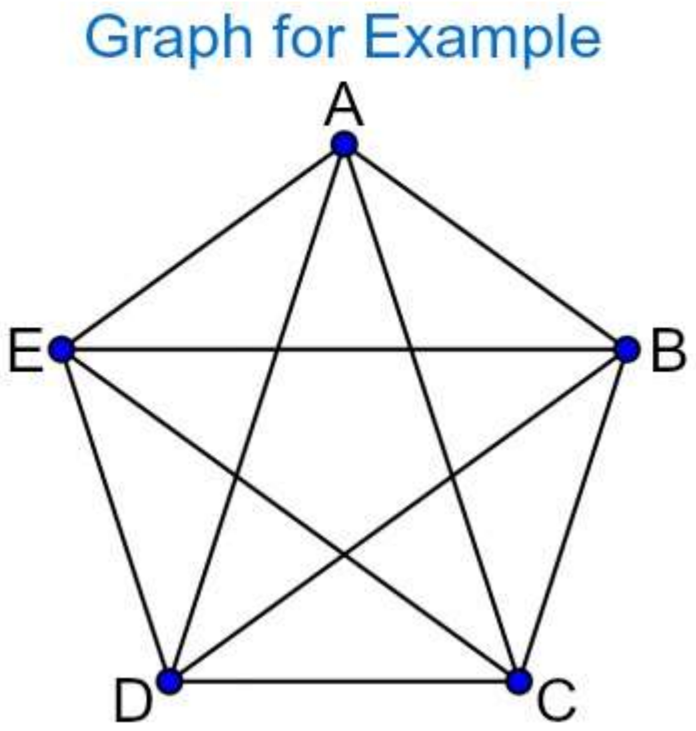
A-B-C-D-E-A

A-B-D-C-E-A

A-C-E-D-B-A

How many Hamiltonian Cycles: $\frac{4!}{2}$

$$\frac{(n-1)!}{2}$$



Practice Problems

Try to solve at your own

7.2.1 Find all complete graphs, paths, and cycles among the graphs in Figures 7.1–7.5.

7.2.2 How many subgraphs does an edgeless graph on n nodes have? How many subgraphs does a triangle have?

7.2.3 Find all graphs that are paths or cycles and whose complements are also paths or cycles.

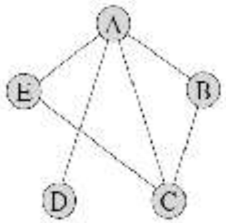


FIGURE 7.1.

FIGURE 7.2.

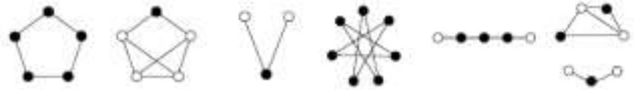


FIGURE 7.3.

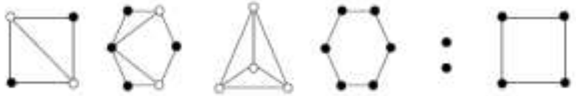


FIGURE 7.4.

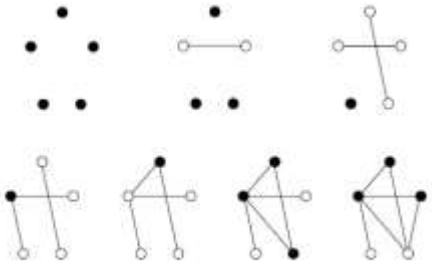


FIGURE 7.5.