COMP 598

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Question 1

First, prove F is not regular:
First, define $F' = \overline{F} \cap ab^*c^* = \{ab^jc^k \mid j, k \geq 0 \land j \neq k\}$
$F'' = \overline{F'} \cap ab^*c^* = \{ab^ic^i \mid i \geq 0\}$ since a is regular but $\{a^nb^n | n \geq 0\}$ is not regular by pumping lemma (proved in previous assignment & lecture notes). Then combine together, By clousre properties of regular languages, F'' is not regular. Similarly, F is not regular.
Secondly, to show that F satisfies the pumping lemma:
Pick $p = 1$. So $p \geq 0$ holds.
We choose $p = 2$. So $\forall w \in L$ such that $|w| \geq p$, $w = xyz$ where $|y| \geq 1$ and $|xy| \leq p$, four cases:
$i = 0$, so $w = b^jc^k$. Then $y = b$ or c is the first letter of the word, and $xy^nz = b^nb^{j-1}c^k \in L$ or $xy^nz = c^nc^{k-1} \in L$.
$i = 1$, so $j = k$, $w = ab^jc^j$. Pick $y = a$ & $x = \epsilon$. $w = a^nb^jc^j \in L$.
$i = 2$, so $w = a^2b^jc^k$. Let $y = aa$ & $x = \epsilon$. $w = (aa)^nb^jc^k \in L$.
$i > 2$, so $w = a^ib^jc^k$. Pick $y = a$ & $x = \epsilon$. $w = a^na^{i-1}b^jc^k \in L$.
And pumping lemma just says that all regular language can be pumped, not all languages that satisfy the pumping lemma are regular. So it's not a contradiction.

Question 2

2.1

False.

Suppose $A = \{\epsilon\}$. Clearly, $A \subseteq B$ but B is not necessarily regular which means it can be any non-regular language.

2.2

False.
let $A = \{a^*\} \subseteq \Sigma^*$ be a regular language. Clearly, $B = \{a^{2^n} \mid n \geq 0\} \subseteq \Sigma^*$ is a non-regular language. But AB and A are regular. So counterexample!

2.3

False.
Let $\Sigma = \{0,1\}$. Define $A_1 = \{01\}$, $A_2 = \{0011\}$... So we can conclude $A_i = \{0^i 1^i | i \geq 0\}$. Clearly, for $\forall i \geq 0$,A_i is regular.
But, $\bigcup_{i=1}^{\infty} A_i = \{0^i 1^i | i \geq 0\}$ is clearly not regular (proved in previous assignment & lecture notes).

2.4

False.
$A = \{0^i 1^i | i \geq 0\}$ is not regular, but $\{ab\} \subseteq A$ is regular.

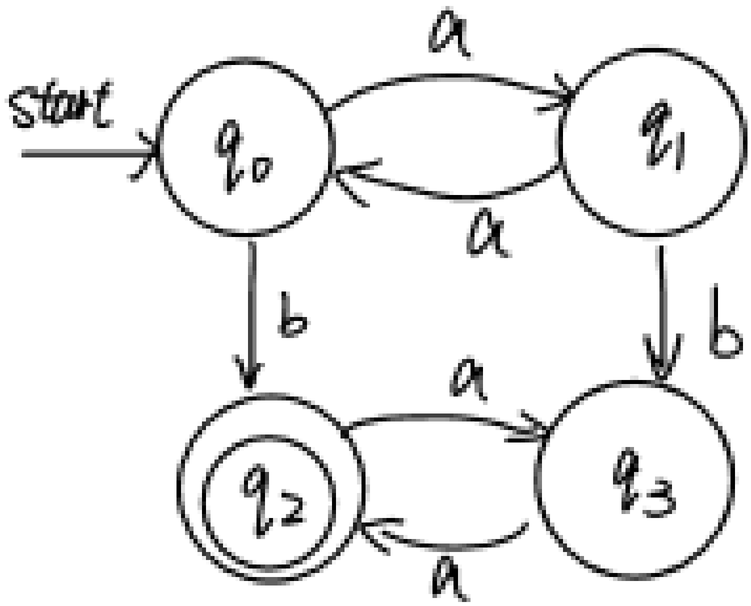
Question 3

Define (S, s_0, F, δ) as a DFA which recognizes L. In order to construct a NFA (Q, Q_0, F', Δ) that recognizes $CYC(L)$.

Let's define $Q = \underbrace{S}_{\text{tracks } v} \times \overbrace{S \times S}^{\text{tracks start and end of } u} \times \underbrace{\{0,1\}}_{\text{if we are reading } u \text{ or } v}$

Define $Q_0 = \{(s_0, s, s, 0)\}$ and $F' = \{(s, t, s, 1) \mid t \in F\}$
If $b = 0$ which means we are reading u & $\delta(t,a) \notin F$ then $\Delta((s,t,t_c,b),a) = \{(s,t',t_c,b) \mid t' = \delta(t,a)\}$.
If $b = 0$ & $\delta(t,a) \in F$ then
$\Delta((s,t,t_c,b),a) = \{(s,t',t_c,b) \mid t' = \delta(t,a)\} \cup \{(s',t',t_c,b') \mid b' = 1, t' = \delta(t,a)\}$
And if $b = 1$ which means we are reading v then $\Delta((s,t,t_c,b),a) = \{(s',t,t_c,b) \mid s' = \delta(s,a)\}$.

This will always works since the end state of v will be the start state of u. And u will end in an accepting state. So $vu \in L$ holds.
Example: Let $\Sigma = \{a, b\}$, $L = \{a^n b a^n \mid n \geq 0\}$. This is clearly non-regular. Since we can never cut b. $CYC(L)$ can just move a from the front to the back or vice vesa.
So $CYC(L) = \{a^n b a^m \mid n + m \text{ is divisible by 2 since } n + n = 2n\}$. It is regular since it can be recognized by a NFA:



Question 4

To show $\equiv_L$ is not a congruence relation. Here comes a counterexample: Consider the finite language $L = \{aa, ac, ba, bc, cb, db\}$. Clearly, $a \sim b$,$c \sim d$. But $ac \not\sim bd$ because $bd \notin L$ but $ab \in L$
To show $\approx_L$ is a congruence relation:
Assume $m \approx_L n$ and $x \approx_L y$.
Since $\forall u, v \in \Sigma^*, umv \in L \iff unv \in L$. Pick $v = xw$, then we will have

$\forall u, w \in \Sigma^*, umxw \in L \iff unxw \in L$.
Similarly for $x \approx_L y$: $\forall u, w \in \Sigma^*, unxw \in L \iff unyw \in L$.
Finally, combine above together, we will get $\forall u, w \in \Sigma^*, umxw \in L \iff unyw \in L$ and this means $mx \approx_L ny$.

Question 5

5.1

Not equivalent
Define $TS = (\phi, \neg\psi) \to (\neg\phi, \neg\psi) \to (\neg\phi, \neg\psi) \to (\neg\phi, \neg\psi) \to \cdots$ Then $TS \models (\Box\Diamond\phi \implies \Box\Diamond\psi)$ holds since the left side of the arrow is false. But, $TS \not\models \Box(\phi \implies \Diamond\psi)$. So not equivalent.

5.2

Not equivalent
Define TS as a transition system where $\phi \wedge \neg\psi$ for all odd steps and $\neg\phi \wedge \psi$ for all even steps.
$TS = (\phi \wedge \neg\psi) \to (\neg\phi \wedge \psi) \to (\phi \wedge \neg\psi) \to (\neg\phi \wedge \psi) \to \cdots$
Then $TS \models \Diamond\phi \wedge \Diamond\psi$ holds. But, $TS \not\models \Diamond(\phi \wedge \psi)$ since in no steps ϕ and ψ both true holds.

5.3

Equivalent
$\bigcirc\Diamond\phi \equiv \bigcirc(\text{ true } U\phi)$
$\equiv (\bigcirc \text{ true })U(\bigcirc\phi)$
$\equiv \text{ true } U(\bigcirc\phi)$
$\equiv \Diamond \bigcirc \phi$
Proved!

Question 6

6.1

Define a formula ϕ as the description of $odd(p)$.

Then we will get $(\phi)_n = p \wedge \bigcirc \bigcirc p \wedge \bigcirc \bigcirc \bigcirc \bigcirc p \wedge ... \wedge \bigcirc^{2n-2}p$ since it satisfies the following: $(\phi)_0 =$ true
$(\phi)_1 = p$
$(\phi)_2 = p \wedge \bigcirc \bigcirc p$
$(\phi)_3 = p \wedge \bigcirc \bigcirc p \wedge \bigcirc \bigcirc \bigcirc \bigcirc p$
...

So we can conclude it as:$(\phi)_n = p \wedge \bigcirc \bigcirc p \wedge \bigcirc \bigcirc \bigcirc \bigcirc p \wedge \cdots \wedge \bigcirc^{2n-2}p$

Rewritten it by using the fixed point operator, Then it will be: $vX.p \wedge \bigcirc \bigcirc X$

6.2

Rewritten it as $\mu X.p \wedge \bigcirc \bigcirc X$ Define a formula $(\heartsuit p)$ to describe it:
$(\heartsuit p)_0 =$ false
$(\heartsuit p)_1 = p \wedge \bigcirc \bigcirc$ false $=$ false
So this formula is just false. So it's false.

6.3

Since after a p, there immediately comes a $\neg p$. And immediately after a $\neg p$, there comes a p.
So clearly, this formula **is** saying that p does **not** hold in any of the even steps.

Question 7

Let ψ_1,ψ_2 be any LTL formula that does not contain any next operators.
First, Define $\phi = \psi_1 U \psi_2$. Since ψ_1,ψ_2 are both LTL formula. So ϕ is also a LTL formula. ϕ holds true on σ if and only if there exists $j \geq 0$ such that $\sigma[j..] \models \psi_2$ and $\forall i < j$ $\sigma[i..] \models \psi_1$ holds.
Since the truth value of ϕ depends on the existence of a state that satisfy ψ_2 after step $j \geq 0$ where all state before that satify ψ_1. So no matter how many U operators we add to ϕ, the requirement for the "true of false" value of ϕ stays the same result. It always holds for some fixed $j \geq 0$.
Here comes an example, let a and b be some states, all these following three cases are the same(the difference is how many a states before the b state):

$(((a) \to (b) \to (a) \to (a) \to \cdots) \models \phi) \equiv (((a) \to (a) \to (a) \to (b) \to (a) \to (a) \to \cdots) \models \phi)$
$\equiv (((a) \to \cdots \to (a) \to (b) \to (a) \to (a) \to \cdots) \models \phi)$
Since other LTL operators such as $\diamond$ and $\Box$ they can be written as U and U satisfies above cases, $\diamond$ and $\Box$ also satisfy these cases. So,in a LTL formula ϕ with $U, \diamond$ and $\Box$ these operations only, ϕ has the same truth value on every σ_i for $i \geq 0$.
Secondly, define $\phi = \bigcirc^i \psi$. ϕ is true on σ if and only if $\sigma[i..] \models \psi$ So we can see the truth value of ϕ only depends on $\sigma[i]$.
So in a LTL formula, if ϕ is with only $\bigcirc$ operators, clearly ϕ has the same result on every σ_j for $j > i$ since ϕ only depends on $\sigma[i]$ and $\sigma[i]$ remains unchanged with variation of j.
Combine above together, a proposition p and any LTL formula ϕ containing n next operators, the fomula ϕ has the same truth value on every σ_i with $i > n$.
Clearly, $odd(p)$ cannot be expressed in LTL. We will prove it by contradiction. Assume it can be expressed in LTL ϕ. Then ϕ must have a finite number of next operators. Assume ϕ has n next operators. Then, by the statement we proved earlier, σ_i is true for all $i > n$ which is a contradiction since one of σ_{n+1} and σ_{n+2} must be true, and the other one must be false.
So proved!

Question 8

8.1

True
$L_\omega((E_1 + E_2) \cdot F^\omega) \equiv L_\omega(E_1 + E_2) \cdot L_\omega(F^\omega)$
$\equiv (L_\omega(E_1) \cup L_\omega(E_2)) \cdot L_\omega(F^\omega)$
$\equiv \{ab \mid a \in (L_\omega(E_1) \cup L_\omega(E_2)) \wedge b \in L_\omega(F^\omega)\}$
$\equiv \{ab \mid a \in L_\omega(E_1) \wedge b \in L_\omega(F^\omega)\} \cup \{ab \mid a \in L_\omega(E_2) \wedge b \in L_\omega(F^\omega)\}$
$\equiv L_\omega(E_1 \cdot F^\omega) \cup L_\omega(E_2 \cdot F^\omega)$
$\equiv L_\omega(E_1 \cdot F^\omega + E_2 \cdot F^\omega)$

8.2

False

Let $E = \epsilon, F_1 = 0$ and $F_2 = 1$.
Then 01010101010101... is recognized by $E \cdot (F_1 + F_2)^\omega$ but is not recognized by $\notin E \cdot F_1^\omega + E \cdot F_2^\omega$.

8.3

False
Let $E = 0$ and $F = 1$.
Then 0101010101010101... is recognized by $(E^* \cdot F)^\omega$ but is not recognized by $E^* \cdot F^\omega$.

Question 9

9.1

$t_0 \models [a]\langle b\rangle$ true , $s_0 \not\models [a]\langle b\rangle$ true
Which is Similar as:
$t_0 \not\models \langle a\rangle\neg(\langle b\rangle$ true), $s_0 \models \langle a\rangle\neg(\langle b\rangle$ true)
The formula says that if we take an action a at t_0, we can always take an action b after. But that doesn't hold for state s_0 since we can do nothing at the state s_3.

9.2

To prove that both t_0 & s_0 agree on all the formulas of the negation-free fragment. we will do this by induction.
Base case: not take any action yet, then $t_0 \models$ true , $s_0 \models$ true
Then, take an action a :$t_0 \models \langle a\rangle$ true , $s_0 \models \langle a\rangle$ true
Then, one more action b: $t_0 \models \langle a\rangle\langle b\rangle$ true , $s_0 \models \langle a\rangle\langle b\rangle$ true
Similarly, $t_0 \not\models \langle b\rangle\langle a\rangle$ true , $s_0 \not\models \langle b\rangle\langle a\rangle$ true holds
Then by induction, they all should agree on any boolean combination of these formulas, for example, $t_0 \models (\langle a\rangle\text{true}) \wedge (\langle a\rangle\langle b\rangle\text{true})$, $s_0 \models (\langle a\rangle\text{true}) \wedge (\langle a\rangle\langle b\rangle\text{true})$
And $t_0 \models (\langle a\rangle\text{true}) \vee (\langle a\rangle\langle b\rangle\text{true})$, $s_0 \models (\langle a\rangle\text{true}) \vee (\langle a\rangle\langle b\rangle\text{true})$ also holds.
Then both t_0 & s_0 agree on all the formulas of the negation-free fragment. So we proved!

Question 10