Definite Integrals

(without using the propertise of definite integrals)

Exp-4, Page-214 :-

Evaluate $\int_{\alpha}^{\beta} \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$ $(\beta > \alpha)$

put $x = \alpha \cos^2\theta + \beta \sin^2\theta$

$\therefore I = \int_0^{\pi/2} 2\,d\theta = 2 \times \pi/2 = \pi$

Exp-5, Page-214 :-

Show that $\int_0^{1/2} \frac{dx}{(1-2x^2)\sqrt{(1-x^2)}} = \frac{1}{2}\log(2+\sqrt{3})$

Put, $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

when $x = 0$, $\theta = 0$ and $x = \frac{1}{2}$, $\theta = \pi/6$

$$\therefore I = \int_0^{\pi/6} \frac{\cos\theta\, d\theta}{\cos 2\theta \cos\theta} = \int_0^{\pi/6} \sec 2\theta\, d\theta$$

$$= \left[\frac{1}{2}\log\tan\left(\pi/4 + \theta\right)\right]_0^{\pi/6}$$

$$= \frac{1}{2}\left[\log\tan 5\pi/12 - \log\tan \pi/4\right]$$

$$= \frac{1}{2}\log(2+\sqrt{3})$$

$\therefore \cos\theta = 0 \quad \therefore \theta = \pi/2$

$$\therefore I = 2(\beta-\alpha)^2 \int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta$$

Now $\sin^2\theta \cos^2\theta = \frac{1}{4} \times 4\sin^2\theta \cos^2\theta = \frac{1}{4}\sin^2 2\theta$

$$= \frac{1}{8}(1-\cos 4\theta)$$

Also, $\int (1-\cos 4\theta)\, d\theta = \theta - \frac{1}{4}\sin\theta$

$$\therefore I = 2(\beta-\alpha)^2 \frac{1}{8} \int_0^{\pi/2} (1-\cos 4\theta)\, d\theta$$

$$= \frac{1}{4}(\beta-\alpha)^2 \left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2}$$

$$= \frac{1}{4}(\beta-\alpha)^2 \left[\pi/2 - \frac{1}{4}\sin 2\pi\right] = \frac{1}{8}\pi(\beta-\alpha)^2$$

Ex- 16(1), Page- 220: Show that, $\int_1^2 \sqrt{(x-1)(2-x)}\, dx = \pi/8$

$$I = \int_1^2 \sqrt{(x-1)(2-x)}\, dx = \int_1^2 \sqrt{(-2+3x-x^2)}\, dx$$

$$= \int_1^2 \sqrt{\left\{\frac{1}{4} - (x-3/2)^2\right\}}\, dx$$

$$= \left[\frac{1}{2}(x-3/2)\sqrt{\{(x-1)(2-x)\}} + \frac{1}{2\cdot 4}\sin^{-1}(2x-3)\right]_1^2$$

$$= 0 + 1/8 \times \pi/2 - 0 - 1/8 \times (-\pi/2) = \pi/8$$

Ex-16, Page-220 :-

~~Evaluate~~ Show that, $\int_8^{15} \frac{dx}{(x-3)\sqrt{(x+1)}} = \frac{1}{2}\log 5/3$

$I = \int_8^{15} \frac{dx}{(x-3)\sqrt{(x+1)}}$

putting $x+1 = t^2$, $dx = 2t\,dt$

when $x = 8$, $t = 3$ and $x = 15$, $t = 4$

$= \int_3^4 \frac{2t\,dt}{(t^2-3-1)t} = 2\int_3^4 \frac{dt}{t^2-4} = 2 \cdot \frac{1}{2\cdot 2}\left[\log \frac{t-2}{t+2}\right]_3^4$

$= \frac{1}{2}\left[\log 2/6 - \log 1/5\right] = \frac{1}{2}\log 2/6 \times 5/1 = \frac{1}{2}\log 5/3$

Ex-20(1), Page-220 :

Show that, $\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \pi/2ab$

$I = \int_0^{\pi/2} \frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x}$

Put $b\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$= \frac{1}{b}\int_0^{\infty} \frac{dt}{a^2+t^2} = \frac{1}{b \cdot a}\left[\tan^{-1}\frac{t}{a}\right]_0^{\infty} = \frac{1}{b \cdot a} \times \pi/2 = \pi/2ab$

Ex-20(ii), Page-220 :-

Show that, $\int_0^{\pi/4} \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx = \frac{1}{6}$

$$I = \int_0^{\pi/4} \frac{\tan^2 x \sec^2 x \, dx}{(\tan^3 x + 1)^2}$$ (Dividing above and below by $\cos^6 x$)

put, $1 + \tan^3 x = t \Rightarrow 3\tan^2 x \sec^2 x \, dx = dt$

$$= \frac{1}{3}\int_1^2 \frac{dt}{t^2} = \frac{1}{3}\left[-\frac{1}{t}\right]_1^2 = \frac{1}{3}\left(-\frac{1}{2}+1\right) = \frac{1}{6}$$

Ex-21(i), Page-220 :-

Show that, $\int_0^{\pi/2} \frac{dx}{4+5\sin x} = \frac{1}{3}\log 2$

$$I = \left[\frac{1}{3}\log \frac{2\tan x/2 + 1}{2\tan x/2 + 4}\right]_0^{\pi/2}$$

$$= \frac{1}{3}\left(\log 3/6 - \log 1/4\right) = \frac{1}{3} \times \log 1/2 \times 4/1 = \frac{1}{3}\log 2$$

Ex-21(ii), Page-220 :-

Show that, $\int_0^{\pi/2} \frac{dx}{5+4\sin x} = 2/3 \tan^{-1} 1/3$

$$I = \left[2/3 \tan^{-1}\left(\frac{5\tan x/2 + 4}{3}\right)\right]_0^{\pi/2}$$

$$= 2/3\left(\tan^{-1} 3 - \tan^{-1} 4/3\right) = 2/3 \tan^{-1} \frac{3 - 4/3}{1 + 3 \cdot 4/3} = 2/3 \tan^{-1}(1/3)$$

Ex-22(i), Page-221 :-

Show that, $\int_0^{\pi/2} \frac{dx}{5+3\cos x} = \frac{1}{2}\tan^{-1} 1/2$

$$I = \int_0^{\pi/2} \frac{\sec^2 x/2\, dx}{5(1+\tan^2 x/2) + 3(1-\tan^2 x/2)}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x/2\, dx}{8+2\tan^2 x/2}$$ Put, $\tan x/2 = t \Rightarrow \frac{1}{2}\sec^2 x/2\, dx = dt$

$$= \int_0^1 \frac{dt}{4+t^2} = \frac{1}{2}\left[\tan^{-1}\frac{t}{2}\right]_0^1 = \frac{1}{2}\tan^{-1}\frac{1}{2}$$

Ex-22(ii), Page-221 :

Show that, $\int_0^{\pi/2} \frac{dx}{3+5\cos x} = \frac{1}{4}\log 3.$

$$I = \left[\frac{1}{4}\log\left(\frac{2+\tan x/2}{2-\tan x/2}\right)\right]_0^{\pi/2}$$

$$= \frac{1}{4}\log 3$$

Ex-25, Page-221 :

Show that, $\int_0^{\pi/4} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx = \pi/4$

$$I = \int_0^{\pi/4} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx$$

$$= \left[\left\{\tan^{-1}(\tan^2 x)\right\}\right]_0^{\pi/4} = \pi/4$$

Ex-28 (i), Page-221 :

Show that, $\int_{2}^{3}\frac{dx}{(x-1)\sqrt{x^2-2x}} = \pi/3$

put $x-1 = 1/t \Rightarrow dx = -1/t^2\, dt$

$$= \int_{1}^{1/2}\frac{(-1/t^2)\,dt}{1/t\sqrt{\{(1/t+1)^2-2\cdot(1/t+1)\}}}$$

$$= \int_{1}^{1/2}\frac{dt}{\sqrt{1-t^2}} = -\left\{\sin^{-1}t\right\}_{1}^{1/2}$$

$$= -\left(\sin^{-1}1/2 - \sin^{-1}1\right) = -\left(\pi/6 - \pi/2\right)$$

$$= \pi/3$$

Ex-28 (ii), Page-221 :-

Show that, $\int_{0}^{1}\frac{dx}{(1+x)\sqrt{(1+2x-x^2)}} = \pi/4\sqrt{2}$

$$I = \int_{0}^{1}\frac{dx}{(1+x)\sqrt{1+2x-x^2}} = \left[\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x\sqrt{2}}{x+1}\right)\right]_{0}^{1}$$

$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{1\sqrt{2}}{1+1}\right) - \sin^{-1}0\right]$$

$$= \frac{1}{\sqrt{2}}\times\pi/4 = \pi/4\sqrt{2}$$

Definite Integrals - 11

Note 1:- Beta function : $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \,.\; m,n>0.$

Note-2:- Grama function : $\int_0^{\infty} e^{-x} x^{n-1}\,dx \quad n>0;$

Note-3:- $B(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}$

Note-4:- $\Gamma n+1 = n\Gamma n = n!$

Note-5:- $\displaystyle\int_0^{\pi/2} \sin^p\theta \cos^q\theta\,dx = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$

Note-6:- $\Gamma 1/2 = \sqrt{\pi}$

Related Preoblem (Note-5)

Exp-3, Page-213: Evalute $\displaystyle\int_\alpha^\beta \sqrt{(x-\alpha)(\beta-x)}\,dx$

put $x = \alpha\cos^2\theta + \beta\sin^2\theta \quad \therefore dx = 2(\beta-\alpha)\sin\theta\cos\theta\,d\theta$

Also, $x-\alpha = \beta\sin^2\theta - \alpha(1-\cos^2\theta) = (\beta-\alpha)\sin^2\theta$

$\beta - x = \beta(1-\sin^2\theta) - \alpha\cos^2\theta = (\beta-\alpha)\cos^2\theta$

when. $x=\alpha,\; (\beta-\alpha)\sin^2\theta = 0$

$\sin\theta = 0$ since $\beta \neq \alpha \quad \therefore \theta = 0$

similarly, when $x=\beta,\; (\beta-\alpha)\cos^2\theta = 0$

Exp-1, Page-235: Evalute $\int_0^1 x^6\sqrt{(1-x^2)}\,dx$

put $x=\sin\theta$ $\therefore dx=\cos\theta\,d\theta$ and $1-x^2=\cos^2\theta$

when $x=0, \theta=0$ and $x=1, \theta=\pi/2$

The given integral then reduces to

$$\int_0^{\pi/2}\sin^6\theta\cos^2\theta\,d\theta=\frac{1\times3\times5\times1}{2\times4\times6\times8}\times\pi/2=\frac{5\pi}{256}$$

Exp-2, Page-236: Evalute $\int_0^1 x^2(1-x)^{3/2}\,dx$

put $x=\sin^2\theta$ $\therefore dx=2\sin\theta\cos\theta\,d\theta$

when $x=0,1$, we have $\theta=0, \pi/2$

$$\therefore I=2\int_0^{\pi/2}\sin^5\theta\cos^4\theta\,d\theta$$

$$=2\times\frac{2\times4}{5\times7\times9}=\frac{16}{315}$$

Ex-17 (iv), Page-238 :- Show that, $\int_0^{\pi/2}\sin^4x\cos^5x\,dx=\frac{8}{315}$

$$\int_0^{\pi/2}\sin^4x\cdot\cos^5x\,dx=\frac{3\times1\times4\times2}{8\times7\times5\times3\times1}=\frac{8}{315}$$

Ex-18(1), Page-239: Show that, $\int_0^1 x^3(1-x)^3\,dx = \frac{1}{140}$

put $x = \sin^2\theta$. $\Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$

$$= \int_0^{\pi/2} \sin^6\theta \cdot (1-\sin^2\theta)^3\, 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^7\theta \cdot \cos^7\theta\,d\theta$$

$$= \frac{6\times4\times2\times6\times4\times2}{14\times12\times10\times8\times6\times4\times2} = \frac{1}{140}$$

Ex-18(2), Page-239:- Show that, $\int_0^1 x^3(1-x^2)^{5/2}\,dx = 2/63$

put $1-x^2 = t \Rightarrow -2x\,dx = dt$

$$= \int_1^0 (1-t)^{5/2}\frac{dt}{-2} = -\frac{1}{2}\int_1^0 (t^{5/2} - t^{7/2})\,dt$$

$$= -\frac{1}{2}\left[2/7\, t^{7/2} - 2/9\, t^{9/2}\right]_1^0$$

$$= \frac{1}{2}(2/7 - 2/9)$$

$$= 2/63$$

Exp - 2, Page - 200 :- Find from the definition, the value of $\int_0^1 x^2 dx$

$\int_0^1 x^2 dx$ exists since x^2 is continuous in $[0, 1]$

From the definition $\int_0^1 x^2 dx = \lim_{h \to 0} \sum (rh)^2$, where $nh = 1$

$= \lim_{h \to 0} h [1^2h^2 + 2^2h^2 + \cdots + n^2h^2]$

$= \lim_{h \to 0} [h^3(1^2 + 2^2 + \cdots n^2)]$

$= \lim_{h \to 0} \frac{n(n+1)(2n+1)}{6}$

$= \frac{1}{6} \lim_{h \to 0} (2n^3h^3 + 3n^2h^2 \cdot h + nh \cdot h^2)$

$= \frac{1}{6} \lim_{h \to 0} (2 + 3h + h^2)$, since $nh = 1$

$= \frac{1}{6} \times 2 = \frac{1}{3}$

Exp-4, Page-202: Prove by summation $\int_a^b \sin x dx = \cos a - \cos b$

$\int_a^b \sin x dx$ which exists since $\sin x$ is continuous in $[a, b]$

$\int_a^b \sin x dx = \lim_{h \to 0} \sum \sin(a + rh)$, where $nh = b - a$

$= \lim_{h \to 0} h [\sin a + \sin(a+h) + \sin(a+2h) + \cdots \text{ n terms}]$

$$= \lim_{h \to 0} h \sin\left\{a + (n-1)\frac{h}{2}\right\} \cdot \frac{\sin nh/2}{\sin h/2}$$

$$= \lim_{h \to 0} \frac{h/2}{\sin h/2} 2 \sin nh/2 \cdot \sin\left\{a + (n-1)h/2\right\}$$

$$= \lim_{h \to 0} \frac{h/2}{\sin h/2} \left[\cos\left(a - h/2\right) - \cos\left(a + (2n-1)h/2\right)\right]$$

$$= \lim_{h \to 0} \left[\cos\left(a - h/2\right) - \cos\left(a + nh - h/2\right)\right]$$

since $\lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 1$

$$= \lim_{\theta \to 0} \left[\cos\left(a - h/2\right) - \cos\left(b - h/2\right)\right] \text{ Since, } a + nh = b$$

$$= \cos a - \cos b$$

<u>Ex-1 (iii), Page-217:</u> $\int_0^1 x^3 \, dx$

$$I = \int_0^1 x^3 \, dx = \left[\frac{1}{4} x^4\right]_0^1$$

$$= \frac{1}{4}$$

<u>Ex-1 (iv),:</u> $\int_0^1 (ax + b) \, dx$

$$I = \int_0^1 (ax+b) \, dx = \left[1/a\left(ax + b\right)\right]_0^1$$

$$= \frac{1}{a}\left(a + b - b\right) = 1$$

Ex-1 (vi), Page-217 :- $\int_a^b \cos\theta \, d\theta$

$$I = \int_a^b \cos\theta \, d\theta = [\sin\theta]_a^b$$

$$= \sin b - \sin a$$

Ex-1 (vii), Page-217 : $\int_0^1 \sqrt{x}\, dx$

$$I = \int_0^1 \sqrt{x}\, dx$$

$$= \left[2/3 \, x^{3/2} \right]_0^1 = 2/3 \, [1-0]$$

$$= 2/3$$

Exp-1, Page-216 :- Evaluate $\lim_{n\to\infty} \left\{ \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right\}$

Dividing the numerator and denominator of each term of the above series by n, the given series becomes

$$\lim_{n\to\infty} \left\{ \frac{\frac{1}{n}}{1+\frac{1}{n}} + \frac{\frac{1}{n}}{1+\frac{2}{n}} + \cdots + \frac{\frac{1}{n}}{1+\frac{n}{n}} \right\}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum \frac{1}{1+\frac{r}{n}} = \lim_{h\to 0} h \sum \frac{1}{1+rh} \quad \left[\text{Putting } h = \frac{1}{n}\right]$$

$$= \int_0^1 \frac{1}{1+x}\, dx = [\log(1+x)]_0^1 = \log 2$$

$+\frac{n^2}{n^2}$

Exp-2, Page-216:

Evaluate $\lim_{n\to\infty}\left\{\left(1+\frac{1}{n^2}\right)^{2/n^2}\left(1+\frac{2^2}{n^2}\right)^{4/n^2}\left(1+\frac{3^2}{n^2}\right)^{6/n^2}\cdots\left(1+\frac{n^2}{n^2}\right)^{2n/n^2}\right\}$

Let A denote the given expression, then

$$\log A = \sum \frac{2r}{n^2}\log\left(1+\frac{r^2}{n^2}\right)$$

$$\therefore \lim_{n\to\infty}\log A = \lim_{n\to\infty}\frac{1}{n}\sum 2^{r}/_{n}\log\left(1+\frac{r^2}{n^2}\right)$$

$$= \int_0^1 2x\log(1+x^2)\,dx$$

$$= \int_1^2 \log z\,dz \quad [\text{putting } 1+x^2=z]$$

$$= \Big[z\log z - z\Big]_1^2 = 2\log 2 - 1$$

$$= \log 4/e$$

Since $\log\lim_{n\to\infty} A = \lim_{n\to\infty}\log A = \log 4/e$

$\therefore \lim_{n\to\infty} A$, the limit $= 4/e$

Ex – 29 (ii), Page – 222: Given series, $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{n^2+r^2}$

$$= \lim_{n\to\infty} \frac{1}{(1+r^2)/n^2} \times \frac{1}{n}$$

$$= \int_0^1 \frac{1}{1+x^2}\,dx = \left\{\tan^{-1}x\right\}_0^1 = \pi/4$$

Ex – 29 (iii), Page – 222: Given series, $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^2}{n^3+r^3}$

$$= \lim_{n\to\infty} \sum \frac{(r/n)^2 \cdot 1/n}{1+(r/n)^3}$$

$$= \int_0^1 \frac{x^2dx}{1+x^3} = \left\{\frac{1}{3}\log(1+x^3)\right\}_0^1 = \frac{1}{3}\log 2$$

Ex – 29 (iv), Page – 222: Given series, $\lim_{n\to\infty} \sum_{r=0}^{n} \frac{n^2}{(n+r)^3}$

$$= \lim \sum \frac{1/n}{(1+r/n)^3}$$

$$= \int_0^1 \frac{dx}{(1+x)^3} = \left\{-\frac{1}{2(x+1)^2}\right\}_0^1$$

$$= -\frac{1}{2}\left[1/4 - \right]$$

$$= 3/8$$

Ex-28 (vii), Page-222: Given series,

$$\lim_{n\to\infty} \sum_{x=0}^{n-1} \frac{\sqrt{n^2-x^2}}{n^2}$$

$$= \lim \sum \sqrt{(1-x^2/n^2)} \times \frac{1}{n}$$

$$= \int_0^1 \sqrt{(1-x^2)}\, dx = \left[\frac{x}{2}\sqrt{(1-x^2)} + \frac{1}{2}\sin^{-1}x\right]_0^1$$

$$= \pi/4$$

Ex-29 (viii), Page-222: Given series

$$\lim_{n\to\infty} \sum_{x=1}^{n-1} \frac{1}{n}\sqrt{\left(\frac{n+x}{n-x}\right)}$$

$$= \lim \sum \sqrt{\left(\frac{1+x/n}{1-x/n}\right)} \times \frac{1}{n} = \int_0^1 \sqrt{\left(\frac{1+x}{1-x}\right)}\, dx$$

$$= \int_0^1 \frac{1+x}{\sqrt{(1-x^2)}}\, dx$$

$$= \int_0^1 \frac{dx}{\sqrt{(1-x^2)}} + \frac{1}{2}\int_0^1 \frac{2x\,dx}{\sqrt{1-x^2}}$$

$$= \left[\sin^{-1}x\right]_0^1 + \frac{1}{2}\left[(-1).\,2\sqrt{(1-x^2)}\right]$$

$$= \pi/2 + 1 = \frac{1}{2}(\pi + 2)$$

Ex – 13 ([illegible]), Page – [illegible] : $\lim\limits_{n \to \infty} \sum\limits_{r=1}^{n} \frac{n+r}{n^2+r^2}$

Let the given expression be A, then

$$\log A = \sum \log(1+r/n) \times \frac{1}{n}$$

$$\therefore \lim_{n \to \infty} \log A = \lim_{n \to \infty} \sum \log(1+r/n) \frac{1}{n}$$

$$= \int_0^1 1 . \log(1+x)\, dx$$

$$= \left\{ x \log(1+x) \right\}_0^1 - \int_0^1 \frac{x . 1}{1+x}\, dx$$

$$= \log 2 - \int_0^1 \frac{x+1-1}{1+x}\, dx = \log$$

$$= \log 2 - \int_0^1 1 . dx + \int_0^1 \frac{1}{1+x}\, dx$$

$$= \log 2 - \left\{ x \right\}_0^1 + \left\{ \log(1+x) \right\}_0^1$$

$$= \log 2 - 1 + \log 2 = 2 \log 2 - 1$$

$$= \log 4 - \log e = \log 4/e$$

Since $\log \lim A = \lim \log A = \log 4/e$

$$\therefore \lim_{n \to \infty} A = 4/e$$

Ex—13(XV), Page-282 :

~~Let $A = \{(1+\frac{1^2}{n^2})$~~

Ex—13(XVII), Page-282 :

Let $A = \left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$

$\therefore \lim\limits_{n\to\infty} \log A = \lim\limits_{n\to\infty} \sum \log\left(1+\frac{r^2}{n^2}\right)\frac{1}{n}$

$= \int_0^1 1\cdot\log(1+x^2)\,dx$

$= \left(x\log(1+x^2)\right)_0^1 - \int_0^1 \frac{x\cdot 2x}{1+x^2}\,dx$

$= \log 2 - 2\int_0^1 \frac{x^2+1-1}{1+x^2}\,dx$

$= \log 2 - 2\int_0^1 dx + 2\int_0^1 \frac{1}{1+x^2}\,dx$

$= \log 2 - 2(x)_0^1 + 2\left[\tan^{-1}x\right]_0^1 = \log 2 - 2 + \pi/2$

$= \log_e 2 + \frac{1}{2}(\pi-4)\log e$

$= \log\left\{2e^{1/2(\pi-4)}\right\}$

Since $\lim \log A = \log \lim A = \log\left\{2\cdot e^{1/2(\pi-4)}\right\}$

$\therefore \lim A = 2\cdot e^{1/2(\pi-4)}$

Ex-29 (ix), Page-222 : Given series,

$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n+r}{n^2+r^2}$$

$$= \lim \sum \frac{1+r/n}{1+r^2/n^2} \times \frac{1}{n} = \int_0^1 \frac{1+x}{1+x^2} dx$$

$$= \int_0^1 \frac{1}{1+x^2} dx + \frac{1}{2}\int_0^1 \frac{2x}{1+x^2} dx$$

$$= \left[\tan^{-1} x\right]_0^1 + \frac{1}{2}\left[\log(1+x^2)\right]_0^1$$

$$= \pi/4 + \frac{1}{2}\log 2$$

Ex-29 (x), Page-222 : Given series

$$\lim \sum \frac{n}{(n+r)\sqrt{2n+r}}$$

$$= \lim \sum \frac{1}{(1+r/n)\sqrt{r/n(2+r/n)}} \times \frac{1}{n}$$

$$= \int_0^1 \frac{1}{(1+x)\sqrt{x(2+x)}} dx \quad \left[\text{Put } 1+x = 1/t,\ dx = -1/t^2 dt\right]$$

$$= \int_1^{1/2} \frac{-1/t^2 dt}{1/t\sqrt{(1/t-1)(2+1/t-1)}}$$

$$= -\int_1^{1/2} \frac{dt}{\sqrt{(1-t)(t+1)}} = -\int_1^{1/2} \frac{dt}{\sqrt{1-t^2}}$$

$= -\left[\sin^{-1} y\right]_1^{1/2} = -\left(\sin^{-1}\frac{1}{2} - \sin^{-1} 1\right)$

$= \pi/3$

Ex-29(xi), Page-222 : Given series,

$$\lim_{n\to\infty} \sum_{x=0}^{2n} \frac{1}{n+x}$$

$$= \lim_{n\to\infty} \sum \frac{1}{1+x/n} \times \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum \frac{2}{1+2x/h} \times h \quad [\text{Putting } 1/2n = h]$$

$$= \int_0^1 \frac{2}{1+2x}\,dx = \left[\log(1+2x)\right]_0^1$$

$= \log 3$

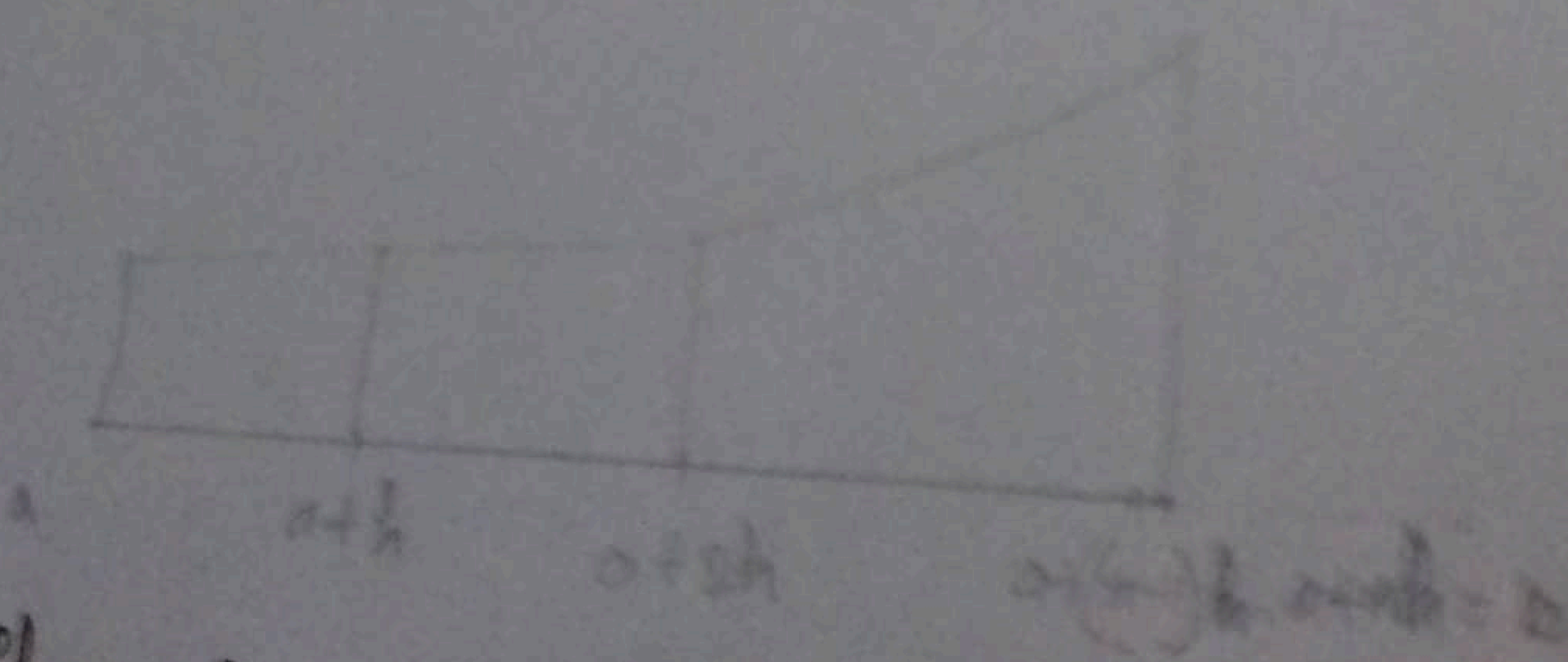
Define $\int_a^b f(x)\,dx$ / integration as the limit of a sum

Let $f(x)$ be a real valued continuous function in $[a, b]$, where $a<b$. Again let the interval $[a, b]$ be divided into n equals sub intervals, each of length h, by the points $a+h, a+2h, \cdots a+(n-1)h$

We now consider the sum

$$n\left[f(a)+f(a+h)+\cdots+f(a+h+n)\right]$$

$$= n\sum_{r=0}^{n-1} f(a+rh) \cdots (1)$$



Now if $n \to \infty$, then $h \to 0$.

Then (1) $\Rightarrow \lim_{h\to 0} \sum_{r=0}^{n+1} f(a+rh)$, the value of the limit is called the definite integral of $f(x)$ in $[a, b]$ and which is denoted by the symbol $\int_a^b f(x)\,dx$.

This $\int_a^b f(x)\,dx = \lim_{h \to 0} \sum_{r=0}^{n-1} f(a+rh)$, where $nh = b-a$ - (11)

if we put $a=0, b=1$, then $nh=1$

From equat. (11) $\Rightarrow \int_a^b f(x)$

Properties of definite integrals with example:-

(1) $\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$

(2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(3) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad a<c<b$

(4) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Exp:- $\int_0^{\pi/2} \sin x = \int_0^{\pi/2} \sin(\pi/2 - x)\,dx = \int_0^{\pi/2} \cos x\,dx$

(6) $\int_0^{na} f(x) = n\int_0^a f(x)\,dx$, if $f(a+x) = f(x)$

Exp:- $\int_0^{4\pi} \sin^2 x\,dx = 4\int_0^{\pi} \sin^2 x\,dx$

(7) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

(8) $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

Exp:- $\int_0^{\pi} \sin^2 x\,dx = \int_0^{2\cdot\pi/2} \sin^2 x\,dx = 2\int_0^{\pi/2} \sin^2 x\,dx$ $[\because \sin(\pi-x) = \sin x]$

Exp:- $\int_0^{\pi} \cos x\,dx = \int_0^{2\pi/2} \cos x\,dx = 0$ $[\because \cos(\pi-x) = -\cos x]$

(9) $\int_{-a}^{a} f(x)\,dx = \int_0^a \{f(x) + f(-x)\}\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(-x) = f(x) \\ 0, & \text{if } f(-x) = -f(x) \end{cases}$

Exp:- $\int_{-2}^{2} x^9(1-x^2)^7\,dx = \int_{-2}^{2} f(x)\,dx$, Here, $f(x) = x^9(1-x^2)^7$

$\therefore f(-x) = -x^9(1-x^2)^7 = -f(x)$

$= 0$

Exp - 1, Page - 229 :- Show that $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \pi/4$

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)}+\sqrt{\cos(\pi/2 - x)}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$\therefore 2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \pi/2$$

$$\therefore I = \frac{\pi}{4}$$

Exp - 2, Page - 229 :- Show that, $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \pi/8 \log 2$

Put $x = \tan\theta \;\therefore dx = \sec^2\theta\, d\theta$

when $x = 0$, $\theta = 0$ and $x = 1$, $\theta = \pi/4$

$$\therefore I = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta = \int_0^{\pi/4} \log\left\{1+\tan(\pi/4 - \theta)\right\} d\theta$$

Now, $1+\tan(\pi/4 - \theta) = 1 + \frac{1-\tan\theta}{1+\tan\theta} = \frac{2}{1+\tan\theta}$

$$\therefore I = \int_0^{\pi/4} \log \frac{2}{1+\tan\theta}\, d\theta = \int_0^{\pi/4} \left\{\log 2 - \log(1+\tan\theta)\right\} d\theta$$

$$= \int_0^{\pi/4} \log 2\, d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta.$$

$$= \pi/4 \cdot \log 2 - I$$

$$\therefore 2I = \pi/4 \log 2$$

$$\Rightarrow I = \pi/8 \log 2$$

<u>Ex-2, Page-237:</u> $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx \cdots (i)$

Also, $I = \int_0^{\pi/2} \frac{\sin(\pi/2 - x)\, dx}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}$ [By prop, (4)]

$$= \int_0^{\pi/2} \frac{\cos x\, dx}{\cos x + \sin x} \cdots (ii)$$

Adding (i) and (ii) we get,

$$2I = \int_0^{\pi/2} \left(\frac{\sin x}{\sin x + \cos x} + \frac{\cos x}{\cos x + \sin x}\right) dx$$

$= \int_0^{\pi/2} dx = [x]_0^{\pi/2}$ and hence $I = \pi/4$

(Showed)

Ex- 7, Page - 237 :- $I = \int_0^{\pi} x \log \sin x \, dx$

Also, $I = \int_0^{\pi} (\pi - x) \log \sin x \, dx$

$\therefore 2I = \int_0^{\pi} \pi \log \sin x \, dx \Rightarrow I = \frac{\pi}{2} \int_0^{\pi} \log \sin x \, dx$

$= \pi/2 . 2 \int_0^{\pi/2} \log \sin x \, dx$

By the (vii) Prop. as $\log \sin(\pi - x) = \log \sin x$

$= \pi . [\pi/2 \log 1/2]$

$= \pi^2/2 \log \frac{1}{2}$ (showed)

Ex- 13, Page - 238 : $I = \int_0^{1} \log \sin \frac{\pi}{2} \theta \, d\theta$,

put $\pi/_2 \theta = x \Rightarrow dx = \pi/_2 d\theta$

$\therefore I = \frac{\pi}{2} \int_0^{\pi/2} \log \sin x \, dx = \frac{2}{\pi} \left(\frac{\pi}{2} \log \frac{1}{2} \right)$

$= \log \frac{1}{2}$ (showed)

Ex-15, Page-238:- $I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$

Also $I = \int_0^{\pi/4} \log\left[1+\tan\left(\pi/4 - \theta\right)\right] d\theta$

$= \int_0^{\pi/4} \log\left[1 + \frac{1-\tan\theta}{1+\tan\theta}\right] d\theta$

$= \int_0^{\pi/4} \log\left[\frac{2}{1+\tan\theta}\right] d\theta$

$\therefore 2I = \int_0^{\pi/4} \log(1+\tan\theta) \frac{2}{1+\tan\theta}\, d\theta$

$= \log 2 \int_0^{\pi/4} d\theta = \pi/4 \log 2$

$\therefore I = \pi/8 \log 2$ (showed)

Ex-20(1), Page-239:- $I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\, dx$

Also $I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\, dx$ [By prop. (iv)]

$\therefore 2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$

put $\cos x = t \Rightarrow dt = -\sin x\, dx$

$2I = \frac{1}{\sqrt{2}} \left[\log\left\{\operatorname{cosec}\left(\pi/4 + x\right) - \cot\left(\pi/4 + x\right)\right\}\right]_0^{\pi/2}$

$$2I = -\pi \int_{1}^{-1} \frac{dt}{1+t^2} = -\pi \left[\tan^{-1} t\right]_{1}^{-1}$$

$$= -\pi \left[-\pi/4 - \pi/4\right] = \frac{\pi^2}{2}$$

$$\therefore I = \pi^2/4$$

(showed)

Ex- 20(ii), Page - 238 : $I = \int_0^{\pi/2} \frac{\sin^2 x \, dx}{\sin x + \cos x}$

Also $I = \int_0^{\pi/2} \frac{\cos^2 x \, dx}{\sin x + \cos x}$ [By prop. (iv)]

$$\therefore 2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx$$

$$= \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{1/\sqrt{2} \sin x + 1/\sqrt{2} \cos x}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\cos \pi/4 \sin x + \sin \pi/4 \cos x}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \operatorname{cosec}(\pi/4 + x) \, dx$$

$$2I = \frac{1}{\sqrt{2}} \left[\log \left\{ \operatorname{cosec}(\pi/4 + x) - \cot(\pi/4 + x) \right\}\right]_0^{\pi/2}$$

$$2I = \frac{1}{\sqrt{2}}\left\{\log\left(\operatorname{Cosec} \pi/4 - \operatorname{Cot} 2\pi/4\right) - \log\left(\operatorname{Cosec} \pi/4 - \operatorname{Cot} \pi/4\right)\right\}$$

$$2I = \frac{1}{\sqrt{2}}\left[\log \frac{\sqrt{2}+1}{\sqrt{2}-1}\right] = \frac{1}{\sqrt{2}} \log \frac{(\sqrt{2}+1)^2}{(\sqrt{2})^2-1}$$

$$2I = \frac{1}{\sqrt{2}} \log \frac{(\sqrt{2}+1)^2}{1} = \frac{2}{\sqrt{2}} \log(1+\sqrt{2})$$

$$\therefore I = \frac{1}{\sqrt{2}} \log(1+\sqrt{2})$$

(Showed)

Ex-20(iii), Page-238: $I = \int_0^\pi \frac{x \tan x}{\sec x + \tan x}$

$$= \int_0^\pi \frac{x \sin x}{1+\sin x}\, dx$$

Also $I = \int_0^\pi \frac{(\pi - x)\sin x}{1+\sin x}\, dx$ [By prop. (iv)]

$$\therefore 2I = \pi \int_0^\pi \frac{\sin x + 1 - 1}{1+\sin x}\, dx$$

$$= \pi \int_0^\pi dx - \pi \int_0^\pi \frac{1}{1+\sin x}\, dx$$

$$= \pi [x]_0^\pi - \pi \int_0^\pi \frac{1}{\cos^2 x/2 + \sin^2 x/2 + 2\sin x/2 \cos x/2}\, dx$$

$$= \pi^2 - \pi \int_0^\pi \frac{\sec^2 x/2\, dx}{(1+\tan x/2)^2}$$

put $1+\tan x/2 = t \Rightarrow dt = \frac{1}{2}\sec^2 x/2 \, dx$

$$2I = \pi^2 - 2\pi \int_1^{\infty} \frac{dt}{t^2} = \pi^2 - 2\pi \left[-\frac{1}{t}\right]_1^{\infty}$$

$$= \pi^2 - 2\pi$$

$$\therefore I = \frac{1}{2}(\pi^2 - 2\pi) = \pi/2 (\pi - 2)$$

(showed)

Ex- 20 (iv), Page- 238 : $I = \int_0^{\pi/2} \frac{x\,dx}{\sin x + \cos x}$

Also, $I = \int_0^{\pi/2} \frac{\pi - x}{\cos x + \sin x} dx$

$$\therefore 2I = \pi/2 \int_0^{\pi/2} \frac{dx}{\cos x + \sin x}$$

$$\Rightarrow I = \pi/4 \int_0^{\pi/4} \frac{dx}{\sin x + \cos x}$$

$$= \pi/4 \cdot \frac{2}{\sqrt{2}} \log(1+\sqrt{2})$$

$$= \frac{\pi}{2\sqrt{2}} \log(1+\sqrt{2})$$

(showed)

Ex-20(v), Page-238:- $I=\int_0^{\pi/2}\frac{x\,dx}{\sec x+\operatorname{cosec} x}$

$$I=\int_0^{\pi/2}\frac{\pi/2-x}{\sec x+\operatorname{cosec} x}\,dx$$

$$2I=\pi/2\int_0^{\pi/2}\frac{dx}{\sec x+\operatorname{cosec} x}=\pi/2\int_0^{\pi/2}\frac{\sin x\cos x}{\sin x+\cos x}\,dx$$

$$=\pi/4\int_0^{\pi/2}\frac{\sin 2x}{\sqrt{(1+\sin 2x)}}\,dx=\pi/4\int_0^{\pi/2}\frac{\sin 2x+1-1}{\sqrt{\sin 2x+1}}\,dx$$

$$=\pi/4\int_0^{\pi/2}\sqrt{(1+\sin 2x)}\,dx-\pi/4\int_0^{\pi/2}\frac{dx}{\sqrt{1+\sin 2x}}$$

$$=\pi/4\int_0^{\pi/2}(\sin x+\cos x)\,dx-\pi/4\int_0^{\pi/2}\frac{dx}{\sin x+\cos x}$$

Now $\int_0^{\pi/2}(\sin x+\cos x)\,dx$

$$=\left[-\cos x+\sin x\right]_0^{\pi/2}=[1+1]=2$$

and $\int_0^{\pi/2}\frac{dx}{\sin x+\cos x}=\frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}$

$$=\frac{1}{\sqrt{2}}\log\frac{(\sqrt{2}-1)^2}{\{(\sqrt{2})^2-1\}}=-\frac{1}{\sqrt{2}}\log(\sqrt{2}-1)^2$$

$$= -\frac{2}{\sqrt{2}}\log(\sqrt{2}-1)$$

$$\therefore 2I = \pi/4\left[2+\frac{2}{\sqrt{2}}\log(\sqrt{2}-1)\right]$$

$$\Rightarrow I = \pi/4\left[1+\frac{1}{\sqrt{2}}\log(\sqrt{2}-1)\right]$$

(showed)

Ex - 20(viii), Page - 239 :- $I=\int_0^{\pi/4}\frac{x\,dx}{1+\cos 2x+\sin 2x}$

Also $I=\int_0^{\pi/4}\frac{(\pi/4-x)\,dx}{1+\sin 2x+\cos 2x}$

$$\therefore 2I = \pi/4\int_0^{\pi/4}\frac{dx}{1+\sin 2x+\cos 2x}$$

$$I = \pi/8\int_0^{\pi/4}\frac{\sec^2 x\,dx}{(1+\tan^2 x)+2\tan x+(1-\tan^2 x)}$$

$$= \pi/8\int_0^{\pi/4}\frac{\sec^2 x\,dx}{2+2\tan x}$$

Put $1+\tan x = t$, $\sec^2 x\,dx = dt$

$$\therefore I = \pi/8\times\frac{1}{2}\int_1^2\frac{dt}{t} = \pi/16\left[\log t\right]_1^2$$

$$= \pi/16\log 2$$ (showed)