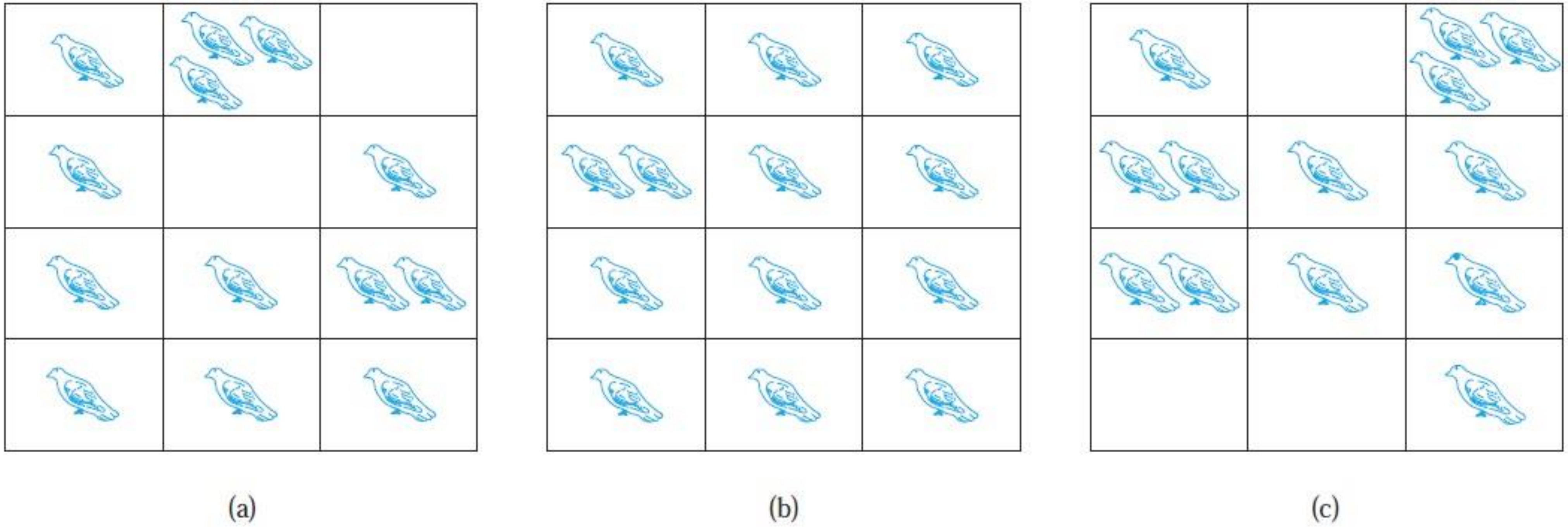
Pigeonhole Principle

- Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost. Because there are 20 pigeons but only 19 pigeonholes, a least one of these 19 pigeonholes must have at least two pigeons in it. To see why this is true, note that if each pigeonhole had at most one pigeon in it, at most 19 pigeons, one per hole, could be accommodated. This illustrates a general principle called the **pigeonhole principle**, which states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it



A function f from a set with $k + 1$ or more elements to a set with k elements is not one-to-one.

EXAMPLE 1 Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays. ◀

EXAMPLE 2 In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet. ◀

EXAMPLE 3 How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

The Generalized Pigeonhole Principle

If k pigeon holes are occupied by kr+1 pigeons, then at least one pigeonhole is occupied by r+1 or more pigeons

If n objects are placed into k boxes, then there is at least one box containing at least $\left\lceil \frac{n}{k} \right\rceil$ objects.

EXAMPLE 5 Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

This principle states that if you have N objects and distribute them among k boxes ($N > k$), then at least one box will contain more than N/k objects (ceiling function not used here).

The inequality $N/k \geq r$ tells us that the number of objects per box (N/k) must be greater than or equal to a specific value (r).

We want to find the minimum number of objects (N) required to satisfy this inequality.

$N = k(r-1) + 1$: This formula represents the smallest possible value of N that guarantees at least r objects in one box. Here's why:

Assuming the Opposite: Suppose we have N objects and none of the boxes contain more than r-1 objects. This implies $N/k <= r-1$ (using the concept of pigeonhole principle).

Contradiction: If the above assumption is true, the total number of objects would be at most $k*(r-1)$. However, we started with N objects (which is supposed to be the total). This creates a contradiction

Since the assumption leads to a contradiction, it means for N/k to be greater than or equal to r (as required by the inequality), the minimum number of objects (N) must be at least $k(r-1) + 1$.

In simpler terms:

Imagine you have boxes (k) and want to ensure at least r objects in one box. The formula tells you the minimum number of total objects (N) needed. It basically says:

- Take the number of boxes (k)
- Multiply it by the minimum number of objects per box you want to avoid (r-1)
- Add 1 (to ensure you go beyond the minimum and have at least r objects in a box)

This formula guarantees that with N objects distributed among k boxes, you'll have at least r objects in one of the boxes.

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Pigeon holes: $k=5$

Pigeons: $r+1 = 6 \rightarrow n = 5$

Minimum students required:

$kr+1 = 5.5+1 = 26$

26 students required to make sure 6 will receive the same grades

Show that in any set of six classes, each meeting regularly once a week on a particular day of the week, there must be two that meet on the same day, assuming that no classes are held on weekends.

n = 6

k = 5

$$\left\lceil \frac{6}{5} \right\rceil = \lceil 1.2 \rceil = 2$$

Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

n = 30

k = 26

$$\left\lceil \frac{30}{26} \right\rceil = \lceil 1.15 \rceil = 2$$

A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

a) How many balls must she select to be sure of having at least three balls of the same color?

b) How many balls must she select to be sure of having at least three blue balls?

a.

k = 2

r+1 = 3 → n = 2

Kr+1 = 2x2+1 = 5

Five balls must be selected because if there are four balls 2 balls can be of different color

b. She needs to withdraw 13 balls to make sure there are at least 3 balls are blue

A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.

a) How many socks must he take out to be sure that he has at least two socks of the same color?

b) How many socks must he take out to be sure that he has at least two black socks?

a.

k = 2

n+1 = 2 → n = 1

Kn+1 = 1x2+1 = 3

3 socks must be selected because if there are 2 socks 1 sock can be of different color

b. He needs to takeout 14 socks to make sure there are at least 2 socks are black

Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

n = 5

k = 4

$$\left\lceil \frac{5}{4} \right\rceil = \lceil 1.25 \rceil = 2$$

What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form NXX-NXX-XXXX, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)

NXX-XXXX: $8 \times 10 \times 10 \times 10^4 = 8 \times 10^6 = 8 \text{ million}$

$$\left\lceil \frac{25000000}{8000000} \right\rceil = \lceil 3.15 \rceil = 4$$

Show that If you pick 5 integers from 1 to 8, you will definitely find two of them must add up to 9.

$$\{1,8\}, \{2,7\}, \{3,6\}, \{4,5\}$$

n = 5
k = 4

$$\left\lceil \frac{5}{4} \right\rceil = \lceil 1.25 \rceil = 2$$

What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

$k = 50$

$n+1 = 100 \rightarrow n = 99$

$Kn+1 = 50(99)+1=4951$

There are *n* people in a party and *n* is more than 1 so they can shake hands with one another at will.

You will always find two people who shake hands with the same number of people.

$$\left\lceil \frac{n}{n-1} \right\rceil = \lceil 1.XXX \rceil = 2$$