Permutations and Combinations

- Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters. Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where the order of the elements selected does not matter. For example, in how many ways can we select three students from a group of five students to stand in line for a picture? How many different committees of three students can be formed from a group of four students? In this section we will develop methods to answer questions such as these.

In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?

- First, note that the order in which we select the students matters. There are five ways to select the first student to stand at the start of the line. Once this student has been selected, there are four ways to select the second student in the line. After the first and second students have been selected, there are three ways to select the third student in the line. By the product rule, there are $\mathbf{5 \cdot 4 \cdot 3 = 60}$ ways to select three students from a group of five students to stand in line for a picture.
- To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way. Consequently, there are $\mathbf{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120}$ ways to arrange all five students in a line for a picture.

Permutations

- A **permutation** of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an ***r*-permutation**

Let $S = \{1, 2, 3\}$.The ordered arrangement 3, 1, 2 is a permutation of S. The ordered arrangement 3, 2 is a 2-permutation of S.

The number of r-permutations of a set with n elements is denoted by $P(n, r)$. We can find $P(n, r)$ using the product rule.

- Let $S = \{a, b, c\}$. The 2-permutations of S are the ordered arrangements a, b; a, c; b, a; b, c; c, a; and c, b. Consequently, there are six 2-permutations of this set with three elements. There are always six 2-permutations of a set with three elements. There are three ways to choose the first element of the arrangement. There are two ways to choose the second element of the arrangement, because it must be different from the first element. Hence, by the product rule, we see that $P(3, 2) = 3 \cdot 2 = 6$. the first element. By the product rule, it follows that $P(3, 2) = 3 \cdot 2 = 6$.

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

If n and r are integers with $0 \leq r \leq n$, then $P(n, r) = \frac{n!}{(n-r)!}$.

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

$$P(100,3) = \frac{100!}{(100-3)!}$$

Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

- $P(8,3) = \frac{8!}{(8-3)!}$

Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

- $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

- Because the letters *ABC* must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block *ABC* and the individual letters *D*, *E*, *F*, *G*, and *H*. Because these six objects can occur in any order, there are $6! = 720$ permutations of the letters *ABCDEFGH* in which *ABC* occurs as a block.

CA1: List all the permutations of $\{a, b, c\}$.

abc, acb, bac, bca, cab, cba

CA2:
How many different permutations are there of the set $\{a, b, c, d, e, f, g\}$?

- There are seven elements in the set so there are **7!** Different permutations

CA3:
How many permutations of $\{a, b, c, d, e, f, g\}$ end with a?

As “a” is fixed so we have to change the positions of remaining 6 elements. So there are 6! Different permutations end with “a”.

CA4: Find the values of the followings

a) $P(6, 3)$

b) $P(6, 5)$

c) $P(8, 1)$

d) $P(8, 5)$

e) $P(8, 8)$

f) $P(10, 9)$

Combinations

We now turn our attention to counting unordered selections of objects.

How many different committees of three students can be formed from a group of four students?

Solution: To answer this question, we need only find the number of subsets with three elements from the set containing the four students. We see that there are four such subsets, one for each of the four students, because choosing three students is the same as choosing one of the four students to leave out of the group. This means that there are four ways to choose the three students for the committee, where the order in which these students are chosen does not matter.

The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$.

Note that $C(n, r)$ is also denoted by $\binom{n}{r}$ and is called a **binomial coefficient**.

- We see that $C(4, 2) = 6$, because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals

$$C(n, r) = \frac{n!}{r!\,(n-r)!}.$$

CA6: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

five cards are dealt from a deck of 52 cards

$$C(52,5) = \frac{52!}{5!\,(52-5)!}$$

Ways to select 47 cards from deck of 52 cards

$$C(52,5) = \frac{52!}{47!\,(52-47)!}$$

CA7: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

$$C(10,5) = \frac{10!}{5!\,(10-5)!}$$

CA8: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission

$$C(30,6) = \frac{30!}{6!\,(30-6)!}$$

CA9: Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

$$C(9,3) \times C(11,4) = \frac{9!}{3!\,(9-3)!} \times \frac{11!}{4!\,(11-4)!}$$

CA10: How many bit strings of length n contain exactly r 1s?

The positions of r 1s in a bit string of length n form an r-combination of the set $\{1, 2, 3, \ldots, n\}$. Hence, there are $C(n, r)$ bit strings of length n that contain exactly r 1s.

The number of k-subsets of an n-set is

$$\frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

It means how many subsets you can make by selecting k elements from the set of n-elements

How many 4-element subsets does {1,2,3,4,5,6,7,8,9} have? The answer is C(9,4)

CA11: In how many ways can a set of two positive integers less than 100 be chosen?

$$C(99,2) = \frac{99!}{2!\,(99-2)!}$$

CA12: How many permutations of the letters *ABCDEFGH* contain

a) the string *ED*?

b) the string *CDE*?

c) the strings *BA* and *FGH*?

d) the strings *AB*, *DE*, and *GH*?

e) the strings *CAB* and *BED*?

f) the strings *BCA* and *ABF*?

CA12: Find the values of the followings

a) $C(5,1)$

b) $C(5,3)$

c) $C(8,4)$

d) $C(8,8)$

e) $C(8,0)$

f) $C(12,6)$