

Stochastic system consideration for EDMD

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1 Notation introduction

A list of all the notation and relevant definition:

1. (Autonomous) SDE in the dynamical system (Ω, μ) :

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

2. Φ^t : Transformation map/flow
3. m : Number of (i.i.d.) data points
4. $n, t = \frac{1}{n}$: sampling frequency and time step between two data snapshots
5. N : Dictionary size
6. $\Psi = \{\psi_i\}_{i=1}^N$: Dictionary or set of basis functions
7. $\mathcal{F} := L^2(\Omega, \mu)$, $\mathcal{F}_N := \text{span}\{\psi_1, \dots, \psi_N\} \subset \mathcal{F}$
8. \mathcal{P}_N : Projection onto the subspace \mathcal{F}_N
9. Data(i.i.d.) snapshots:

$$X = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_m \\ | & & | \end{bmatrix}, \quad Y = \begin{bmatrix} | & & | \\ y_1 & \cdots & y_m \\ | & & | \end{bmatrix}$$

10. (Row)Vector of basis functions:

$$\Psi_N(x) := [\psi_1(x), \dots, \psi_N(x)]$$

11. (Empirical)Matrices of dictionary $\{\psi_j\}_{j=1}^N$ evaluated on data $\{x_i, y_i\}_{i=1}^m$:

$$\Psi_X = \begin{bmatrix} \text{---} & \Psi_N(x_1) & \text{---} \\ & \vdots & \\ \text{---} & \Psi_N(x_m) & \text{---} \end{bmatrix} = \begin{bmatrix} \psi_1(x_1) & \cdots & \psi_N(x_1) \\ \vdots & \ddots & \vdots \\ \psi_1(x_m) & \cdots & \psi_N(x_m) \end{bmatrix}$$

$$\Psi_Y = \begin{bmatrix} \text{---} & \Psi_N(y_1) & \text{---} \\ & \vdots & \\ \text{---} & \Psi_N(y_m) & \text{---} \end{bmatrix} = \begin{bmatrix} \psi_1(y_1) & \cdots & \psi_N(y_1) \\ \vdots & \ddots & \vdots \\ \psi_1(y_m) & \cdots & \psi_N(y_m) \end{bmatrix}$$

12. Definition of Koopman operator in deterministic system:

$$\mathcal{K}f = f \circ \Phi$$

where Φ is the transformation map.

Definition of stochastic Koopman operator (semigroup):

$$\mathcal{K}(t)f = \mathbb{E}[f \circ \Phi^t]$$

where Φ^t is the flow map.

Remark: In the deterministic system, we can still use \mathcal{K}^t, Φ^t where t is the (fixed) time interval between two sampling snapshots.

13. Definition of Koopman generator:

$$\begin{aligned} \mathcal{L}f(x) &:= \lim_{t \rightarrow 0} \frac{[\mathcal{K}^t f(x)] - f(x)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\mathbb{E}[f \circ \Phi^t(x)] - f(x)}{t} \end{aligned} \quad (1)$$

EDMD framework review

Let $f \in \text{span}\{\psi_i\}_{i=1}^N$ (Notice that $f \circ \Phi$ is not necessarily in $\text{span}\{\psi_i\}_{i=1}^N$). Then

$$\mathcal{K}f(x) = (f \circ \Phi)(x) = \Psi_N(x)Ka + r(x)$$

where $r(x)$ is the residual. Notice that the span of $\{\psi_1, \dots, \psi_N\}$ is not necessarily invariant under \mathcal{K} .

In the EDMD framework, given a dataset $\{(x_i, y_i)\}_{i=1}^m$, we want to minimize the residual over the finite data set:

$$J := \sum_{i=1}^m |r(x_i)|^2 = \sum_{i=1}^m |(\Psi_N(y_i) - \Psi_N(x_i)K)\mathbf{a}|^2,$$

so, this is equivalent to $\min_K \|\Psi_Y - \Psi_X K\|_F$ and the minimal K is

$$K = K_{N,m} = \Psi_X^\dagger \Psi_Y = (\Psi_X^T \Psi_X)^\dagger (\Psi_X^T \Psi_Y) = G^\dagger A$$

where \dagger is the pseudoinverse and $G = \frac{1}{m} \Psi_X^T \Psi_X$, $A = \frac{1}{m} \Psi_X^T \Psi_Y$.

Remark. *Regularization through truncated SVD or by adding a small perturbation is typically applied.*

S-EDMD framework

First, we can expand the Koopman operator (semigroup) $\mathcal{K}(t)$ similar as in EDMD:

$$\begin{aligned}\mathcal{K}(t)f(x) &= \mathbb{E}[f \circ \Phi^t(x)] \\ &= \Psi_N(x)K(t)\mathbf{a} + r(x)\end{aligned}\quad (2)$$

Alternatively, we approximate the Koopman operator(semigroup) $\mathcal{K}(t)$ analogous to Taylor expansion using the definition of generator(1):

$$\begin{aligned}\mathcal{K}(t)f(x) &= \mathbb{E}[f \circ \Phi^t(x)] \\ &= \mathbb{E}[\Psi_N \circ \Phi^t(x)]\mathbf{a} \\ &= [\Psi_N(x) + t \cdot \mathcal{L}\Psi_N(x) + O(t^2)]\mathbf{a}\end{aligned}\quad (3)$$

where $\mathcal{L}\Psi_N(x) = [\mathcal{L}\psi_1(x), \dots, \mathcal{L}\psi_N(x)]$ and

$$\mathcal{L}\psi_j(x) = b(x) \cdot \nabla \psi_j(x) + \frac{1}{2}\sigma^2(x) : \nabla^2 \psi_j(x) \quad (4)$$

Remark. We can get explicit higher order term $O(t^2)$ in (3) by applying Itô's formula recursively. See A for more details. Or we can simply apply Taylor expansion here to obtain $O(t^2)$.

Remark. In order to obtain (4), we can use Itô's formula. From Itô's formula, we know that given $f \in C_b^2(\Omega)$ and $X_0 = x$,

$$df(X_t) = \left(b(X_t) \cdot \nabla f(X_t) + \frac{1}{2}\sigma^2(x) : \nabla^2 f(X_t) \right) dt + \sigma(X_t) \cdot \nabla f(X_t) dW_t$$

Taking expectation on its integral form, we have:

$$\mathbb{E}[f(X_t)|X_0 = x] - f(x) = \int_0^t \left(b(X_s) \cdot \nabla f(X_s) + \frac{1}{2}\sigma^2(X_s) : \nabla^2 f(X_s) \right) ds$$

Let $t \rightarrow 0$, we have

$$\begin{aligned}\mathcal{L}f(x) &:= \lim_{t \rightarrow 0} \frac{\mathbb{E}[f(X_t) - f(x)|X_0 = x]}{t} \\ &= b(X_t) \cdot \nabla f(X_t) + \frac{1}{2}\sigma^2(x) : \nabla^2 f(X_t)\end{aligned}$$

In our case, combining (3) and (2), we can minimize the following:

$$J := \sum_{i=1}^m |r(x_i)|^2 = \sum_{i=1}^m \left| [\Psi_N(x_i) + t \cdot \mathcal{L}\Psi_N(x_i) + O(t^2) - \Psi_N(x_i)K(t)]\mathbf{a} \right|^2$$

which is equivalent to minimize

$$J = \min_{K(t)} \|\Psi_X + t \cdot \mathcal{L}\Psi_X + O(t^2) - \Psi_X K(t)\|_F^2 \quad (5)$$

and the minimal $K(t)$ is

$$K(t) = K_{N,n,m}(t) = \Psi_X^\dagger(\Psi_X + t \cdot \mathcal{L}\Psi_X + O(t^2)) \quad (6)$$

where the generator(time differential) matrix approximation $[\mathcal{L}\Psi_X]_{ij} = \mathcal{L}\psi_j(x_i)$.

Remark. $O(t^2)$ in (7), (3) and (5) are different. They are scalar, vector and matrix respectively. For example, each element of $O(t^2)$ in (3) and (5) is a scalar $O(t^2)$ in (7).

Empirically, if t is very small, we update the Koopman matrix approximation in the following way with $O(t^2)$ omitted:

$$\widehat{K}_{N,n,m}(t) = \widehat{\Psi}_X^\dagger(\widehat{\Psi}_X + t \cdot \widehat{\mathcal{L}}\widehat{\Psi}_X)$$

Remark. For the case of trained basis by NN, we replace $\Psi_X + t \cdot \mathcal{L}\Psi_X + O(t^2)$ by Ψ_Y in (5),

2 Convergence in the limit of large data

We fix the dictionary size N and sampling time t (sampling frequency $n = \frac{1}{t}$). Let $K_{N,n}$ denote the matrix that represents the large-data limit of $\widehat{K}_{N,n,m}$.

(1) i.i.d. data (middle 3 steps in the following has confusing notations. You can ignore and move to the last equality):

$$\begin{aligned}
& P\left(\|\widehat{K}_{N,n,m} - K_{N,n}\|_F > \epsilon\right) \\
&= P\left(\|\widehat{\Psi}_X^\dagger(\widehat{\Psi}_X + t\widehat{\mathcal{L}}\widehat{\Psi}_X) - \Psi_X^\dagger(\Psi_X + t\mathcal{L}\Psi_X + O(t^2))\|_F > \epsilon\right) \\
&= P\left(\|t\widehat{\Psi}_X^\dagger\widehat{\mathcal{L}}\widehat{\Psi}_X - t\Psi_X^\dagger\mathcal{L}\Psi_X + O(t^2)\|_F > \epsilon\right) \\
&\leq P\left(t\|\widehat{\Psi}_X^\dagger\widehat{\mathcal{L}}\widehat{\Psi}_X - \Psi_X^\dagger\mathcal{L}\Psi_X\|_F + O(t^2) > \epsilon\right) \\
&= P\left(t\|\widehat{G}^{-1}\widehat{A} - G^{-1}A\|_F + O(t^2) > \epsilon\right)
\end{aligned}$$

where $\widehat{G} = \frac{1}{m}\Psi_X^\top\Psi_X$, $\widehat{A} = \frac{1}{m}\Psi_X^\top\mathcal{L}\Psi_X$ and $G = \lim_{m \rightarrow \infty} \frac{1}{m}\Psi_X^\top\Psi_X$, $A = \lim_{m \rightarrow \infty} \frac{1}{m}\Psi_X^\top\mathcal{L}\Psi_X$, $K_{N,n}$ is the Koopman matrix approximation constructed by sufficient large amount of data.

A function $f : \mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_n \rightarrow \mathbb{R}$ satisfies the *bounded differences property* if substituting the value of the i th coordinate x_i changes the value of f by at most c_i . More formally, if there are constants c_1, c_2, \dots, c_n such that for all $i \in [n]$, and all $x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2, \dots, x_n \in \mathcal{X}_n$,

$$\sup_{x'_i \in \mathcal{X}_i} |f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)| \leq c_i.$$

Lemma 1 (McDiarmid's Inequality). *Let $f : \mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_n \rightarrow \mathbb{R}$ satisfy the bounded differences property with bounds c_1, c_2, \dots, c_n . Consider independent random variables X_1, X_2, \dots, X_n where $X_i \in \mathcal{X}_i$ for all i . Then, for any $\epsilon > 0$,*

$$\mathbb{P}(|f(X_1, X_2, \dots, X_n) - \mathbb{E}[f(X_1, X_2, \dots, X_n)]| \geq \epsilon) \leq 2 \exp\left(-\frac{2\epsilon^2}{\sum_{i=1}^n c_i^2}\right).$$

Proof. Omitted. \square

Lemma 2 (Hoeffding's Inequality). *Assume $X_i \in [a_i, b_i]$. Then, for any $\epsilon > 0$,*

$$\mathbb{P}\left(\sum_{i=1}^n X_i - \mathbb{E}\left[\sum_{i=1}^n X_i\right] \geq \epsilon\right) \leq e^{-\frac{2\epsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}}$$

and

$$\mathbb{P}\left(\left|\sum_{i=1}^n X_i - \mathbb{E}\left[\sum_{i=1}^n X_i\right]\right| \geq \epsilon\right) \leq 2e^{-\frac{2\epsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}}$$

Proof. The McDiarmid's Inequality(1) directly implies Hoeffding's inequality by taking $f(X_1, \dots, X_n) = \sum_{i=1}^n X_i$. \square

Theorem 3. *Let $|\psi_i(X_k)| \leq C$ \mathbb{P} -a.e. for all $1 \leq i \leq N$ and $1 \leq k \leq m$. Suppose $\|\mathcal{L}_{N,n,m}\| \leq L$. Then, for any $\epsilon > 0$,*

$$\begin{aligned}\mathbb{P}\left(\left\|\widehat{G} - G\right\|_F \geq \epsilon\right) &\leq 2N^2 \exp\left(-\frac{m\epsilon^2}{8N^2C^4}\right), \\ \mathbb{P}\left(\left\|\widehat{A} - A\right\|_F \geq \epsilon\right) &\leq 2N^2 \exp\left(-\frac{m\epsilon^2}{8N^2C^4L^2}\right).\end{aligned}$$

Proof. Define the random matrix $\eta(X) := \Psi(X)^T \Psi(X) \in \mathbb{R}^{N \times N}$, i.e., $\eta_{ij}(X) = \psi_i(X)\psi_j(X)$. Let $G = \mathbb{E}[\eta(X_1)]$, $\widehat{G} = \frac{1}{m} \sum_{k=1}^m \eta(X_k)$. Then,

$$\begin{aligned}\|\widehat{G} - G\|_F^2 &= \sum_{i=1}^N \sum_{j=1}^N \left| \widehat{G}_{ij} - G_{ij} \right|^2 \\ &= \sum_{i=1}^N \sum_{j=1}^N \left| \frac{1}{m} \sum_{k=1}^m \eta_{ij}(X_k) - \mathbb{E}[\eta_{ij}(X_k)] \right|^2\end{aligned}$$

where $\tilde{\eta}(X) := \eta(X) - \mathbb{E}[\eta(X_1)]$ and thus $|\tilde{\eta}_{ij}(X)| \leq 2C^2$ for all $1 \leq i, j \leq N$. Next, applying Hoeffding's inequality (2) and union bound, we have

$$\begin{aligned}\mathbb{P}\left(\left\|\widehat{G} - G\right\|_F \geq \epsilon\right) &= \mathbb{P}\left(\left\|\widehat{G} - G\right\|_F^2 \geq \epsilon^2\right) \\ &\leq \mathbb{P}\left(\sum_{i=1}^N \sum_{j=1}^N \left| \frac{1}{m} \sum_{k=1}^m \tilde{\eta}_{ij}(X_k) \right|^2 \geq \epsilon^2\right) \\ &\leq N^2 \mathbb{P}\left(\left| \frac{1}{m} \sum_{k=1}^m \tilde{\eta}_{ij}(X_k) \right|^2 \geq \epsilon^2/N^2\right) \\ &= N^2 \mathbb{P}\left(\left| \frac{1}{m} \sum_{k=1}^m \tilde{\eta}_{ij}(X_k) \right| \geq \epsilon/N\right) \\ &\leq 2N^2 \exp\left(-\frac{2(m\epsilon/N)^2}{m(4C^2)^2}\right) \\ &= 2N^2 \exp\left(-\frac{m\epsilon^2}{8N^2C^4}\right)\end{aligned}$$

Similarly, define $\xi(X) := \Psi(X)^T \mathcal{L}_{N,n,m} \Psi(X)$. Since

$$|\psi_i(X) \mathcal{L} \psi_j(X)| \leq C^2 \|\mathcal{L}_{N,n,m}\| \leq C^2 L,$$

we have

$$\mathbb{P}\left(\left\|\widehat{A} - A\right\|_F \geq \epsilon\right) \leq 2N^2 \exp\left(-\frac{m\epsilon^2}{8N^2C^4L^2}\right),$$

\square

Lemma 4 (Philipp, Lemma C.5). *Let $G, A \in \mathbb{R}^{N \times N}$ be such that G is invertible and $A \neq 0$. Let $\widehat{G}, \widehat{A} \in \mathbb{R}^{N \times N}$ be random matrices such that \widehat{G} is invertible a.s. Then for any sub-multiplicative matrix norm $\|\cdot\|$ on $\mathbb{R}^{N \times N}$ and any $\epsilon > 0$ we have*

$$\mathbb{P}\left(\|G^{-1}A - \widehat{G}^{-1}\widehat{A}\| > \epsilon\right) \leq \mathbb{P}\left(\|A - \widehat{A}\| > \frac{\epsilon}{\tau}\|A\|\right) + \mathbb{P}\left(\|G - \widehat{G}\| > \frac{\epsilon}{\tau}\|G^{-1}\|^{-1}\right),$$

where $\tau = 2\|G^{-1}\|\|A\| + \epsilon$.

Theorem 5. *Let $\epsilon > 0$. Define $\epsilon_t = (\epsilon - O(t^2)) / t$. Then, with same conditions defined in Theorem 3 and Lemma 4, we have*

$$P\left(\|\widehat{K}_{N,n,m} - K_{N,n}\|_F > \epsilon\right) \leq 2N^2 \left[\exp\left(-\frac{m}{8} \left(\frac{\epsilon_t}{NC^2\tau\|G^{-1}\|}\right)^2\right) + \exp\left(-\frac{m}{8} \left(\frac{\epsilon_t\|A\|}{NC^2\tau L}\right)^2\right) \right]$$

Proof.

$$\begin{aligned} P\left(\|\widehat{K}_{N,n,m} - K_{N,n}\|_F > \epsilon\right) &= P\left(t\|\widehat{G}^{-1}\widehat{A} - G^{-1}A\|_F + O(t^2) > \epsilon\right) \\ &= P\left(\|\widehat{G}^{-1}\widehat{A} - G^{-1}A\|_F > \epsilon_t\right) \\ &\leq \mathbb{P}\left(\|G - \widehat{G}\| > \frac{\epsilon_t}{\tau}\|G^{-1}\|^{-1}\right) + \mathbb{P}\left(\|A - \widehat{A}\| > \frac{\epsilon_t}{\tau}\|A\|\right) \\ &\leq 2N^2 \exp\left(-\frac{m(\frac{\epsilon_t}{\tau}\|G^{-1}\|^{-1})^2}{8N^2C^4}\right) + 2N^2 \exp\left(-\frac{m(\frac{\epsilon_t}{\tau}\|A\|)^2}{8N^2C^4L^2}\right) \\ &= 2N^2 \left[\exp\left(-\frac{m}{8} \left(\frac{\epsilon_t\|G^{-1}\|^{-1}}{NC^2\tau}\right)^2\right) + \exp\left(-\frac{m}{8} \left(\frac{\epsilon_t\|A\|}{NC^2\tau L}\right)^2\right) \right] \end{aligned}$$

□

3 Convergence in zero-limit of sampling time

Now we have dimensional Koopman matrix approximation $K_{N,n}(1/n)$ at discrete time $t = 1/n$. We will use it to construct the matrix $A_{N,n}$, which is the matrix representation of the discrete projected Koopman generator approximant $\mathcal{A}_{N,n}$ (Is this correct?). Then we find its limit in strong operator topology as $n \rightarrow \infty$.

First, for each $N > 0$, construct the following sequence of matrices $\{A_{N,n}\}_{n \geq 1}$:

$$\begin{aligned} A_{N,n} &:= \frac{K_{N,n}(\frac{1}{n}) - I}{\frac{1}{n}} \\ &= \frac{\lim_{m \rightarrow \infty} \left[\Psi_X^\dagger \left(\Psi_X + \frac{1}{n} \cdot \mathcal{L}\Psi_X + O(\frac{1}{n^2}) \right) \right] - I}{\frac{1}{n}} \\ &= \frac{\lim_{m \rightarrow \infty} \left[I + \frac{1}{n} \Psi_X^\dagger \mathcal{L}\Psi_X + O(\frac{1}{n^2}) \right] - I}{\frac{1}{n}} \\ &= \lim_{m \rightarrow \infty} \Psi_X^\dagger \mathcal{L}\Psi_X + O(\frac{1}{n}) \\ &= A_N + O(\frac{1}{n}) \end{aligned}$$

where $A_N := \lim_{m \rightarrow \infty} \Psi_X^\dagger \mathcal{L}\Psi_X$.

Remark. Here the first term $\Psi_X^\dagger \mathcal{L}\Psi_X$ is not dependent on n since we choose the basis functions manually.

Choose $f \in \text{span}\{\psi_i\}_{i=1}^N$ so that $f = \Psi_N \mathbf{a}$. Then we have

$$\begin{aligned} &\Psi_N A_{N,n} \mathbf{a} \\ &= \Psi_N (A_N + O(\frac{1}{n})) \mathbf{a} \\ &= \Psi_N A_N \mathbf{a} + \Psi_N O(\frac{1}{n}) \mathbf{a} \quad (\text{Here the } O(\frac{1}{n}) \text{ is a matrix}) \end{aligned}$$

Notice that, the first term is independent of n . Now, we consider the second term:

$$\begin{aligned} &\|\Psi_N O(\frac{1}{n}) \mathbf{a}\| \\ &= \left\| \begin{bmatrix} \psi_1 & \cdots & \psi_N \end{bmatrix} \begin{bmatrix} O(\frac{1}{n}) & \cdots & O(\frac{1}{n}) \\ \vdots & \ddots & \vdots \\ O(\frac{1}{n}) & \cdots & O(\frac{1}{n}) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} \psi_1 & \cdots & \psi_N \end{bmatrix} \begin{bmatrix} a'_1 \\ \vdots \\ a'_N \end{bmatrix} \right\| \\ & \quad (\text{where } a'_i = \sum_{j=1}^N [O(\frac{1}{n})]_{ij} a_j) \end{aligned}$$

$$\begin{aligned}
&= \left\| \sum_{i=1}^N \psi_i a'_i \right\| \\
&= \left\| \sum_{i=1}^N \psi_i \left(\sum_{j=1}^N [O(\frac{1}{n})]_{ij} a_j \right) \right\| \\
&\leq \left\| \sum_{i=1}^N \psi_i \left((N \cdot O(\frac{1}{n})) \sum_{j=1}^N a_j \right) \right\| \\
&= \left\| (N \cdot O(\frac{1}{n})) \sum_{i,j=1}^N \psi_i \left(\sum_{i,j=1}^N a_j \right) \right\| \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

Therefore, for $f = \Psi_N \mathbf{a}$, we have

$$\|\Psi_N A_{N,n} \mathbf{a} - \Psi_N A_N \mathbf{a}\| = \|\Psi_N O(\frac{1}{n}) \mathbf{a}\| \rightarrow 0,$$

In other words,

$$\lim_{n \rightarrow \infty} \mathcal{A}_{N,n} f = \mathcal{A}_N f,$$

where $\mathcal{A}_N = \mathcal{P}_N \mathcal{A}|_{\mathcal{F}_N}$.

The above content is summarized in the next theorem:

Theorem 6. *For each $N > 0$, let $\mathcal{A}_N = \mathcal{P}_N \mathcal{A}|_{\mathcal{F}_N}$ be the projected Koopman generator with its matrix representation $A_N = \Psi_X^\dagger \mathcal{L} \Psi_X$ and $\mathcal{A}_{N,n}$ be the discrete projected Koopman generator with its matrix representation $A_{N,n}$ defined in the following:*

$$A_{N,n} := \frac{K_{N,n}(\frac{1}{n}) - I}{\frac{1}{n}}$$

Then, $\mathcal{A}_{N,n} \rightarrow \mathcal{A}_N$ in strong operator topology as $n \rightarrow \infty$.

Proof.

□

4 Convergence in large dictionary size N

In this section, we want to show that $\mathcal{P}_N \mathcal{A}|_{\mathcal{F}_N} = \mathcal{A}_N \rightarrow \mathcal{A}$ in strong operator topology as $N \rightarrow \infty$, under a similar mathematical framework established in gEDMD analysis paper . In addition, if we can assume $e^{t\mathcal{A}_N}, e^{t\mathcal{A}}$ are exponentially bounded for all $t \geq 0$, $N \in \mathbb{N}$, then we can use the Trotter-Kato Approximation theorem to obtain $e^{t\mathcal{A}_N} \rightarrow e^{t\mathcal{A}}$ in strong operator topology as $N \rightarrow \infty$, uniformly for t in a compact interval.

4.1 Convergence of \mathcal{A}_N to \mathcal{A}

Consider the state space $(\Omega, \mathcal{B}, \mu)$, where the set of observables is given by $\mathcal{F} := L^2(\Omega, \mu)$. The domain of the operator \mathcal{A} is defined as:

$$\mathcal{D} := \{f \in \mathcal{F} : \mathcal{A}f \in \mathcal{F}\}.$$

The operator \mathcal{A} is referred to as a *closed operator* if the graph of \mathcal{A} , defined by:

$$\{(f, \mathcal{A}f) \in \mathcal{F} \times \mathcal{F} : f \in \mathcal{D}\},$$

is a closed subspace of $\mathcal{F} \times \mathcal{F}$. In this context, if \mathcal{A} is closed, then \mathcal{D} becomes a Hilbert space equipped with the inner product:

$$\langle f, g \rangle_{\mathcal{D}} := \langle f, g \rangle_{\mathcal{F}} + \langle \mathcal{A}f, \mathcal{A}g \rangle_{\mathcal{F}}, \quad \forall f, g \in \mathcal{D},$$

and the corresponding norm:

$$\|f\|_{\mathcal{D}}^2 := \langle f, f \rangle_{\mathcal{D}}.$$

It follows directly from this structure that $\|f\|_{\mathcal{F}} \leq \|f\|_{\mathcal{D}}$, and the operator $\mathcal{A} : \mathcal{D} \rightarrow \mathcal{F}$ is continuous with $\|\mathcal{A}\| \leq 1$.

In practice, if \mathcal{A} is the infinitesimal generator of a strongly continuous semi-group, \mathcal{D} is dense in \mathcal{F} .

An example is that when \mathcal{D} is a weighted Sobolev space $H^2(\Omega, \mu)$, containing all measurable functions $\psi : \Omega \rightarrow \mathbb{R}$ with finite norm:

$$\|\psi\|_{H^2(\Omega, \mu)} = \left(\sum_{|\alpha| \leq 2} \int_{\Omega} |D^{\alpha} \psi|^2 d\mu \right)^{1/2},$$

where $D^{\alpha} \psi$ represents the weak derivative of order α .

Assumption 1. *We assume the following:*

1. The basis functions $\Psi = \{\psi_1, \dots, \psi_N\} \subset \mathcal{D}$ are linearly independent.
2. The functions $\{\psi_i, \mathcal{A}\psi_i\}_{i=1}^N$ are continuous μ -a.e..
3. The points $\{x_i\}_{i=1}^m \subset \Omega$ are i.i.d. samples from μ .

Assumption 2. *Assumption 1 holds and*

$$\lim_{N \rightarrow \infty} \|\mathcal{P}_N \phi - \phi\|_{\mathcal{F}} = 0, \quad \forall \phi \in \mathcal{F},$$

where \mathcal{P}_N is the projection of \mathcal{F} onto \mathcal{F}_N using the inner product on \mathcal{F} .

Assumption 3. *Assumption 1 holds, \mathcal{A} is a closed operator, and*

$$\lim_{N \rightarrow \infty} \|\mathcal{P}_{\mathcal{D}_N} f - f\|_{\mathcal{D}} = 0, \quad \forall f \in \mathcal{D}.$$

Here, $\mathcal{P}_{\mathcal{D}_N}$ is the projection of \mathcal{D} onto \mathcal{F}_N using the inner product on \mathcal{D} .

Theorem 7. *Let Ψ satisfy Assumption 2 and 3, then*

$$\lim_{N \rightarrow \infty} \|\mathcal{A}_N \mathcal{P}_{\mathcal{D}_N} f - \mathcal{A}f\|_{\mathcal{F}} = 0, \quad \forall f \in \mathcal{D}.$$

Proof. By definition $\mathcal{A}_N = \mathcal{P}_N \mathcal{A}|_{\mathcal{F}_N}$, we obtain

$$\begin{aligned} \mathcal{A}_N \mathcal{P}_{\mathcal{D}_N} - \mathcal{A} &= (\mathcal{A}_N - \mathcal{A}) \mathcal{P}_{\mathcal{D}_N} + \mathcal{A} \mathcal{P}_{\mathcal{D}_N} - \mathcal{A} \\ &= (\mathcal{P}_N - \text{Id}) \mathcal{A} \mathcal{P}_{\mathcal{D}_N} + \mathcal{A} (\mathcal{P}_{\mathcal{D}_N} - \text{Id}) \\ &= (\mathcal{P}_N - \text{Id}) \mathcal{A} + (\mathcal{P}_N - \text{Id}) \mathcal{A} (\mathcal{P}_{\mathcal{D}_N} - \text{Id}) + \mathcal{A} (\mathcal{P}_{\mathcal{D}_N} - \text{Id}). \end{aligned}$$

Consider now $f \in \mathcal{D}$. By Assumptions 2, 3 and the fact that \mathcal{A} is continuous on its domain, we have

$$\begin{aligned} \|\mathcal{A}_N \mathcal{P}_{\mathcal{D}_N} f - \mathcal{A}f\|_{\mathcal{F}} &\leq \|(\mathcal{P}_N - \text{Id}) \mathcal{A}f\|_{\mathcal{F}} + \|(\mathcal{P}_N - \text{Id}) \mathcal{A}\| \|\mathcal{P}_{\mathcal{D}_N} f - f\|_{\mathcal{D}} \\ &\quad + \|\mathcal{A}\| \|\mathcal{P}_{\mathcal{D}_N} f - f\|_{\mathcal{D}} \rightarrow 0. \end{aligned}$$

□

4.2 Convergence of $\mathcal{K}_N(t)$ to $\mathcal{K}(t)$

Define the semigroup generated by $\{\mathcal{A}_N\}_{N \geq 1}$ and \mathcal{A} :

$$\mathcal{K}_N(t) := e^{t\mathcal{A}_N}, \quad \mathcal{K}(t) := e^{t\mathcal{A}}$$

and assume that

$$\|\mathcal{K}_N(t)\|, \|\mathcal{K}(t)\| \leq M e^{wt} \quad \text{for all } t \geq 0, N \in \mathbb{N}$$

and some constants $M \geq 1, w \in \mathbb{R}$.

Corollary 8. *Let $(\mathcal{K}_N(t))_{t \geq 0}$ and $(\mathcal{K}(t))_{t \geq 0}$ be strongly continuous semigroups with generators \mathcal{A}_N and \mathcal{A} , respectively. Assume that for some constants $M \geq 1, w \in \mathbb{R}$, the semigroups satisfy*

$$\|\mathcal{K}_N(t)\|, \|\mathcal{K}(t)\| \leq M e^{wt} \quad \text{for all } t \geq 0, N \in \mathbb{N}.$$

Furthermore, assume that for each $x \in \mathcal{D}(\mathcal{A})$, $\mathcal{A}_N x \rightarrow \mathcal{A}x$ as $N \rightarrow \infty$. Then

$$\mathcal{K}_N(t) \rightarrow \mathcal{K}(t)$$

in the strong operator topology, uniformly for t in compact intervals.

Proof. By applying the First Trotter–Kato Approximation Theorem 9, the convergence result follows directly. □

First Trotter–Kato Approximation Theorem

Theorem 9 (Trotter 1958, Kato 1959). *Let $(T(t))_{t \geq 0}$ and $(T_n(t))_{t \geq 0}$, $n \in \mathbb{N}$, be strongly continuous semigroups on X with generators \mathcal{A} and \mathcal{A}_n , respectively, and assume that they satisfy the estimate*

$$\|T(t)\|, \|T_n(t)\| \leq Me^{wt} \quad \text{for all } t \geq 0, n \in \mathbb{N},$$

and some constants $M \geq 1$, $w \in \mathbb{R}$. Take \mathcal{D} to be a core for \mathcal{A} and consider the following assertions.

- (a) $\mathcal{D} \subset \mathcal{D}(\mathcal{A}_n)$ for all $n \in \mathbb{N}$ and $\mathcal{A}_n x \rightarrow \mathcal{A}x$ for all $x \in \mathcal{D}$.
- (b) For each $x \in \mathcal{D}$, there exists $x_n \in \mathcal{D}(\mathcal{A}_n)$ such that $x_n \rightarrow x$ and $\mathcal{A}_n x_n \rightarrow \mathcal{A}x$.
- (c) $R(\lambda, \mathcal{A}_n)x \rightarrow R(\lambda, \mathcal{A})x$ for all $x \in X$ and some/all $\lambda > w$.
- (d) $T_n(t)x \rightarrow T(t)x$ for all $x \in X$, uniformly for t in compact intervals.

Then the implications

$$(a) \implies (b) \iff (c) \iff (d)$$

hold, while (b) does not imply (a).

A "Taylor" expansion of stochastic Koopman operator

By applying Itô's formula to both $f(X_t)$ and $\mathcal{A}f(X_t)$, we can derive a "Taylor expansion" for $\mathbb{E}[f(X_t)]$ as in (3).

First, we apply Itô's formula to $f(X_t)$:

$$f(X_t) = f(x) + \int_0^t (\mathcal{A}f)(X_s) ds + \int_0^t f'(X_s) \sigma(X_s) dW_s.$$

Next, we treat $\mathcal{A}f$ as a function and apply Itô's formula to $\mathcal{A}f(X_s)$:

$$(\mathcal{A}f)(X_t) = (\mathcal{A}f)(x) + \int_0^t [\mathcal{A}(\mathcal{A}f)](X_s) ds + \int_0^t (\mathcal{A}f')(X_s) \sigma(X_s) dW_s.$$

Then, we substitute this expression for $\mathcal{A}f(X_t)$ back into the formula for $f(X_t)$:

$$\begin{aligned} f(X_t) &= f(x) + \int_0^t (\mathcal{A}f)(X_s) ds + \int_0^t f'(X_s) \sigma(X_s) dW_s \\ &= f(x) + \int_0^t \left[(\mathcal{A}f)(x) + \int_0^s [\mathcal{A}(\mathcal{A}f)](X_u) du + \int_0^s (\mathcal{A}f')(X_u) \sigma(X_u) dW_u \right] ds \\ &\quad + \int_0^t f'(X_s) \sigma(X_s) dW_s. \end{aligned}$$

After rearranging terms, we have:

$$\begin{aligned} f(X_t) &= f(x) + (\mathcal{A}f)(x) t + \int_0^t \int_0^s [\mathcal{A}(\mathcal{A}f)](X_u) du ds \\ &\quad + \int_0^t \int_0^s (\mathcal{A}f')(X_u) \sigma(X_u) dW_u ds + \int_0^t f'(X_s) \sigma(X_s) dW_s. \end{aligned}$$

Taking expectations on both sides and noting that the stochastic integrals have zero mean (assuming appropriate integrability conditions), we get:

$$\mathbb{E}[f(X_t)] = f(x) + (\mathcal{A}f)(x) t + \int_0^t \int_0^s \mathbb{E}[\mathcal{A}(\mathcal{A}f)(X_u)] du ds.$$

The double integral term represents the accumulated effect of the higher-order derivatives of f over time. To understand its order, consider that if $\mathbb{E}[\mathcal{A}(\mathcal{A}f)(X_u)]$ is bounded by some constant M , then:

$$\left| \int_0^t \int_0^s \mathbb{E}[\mathcal{A}(\mathcal{A}f)(X_u)] du ds \right| \leq M \int_0^t \int_0^s du ds = M \int_0^t s ds = M \frac{t^2}{2}.$$

This shows that the double integral is of order $O(t^2)$ when t is small.

Therefore, for small t , the expected value simplifies to:

$$\mathbb{E}[f(X_t)] = f(x) + (\mathcal{A}f)(x) t + O(t^2). \quad (7)$$