Category theory course
Lecture 4
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1 Functors

1.1 Morphisms of categories

Definition 1 (Functor)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories; we define a* functor $F : \mathcal{C} \to \mathcal{D}$ *as a pair (F_0, F_1) consisting of the following data:*

- *F_0 is a function $\mathcal{C}_o \to \mathcal{D}_o$ sending an object $C \in \mathcal{C}_o$ to an object $FC \in \mathcal{D}_o$;*
- *F_1 is a family of functions $F_{AB} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$, one for each pair of objects $A, B \in \mathcal{C}_o$, sending each arrow $f : A \to B$ into an arrow $Ff : FA \to FB$, and such that:*
  - *$F_{AA}(1_A) = 1_{FA}$;*
  - *$F_{AC}(g \circ_{\mathcal{C}} f) = F_{BC}(g) \circ_{\mathcal{D}} F_{AB}(f)$.*

```haskell
-- | All instances of the `Functor` type-class must satisfy two laws.
-- These laws are not checked by the compiler.
--
-- * The law of identity
--   `forall x. (id <$> x) ~ x`
--
-- * The law of composition
--   `forall f g x.(f . g <$> x) ~ (f <$> (g <$> x))`
class Functor k where
-- Pronounced, eff-map.
(<$>) :: (a -> b) -> k a -> k b
```

If $F : \mathcal{C} \to \mathcal{D}$ is a functor, it corresponds to a data constructor `f :: * -> *` (this is the F_0 part, a *function on objects* from the collection of object $\mathcal{C}_o$ of $\mathcal{C}$ to the collection of objects $\mathcal{D}_o$[1]) and a *function on morphisms*

```haskell
<$> :: (a -> b) -> f a -> f b
```

If you recall how the associativity of `->` works, it is evident that this is a function from the type of maps `a -> b` to the type of maps `f a -> f b`; or rather, this is a function that given a function `u :: a -> b`, and an "element" of type `f a`, yields an element `x :: f b`.

Remark 1. *There is a category whose objects are (small) categories, and whose morphisms $\mathcal{C} \to \mathcal{D}$ are functors. The* identity *functor $1_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ of a category $\mathcal{C}$, and the composition $G \circ F : \mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ of two functors are defined in the obvious way; all category axioms follow.*[2]

1.2 Examples of functors

As you can imagine, both Haskell and mathematics are crawling with examples of functors;

Example 1 (Examples of functors).

1. *Let's rule out a few edge examples: for every category $\mathcal{C}$, there is a unique functor $F : \varnothing \to \mathcal{C}$, where $\varnothing$ is the empty category with no objects an no morphisms; for every category $\mathcal{C}$, there is an obvious*

[1] Although in Haskell, $\mathcal{C} = \mathcal{D}$ is always the same nameless category `*`.

[2] The reader might now wonder if there is a category of Haskell types; unfortunately there is no such thing, but better languages have a better behaviourin this respect (for example `Agda` or `Idris`).

definition of the identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$*, whose correspondences on objects and on arrows* $1_{\mathcal{C},AB} : \mathcal{C}(A,B) \to \mathcal{C}(A,B)$ *are all identity functions.*

2. *As we have hinted in the preceding lesson, all correspondences that* forget *a structure, yielding for example the underlying set* $U\mathbb{M}$ *of a thingum* $\mathbb{M}$*, are functors* $\mathsf{Thng} \to \mathsf{Set}$ *from their structured categories of definition to the category of sets and functions.*

3. *All correspondences that* regard *a structure as an example of another are -more or less tautological- examples of functors: a monoid can be seen as a one-object category, and this yields a functor* $\boldsymbol{B} : \mathsf{Mon} \to \mathsf{Cat}$ *because a functor* $F : \boldsymbol{B}M \to \boldsymbol{B}N$ *between monoids is but a monoid homomorphism* $f : M \to N$*; a preset can be seen as a category with at most one arrow between any two objects, and this defines a functor* ${}^{\chi}(\_) : \mathsf{Pres} \to \mathsf{Cat}$*, because a functor* $F : {}^{\chi}P \to {}^{\chi}Q$ *between presets is but a monotone function* $f : P \to Q$*; a set can be seen as a discrete topological space* A^{δ} *or as a category having only identity arrows, and these are functors* $\mathsf{Set} \to \mathsf{Top}, \mathsf{Cat}$*. (Continue the list at your will.)*

4. *Let* M *be a monoid, and* $\boldsymbol{\Delta}$ *be the category having objects nonempty, finite, totally ordered sets and monotone functions as morphisms. The correspondence that sends* $[n] \in \boldsymbol{\Delta}$ *to the set* $M^{[n]}$ *of ordered* n*-tuples* $[a_1|\dots|a_n]$ *of elements of* M *and a monotone function* $f : [m] \to [n]$ *to the function* $M^f : M^{[n]} \to M^{[m]}$ *defined by*

$$M^f[a_1|\dots|a_n] = [a_{f1}|\dots|a_{fn}]$$

is a functor $\boldsymbol{\Delta}^{op} \to \mathsf{Set}$*. This functor is called the* classifying space *of the monoid.*

5. *Let* Set_* *be the category of* pointed sets*: objects are pairs* (A, a) *where* $a \in A$ *is a distinguished element, and a morphism* $f : (A, a) \to (B, b)$ *is a function* $f : A \to B$ *such that* $fa = b$*. It is easy to prove that this category is isomorphic to the coslice* $*/\mathsf{Set}$ *of functions* $a : * \to A$*.*

6. *Let* $\partial\mathsf{Set}$ *be the category so defined: objects are sets, and a morphism* $(f, S) : A \to B$ *is a pair* $(S \subseteq A, f : S \to B)$*. This is called a* partial function *from* A *to* B*. It is evident how to define the composition of morphisms, and the identity arrow of* $A \in \partial\mathsf{Set}$*. There is a functor* $(\_)_{\bullet} : \partial\mathsf{Set} \to \mathsf{Set}_*$ *defined as* $A \mapsto (A \sqcup \{\infty\}, \infty)$*, where* ∞ *is an element that does not belong to* A*, and a function* (f, S) *goes to* $(f, S)_{\bullet} : A \sqcup \{\infty_A\} \to B \sqcup \{\infty_B\}$*, defined sending* $S^c \cup \{\infty_A\}$ *to* $\{\infty_B\}$*.*

7. *Every directed graph $\underline{G}$, with set of vertices V and set of edges E, defines a quotient set of V by the equivalence relation $\approx$ generated by the subset of those (A, B) for which there is an arrow $A \to B$; the symmetric and transitive closure of this relation yields a quotient V/R that is usually denoted as the set of* connected components $\pi_0(\underline{G})$. *This is a functor* $\mathsf{Gph} \to \mathsf{Set}$*, and if we now regard a category $\mathcal{C}$ as a mere directed graph we obtain a well-defined set $\pi_0(\mathcal{C})$ of connected components of a category.*

1.3 Diagrams or: functors as pictures

Most of category theory is based on the intuition that a diagram of a certain shape in a category $\mathcal{C}$ is a functor $F : \mathcal{J} \to \mathcal{C}$; the definition of a diagram itself is engineered to blur the distinction between a functor as a morphism of categories, and a functor as a correspondence that "pictures" a small category $\mathcal{J}$ inside a big category $\mathcal{C}$. After having recalled the definition of diagram, and the definition of commutative diagram, we collect a series of examples to convince the reader that this identification of "diagrams as pictures" is fruitful and suggestive.

The recommended soundtrack while reading this section is, of course, Emerson, Lake & Palmer's *Pictures at an Exhibition*.

1.3.1 Diagrams, formally

Arrangements of objects and arrows in a category are called *diagrams*; to some extent, category theory is the art of making diagrams *commute*, i.e. the art of proving that two paths $X \to A_1 \to A_2 \to \cdots \to A_n \to Y$ and $X \to B_1 \to \cdots \to B_m \to Y$ result in the same arrow when they are fully composed.

Remark 2 (Diagrams and their commutation)**.** *Depicting morphisms as arrows allows to draw regions of a given category $\mathcal{C}$ as parts of a (possibly non planar) graph; we call a* diagram *such a region in $\mathcal{C}$, the graph whose vertices are objects of $\mathcal{C}$ and whose edges are morphisms of suitable domains* *and codomains. For example, we can consider the diagram*

$$
\begin{tikzcd}
 & & X_2 \arrow[drr, "v"] & & \\
X_1 \arrow[urr, "u"] \arrow[rrrr, "h"'] \arrow[dr, "g"'] & & & & X_3 \\
 & X_0 \arrow[uur, "q"'] \arrow[rr, "p"'] & & X_4 \arrow[ur, "k"'] &
\end{tikzcd}
$$

The presence of a composition rule in $\mathcal{C}$ entails that we can meaningfully compose paths $[u_0, \dots, u_n]$ *of morphisms of $\mathcal{C}$. In particular, we can consider diagrams having distinct paths between a fixed source and a fixed 'sink' (say, in the diagram above, we can consider two different paths $\mathfrak{P} = [k, p, g]$ and $\mathfrak{Q} = [v, u]$); both paths go from X_1 to X_3, and we can ask the two compositions $\circ[k, p, g] = k \circ p \circ g$ and $v \circ u$ to be the same arrow $X_1 \to X_3$; we say that a diagram* commutes at $\mathfrak{P}, \mathfrak{Q}$ *if this is the case; we say that a diagram* commutes *(without mention of $\mathfrak{P}, \mathfrak{Q}$) if it commutes for every choice of paths for which this is meaningful.*

Searching a formalisation of this intuitive pictorial idea leads to the following:

Definition 2. *A* diagram *is a map of directed graphs ('digraphs') $D : J \to \mathcal{C}$ where J is a digraph and $\mathcal{C}$ is the digraph underlying a category.*[3] *Such a diagram D* commutes *if for every pair of parallel edges $f, g : i \rightrightarrows j$ in J one has $Df = Dg$.*

1.3.2 Diagrams, everywhere

- An object is a diagram: the terminal object in the category of categories is the category
  $$[0] = \left\{ \begin{tikzcd} \bullet \arrow[loop right, "1"] \end{tikzcd} \right\}$$
  A functor $C : \mathbf{1} \to \mathcal{C}$ consists of an object of $\mathcal{C}$.
- An arrow is a diagram: let
  $$[1] = \left\{ \begin{tikzcd} 0 \arrow[r] & 1 \end{tikzcd} \right\}$$

[3] Every small category has an underlying graph, obtained keeping objects and arrows and forgetting all compositions; there is of course a category of graphs, and regarding a category as a graph is another example of forgetful functor. Of course, making this precise means that the collection of categories and functors form a category on its own right.

be the category with two objects $0, 1$ and a single nonidentity morphism $0 \to 1$. A functor $f : \mathbf{2} \to \mathcal{C}$ consists of two objects $X_i = f(i)$ for $i = 0, 1$, and a morphism $X_0 \to X_1$.

- A chain is a diagram: let $n > 2$ be a positive integer. Let

$$[n] = \left\{ 0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n \right\}$$

be the category with exactly $n+1$ objects and exactly one nonidentity arrow $j-1 \to j$ for $j = 1, \ldots, n$. A functor $c : [n] \to \mathcal{C}$ consists of a tuple of objects $X_0 \to X_1 \to \cdots \to X_n$, and morphisms $X_{j-1} \to X_j$ for $j = 1, \ldots, n$.

- A span is a diagram: let

$$\mathcal{S} = \left\{ \begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 1 & & 2 \end{tikzcd} \right\}$$

be the category with three objects and arrows as depicted. A functor $F : \mathcal{S} \to \mathcal{C}$ is a diagram of the same shape, labeled in the same way…

So, you get the idea. Define a *cospan* in $\mathcal{C}$ to be a functor F with domain the opposite category $\mathcal{S}^{\mathrm{op}}$.

- A commutative square is a diagram: let

$$\begin{tikzcd} (00) \arrow[r] \arrow[d] & (10) \arrow[d] \\ (01) \arrow[r] & (11) \end{tikzcd}$$

be the category with four objects and nonidentity morphisms forming a commutative square. A functor F with this domain, and values in $\mathcal{C}$, consists of a commutative square of objects in $\mathcal{C}$.

- In a similar fashion, a commutative cube is a diagram: a cube is a morphism between the square of sources and the square of targets. Draw a picture, and formalise this statement (what is the identity cube of a square? How do you compose cubes?).

- A diagram is a diagram is a diagram,…, is a diagram: generalise to n-dimensional cubes.

1.4 General definition of limits and colimits

Definition 3 (Cone completions of $\mathcal{J}$)**.** *Let $\mathcal{J}$ be a small category; we denote $\mathcal{J}^{\rhd}$ the category obtained adding to $\mathcal{J}$ a single terminal object ∞; more in detail, $\mathcal{J}^{\rhd}$ has objects $\mathcal{J}_o \cup \{\infty\}$, where $\infty \notin \mathcal{J}$, and it is defined by*

$$\begin{aligned} \mathcal{J}^{\rhd}(J, J') &= \mathcal{J}(J, J') \\ \mathcal{J}^{\rhd}(J, \infty) &= \{*\} \end{aligned}$$

and it is empty otherwise. This category is called the right cone *of $\mathcal{J}$.*

Dually, we define a category $\mathcal{J}^{\lhd}$, the left cone *of $\mathcal{J}$, as the category obtained adding to J a single* initial *object $-\infty$; this means that $\mathcal{J}^{\lhd}(J, J') = \mathcal{J}(J, J')$, $\mathcal{J}^{\lhd}(-\infty, J) = \{*\}$, and it is empty otherwise.*

Example 2. *Let ω be the ordinal number obtained from the union of all finite ordinals; then, when regarded as a category, $\omega^{\lhd}$ is the category $\{-\infty \to 0 \to 1 \to \dots\}$ (and thus is isomorphic to ω in an evident way), whereas $\omega^{\rhd}$ is $\omega + 1$, in the sense of ordinal sum.*

For those willing to embark in an extra exercise: what is $G^{\rhd}$ if G is a monoid regarded as a category?

Remark 3. *The correspondences $\mathcal{J} \mapsto \mathcal{J}^{\rhd}$ and $\mathcal{J} \mapsto \mathcal{J}^{\lhd}$ are functorial. As an easy exercise, define them on morphisms and prove their functoriality. Prove also that there is a natural embedding $i_{\rhd}$ of $\mathcal{J}$ into $\mathcal{J}^{\rhd}$ and one $i_{\lhd}$ of $\mathcal{J}$ into $\mathcal{J}^{\lhd}$ (that we invariably denote i in the following discussion).*

Definition 4 (Cone of a diagram)**.** *Let $\mathcal{J}$ be a small category, $\mathcal{C}$ a category, and $D : \mathcal{J} \to \mathcal{C}$ a functor; all along this section, an idiosyncratic way to refer to D will be as a* diagram *of shape $\mathcal{J}$. We call a* cone *for D any extension of the diagram D to the left cone category of $\mathcal{J}$ defined in , so that the diagram*

$$\begin{tikzcd} \mathcal{J} \arrow[r, "D"] \arrow[d, "i_{\lhd}"'] & \mathcal{C} \\ \mathcal{J}^{\lhd} \arrow[ur, dotted, "\bar{D}"'] & \end{tikzcd}$$

commutes.

Every such extension is thus forced to coincide with D on all objects in $\mathcal{J} \subseteq \mathcal{J}^{\lhd}$; the value of $\bar{D}$ on $-\infty$ is called the base of the cone; dually, the value of an extension of D to $\mathcal{J}^{\rhd}$ coincides with D on $\mathcal{J} \subseteq \mathcal{J}^{\rhd}$, and $\bar{D}(\infty)$ is called the *tip* of the cone.

There is of course a similar definition of a *cocone* for D: it is an extension of the diagram D to the right cone category of $\mathcal{J}$ so that the diagram

$$\begin{tikzcd}
\mathcal{J} \arrow[r, "D"] \arrow[d, "i_{\triangleright}"'] & \mathcal{C} \\
\mathcal{J}^{\triangleright} \arrow[ur, dotted, "\bar{D}"'] &
\end{tikzcd}$$

commutes. Exercise sheds a light on and it substantiates the quote from [**?**] therein: cones for D are exactly cocones for the opposite functor D^{op}.

Remark 4.

- *The class of cones for D forms a category $Cn(D)$, whose morphisms are the natural transformations $\alpha : D' \Rightarrow D'' : \mathcal{J} \to \mathcal{C}$ such that the right whiskering of α with $i : \mathcal{J} \to \mathcal{J}^{\triangleleft}$ coincides with the identity natural transformation of D; this means that a morphism α of this sort is a natural transformation such that*

$$\begin{tikzcd}
& \mathcal{J} \arrow[dl, "i"'] \arrow[dr, "D"] & \\
\mathcal{J}^{\triangleleft} \arrow[rr, bend left=15, "D'"] \arrow[rr, bend right=15, "D''"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & \mathcal{C}
\end{tikzcd} \qquad = 1_D$$

  as a 2-cell $D \Rightarrow D$.

- *Dually, the class of cocones for D forms a category $Ccn(D)$, whose morphisms are the natural transformations $\alpha : D' \Rightarrow D'' : \mathcal{J} \to \mathcal{C}$ such that the right whiskering of α with $i : \mathcal{J} \to \mathcal{J}^{\triangleright}$ coincides with the identity natural transformation of D; this means that a morphism α of this sort is a natural transformation such that*

$$\begin{tikzcd}
& \mathcal{J} \arrow[dl, "i"'] \arrow[dr, "D"] & \\
\mathcal{J}^{\triangleright} \arrow[rr, bend left=15, "D'"] \arrow[rr, bend right=15, "D''"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & \mathcal{C}
\end{tikzcd} \qquad = 1_D$$

  as a 2-cell $D \Rightarrow D$.

Definition 5 (Colimit, limit)**.** *The* limit *of a diagram $D : \mathcal{J} \to \mathcal{C}$ is the terminal object denoted '$\lim_{\mathcal{J}} D$' in the category of cones for D; dually, the* colimit *of D is the initial object denoted '$\mathrm{colim}_{\mathcal{J}} D$' in the category of cocones for D.*

It is a good idea to unwind definitions and in order to obtain the more classically explained notion of limit and colimit: the key for this unwinding operation is that a co/cone for $D : \mathcal{J} \to \mathcal{C}$ amounts to a natural family of maps from a constant object (the base) or to a constant object (the tip).

- A *cone* for a diagram $D : \mathcal{J} \to \mathcal{C}$ is a natural transformation from a constant functor $\Delta_c : \mathcal{J} \to \mathcal{C}$ to $D(\_)$;
- there is a category of cones for $D(\_)$, where morphisms between a cone $c \to D(\_)$ and a cone $C' \to D(\_)$ are arrows $k : C \to C'$ such that the diagram

$$\begin{tikzcd} DI \arrow[dd, "D\phi"'] & & \\ & C \arrow[ul, "l_I"'] \arrow[dl, "l_J"] \arrow[r, dotted, "k"] & C' \arrow[ull, "l'_I"'] \arrow[dll, "l'_J"] \\ DJ & & \end{tikzcd}$$

  is commutative;
- a limit for D is a terminal object in the category of cones for D. This means that given a cone for D, there is a unique arrow k which is a morphism of cones.

Of course, a straightforward dualisation yields the definition of a cocone, and a colimit for D.

1.5 Examples of limits and colimits

We start with a classical edge example: a terminal object is the limit of the empty diagram.

Example 3. *Let $J = \varnothing$ be the empty set; the limit of the unique diagram $D : \mathcal{J} \to \mathcal{C}$ is the* terminal object *of $\mathcal{C}$.*

The universal property exhibited by the terminal object $*$ of $\mathcal{C}$ is the following: there is a unique morphism $C \to *$ for every $C \in \mathcal{C}$ (with no other condition, since $\mathcal{J}$ is empty).

Example 4 (Product)**.** *Let $\mathcal{J}$ be a set, and $\{X_i \mid i \in \mathcal{J}\}$ a family of objects of a category $\mathcal{C}$; the* product *of the $\{X_i\}$'s, denoted $\prod_{i\in\mathcal{J}} X_i$, is the limit of the diagram $D : \mathcal{J} \to \mathcal{C}$, when the set $\mathcal{J}$ is regarded as a discrete category.*

The universal property exhibited by the object $\prod_{i\in\mathcal{J}} X_i$ is the following: there is a cone $\bar{p} = \{p_i : \prod X_j \to X_i \mid i \in \mathcal{J}\}$ such that $(\bar{p}(-\infty) = \prod X_j$

and) for every other cone $\{f_i : E \to X_i \mid i \in \mathcal{J}\}$ *there exists a unique dotted* $\bar{f} : E \to \prod_{i\in\mathcal{J}} X_i$ *such that*

$$\begin{tikzcd}
E \arrow[d, dotted, "\bar{f}"'] \arrow[dr, "f_{\bar{\imath}}"] & \\
\prod X_i \arrow[r, "p_{\bar{\imath}}"'] & X_{\bar{\imath}}
\end{tikzcd}$$

commutes for every $\bar{\imath} \in \mathcal{J}$*.*

Example 5 (Pullback)**.** *Let* $\mathcal{J}$ *be the category* $0 \to 2 \leftarrow 1$*, and* $\{X_0 \to X_2 \leftarrow X_1\}$ *the corresponding diagram* $X : \mathcal{J} \to \mathcal{C}$*; the* pullback *of the diagram* X*, denoted* $X_0 \times_{X_2} X_1$*, is the limit of* X*; the universal property exhibited by the object* $X_0 \times_{X_2} X_1$ *is the following: there is a cone* $X_0 \xleftarrow{p_0} X_0 \times_{X_2} X_1 \xrightarrow{p_1} X_1$ *such that for every other cone* $X_0 \xleftarrow{f_0} E \xrightarrow{f_1} X_1$ *there exists a unique dotted* $\langle f_0, f_1\rangle$ *such that*

$$\begin{tikzcd}
E \arrow[dr, dotted] \arrow[drr, bend left, "f_0"] \arrow[ddr, bend right, "f_1"'] & & \\
& X_0 \times_{X_2} X_1 \arrow[r, "p_0"] \arrow[d, "p_1"'] & X_0 \arrow[d, "q_{02}"] \\
& X_1 \arrow[r, "q_{12}"'] & X_2
\end{tikzcd}$$

In the same notation above, when we want to stress the dependence of the pullback from the maps q_{02}, q_{12}*, the object is sometimes denoted as* $q_{02} \times q_{12}$ *instead of* $X_0 \times_{X_2} X_1$*. This is not a real clash of notation, as it is possible to prove that* $q_{02} \times q_{12}$ *is the product (in the sense of 4 above) of* q_{02}, q_{12} *regarded as objects of the slice category* $\mathcal{C}/X_2$*.*

In the category of sets, the pullback $X_0 \times_{X_2} X_1$ of a pair f_0, f_1 can be easily characterised as the subset of $X_0 \times X_1$ made by all pairs (x_0, x_1) such that $f_0(x_0) = f_1(x_1)$.

Example 6 (Equaliser)**.** *Let* $\mathcal{J}$ *be the category* $0 \rightrightarrows 1$*, and* $\{X_0 \overset{u}{\underset{v}{\rightrightarrows}} X_1\}$ *the corresponding diagram* $X : \mathcal{J} \to \mathcal{C}$*; the* equaliser *of the diagram* X*, denoted* $eq(u, v)$*, is the limit of* X*; the universal property exhibited by the object is the following: there is an cone* $e : eq(u, v) \to X_0$ *and for every other cone*

$k : E \to X_0$ there is a unique dotted $\bar{k} : E \to eq(u,v)$ such that

$$
\begin{tikzcd}
eq(u,v) \arrow[r, "e"] & X_0 \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & X_1 \\
E \arrow[u, dotted, "\bar{k}"] \arrow[ur, "k"] & &
\end{tikzcd}
$$

In the category of sets, the equaliser of a pair of maps u, v can be easily characterised as the subset of X_0 made by all elements such that $u(x) = v(x)$; it is 'the largest subset of X_0 where $u = v$'.

1.5.1 in monoids and Mon

1.5.2 in posets and Pos

1.5.3 in Set, and other algebraic categories

1.5.4 Exercises

Exercises denoted with a star symbol are supposed to be difficult. Don't be put off, and enjoy!

1.