Fahim

Integral Calculus :

Indefinite integral :

⇒ If $f(x)$ be a given function and if $F(x)$ be another function such that $\frac{d}{dx} F(x) = f(x)$ then $F(x)$ is defined as an indefinite integral of $f(x)$ w.r.t x.

Symbolically, if $\frac{d}{dx} F(x) = f(x)$, then

$$\int f(x)\, dx = F(x)$$

Hence, $\int f(x)\, dx$ is called an integral of $f(x)$ w.r.t x.

In addition, if $\frac{d}{dx}\{F(x)+c\} = f(x)$, then

$$\int f(x)\, dx = F(x) + C$$

Here, c is called the constant of integration.

Again if $f(x)$ is continuous in $[a,b]$, then the definite integral of $f(x)$ w.r.t x in $[a,b]$ is denoted and defined by $\int_a^b f(x)\, dx = [F(x)+c]_a^b$

$$= F(b) - F(a)$$

2

Page-19 : Example-7 :

Prob : Integrate $\int \frac{dx}{\sqrt{(x+a)}+\sqrt{(x+b)}}$

Sol^n :

Multiplying the numerator and denominator by $\sqrt{x+a}-\sqrt{x+b}$, we have

$$I = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}\, dx$$

$$= \frac{1}{a-b}\left[\int \sqrt{x+a}\, dx - \int \sqrt{x+b}\, dx\right]$$

Putting $x+a = z$, so that $dx = z$

$$\int \sqrt{x+a}\, dx = \int \sqrt{z}\, dz = \frac{2}{3} z^{3/2} = \frac{2}{3}(x+a)^{3/2}$$

Similarly, $\int \sqrt{x+b}\, dx = \frac{2}{3}(x+b)^{3/2}$

$$\therefore I = \frac{2}{3} \cdot \frac{1}{a-b}\left[(x+a)^{3/2} - (x+b)^{3/2}\right]$$

Page- 22 : Excercise -13 (i)

Prob:

Integrate $\int \frac{3x-1}{\sqrt{(3x^2-2x+7)}} dx$

Sol^n:

Let, $I = \int \frac{3x-1}{\sqrt{(3x^2-2x+7)}} dx$

Put, $3x^2-2x+7 = t$

$\Rightarrow (6x-2)dx = dt$

$\Rightarrow (3x-1)dx = \frac{1}{2}dt$

$\therefore I = \int \frac{1}{2} \cdot \frac{1}{\sqrt{t}} dt$

$= \frac{1}{2} \cdot 2\sqrt{t}$

$= \frac{1}{2} \cdot 2\sqrt{3x^2-2x+7}$

$= \sqrt{3x^2-2x+7}$

Page - 23 : Excercise - 17 (i):

Prob:

Integrate $\int \frac{dx}{\sqrt{(x+1)} - \sqrt{(x-1)}}$

Solⁿ:

Let, $I = \int \frac{dx}{\sqrt{(x+1)} - \sqrt{(x-1)}}$

Multiplying the numerator and denominor by $\sqrt{x+1} + \sqrt{x-1}$, we have

$$I = \frac{1}{2}\left[\int \sqrt{x+1}\,dx + \int \sqrt{x-1}\,dx\right]$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3}\left[(x+1)^{3/2} + (x-1)^{3/2}\right]$$

$$= \frac{1}{3}\left[(x+1)^{3/2} + (x-1)^{3/2}\right] \text{ (Ans).}$$

Page - 23: Excercise - 17 (ii):

Prob:

Integrateq: $\int \frac{dx}{\sqrt{(2x+5)} + \sqrt{(2x-3)}}$

Let, $I = \int \frac{dx}{\sqrt{(2x+5)} + \sqrt{(2x-3)}}$

Multiplying the numerator and denominator by $\sqrt{(2x+5)} - \sqrt{(2x-3)}$, we have

$$I = \frac{1}{8}\left[\int \sqrt{2x+5}\,dx - \int \sqrt{2x-3}\,dx\right]$$

$$= \frac{1}{8}\cdot\frac{2}{3}\left[(2x+5)^{3/2} - (2x-3)^{3/2}\right]$$

$$= \frac{1}{12}\left[(2x+5)^{3/2} - (2x-3)^{3/2}\right]$$

Page-23: Excercise- 20 (i)

<u>Prob:</u>

Integrate $\int (3x+2)\sqrt{2x+1}\,dx$

<u>Sol^n:</u>

Let, $I = \int (3x+2)\sqrt{2x+1}\,dx$

Put, $2x+1 = t$

$\therefore \; 2dx = dt$

$$I = \int \left[\left(\frac{3(t-1)}{2} + 2\right)\frac{\sqrt{t}\,dt}{2}\right] = \frac{1}{4}\int (3t+1)\sqrt{t}\,dt$$

$$= \frac{1}{4}\int \left(3t^{3/2} + t^{1/2}\right)dt = \frac{1}{4}\left(3\cdot\tfrac{2}{5}\cdot t^{5/2} + \tfrac{2}{3}t^{3/2}\right)$$

$$= \frac{3}{10}(2x+1)^{5/2} + \tfrac{1}{16}(2x+1)^{3/2} \text{ (Ans)}$$

6.

Page-23: Exercise- 23(i):

Prob:

Integrate $\int \frac{2x+1}{\sqrt{3x+2}} dx$

Sol^n:

Let, $I = \int \frac{2x+1}{\sqrt{3x+2}} dx$

Put, $3x+2 = t$, $3dx = dt$

$\therefore I = \frac{1}{3} \int \frac{2 \cdot \frac{t-2}{3} + 1}{\sqrt{t}} dt$

$= \frac{1}{9} \int \frac{2t-1}{\sqrt{t}} dt$

$= \frac{1}{9} \int \left[2\sqrt{t} - \frac{1}{\sqrt{t}} \right] dt$

$= \frac{1}{9} \left[2 \cdot \frac{2}{3} t^{3/2} - 2\sqrt{t} \right]$

$= \frac{4}{27} (3x+2)^{3/2} - \frac{2}{9} (3x+2)^{1/2}$ (Ans).

Page-23: Exercise- 24

Prob:

Integrate $\int \sqrt{\frac{a+x}{a-x}} dx$

Sol^n: Let, $I = \int \sqrt{\frac{a+x}{a-x}} dx$

put $x = a\cos 2\theta$, $dx = -2a\sin 2\theta\, d\theta$

$$\therefore\ I = -\int \sqrt{\frac{a + a\cos 2\theta}{a - a\cos 2\theta}} \cdot 2a\sin 2\theta\, d\theta$$

$$= -2a \int \frac{\cos\theta}{\sin\theta} \cdot 2\sin\theta \cdot \cos\theta\, d\theta$$

$$= -2a \int 2\cos^2\theta\, d\theta$$

$$= -2a \int (1 + \cos 2\theta)\, d\theta$$

$$= -2a\left(\theta + \frac{\sin 2\theta}{2}\right)$$

$$= -a \cdot \cos^{-1} x/a - \sqrt{a^2 - x^2} \quad \text{(Ans)}$$

Page- 27 : Exercise - 27 :

Prob :

Integrate $\int \sqrt{\frac{x}{a-x}}\, dx$

Sol^n:

Let. $I = \int \sqrt{\frac{x}{a-x}}\, dx$

⑧

Put, $x = a\sin^2\theta$, $dx = 2a\sin\theta\cos\theta\,d\theta$

$$I = \int \sqrt{\left(\frac{a\sin^2\theta}{a(1-\sin^2\theta}\right)}\,2a\sin\theta\cos\theta\,d\theta$$

$$= a\int 2\sin^2\theta\,d\theta$$

$$= a\int (1-\cos2\theta)\,d\theta$$

$$= a(\theta - \sin 2\theta/2)$$

$$= a(\theta - \sin\theta\cos\theta)$$

$$= a\sin^{-1}\sqrt{\frac{x}{a}} - \sqrt{[x(a-x)]} \quad \text{(Ans)}$$

Page-24: Exercise- 38 (i)

Prob:

Integrate $\int \frac{dx}{x\sqrt{x^4-1}}$

Sol^n÷

Put, $x^2 = \sec\theta$, $x\,dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$

$$I = \int \frac{x\,dx}{x^2\sqrt{x^4-1}}$$

$$= \frac{1}{2}\int \frac{\sec\theta\tan\theta}{\sec\theta.\tan\theta}\,d\theta$$

8

$= \frac{1}{2} \int d\theta$

$= \frac{1}{2} \cdot \theta$

$= \frac{1}{2} \sec^{-1}(x^2)$ (Ans).

Rule – I : $\int \frac{dx}{ax+bx+c}$

* Page – 39 : Example – 9

Prob : Integrate $\int \frac{x\,dx}{x^4-x^2-2}$

Sol^n :

Put, $x^2 = z \Rightarrow 2x\,dx = dz \Rightarrow x\,dx = \frac{1}{2}dz$

Let,

$$I = \int \frac{x\,dx}{x^4-x^2-2}$$

$$= \frac{1}{2}\int \frac{dz}{z^2-z-2}$$

$$= \frac{1}{2}\int \frac{dz}{(z^2-\frac{1}{2})^2-(\frac{3}{2})^2}$$

$$= \frac{1}{2} \cdot \frac{1}{2 \cdot \frac{3}{2}} \log\left|\frac{z-\frac{1}{2}-\frac{3}{2}}{z-\frac{1}{2}+\frac{3}{2}}\right| + c$$

$$= \frac{1}{6}\log\left|\frac{x^2-2}{x^2+1}\right| + c$$

Page- 49 : Exercise- 7 (i).

Prob:

Integrate $\int \frac{dx}{1+x+x^2}$

Soln ÷

Let, $I = \int \frac{dx}{1+x+x^2} = \int \frac{dx}{(x+\frac{1}{2})^2 + 3/4}$

put, $x + \frac{1}{2} = t$, $dx = dt$

$$I = \int \frac{dt}{t^2 + (\sqrt{\frac{3}{2}})^2}$$

$$= \frac{1}{(\sqrt{3/2})^2} \tan^{-1}\left(\frac{2t}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \quad \text{(Ans).}$$

Page- 49 : Exercise- 8(i) ÷

Prob: Integrate $\int \frac{dx}{1+x-x^2}$

Soln:

Let, $I = \int \frac{dx}{1+x-x^2} = \int \frac{dx}{1+\frac{1}{4}-\frac{1}{4}+x-x^2}$

$$= \int \frac{dx}{\frac{5}{4} - (x-\frac{1}{2})^2}$$

put, $x-\frac{1}{2}=t$, $dx=dt$

$$I=\int\frac{dt}{(\sqrt{5}/2)^2-t^2}$$

$$=\frac{1}{2\cdot(\sqrt{5}/2)}\log\left[\frac{\sqrt{5}/2+t}{\sqrt{5}/2-t}\right]$$

$$=\frac{1}{\sqrt{5}}\log\left[\frac{\sqrt{5}+2x-1}{\sqrt{5}-2x+1}\right] \quad \text{(Ans).}$$

Page-49: Exercise- 9:

Prob: Integrate $\int\frac{x\,dx}{x^4+2x^2+2}$

Sol^n:

Let, $I=\int\frac{x\,dx}{x^4+2x^2+2}$

put, $x^2=t$, $2x\,dx=dt$

$$I=\frac{1}{2}\int\frac{dt}{t^2+2t+1+1}$$

$$=\frac{1}{2}\int\frac{dt}{(t+1)^2+1}$$

put, $t+1=z$, $dt=dz$

$$I=\frac{1}{2}\int\frac{dz}{z^2+1}$$

$$=\frac{1}{2}\tan^{-1}(z)=\frac{1}{2}\tan^{-1}(x^2+1)$$

Page-49: Exercise-10:

Prob: Integrate $\int \frac{\cos x \, dx}{\sin^2 x + 4\sin x + 3}$

Sol^n:

put, $\sin x = t$, $\cos x dx = dt$

let, $I = \int \frac{\cos x \, dx}{\sin^2 x + 4\sin x + 3}$

$= \int \frac{dt}{t^2 + 4t + 4 - 1}$

$= \int \frac{dt}{(t+2)^2 - 1}$

put, $t+2 = z$, $dt = dz$

$I = \int \frac{dt}{z^2 - 1}$

$= \frac{1}{2} \log \frac{z-1}{z+1}$

$= \frac{1}{2} \log \frac{t+1}{t+3}$

$= \frac{1}{2} \log \frac{\sin x + 1}{\sin x + 3}$

Rule - II : $\int \frac{px+q}{ax^2+bx+c}$

Page-49: Exercise- 15 (i)

Prob: Integrate $\int \frac{x\,dx}{x^2+2x+1}$

Soln:

Let, $I = \int \frac{x\,dx}{x^2+2x+1}$

$= \frac{1}{2} \int \frac{2x+2-2}{x^2+2x+1}\,dx$

$= \frac{1}{2} \int \frac{2x+2}{x^2+2x+1}\,dx - \int \frac{dx}{x^2+2x+1}$

$= \frac{1}{2} \log (x^2+2x+1) - \int \frac{dx}{(x+1)^2}$

$= \frac{1}{2} \log (x^2+2x+1) + \frac{1}{x+1}$ (Ans)

Page- 49: Exercise - 16 (i) —

Prob: Integrate $\int \frac{x+1}{x^2+4x+5}\,dx$

Soln:

Let, $I = \int \frac{x+1}{x^2+4x+5}\,dx$

$$= \frac{1}{2}\int \frac{2x+2+2-2}{x^2+4x+5}\,dx$$

$$= \frac{1}{2}\int \frac{2x+4}{x^2+4x+5}\,dx - \frac{1}{2}\int \frac{2}{(x+2)^2+1}\,dx$$

$$= \frac{1}{2}\log(x^2+4x+5) - \tan^{-1}(x+2) \quad \text{(Ans)}.$$

Page-49: Exercise - 17 (i)*

Prob: Integrate $\int \frac{4x+3}{3x^2+3x+1}\,dx$

Sol^n:

Let, $I = \int \frac{4x+3}{3x^2+3x+1}\,dx$

Let, $4x+3 = l(6x+3)+m$

$\therefore\ 4 = 6l,\ 3 = 3l+m \Rightarrow l = \frac{2}{3},\ m = 1$

$$I = \frac{2}{3}\int \frac{6x+3}{3x^2+3x+1}\,dx + 1.\int \frac{dx}{3x^2+3x+1}$$

$$= \frac{2}{3}\log(3x^2+3x+1) + \frac{1}{3}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{1}{12}}$$

$$= \frac{2}{3}\log(3x^2+3x+1) + \frac{2}{\sqrt{3}}\tan^{-1}\left\{\sqrt{3}(2x+1)\right\}$$

(Ans).

Page – 51 : Exercise – 37(ii) : *

Prob: Integrate $\int \frac{dx}{x+\sqrt{(x-1)}}$

Sol^n:

Let, $I = \int \frac{dx}{x+\sqrt{(x-1)}}$

put, $x-1 = t^2$, $dx = 2t\,dt$

$$I = \int \frac{2t\,dt}{(t^2+1)+t} = \int \frac{(2t+1-1)}{t^2+t+1}\,dt$$

$$= \int \frac{2t+1}{t^2+t+1}\,dt - \int \frac{dt}{\left(t+\frac{1}{2}\right)^2+3/4}$$

$$= \log(t^2+t+1) - \frac{1}{\sqrt{3}/2}\tan^{-1}\frac{t+1/2}{\sqrt{3}/2}$$

$$= \log(x+\sqrt{x-1}) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\sqrt{x-1}+1}{\sqrt{3}} \quad \text{(Ans).}$$

Rule – III : $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

Page – 50 : Exercise – 22(i)

Prob: Integrate $\int \frac{dx}{\sqrt{1-x-x^2}}$

Sol^n:

Let, $I = \int \frac{dx}{\sqrt{(1-x-x^2)}}$

$$= \int \frac{dx}{\left\{\frac{5}{4} - \left(\frac{3}{4} + x + x^2\right)\right\}^{1/2}}$$

$$= \int \frac{dx}{\sqrt{(\sqrt{5}/2)^2 - (x+\frac{1}{2})^2}}$$

$$= \sin^{-1}\left(\frac{x + 1/2}{\sqrt{5}/2}\right) = \sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) \text{ (Ans).}$$

Page - 50 : Exercise - 23 :

Prob: Integrate $\int \frac{dx}{\sqrt{(2x^2+3x+4)}}$

Sol^n: Let, $I = \int \frac{dx}{\sqrt{(2x^2+3x+4)}}$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{x^2 + \frac{3}{2}x + 2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\{(x+\frac{3}{2})^2 + \frac{23}{16}\}}}$$

$$= \frac{1}{\sqrt{2}} \log\left[(x+\frac{3}{4}) + \sqrt{\{(x+\frac{3}{4})^2 + 23/16\}}\right.$$

$$= \frac{1}{\sqrt{2}} \log \left[\left(x+\frac{3}{4}\right) + \sqrt{\left(x^2+\frac{3}{2}x+2\right)}\right] \text{(Ans)}.$$

Page- 50: Exercise- 24:

Prob: Integrate $\int \frac{dx}{\sqrt{x^2-7x+12}}$

Sol^n:

Let, $I = \int \frac{dx}{\sqrt{x^2-7x+12}}$

$$= \int \frac{dx}{\sqrt{\left\{\left(x-\frac{7}{2}\right)^2 - \frac{1}{4}\right\}}}$$

$$= \log\left[x - 7/2 + \sqrt{\left\{(x-7/2)^2 - 1/4\right\}}\right]$$

$$= \log\left[(2x-7) + 2\sqrt{(x^2-7x+12)}\right]$$

$$= 2\log\left[\sqrt{(x-3)} + \sqrt{(x-4)}\right] \text{ Ans}$$

Page* 50: Exercise-25:

Prob: Integrate $\int \frac{dx}{\sqrt{6+11x-10x^2}}$

Soln:

Let, $I = \int \frac{dx}{\sqrt{6+11x-10x^2}}$

$= \int \frac{dx}{\sqrt{(5x+2)(-2x+2)}}$

put, $5x+2 = t^2$, $5dx = 2t\,dt$

$I = \frac{2}{5} \int \frac{t\,dt}{\sqrt{t^2[-2\frac{t^2-2}{5}+3]}}$

$= \frac{2}{5} \int \frac{\sqrt{5}}{\sqrt{(-2t^2+19)}}\,dt$

$= \frac{2}{\sqrt{5}} \int \frac{dt}{\sqrt{-(\sqrt{2}t)^2+(\sqrt{19})^2}}$

$= \frac{2}{\sqrt{5}} \sin^{-1}\frac{\sqrt{2}t}{\sqrt{19}} \cdot \frac{1}{\sqrt{2}}$

$= \sqrt{\frac{2}{5}} \cdot \sin^{-1}\left(\sqrt{\frac{2(5x+2)}{19}}\right)$

$= \sqrt{\frac{2}{5}} \sin^{-1}\left[\sqrt{\left(\frac{10x+4}{19}\right)}\right]$

Page-50: Exercise-2.6:

Prob: Integrate $\int \frac{\cos x \, dx}{\sqrt{(5\sin^2 x - 12\sin x + 4}}$

Soln:

let, $I = \int \frac{\cos x \quad dx}{\sqrt{(5\sin^2 x - 12\sin x + 4)}}$

put, $\sin x = t$, $\cos x dx = dt$

$$I = \int \frac{dt}{\sqrt{5t^2 - 12t + 4}} = \int \frac{dt}{\sqrt{\{(-5t+2)(-t+2)}}$$

put, $-5t+2 = z^2$; $dt = -\frac{2}{5} z dz$

$$\therefore I = \int \frac{-2/5 . z dz}{z\sqrt{z^2 + 8/5}}$$

$$= \frac{-2}{\sqrt{5}} \int \frac{dz}{\sqrt{z^2+8}} = \frac{-2}{5} \log\{z + \sqrt{z^2+8}\}$$

$$= \frac{-2}{\sqrt{5}} \log\{\sqrt{2-5t} + \sqrt{5(2-t)}\}$$

$$= \frac{-2}{\sqrt{5}} \log\{\sqrt{2-5\sin x} + \sqrt{5(2-\sin x)}\}$$ (Ans).

Page- 50 : Exercise- 27 :*

Prob: Integrate $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$

Sol^n:

Let, $I = \int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$

put, $x - \alpha = z^2$, $dx = 2z\, dz$

$$I = \int \frac{2z\, dz}{\sqrt{\{z^2(z^2+\alpha-\beta)\}}}$$

$$= 2\int \frac{dz}{\sqrt{z^2+(\alpha-\beta)}}$$

$$= 2\log\{z+\sqrt{z^2+(\alpha-\beta)}\}$$

$$= 2\log\{\sqrt{x-\alpha}+\sqrt{x-\alpha+\alpha-\beta}\}$$

$$= 2\log\{\sqrt{x-\alpha}+\sqrt{x-\beta}\} \text{ (Ans).}$$

P.39/Ex-3

Rule- IV : $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\, dx$

Page- 50 :* Exercise- 30 :

Prob: Integrate $\int \frac{x-2}{\sqrt{(2x^2-8x+5)}}$

Sol^n:

Let, $I = \int \frac{x-2}{\sqrt{2x^2-8x+5}} dx$

$= \frac{1}{4} \int \frac{4x-8}{\sqrt{2x^2-8x+5}}$

put, $2x^2-8x+5 = t$, $(4x-8)dx = dt$

$\therefore I = \frac{1}{4} \cdot \frac{dt}{\sqrt{t}} = \frac{1}{4} \cdot 2\sqrt{t} = \frac{1}{2}\sqrt{2x^2-8x+5}$ (Ans).

Page-50: Exercise- 31 (i):

Prob: Integrate $\int \frac{x+1}{\sqrt{4+8x-5x^2}} dx$

Sol^n:

Let, $I = \int \frac{x+1}{\sqrt{4+8x-5x^2}} dx$

Let, $x+1 = l(8-10x)+m$

Comparing the coefficients of like powers on the both sides, we get, $l = \frac{1}{10}$, $m = 9/5$

$\therefore I = -1/10 \int \frac{8-10x}{\sqrt{4+8x-5x^2}} + \frac{9}{5} \int \frac{dx}{\sqrt{4+8x-5x^2}}$

$= -\frac{1}{10} \cdot 2\sqrt{4+8x-5x^2} \cancel{dx} + \frac{9}{5\sqrt{5}} \int \frac{dx}{\sqrt{(4/5 + \frac{8}{5}x - x^2)}}$

$$= -\frac{1}{5}\cdot\sqrt{4+8x-5x^2} + \frac{9}{5\sqrt{5}}\int\frac{dx}{\sqrt{\left\{\frac{36}{25}-\left(x-\frac{4}{5}\right)^2\right\}}}$$

$$= -\frac{1}{5}\cdot\sqrt{4+8x-5x^2} + \frac{9}{5\sqrt{5}}\sin^{-1}\left(\frac{5x-4}{6}\right) \text{ (Ans).}$$

Rule – V : $\int\frac{dx}{(ax+b)\sqrt{cx+d}}$

Page- 50: Exercise- 32 (ii)

Prob: Integrate $\int\frac{dx}{(2x+1)\sqrt{(4x+3)}}$

Sol^n:

Let, $I = \int\frac{dx}{(2x+1)\sqrt{(4x+3)}}$ $\rightarrow \sqrt{(4x+3)}$

put, $4x+3 = t^2$, $4dx = 2t\,dt$

$$\therefore I = \frac{1}{2}\int\frac{t\,dt}{\left\{2\cdot\frac{(t^2-3)}{4}+1\right\}t}$$

$$= \frac{1}{2}\int\frac{2\,dt}{t^2-1} = \frac{1}{2}\log\frac{t-1}{t+1}$$

$$= \frac{1}{2}\log\left[\frac{\sqrt{4x+3}-1}{\sqrt{4x+3}+1}\right] \text{ (Ans).}$$

Rule- VI : $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$

Page-51; Exercise-36(II)

Prob: Integrate $\int \frac{dx}{(1+x)\sqrt{1-x^2}}$

Sol^n :

Let, $I = \int \frac{dx}{(1+x)\sqrt{1-x^2}}$

put, $1+x = \frac{1}{t}$, $dx = -\frac{1}{t^2}dt$

$$I = \int \frac{-dt/t^2}{\frac{1}{t}\sqrt{1-(\frac{1}{t}-1)^2}}$$

$$= -\int \frac{dt}{\sqrt{t^2-(1-t)^2}}$$

$$= -\int \frac{dt}{\sqrt{2t-1}}$$

$$= -2\cdot\frac{\sqrt{2t-1}}{2}$$

$$= \sqrt{2\cdot\frac{1}{x+1}-1}$$

$$= -\sqrt{\frac{1-x}{1+x}}$$ (Ans).

Page-51; Exercise- 36(iv):

Prob: Integrate $\int \frac{dx}{(1+x)\sqrt{(1+2x-x^2)}}$

Soln: Let, $I = \int \frac{dx}{(1+x)\sqrt{1+2x-x^2}}$

put, $1+x = \frac{1}{t}$, $dx = -\frac{1}{t^2}\, dt$

$$I = \int \frac{dt}{\frac{1}{t}\sqrt{1+2(1/t-1)-(1/t-1)^2}}$$

$$= -\int \frac{dt}{\sqrt{-1+4t-2t^2}} = -\frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{-\frac{1}{2}+2t-t^2-\frac{1}{2}+\frac{1}{2}}}$$

$$= -\frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{\frac{1}{2}-(t-1)^2}}$$

$$= -\frac{1}{\sqrt{2}} \sin^{-1} \frac{(t-1)}{1/\sqrt{2}}$$

$$= -\frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{1}{x+1}-1\right)\sqrt{2}$$

$$= 0 - \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x\sqrt{2}}{x+1}\right) \text{ (Ans).}$$

Page-51: Exercise-36(vi)

Prob: Integrate $\int \frac{dx}{(1+x)\sqrt{(1+x-x^2)}}$

Sol^n:

Let, $I = \int \frac{dx}{(1+x)\sqrt{(1+x-x^2)}}$

put, $1+x = \frac{1}{t}$, $dx = dt \cdot -\frac{1}{t^2}$

$$\therefore I = \int \frac{\left(-\frac{1}{t^2}\right)dt}{\frac{1}{t}\sqrt{1+2(1/t-1)-(\frac{1}{t}-1)^2}}$$

$$= -\int \frac{dt}{\sqrt{-1+4t-2t^2}} = -\frac{1}{\sqrt{2}}\sin^{-1}\frac{t-1}{1/\sqrt{2}}$$

$$= -\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{1}{x+1}-1\right)\sqrt{2}$$

$$= -\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x\sqrt{2}}{x+1}\right) \text{ (Ans).}$$

Page-34: Example-9:

Prob: Integrate $\int \frac{dx}{(2x+3)\sqrt{x^2+3x+2}}$

Sol^n: Let, $I = \int \frac{dx}{(2x+3)\sqrt{x^2+3x+2}}$

put, $2x+3 = \frac{1}{z}$ $\therefore 2dx = -\frac{1}{z^2} dz$

$\Rightarrow dx = -\frac{1}{2} \cdot \frac{dz}{z^2}$

And, $x = \frac{1}{2}\left(\frac{1}{z} - 3\right)$; $z = \frac{1}{2x+3}$

$$\therefore I = -\frac{1}{2} \int \frac{dz}{z^2 \cdot \frac{1}{z} \sqrt{\left\{ \frac{1}{z^2}\left(\frac{1}{z} - 3\right)^2 + \frac{3}{2}\left(\frac{1}{z} - 3\right) + 2 \right.}}$$

$$= -\frac{1}{2} \int \frac{dz}{\sqrt{(1-z^2)}} = -\sin^{-1} z = -\sin^{-1}\left(\frac{1}{2x+3}\right)$$

Alternatively,

$$I = \int \frac{dx}{(2x+3)\sqrt{\left\{ \frac{1}{4}(4x^2+12x+8) \right\}}}$$

$$= \int \frac{dx}{(2x+3)\frac{1}{2}\sqrt{\left\{ (2x+3)^2 - 1 \right.}}$$

put, $2x+3 = z$, $\therefore 2dx = dz$ $\therefore dx = \frac{1}{2} dz$

$$\therefore I = \int \frac{dz}{z\sqrt{z^2-1}} = \sec^{-1} z = \sec^{-1}(2x+3)$$

(Ans).

Page - 41 : Exercise - 11 :

Prob : Integrate $\int \frac{x^2-1}{x^4-x^2-1}\,dx$

Soln :

Let, $I = \int \frac{x^2-1}{x^4-x^2-1}\,dx$

$$= \int \frac{x^2-1}{x^2\left(x^2-1-\frac{1}{x^2}\right)}\,dx$$

$$= \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-3}\,dx$$

Now, we substitute, $x+\frac{1}{x} = z \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dz$

So, $I = \int \frac{dz}{z^2-(\sqrt{3})^2}$

$$= \frac{1}{2\sqrt{3}} \log\left|\frac{z-\sqrt{3}}{z+\sqrt{3}}\right| + c$$

$$= \frac{1}{2\sqrt{3}} \log\left|\frac{x+\frac{1}{x}-\sqrt{3}}{x+\frac{1}{x}+\sqrt{3}}\right| + c$$

$$= \frac{1}{2\sqrt{3}} \log\left|\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1}\right| + c$$

(Ans).

Page – 51 : Exercise – 39 (i) :

Prob: Integrate $\int \sqrt{\frac{a+x}{x}}\,dx$

Soln :

Let, $I = \int \sqrt{\frac{a+x}{x}}\,dx = \int \frac{a+x}{\sqrt{x(a+x)}}\,dx$

$$= \frac{1}{2}\int \frac{(2x+a)\,dx}{\sqrt{x^2+ax}} + \frac{1}{2}\int \frac{a\,dx}{\sqrt{x^2+ax+a^2/4-a^2/4}}$$

$$= \frac{1}{2}\cdot 2\sqrt{x^2+ax} + \frac{a}{2}\int \frac{dx}{\sqrt{(x+a/2)^2-a^2/4}}$$

$$= \sqrt{x^2+ax} + a/2 \log\left\{(x+a/2) + \sqrt{(x+a/2)^2 - a^2/4}\right\}$$

$$= \frac{1}{2}\cdot 2\sqrt{x^2+ax} + a/2 \log\left\{x+(x+a) + 2\sqrt{x(x+a)}\right\}$$

$$= \sqrt{x(x+a)} + a\log\left[\sqrt{x+a} + \sqrt{x}\right] \text{ (Ans).}$$

imp – P-64 : EXP. 3*

1 exercise 29**

Exercise 3.1 (i) (ii)*

P-69 Exp 3.2

P-89 Ex: 2.1 (i), 2.2 (i, ii)

P-87 : Exp 30

Special Trigonometric Function

Page - 106 : Example - 1 :

Prob: Integrate $\int \frac{dx}{\sin x + \cos x}$

Sol^n:

Let, $I = \int \frac{dx}{2\sin\frac{1}{2}x\cos\frac{1}{2}x + \cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x}$

$$= \int \frac{\sec^2\frac{1}{2}x \, dx}{2\tan\frac{1}{2}x + 1 - \tan^2\frac{1}{2}x}$$

on multiplying the numerator and denominator by $\sec^2\frac{1}{2}x$

$$= \int \frac{2\,dz}{2z+1-z^2} \quad \left[\text{putting } \tan\frac{1}{2}x = z\right]$$

$$= \int \frac{2\,dz}{2-(z^2-2z+1)}$$

$$= 2\int \frac{dz}{(\sqrt{2})^2 - (z-1)^2}$$

$$= 2\int \frac{dy}{a^2-y^2} \quad \text{where, } a=\sqrt{2}, y=z-1$$

$$= 2 \cdot \frac{1}{2a}\log\frac{a+y}{a-y} = \frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+(\tan\frac{1}{2}x-1)}{\sqrt{2}-(\tan\frac{1}{2}x-1)}$$

(Ans).

Page - 107 : Example - 2 :

Prob : Integrate $\int \frac{dx}{a\sin x + b\cos x}$

Sol^n :

put, $a = r\cos\theta$, $b = r\sin\theta$, then $a\sin x + b\cos x$
$= r\sin(x+\theta)$

Here, $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$

$$I = \int \frac{dx}{r\sin(x+\theta)} = \frac{1}{r}\int \operatorname{cosec}(x+\theta)\,dx$$

$$= \frac{1}{r}\int \operatorname{cosec} z\,dz \text{ ; } \quad \text{where, } z = x+\theta$$

$$= \frac{1}{r}\log\left|\tan\frac{z}{2}\right|$$

$$= \frac{1}{\sqrt{a^2+b^2}}\log\left|\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{b}{a}\right)\right|$$

Note. Since, as above, $\sin x + \cos x = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$

$$\therefore \int \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}}\int \operatorname{cosec}\left(x+\frac{\pi}{4}\right)dx$$

$$= \frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right.$$

(Ans)

Page – 108 : Example – 4 :

Prob: Integrate $\int \frac{dx}{5-13\sin x}$

Soln:

Let, $I = \int \frac{dx}{5(\sin^2\frac{1}{2}x+\cos^2\frac{1}{2}x) - 13.2\sin\frac{1}{2}x\cos\frac{1}{2}x}$

Multiplying the numerator and denominator by $\sec^2\frac{1}{2}x$, this

$$= \int \frac{\sec^2\frac{1}{2}x\,dx}{5(\tan^2\frac{1}{2}x+1) - 26\tan\frac{1}{2}x}$$

$$= \int \frac{2dz}{5z^2-26z+5} \qquad [\text{putting } \tan\tfrac{1}{2}x = z]$$

$$= \frac{2}{5}\int \frac{dz}{(z-\frac{13}{5})^2-(\frac{12}{5})^2}$$

$$= \frac{2}{5}\int \frac{du}{u^2-a^2}, \quad \text{where, } u = z-\frac{13}{5} \text{ and } a = \frac{12}{5}$$

$$= \frac{2}{5}\cdot\frac{1}{2a}\log\frac{u-a}{u+a} = \frac{1}{12}\log\frac{z-5}{z-\frac{1}{5}}$$

$$= \frac{1}{12}\log\left|\frac{5\tan\frac{1}{2}x-25}{5\tan\frac{1}{2}x-1}\right| \text{ (Ans).}$$

Page-109: Example-5:

(32)

Prob: Integrate $\int \frac{dx}{13+3\cos x+4\sin x}$

Soln:

$$I = \int \frac{dx}{13(\sin^2\frac{1}{2}x+\cos^2\frac{1}{2}x)+3(\cos^2\frac{1}{2}x-\sin^2\frac{1}{2}x)+4.2\sin\frac{1}{2}x.\cos\frac{1}{2}x}$$

Multiplying the numerator and denominator by $\sec^2\frac{1}{2}x$

$$= \int \frac{\sec^2\frac{1}{2}x\ dx}{10\tan^2\frac{1}{2}x+8\tan\frac{1}{2}x+16}$$

$$= \int \frac{2\,dz}{10z^2+8z+16} \quad [\text{putting } z=\tan\tfrac{1}{2}x]$$

$$= \frac{1}{5}\int \frac{dz}{\left(z+\frac{2}{5}\right)^2+\left(\frac{6}{5}\right)^2} = \frac{1}{5}\int \frac{du}{u^2+a^2},$$

where, $u = z+\frac{2}{5}$, $a=\frac{6}{5}$

$$= \frac{1}{5}\cdot\frac{1}{a}\tan^{-1}\frac{u}{a} = \frac{1}{6}\tan^{-1}\frac{5z+2}{6}$$

$$= \frac{1}{6}\tan^{-1}\frac{5\tan\frac{1}{2}x+2}{6} \quad \text{(Ans).}$$

Page-111 : Example- 9:

Prob: Integrate $\int \frac{dx}{3+4\cosh x}$.

Sol^n:

$$I = \int \frac{dx}{3(\cosh^2\frac{1}{2}x - \sinh^2\frac{1}{2}x) + 4(\cosh^2\frac{1}{2}x + \sinh^2\frac{1}{2}x)}$$

$$= \int \frac{dx}{7\cosh^2\frac{1}{2}x + \sinh^2\frac{1}{2}x} = \int \frac{\operatorname{sech}^2\frac{1}{2}x}{7+\tanh^2\frac{1}{2}x}$$

put $\tanh\frac{1}{2}x = z$; then $\frac{1}{2}\sec^2\frac{1}{2}x\,dx = dz$

$$\therefore I = 2\int \frac{dz}{7+z^2} = \frac{2}{\sqrt{7}}\tan^{-1}\frac{z}{\sqrt{7}} = \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{1}{\sqrt{7}}\tanh\frac{1}{2}x\right)$$

(Ans).

Page-111 : Example-1 :

Prob: Integrate $\int \frac{dx}{4+3\sin x}$

Sol^n:

$$I = \int \frac{dx}{4+3\sin x} = \int \frac{dx}{4+3x\frac{2\tan\frac{1}{2}x}{1+\tan^2\frac{1}{2}x}} = \int \frac{\sec^2\frac{1}{2}x\,dx}{4+4\tan^2\frac{1}{2}x+6\tan\frac{1}{2}x}$$

put, $\tan\frac{x}{2} = z, \Rightarrow \frac{1}{2}\sec^2\frac{1}{2}x.dx = dz \Rightarrow \sec^2\frac{1}{2}x\,dx = 2dz$

$$\therefore I = \int \frac{2dz}{4z^2+6z+4} = \frac{1}{2}\int \frac{dz}{z^2+2.z.\frac{3}{4}+\frac{9}{16}-\frac{9}{16}+1}$$

$$= \frac{1}{2}\int \frac{dz}{(z+\frac{3}{4})^2+(\frac{\sqrt{7}}{4})^2} = \frac{1}{2}.\frac{4}{\sqrt{7}}.\tan^{-1}\frac{z+\frac{3}{4}}{\frac{\sqrt{7}}{4}} + c$$

$$= \frac{2}{\sqrt{7}} \cdot \tan^{-1}\left(\frac{4\tan\frac{1}{2}x + 3}{\sqrt{7}} \right) + c$$

Page-112: Example-2:

Prob: Integrate $\int \frac{dx}{5+4\cos x}$

Sol^n:

$$I = \int \frac{dx}{5+4\cos x} = \int \frac{dx}{5 + \frac{4(1-\tan^2\frac{1}{2}x)}{1+\tan^2\frac{1}{2}x}}$$

$$= \int \frac{\sec^2\frac{1}{2}x\, dx}{9+\tan^2\frac{1}{2}x} = 2\int \frac{dz}{9+z^2}, \text{ where } z = \tan\tfrac{1}{2}x$$

$$= 2 \cdot \frac{1}{3} \tan^{-1}\frac{z}{3} + C$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{1}{2}x\right) + c \quad \text{(Ans)}.$$

Page-113: Prob-3

Prob: Integrate $\int \frac{dx}{3+2\sin x+\cos x}$

Sol^n:

$$I = \int \frac{dx}{3+2\sin x+\cos x}$$

$$= \int \frac{dx}{3 + 2\cdot\frac{2\tan\frac{1}{2}x}{1+\tan^2\frac{1}{2}x} + \frac{1-\tan^2\frac{1}{2}x}{1+\tan^2\frac{1}{2}x}}$$

$$= \int \frac{(1+\tan^2\frac{1}{2}x)\,dx}{3+3\tan^2\frac{1}{2}x + 4\tan\frac{1}{2}x + 1 - \tan^2\frac{1}{2}x}$$

$$= \int \frac{\sec^2 \frac{1}{2}x \, dx}{2\tan^2 \frac{1}{2}x + 4\tan\frac{1}{2}x + 4} = \int \frac{2\,dz}{2z^2+4z+4}, \text{ where, } z = \tan\frac{1}{2}x$$

$$= \int \frac{dz}{z^2+2z+2} = \int \frac{dz}{(z+1)^2+1} = \tan^{-1}(z+1) + c$$

$$= \tan^{-1}\left(1+\tan\frac{1}{2}x\right) + c$$

Page- 114 : Example- 4 :

Prob: Integrate $\int \frac{dx}{2\sin x + 3\cos x + 4}$

Soln :

$$I = \int \frac{dx}{2\sin x + 3\cos x + 4}$$

$$= \int \frac{dx}{2 \cdot \frac{2\tan\frac{1}{2}x}{1+\tan^2\frac{1}{2}x} + \frac{3(1-\tan^2\frac{1}{2}x)}{1+\tan^2\frac{1}{2}x} + 4}$$

$$= \int \frac{\sec^2\frac{1}{2}x \, dx}{4\tan\frac{1}{2}x + 3 - 3\tan^2\frac{1}{2}x + 4 + 4\tan^2\frac{1}{2}x}$$

$$= \int \frac{\sec^2\frac{1}{2}x \, dx}{\tan^2\frac{1}{2}x + 4\tan\frac{1}{2}x + 7}$$

$$= 2\int \frac{dz}{z^2+4z+7}, \text{ where, } z = \tan\frac{1}{2}x$$

$$= 2\int \frac{dz}{(z+2)^2+(\sqrt{3})^2} = 2\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{z+2}{\sqrt{3}}+c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left\{\frac{1}{\sqrt{3}}\left(2+\tan\frac{1}{2}x\right)\right\}+c \text{ (Ans).}$$

Page-116: Example – 6:

Prob: Integrate $\int \frac{dx}{1+2\tan x}$

Solⁿ:

$$I = \int \frac{dx}{1+2\tan x} = \int \frac{\cos x}{\cos x + 2\sin x}$$

Let, $\cos x = l(\cos x + 2\sin x) + m(-\sin x + 2\cos x)$

$$= (l+2m)\cos x + (2l-m)\sin x$$

Equating the coeficients of $\cos x$ and $\sin x$ of both the sides, we have

$$l + 2m = 1 \quad \text{——— (i)}$$

$$2l - m = 0 \quad \text{——— (ii)}$$

Solving (i) and (ii), $l = \frac{1}{5}$, $m = \frac{2}{5}$

$$\therefore I = \int \frac{l(\cos x + 2\sin x) + m(-\sin x + 2\cos x)}{\cos x + 2\sin x}\,dx$$

$$= l\int dx + m\int \frac{dz}{z} \quad \text{, where, } z = \cos x + 2\sin x$$

$$= \frac{1}{5}x + \frac{2}{5}\log|z| + c = \frac{1}{5}x + \frac{2}{5}\log|\cos x + 2\sin x| + c$$

Page - 116 : Example - 7 :

Prob: Integrate $\int \frac{dx}{\tan x(1+\tan x)}$

Sol^n:

$$I = \int \frac{dx}{\tan x} - \int \frac{dx}{1+\tan x}$$

$$= \int \frac{\cos x\, dx}{\sin x} - \int \frac{\cos x}{\cos x+\sin x} dx$$

$$= \int \frac{dz}{z} - \frac{1}{2}\int \frac{(\cos x+\sin x)-(\sin x-\cos x)}{\sin x+\cos x} dx$$

where, $z = \sin x$, $dz = \cos x\, dx$

$$= \int \frac{dz}{z} - \frac{1}{2}\int dx - \frac{1}{2}\int \frac{\cos x-\sin x}{\sin x+\cos x} dx$$

$$= \int \frac{dz}{z} - \frac{1}{2}\int dx - \frac{1}{2}\int \frac{du}{u}, \text{ where, } u = \sin x+\cos x$$

$$= \log|z| - \frac{1}{2}x - \frac{1}{2}\log|u| + c$$

$$= \log|\sin x| - \frac{1}{2}x - \frac{1}{2}\log|\sin x+\cos x| + c \text{ (Ans).}$$

Page-126 : Exercise - 25 (i) :

Prob: Integrate $\int \frac{dx}{5+4\sin x}$

Sol^n:

$$I = \int \frac{dx}{5+4\sin x} = \int \frac{dx}{5\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+4.2\sin\frac{x}{2}.\cos\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}\, dx}{5+5\tan^2 x/2 + 8\tan x/2}$$

38

put, $\tan x/2 = t$, $\frac{1}{2}\sec^2 x/2 \, dx = dt$

$$I = \int \frac{2\,dt}{5t^2+8t+5} = \frac{2}{5}\int \frac{dt}{t^2+\frac{8}{5}t+1}$$

$$= \frac{2}{3}\int \frac{dt}{(t+4/5)^2+9/25} = \frac{2}{5}\cdot\frac{1}{3/5}\cdot\tan^{-1}\left(\frac{t+4/5}{3/5}\right)$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{5\tan x/2+4}{3}\right)$$

P 126: Example- 25 (iii)

Prob: Integrate $\int \frac{dx}{4+3\sin hx}$

Soln:

$$I = \int \frac{dx}{4+3\sin hx}$$

$$= \int \frac{dx}{4(\sinh^2\frac{x}{2}+\cosh^2\frac{x}{2})+3.2\sinh x/2 . \cosh x/2}$$

$$= \int \frac{\sec h^2 x/2 \quad dx}{4(1-\tan h^2 x/2)+6\tan h\, x/2}$$

$$= \int \frac{\frac{1}{2}\operatorname{sech}^2 x/2 \quad dx}{2+3\tan h\, x/2 - 2\tanh^2 x/2}$$

put, $\tanh x/2 = t$, $\frac{1}{2}$ ~~tan~~ $\operatorname{sech}^2 x/2 \, dx = dt$

$$\therefore I = \int \frac{dt}{2+3t-2t^2} = \frac{1}{2}\int \frac{dt}{1+\frac{3}{2}t - t^2 - \frac{9}{16} + \frac{9}{16}}$$

$$= \frac{1}{2}\int \frac{dt}{\frac{25}{16} - (t-\frac{3}{4})^2} = \frac{1}{2}\cdot\frac{1}{2\cdot 5/4} \log\left\{\frac{5/4 + (t-3/4)}{5/4 - (t-3/4)}\right\}$$

$$= \frac{1}{5}\log\left\{\frac{1+2\tanh x/2}{4-2\tanh x/2}\right\} \text{ (Ans).}$$

Page–126 : Exercise-26(i):

<u>Prob:</u> Integrate $\int \frac{dx}{5+4\cos x}$

<u>Sol^n:</u>

$$I = \int \frac{dx}{5+4\cos x}$$

$$= \int \frac{dx}{5(\cos^2 x/2 + \sin^2 x/2) + 4(\cos^2 x/2 - \sin^2 x/2)}$$

$$= \int \frac{\sec^2 x/2 \, dx}{5(1+\tan^2 x/2) + 4(1-\tan^2 x/2)} = \int \frac{\sec^2 x/2 \, dx}{9+\tan^2 x/2}$$

put $\tan x/2 = t$, $\frac{1}{2}\sec^2 x/2 \, dx = dt$

$$= \int \frac{2\,dt}{9+t^2} = \frac{2}{3}\tan^{-1}(t/3) = \frac{2}{3}\tan^{-1}\left(\frac{1}{2}\tan x/2\right)$$

(Ans).

40

Page-126 : Exercise-26(ii):

Prob: Integrate $\int \frac{dx}{3+5\cos x}$

Sol^n ÷

$$I = \int \frac{dx}{3+5\cos x}$$

$$= \int \frac{dx}{3+5(\cos^2 x/2 - \sin^2 x/2)}$$

$$= \int \frac{\sec^2 x/2 \; dx}{3(1+\tan^2 x/2)+5(1-\tan^2 x/2)}$$

$$= \int \frac{\sec^2 x/2}{8-2\tan^2 x/2} = \int \frac{\frac{1}{2}\sec^2 x/2 \, dx}{4-\tan^2 x/2}$$

$$= \int \frac{dt}{4-t^2}$$ put, $\tan x/2 = t$, $\frac{1}{2}\sec^2 x/2 \, dx = dt$

$$= \frac{1}{2.2}\log\frac{2+t}{2-t} = \frac{1}{2}\log\left\{\frac{2+\tan x/2}{2-\tan x/2}\right\}$$

Page-126 : Exercise-27(i):

Prob: Integrate $\int \frac{dx}{\cos\alpha + \cos x}$

Sol^n ÷

$$I = \int \frac{dx}{\cos\alpha + \cos x}$$

$$= \int \frac{\sec^2 x/2}{\cos\alpha(1+\tan^2 x/2)+(1-\tan^2 x/2)}$$

$$= \int \frac{\sec^2 x/2 \, dx}{(1+\cos\alpha) - (1-\cos\alpha)\tan^2 x/2}$$ (41)

put, $\tan x/2 = t$, $\frac{1}{2}\sec^2 x/2 \, dx = dt$

$$= \frac{1}{1-\cos\alpha}\int \frac{2\,dt}{\left(\frac{1+\cos\alpha}{1-\cos\alpha}\right) - t^2}$$

$$= \frac{2}{1-\cos\alpha} \cdot \frac{1}{2\sqrt{\left(\frac{1+\cos\alpha}{1-\cos\alpha}\right)}} \cdot \log\left\{\frac{\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}+t}{\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}-t}\right\}$$

$$= \frac{1}{\sqrt{(1-\cos^2\alpha)}} \log\left\{\frac{\cot\alpha/2 + \tan x/2}{\cot\alpha/2 - \tan x/2}\right\}$$

Page-126: Exercise- 32 (i):

Prob:- Integrate $\int \frac{dx}{1-\cos x+\sin x}$

Sol^n:

$$I = \int \frac{dx}{1-\cos x+\sin x}$$

$$= \int \frac{dx}{2\sin^2 x/2 + 2\sin x/2 \cos x/2}$$

42

$$= \int \frac{\frac{1}{2}\sec^2 x/2}{\tan^2 x/2 + \tan x/2}\, dx$$

put, $\tan x/2 = t$, $\frac{1}{2}\sec^2 x/2 \, dx = dt$

$$= \int \frac{dt}{t^2+t+\frac{1}{4}-\frac{1}{4}} = \int \frac{dt}{(t+\frac{1}{2})^2-\frac{1}{4}}$$

$$= \frac{1}{2\cdot 1/2} \log\left(\frac{t+\frac{1}{2}-\frac{1}{2}}{t+\frac{1}{2}+\frac{1}{2}}\right)$$

$$= \log\left(\frac{\tan^2 x/2}{1+\tan x/2}\right) = \log\left(\frac{1}{1+\cot x/2}\right)$$

$$= \log(1+\cot x/2)$$

Page-126: Exercise - 32 (ii)

Prob: Integrate $\int \frac{dx}{3+2\sin x+\cos x}$

Sol^n:

$$I = \int \frac{dx}{3+2\sin x+\cos x}$$

$$= \int \frac{dx}{3(\cos^2 x/2+\sin^2 x/2)+2\cdot 2\sin x/2\cdot\cos x/2 + (\cos^2 x/2-\sin^2 x/2)}$$

$$= \int \frac{\sec^2 x/2}{1+4\tan x/2 + 2\tan^2 x/2}\, dx.$$

put, $\tan x/2 = t$, $\frac{1}{2}\sec^2 x/2\, dx = dt$

$$\therefore = \int \frac{dt}{2+2t+t^2} = \int \frac{dt}{(t+1)^2+1}$$

$$= \tan^{-1}(t+1) = \tan^{-1}(\tan x/2 + 1)$$

Page-126: Exercise- 33

Prob: Integrate $\int \frac{6+3\sin x+14\cos x}{3+4\sin x+5\cos x}\, dx$

Soln:

Let, $6+3\sin x+14\cos x = t(3+4\sin x+5\cos x) + m(4\cos x - 5\sin x) + n$

Comparing both sides, we get,

$6 = 3t+n,\ 3 = 4t-5m,\ 14 = 5t+4m$

$t = 2,\ m = 1,\ n = 0$

$$I = \int \frac{2(3+4\sin x+5\cos x) + \frac{1}{2}(4\cos x - 5\sin x)\, dx}{3+4\sin x+5\cos x}$$

$$= 2\int dx + \int \frac{4\cos x - 5\sin x}{3 + 4\sin x + 5\cos x}$$

$$= 2x + \log(3 + 4\sin x + 5\cos x) \text{ (Ans).}$$

Page-126 : Exercise-35 (i) :

Prob: Integrate $\int \frac{1}{\sec x + \operatorname{cosec} x} dx$

Sol^n:

$$I = \int \frac{1}{\sec x + \operatorname{cosec} x} dx$$

$$= \frac{1}{2}\int \frac{1 + 2\sin x\cos x - 1}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int \frac{(\cos x + \sin x)^2}{\sin x + \cos x} dx - \frac{1}{2}\int \frac{dx}{\sin x + \cos x}$$

$$= \frac{1}{2}\left[\int (\cos x + \sin x) dx - \frac{1}{\sqrt{2}}\int \frac{dx}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x}\right]$$

$$= \frac{1}{2}\left[\sin x - \cos x - \frac{1}{\sqrt{2}}\int \frac{dx}{\cos \pi/4 \sin x + \cos x \sin \pi/4}\right]$$

$$= \frac{1}{2}\left[\sin x - \cos x - 1/\sqrt{2} \log \tan(\pi/8 + x/2)\right]$$

(Ans).

Page-127 : Exercise - 35(ii):

Prob: Integrate $\int \frac{dx}{\sin x + \tan x}$

Sol^n:

$$I = \int \frac{dx}{\sin x + \tan x} = \int \frac{\cos x}{\sin x(\cos x + 1)} dx$$

$$= \int \frac{2\cos^2 x/2 - 1}{\sin x(2\cos^2 x/2)} dx = \int \frac{1}{\sin x} dx -$$

$$\int \frac{dx}{2\sin x/2 \cos x/2 \cdot 2\cos^2 x}$$

$$= \int \operatorname{cosec} x \, dx - \frac{1}{2}\int \frac{\frac{1}{2}\sec^2 x/2 \cdot \sec^2 x/2}{\tan x/2}$$

(Deviding above and below by $\cos^2 x/2$)

$$= \log \tan x/2 - \frac{1}{2}\int \frac{(1+t^2)}{t} dt$$

putting. $\tan x/2 = t$, $\frac{1}{2}\sec^2 x/2 \, dx = dt$

$$= \log \tan x/2 - \frac{1}{2}\int (1/t + t) dt$$

$$= \log \tan x/2 - \frac{1}{2}[\log t + t^2/2]$$

$$= \frac{1}{2}\log \tan x/2 - \frac{1}{4}\tan^2 x/2 \text{ (Ans)}.$$

Rational Fraction

Page-139 : Exercise- Example - 2 :

Prob: Integrate $\int \frac{e^{-x}\,dx}{e^{x}+2e^{-x}+3}$

Soln:

$$I = \int \frac{e^{-x}\,dx}{e^{x}+2e^{-x}+3}$$

$$= \int \frac{e^{x}}{e^{3x}+2e^{x}+3e^{2x}}\,dx$$

If we substitute, $e^{x}=z$, $e^{x}dx = dz$ then

$$I = \int \frac{dz}{z^3+3z^2+2z} = \int \frac{dz}{z(z+1)(z+2)}$$

Let, $\frac{1}{z(z+1)(z+2)} = \frac{a}{z} + \frac{b}{z+1} + \frac{c}{z+2}$

Then, $1 = a(z+1)(z+2)+bz(z+2)+cz(z+1)$ — (1)

Putting, $z=0,-1,-2$ successively in (1), we get

$a=\frac{1}{2}$, $b=-1$, $c=\frac{1}{2}$

$$\therefore I = \frac{1}{2}\int\frac{dz}{z} - \int\frac{dz}{z+1} + \frac{1}{2}\int\frac{dz}{z+2}$$

$$= \frac{1}{2}\log|z| - \log|z+1| + \frac{1}{2}\log|z+2| + c$$

$$= \frac{1}{2}\log|e^{x}| - \log|e^{x}+1| + \frac{1}{2}\log|e^{x}+2| + c$$

$$= \frac{1}{2}x - \log|e^{x}+1| + \frac{1}{2}\log|e^{x}+2| + c$$

Page - 149 : Exercise - 22(i) :

Prob : Integrate $\int \frac{x^2\,dx}{x^4 - x^2 - 12}$

Sol^n :

$$I = \int \frac{x^2\,dx}{x^4 - x^2 - 12} = \int \frac{x^2\,dx}{(x^2 - 4)(x^2+3)}$$

Let, $x^2 = t$

$$= \int \left[\frac{t}{(t-4)(t+3)}\right] dx = \left[\frac{4}{7(t-8)} + \frac{3}{t(t+3)}\right] dx$$

$$= 4/7 \int \frac{dx}{x^2 - 4} + 3/7 \int \frac{dx}{x^2+3}$$

$$= 4/7 \cdot \frac{1}{2.2} \log\left[\frac{x-2}{x+2}\right] + \frac{3}{7} \cdot \frac{1}{\sqrt{3}} \tan^{-1}(x/\sqrt{3})$$

$$= \frac{1}{7}\left[\log \frac{x-2}{x+2} + \sqrt{3} \tan^{-1} x/\sqrt{3}\right] \text{ Ans)}$$

Page-149 : Exercise - 22(ii)

Prob : Integrate $\int \frac{x\,dx}{x^4 - x^2 - 2}$

Sol^n :

$$I = \int \frac{x\,dx}{x^4 - x^2 - 2} = \frac{1}{2}\int \frac{dt}{t^2 - t - 2}$$

putting, $x^4 = t$, $2x\,dx = dt$

$$\therefore I = \frac{1}{2}\int \frac{dt}{(t-\frac{1}{2})^2 - 9/4}$$

$$= \frac{1}{2} \cdot \frac{1}{2 \cdot 3/2} \log \frac{(t - 1/2 - 3/2)}{t - 1/2 + 3/2}$$

$$= \frac{1}{6} \log \left(\frac{x^2-2}{x^2+1}\right) \text{ (Ans).}$$

Page-150: Exercise- 29(i):

<u>Prob:</u> Integrate $\int \frac{dx}{1+3e^x+2e^{2x}}$

<u>Solⁿ:</u>

put, $e^x = t$, $e^x dx = dt$

$$\therefore I = \int \frac{dt}{t(1+3t+2t^2)} = \int \left[\frac{1}{t(1+t)(1+2t)}\right] dt$$

$$= \int \left[\frac{1}{t} + \frac{1}{t+1} - \frac{4}{1+2t}\right] dt$$

$$= \log t + \log|1+t| - 4/2 \cdot \log(1+2t)$$

$$= \log e^x + \log(1+e^x) - 2\log(1+2e^x) \text{ (Ans).}$$

Page-150: Exercise- 30:

Prob: Integrate $\int \frac{dx}{\sin x(3+2\cos x)}$

Soln:

$$I = \int \frac{dx}{\sin x(3+2\cos x)}$$

$$= \int \frac{\sin x}{(1-\cos^2 x)(3+2\cos x)} dx$$

put. $\cos x = t$, $\sin x \, dx = dt$

$$I = \int \frac{-dt}{(1+t)(1-t)(3+2t)} = \int \frac{dt}{(1+t)(t-1)(3+2t)}$$

$$= \int \left\{ -\frac{1}{2(1+t)} + \frac{1}{10(t-1)} + \frac{4}{5(3+2t)} \right\} dt$$

$$= -\frac{1}{2}\log(1+t) + \frac{1}{10}\log(t-1) + \frac{4}{5 \cdot 2}\log(3+2t)$$

$$= -\frac{1}{2}\log(1+\cos x) + \frac{1}{10}\log(\cos x - 1) + \frac{2}{5}\log(3+2\cos x)$$

(Ans).

Irrational Fraction

Page- 160 : Exercise-8 (i) :

Prob: Integrate $\int \frac{dx}{(x^2+1)\sqrt{x^2+4}}$

Soln:

$$I = \int \frac{dx}{(x^2+1)\sqrt{x^2+4}}$$

put, $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$

$$I = \int \frac{2\sec^2\theta\, d\theta}{(1+4\tan^2\theta)\cdot 2\sec\theta}$$

$$= \int \frac{\sec\theta\, d\theta}{1+4\tan^2\theta}$$

$$= \int \frac{\sec\theta \cdot \cos^2\theta}{[\cos^2\theta + 4\sin^2\theta]} = \int \frac{\cos\theta}{1+3\sin^2\theta}\, d\theta$$

put, $\sqrt{3}\sin\theta = t$, $\sqrt{3}\cos\theta\, d\theta = dt$

$$= \frac{1}{\sqrt{3}} \int \frac{dt}{[1+t^2]} = \frac{1}{\sqrt{3}} \tan^{-1}(t) = \frac{1}{\sqrt{3}} \tan^{-1}(\sqrt{3}\sin\theta)$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{(\sqrt{3}\cdot x)}{\sqrt{(x^2+4)}} \quad \text{(Ans)}$$

Page-161 : Exercise-17 :

Prob : Integrate $\int \frac{x^2}{x^4+x^2+1}\,dx$

Soln :

$$I = \int \frac{x^2}{x^4+x^2+1}\,dx$$

$$= \frac{1}{2}\int \frac{x^2+1+x^2-1}{(x^4+x^2+1)}\,dx$$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+x^2+1}\,dx + \frac{1}{2}\int \frac{x^2-1}{x^4+x^2+1}\,dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{x^2-1}{x\sqrt{3}} + \frac{1}{2}\log\left\{\frac{x^2-x+1}{x^2+x+1}\right\}\right.$$

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