Relations

$5 < 10$	$5 \leq 5$	$6 = \frac{30}{5}$	$5 \mid 80$	$7 > 4$	$x \neq y$	$8 \nmid 3$
$a \equiv b \pmod{n}$	$6 \in \mathbb{Z}$	$X \subseteq Y$	$\pi \approx 3.14$	$0 \geq -1$	$\sqrt{2} \notin \mathbb{Z}$	$\mathbb{Z} \nsubseteq \mathbb{N}$

- mathematics there are endless ways that two entities can be related to each other
- In each case two entities appear on either side of a symbol, and we interpret the symbol as expressing some relationship between the two entities. Symbols such as $<$, $>$, $\gg$, $\geq$, $\mp$, $\sim$and ½, etc., are called *relations* because they convey relationships among things.

A **relation** on a set A is a subset $R \subseteq A \times A$. We often abbreviate the statement $(x, y) \in R$ as xRy. The statement $(x, y) \notin R$ is abbreviated as $x \not R y$.

Notice that a relation is a set

- Let A be the set of students in your school, and let B be the set of courses. Let R be the relation that consists of those pairs (a, b), where a is a student enrolled in course b. For instance, if Jason Goodfriend and Deborah Sherman are enrolled in CS518, the pairs. **(Jason Goodfriend, CS518)** and **(Deborah Sherman, CS518)** belong to R.
- If Jason Goodfriend is also enrolled in CS510, then the pair **(Jason Goodfriend, CS510)** is also in R. However, if Deborah Sherman is not enrolled in CS510, then the pair **(Deborah Sherman, CS510)** is not in R.

Let $A = \{1,2,3,4\}$, and consider the following set:

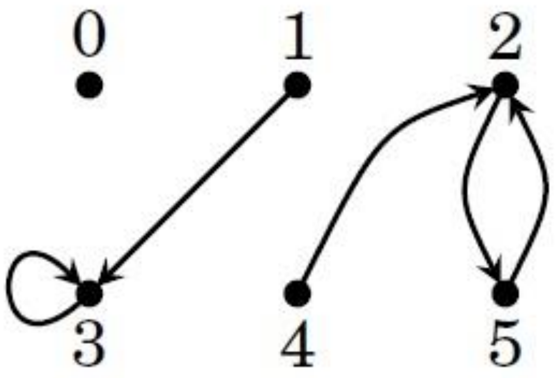
$$R = \{(1,1),(2,1),(2,2),(3,3),(3,2),(3,1),(4,4),(4,3),(4,2),(4,1)\} \subseteq A \times A$$

The set R is a relation on A, by Definition 11.1. Since $(1,1) \in R$, we have $1R1$. Similarly $2R1$ and $2R2$, and so on. However notice that (for example) $(3,4) \notin R$, so $3 \not R 4$. Observe that R is the familiar relation $\geq$ for the set A.

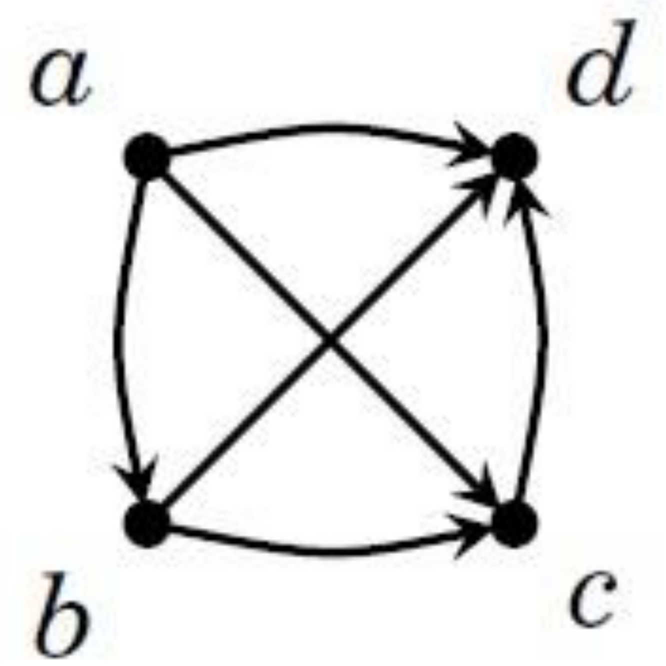
Relations can be described as graphs

Let $A = \{1, 2, 3, 4\}$, and consider the following set:

$$S = \{(1,1), (1,3), (3,1), (3,3), (2,2), (2,4), (4,2), (4,4)\} \subseteq A \times A$$



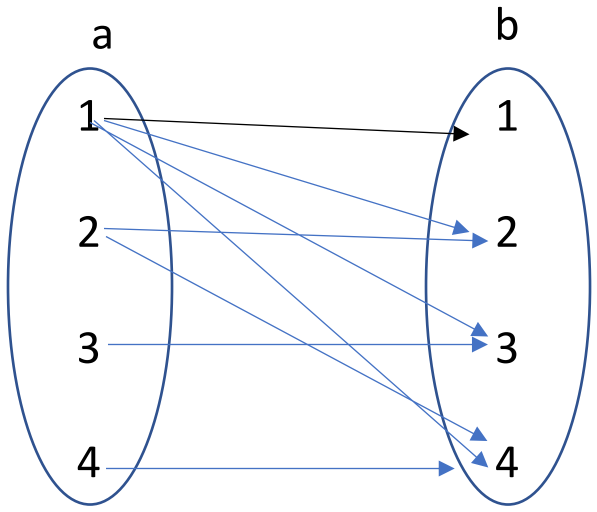
R= {(a,b),(a, c),(a,d),(b, c),(b,d),(c,d)}



Relations on Sets

Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Think of it like this



R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}

How many relations are there on a set with *n* elements?

Solution: A relation on a set A is a subset of $A \times A$. Because $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are 2^{n^2} subsets of $A \times A$. Thus, there are 2^{n^2} relations on a set with n elements. For example, there are $2^{3^2} = 2^9 = 512$ relations on the set $\{a, b, c\}$. ◀

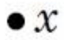
Relation Properties

Definition Suppose R is a relation on a set A.

1. Relation R is **reflexive** if xRx for every $x \in A$. Every element is related to itself
   (That is, R is reflexive if $\forall x \in A, xRx$.)
2. Relation R is **symmetric** if xRy implies yRx for all $x, y \in A$
   (That is, R is symmetric if $\forall x, y \in A, xRy \Rightarrow yRx$.)
3. Relation R is **transitive** if whenever xRy and yRz, then also xRz.
   (That is, R is transitive if $\forall x, y, z \in A, \big((xRy) \wedge (yRz)\big) \Rightarrow xRz$.)

A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

1. A relation is **reflexive** if for each point x ...

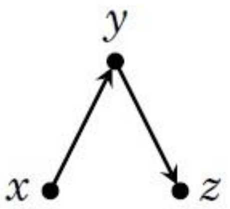


...there is a loop at x.

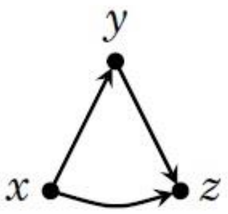
2. A relation is **symmetric** if whenever there is an arrow from x to y ...

...there is also an arrow from y back to x:

3. A relation is **transitive** if whenever there are arrows from x to y and y to z ...



...there is also an arrow from x to z:



(This also means that whenever there is an arrow from x to y and from y to x ...

...there is also a loop from x back to x.)

Reflexive Property Examples

Example: Let A= {1,2,3,4}

R1={(1,1),(3,3),(2,2),(4,4)}

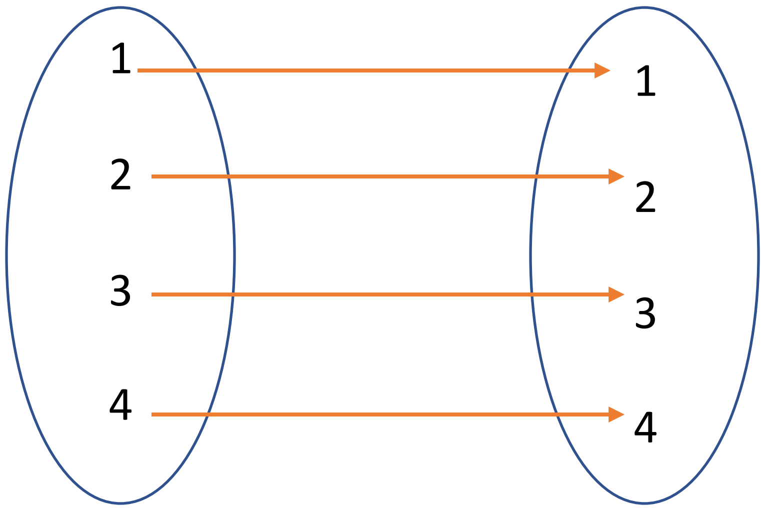
R2={(1,1),(1,4),(2,2),(3,3),(4,3)}

R3={(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)}

R4={(1,3),(2,2),(2,4),(3,1),(4,4)}

Which relation is reflexive?

Check this in every R



Is the “divides” relation on the set of positive integers reflexive?

Solution: Because $a \mid a$ whenever a is a positive integer, the “divides” relation is reflexive. (Note that if we replace the set of positive integers with the set of all integers the relation is not reflexive because by definition 0 does not divide 0.)

Symmetric Property Examples

Example: Let A= {1,2,3,4}

R1={(1,1),(1,3),(2,4),(3,1),(4,2)}

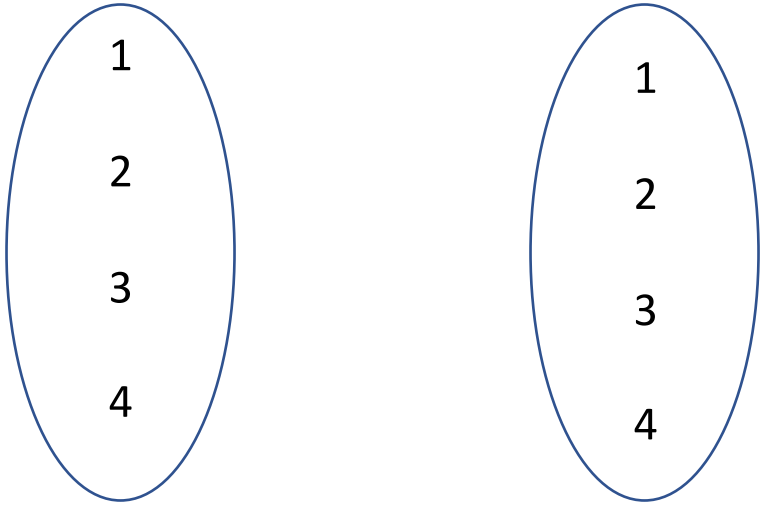
R2={(1,1),(2,2),(3,3),(4,4)}

R3={(2,2),(2,3),(3,4)}

R4={(1,1),(2,2),(3,3),(4,3),(4,4)}

Which relation is symmetric?

Apply arrow diagram for R



Is the “divides” relation on the set of positive integers symmetric? Is it antisymmetric?

Solution: This relation is not symmetric because $1 \mid 2$, but $2 \nmid 1$. It is antisymmetric, for if a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$ (the verification of this is left as an exercise for the reader).

- R1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)},
- R2 = {(1, 1), (1, 2), (2, 1)},
- R3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)},
- R4 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)},
- R5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)},
- R6 = {(3, 4)}.

Equivalence Relations

- The relation = on the set Z (or on any set A) is reflexive, symmetric and transitive. There are many other relations that are also reflexive, symmetric and transitive. Relations which have all three of these properties occur very frequently in mathematics and often play quite significant roles. (For instance, this is certainly true of the relation =) Such relations are given a special name. They are called *equivalence relations*.

For set A={-1,1,2,3,4}

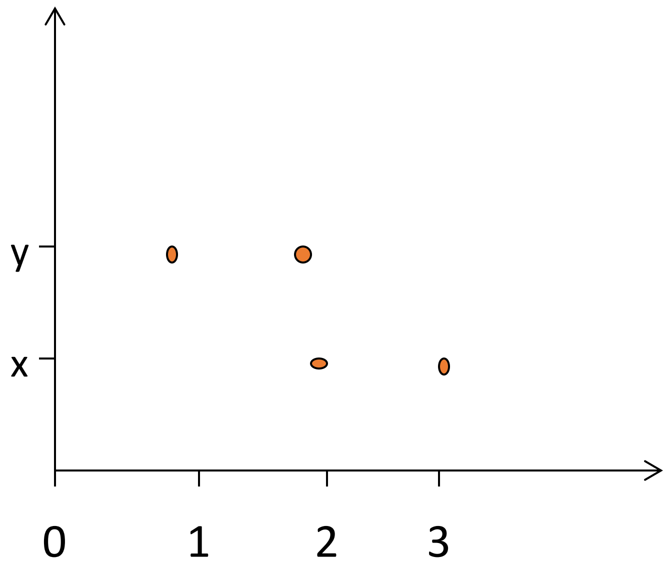
R={(-1,-1),(1,1),(2,2),(3,3),(4,4),(-1,1),(1,-1),(-1,3),(3,-1), (1,3),(3,1),(2,4),(4,2)}

The R has equivalence relation

Relations – Representations

1 Coordinate diagram:

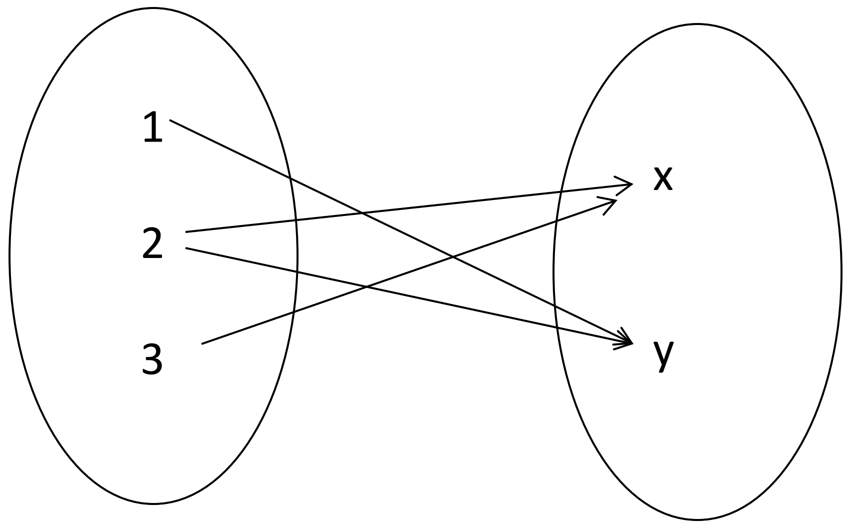
Let A={1,2,3} and B={x , y}. Let R be a relation from A to B defined as R={(1,y),(2,x),(2,y),(3,x)}



Relations – Representations

2 Arrow diagram:

Let A={1,2,3} and B={x , y}. Let R be a relation from A to B defined as R={(1,y),(2,x),(2,y),(3,x)}



Relations – Representations

3 Matrix Representation:

Let A={1,2,3} and B={x , y}. Let R be a relation from A to B defined as R={(1,y),(2,x),(2,y),(3,x)}

$$\begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$