Binomial Coefficients and Identities

The Binomial Theorem

- As we remarked in Section 6.3, the number of r-combinations from a set with n elements is often denoted by $\binom{n}{r}$. This number is also called a **binomial coefficient** because these numbers occur as coefficients in the expansion of powers of binomial expressions such as $(a + b)^n$
- The binomial theorem gives the coefficients of the expansion of powers of binomial expressions. A **binomial** expression is simply the sum of two terms, such as $a+b$

The expansion of $(x+y)^3$ can be found using combinatorial reasoning instead of multiplying the three terms out. When $(x+y)^3 = (x+y)(x+y)(x+y)$ is expanded, all products of a term in the first sum, a term in the second sum, and a term in the third sum are added. Terms of the form x^3, x^2y, xy^2, and y^3 arise. To obtain a term of the form x^3, an x must be chosen in each of the sums, and this can be done in only one way. Thus, the x^3 term in the product has a coefficient of 1. To obtain a term of the form x^2y, an x must be chosen in two of the three sums (and consequently a y in the other sum). Hence, the number of such terms is the number of 2-combinations of three objects, namely, $\binom{3}{2}$. Similarly, the number of terms of the form xy^2 is the number of ways to pick one of the three sums to obtain an x (and consequently take a y from each of the other two sums). This can be done in $\binom{3}{1}$ ways. Finally, the only way to obtain a y^3 term is to choose the y for each of the three sums in the product, and this can be done in exactly one way. Consequently, it follows that

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y) = (xx+xy+yx+yy)(x+y)\\ &= xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy\\ &= x^3+3x^2y+3xy^2+y^3.\end{aligned}$$

THE BINOMIAL THEOREM Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

What is the expansion of $(x+y)^4$?

$$\begin{aligned}(x+y)^4 &= \sum_{j=0}^{4}\binom{4}{j}x^{4-j}y^j\\ &= \binom{4}{0}x^4+\binom{4}{1}x^3y+\binom{4}{2}x^2y^2+\binom{4}{3}xy^3+\binom{4}{4}y^4\\ &= x^4+4x^3y+6x^2y^2+4xy^3+y^4.\end{aligned}$$

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$?

From the binomial theorem it follows that this coefficient is

$$\binom{25}{13} = \frac{25!}{13!\,12!} = 5{,}200{,}300.$$

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

First, note that this expression equals $(2x + (-3y))^{25}$. By the binomial theorem, we have

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$, namely,

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!\,12!} 2^{12} 3^{13}.$$

CA1: Find the coefficient of $x^{5}y$^8 in $(x + y)$^13.

CA2: What is the coefficient of $x^{101}y$^99 in the expansion of $(2x - 3y)$^200?

CA3: What is the coefficient of x^7 in $(1 + x)$^11?

CA4: What is the coefficient of x^9 in $(2 - x)$^19?