Category theory course
Lecture 4
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1 Functors

1.1 Morphisms of categories

Definition 1 (Functor)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories; we define a* functor *$F : \mathcal{C} \to \mathcal{D}$ as a pair (F_0, F_1) consisting of the following data:*

- *F_0 is a function $\mathcal{C}_o \to \mathcal{D}_o$ sending an object $C \in \mathcal{C}_o$ to an object $FC \in \mathcal{D}_o$;*
- *F_1 is a family of functions $F_{AB} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$, one for each pair of objects $A, B \in \mathcal{C}_o$, sending each arrow $f : A \to B$ into an arrow $Ff : FA \to FB$, and such that:*
  - $F_{AA}(1_A) = 1_{FA}$;
  - $F_{AC}(g \circ_{\mathcal{C}} f) = F_{BC}(g) \circ_{\mathcal{D}} F_{AB}(f)$.

```haskell
-- | All instances of the `Functor` type-class must satisfy two laws.
-- These laws are not checked by the compiler.
--
-- * The law of identity
--   `forall x. (id <$> x) ~ x`
--
-- * The law of composition
--   `forall f g x.(f . g <$> x) ~ (f <$> (g <$> x))`
class Functor k where
-- Pronounced, eff-map.
(<$>) :: (a -> b) -> k a -> k b
```

If $F : \mathcal{C} \to \mathcal{D}$ is a functor, it corresponds to a data constructor `f :: * -> *` (this is the F_0 part, a *function on objects* from the collection of object $\mathcal{C}_o$ of $\mathcal{C}$ to the collection of objects $\mathcal{D}_o$[1]) and a *function on morphisms*

```haskell
<$> :: (a -> b) -> f a -> f b
```

If you recall how the associativity of `->` works, it is evident that this is a function from the type of maps `a -> b` to the type of maps `f a -> f b`; or rather, this is a function that given a function `u :: a -> b`, and an "element" of type `f a`, yields an element `x :: f b`.

Remark 1. *There is a category whose objects are (small) categories, and whose morphisms $\mathcal{C} \to \mathcal{D}$ are functors. The* identity *functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ of a category $\mathcal{C}$, and the composition $G \circ F : \mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ of two functors are defined in the obvious way; all category axioms follow.*[2]

1.2 Examples of functors

As you can imagine, both Haskell and mathematics are crawling with examples of functors;

Example 1 (Examples of functors)**.**

1. *Let's rule out a few edge examples: for every category $\mathcal{C}$, there is a unique functor $F : \varnothing \to \mathcal{C}$, where $\varnothing$ is the empty category with no objects an no morphisms; for every category $\mathcal{C}$, there is an obvious*

[1] Although in Haskell, $\mathcal{C} = \mathcal{D}$ is always the same nameless category `*`.

[2] The reader might now wonder if there is a category of Haskell types; unfortunately there is no such thing, but better languages have a better behaviourin this respect (for example **Agda** or **Idris**).

definition of the identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$*, whose correspondences on objects and on arrows* $1_{\mathcal{C},AB} : \mathcal{C}(A,B) \to \mathcal{C}(A,B)$ *are all identity functions.*

2. *As we have hinted in the preceding lesson, all correspondences that* forget *a structure, yielding for example the underlying set* $U\mathbb{M}$ *of a thingum* $\mathbb{M}$*, are functors* $\mathsf{Thng} \to \mathsf{Set}$ *from their structured categories of definition to the category of sets and functions.*

3. *All correspondences that* regard *a structure as an example of another are -more or less tautological- examples of functors: a monoid can be seen as a one-object category, and this yields a functor* $\boldsymbol{B} : \mathsf{Mon} \to \mathsf{Cat}$ *because a functor* $F : \boldsymbol{B}M \to \boldsymbol{B}N$ *between monoids is but a monoid homomorphism* $f : M \to N$*; a preset can be seen as a category with at most one arrow between any two objects, and this defines a functor* ${}^{\chi}(\_) : \mathsf{Pres} \to \mathsf{Cat}$*, because a functor* $F : {}^{\chi}P \to {}^{\chi}Q$ *between presets is but a monotone function* $f : P \to Q$*; a set can be seen as a discrete topological space* A^{δ} *or as a category having only identity arrows, and these are functors* $\mathsf{Set} \to \mathsf{Top}, \mathsf{Cat}$*. (Continue the list at your will.)*

4. *Let* M *be a monoid, and* $\boldsymbol{\Delta}$ *be the category having objects nonempty, finite, totally ordered sets and monotone functions as morphisms. The correspondence that sends* $[n] \in \boldsymbol{\Delta}$ *to the set* $M^{[n]}$ *of ordered* n*-tuples* $[a_1|\dots|a_n]$ *of elements of* M *and a monotone function* $f : [m] \to [n]$ *to the function* $M^f : M^{[n]} \to M^{[m]}$ *defined by*
$$M^f[a_1|\dots|a_n] = [a_{f1}|\dots|a_{fn}]$$
is a functor $\boldsymbol{\Delta}^{op} \to \mathsf{Set}$*. This functor is called the* classifying space *of the monoid.*

5. *Let* Set_* *be the category of* pointed sets*: objects are pairs* (A,a) *where* $a \in A$ *is a distinguished element, and a morphism* $f : (A,a) \to (B,b)$ *is a function* $f : A \to B$ *such that* $fa = b$*. It is easy to prove that this category is isomorphic to the coslice* $*/\mathsf{Set}$ *of functions* $a : * \to A$*.*

6. *Let* $\partial\mathsf{Set}$ *be the category so defined: objects are sets, and a morphism* $(f,S) : A \to B$ *is a pair* $(S \subseteq A, f : S \to B)$*. This is called a* partial function *from* A *to* B*. It is evident how to define the composition of morphisms, and the identity arrow of* $A \in \partial\mathsf{Set}$*. There is a functor* $(\_)_{\bullet} : \partial\mathsf{Set} \to \mathsf{Set}_*$ *defined as* $A \mapsto (A \sqcup \{\infty\}, \infty)$*, where* ∞ *is an element that does not belong to* A*, and a function* (f,S) *goes to* $(f,S)_{\bullet} : A \sqcup \{\infty_A\} \to B \sqcup \{\infty_B\}$*, defined sending* $S^c \cup \{\infty_A\}$ *to* $\{\infty_B\}$*.*

7. *Every directed graph $\underline{G}$, with set of vertices V and set of edges E, defines a quotient set of V by the equivalence relation $\approx$ generated by the subset of those (A, B) for which there is an arrow $A \to B$; the symmetric and transitive closure of this relation yields a quotient V/R that is usually denoted as the set of* connected components $\pi_0(\underline{G})$. *This is a functor* $\mathsf{Gph} \to \mathsf{Set}$*, and if we now regard a category $\mathcal{C}$ as a mere directed graph we obtain a well-defined set $\pi_0(\mathcal{C})$ of connected components of a category.*

1.3 Diagrams or: functors as pictures

Most of category theory is based on the intuition that a diagram of a certain shape in a category $\mathcal{C}$ is a functor $F : \mathcal{J} \to \mathcal{C}$; the definition of a diagram itself is engineered to blur the distinction between a functor as a morphism of categories, and a functor as a correspondence that "pictures" a small category $\mathcal{J}$ inside a big category $\mathcal{C}$. After having recalled the definition of diagram, and the definition of commutative diagram, we collect a series of examples to convince the reader that this identification of "diagrams as pictures" is fruitful and suggestive.

The recommended soundtrack while reading this section is, of course, Emerson, Lake & Palmer's *Pictures at an Exhibition*.

1.3.1 Diagrams, formally

Arrangements of objects and arrows in a category are called *diagrams*; to some extent, category theory is the art of making diagrams *commute*, i.e. the art of proving that two paths $X \to A_1 \to A_2 \to \cdots \to A_n \to Y$ and $X \to B_1 \to \cdots \to B_m \to Y$ result in the same arrow when they are fully composed.

Remark 2 (Diagrams and their commutation)**.** *Depicting morphisms as arrows allows to draw regions of a given category $\mathcal{C}$ as parts of a (possibly non planar) graph; we call a* diagram *such a region in $\mathcal{C}$, the graph whose vertices are objects of $\mathcal{C}$ and whose edges are morphisms of suitable domains* *and codomains. For example, we can consider the diagram*

$$\begin{tikzcd}
 & & X_2 \arrow[drr, "v"] & & \\
X_1 \arrow[urr, "u"] \arrow[rrrr, "h"'] \arrow[dr, "g"'] & & & & X_3 \\
 & X_0 \arrow[uur, "q"] \arrow[rr, "p"'] & & X_4 \arrow[ur, "k"'] &
\end{tikzcd}$$

The presence of a composition rule in $\mathcal{C}$ entails that we can meaningfully compose paths $[u_0, \dots, u_n]$ *of morphisms of $\mathcal{C}$. In particular, we can consider diagrams having distinct paths between a fixed source and a fixed 'sink' (say, in the diagram above, we can consider two different paths $\mathfrak{P} = [k, p, g]$ and $\mathfrak{Q} = [v, u]$); both paths go from X_1 to X_3, and we can ask the two compositions $\circ[k, p, g] = k \circ p \circ g$ and $v \circ u$ to be the same arrow $X_1 \to X_3$; we say that a diagram* commutes at $\mathfrak{P}, \mathfrak{Q}$ *if this is the case; we say that a diagram* commutes *(without mention of $\mathfrak{P}, \mathfrak{Q}$) if it commutes for every choice of paths for which this is meaningful.*

Searching a formalisation of this intuitive pictorial idea leads to the following:

Definition 2. *A* diagram *is a map of directed graphs ('digraphs') $D : J \to \mathcal{C}$ where J is a digraph and $\mathcal{C}$ is the digraph underlying a category.*[3] *Such a diagram D* commutes *if for every pair of parallel edges $f, g : i \rightrightarrows j$ in J one has $Df = Dg$.*

1.3.2 Diagrams, everywhere

- An object is a diagram: the terminal object in the category of categories is the category

  $$[0] = \left\{ \begin{tikzcd} \bullet \arrow[loop right, "1"] \end{tikzcd} \right\}$$

  A functor $C : \mathbf{1} \to \mathcal{C}$ consists of an object of $\mathcal{C}$.
- An arrow is a diagram: let

  $$[1] = \left\{ 0 \longrightarrow 1 \right\}$$

[3] Every small category has an underlying graph, obtained keeping objects and arrows and forgetting all compositions; there is of course a category of graphs, and regarding a category as a graph is another example of forgetful functor. Of course, making this precise means that the collection of categories and functors form a category on its own right.

be the category with two objects $0, 1$ and a single nonidentity morphism $0 \to 1$. A functor $f : \mathbf{2} \to \mathcal{C}$ consists of two objects $X_i = f(i)$ for $i = 0, 1$, and a morphism $X_0 \to X_1$.

- A chain is a diagram: let $n > 2$ be a positive integer. Let

$$[n] = \left\{ \begin{tikzcd} 0 \arrow[r] & 1 \arrow[r] & \cdots \arrow[r] & n \end{tikzcd} \right\}$$

  be the category with exactly $n+1$ objects and exactly one nonidentity arrow $j - 1 \to j$ for $j = 1, \dots, n$. A functor $c : [n] \to \mathcal{C}$ consists of a tuple of objects $X_0 \to X_1 \to \cdots \to X_n$, and morphisms $X_{j-1} \to X_j$ for $j = 1, \dots, n$.

- A span is a diagram: let

$$\mathcal{S} = \left\{ \begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 1 & & 2 \end{tikzcd} \right\}$$

  be the category with three objects and arrows as depicted. A functor $F : \mathcal{S} \to \mathcal{C}$ is a diagram of the same shape, labeled in the same way...

  So, you get the idea. Define a *cospan* in $\mathcal{C}$ to be a functor F with domain the opposite category $\mathcal{S}^{\mathrm{op}}$.

- A commutative square is a diagram: let

$$\begin{tikzcd} (00) \arrow[r] \arrow[d] & (10) \arrow[d] \\ (01) \arrow[r] & (11) \end{tikzcd}$$

  be the category with four objects and nonidentity morphisms forming a commutative square. A functor F with this domain, and values in $\mathcal{C}$, consists of a commutative square of objects in $\mathcal{C}$.

- In a similar fashion, a commutative cube is a diagram: a cube is a morphism between the square of sources and the square of targets. Draw a picture, and formalise this statement (what is the identity cube of a square? How do you compose cubes?).

- A diagram is a diagram is a diagram,..., is a diagram: generalise to n-dimensional cubes.

1.4 General definition of limits and colimits

Definition 3 (Cone completions of $\mathcal{J}$)**.** *Let $\mathcal{J}$ be a small category; we denote $\mathcal{J}^{\triangleright}$ the category obtained adding to $\mathcal{J}$ a single terminal object ∞; more in detail, $\mathcal{J}^{\triangleright}$ has objects $\mathcal{J}_o \cup \{\infty\}$, where $\infty \notin \mathcal{J}$, and it is defined by*

$$\begin{aligned} \mathcal{J}^{\triangleright}(J, J') &= \mathcal{J}(J, J') \\ \mathcal{J}^{\triangleright}(J, \infty) &= \{*\} \end{aligned}$$

and it is empty otherwise. This category is called the right cone *of $\mathcal{J}$.*

Dually, we define a category $\mathcal{J}^{\triangleleft}$, the left cone *of $\mathcal{J}$, as the category obtained adding to J a single* initial *object $-\infty$; this means that $\mathcal{J}^{\triangleleft}(J, J') = \mathcal{J}(J, J')$, $\mathcal{J}^{\triangleleft}(-\infty, J) = \{*\}$, and it is empty otherwise.*

Example 2. *Let ω be the ordinal number obtained from the union of all finite ordinals; then, when regarded as a category, $\omega^{\triangleleft}$ is the category $\{-\infty \to 0 \to 1 \to \dots\}$ (and thus is isomorphic to ω in an evident way), whereas $\omega^{\triangleright}$ is $\omega + 1$, in the sense of ordinal sum.*

For those willing to embark in an extra exercise: what is $G^{\triangleright}$ if G is a monoid regarded as a category?

Remark 3. *The correspondences $\mathcal{J} \mapsto \mathcal{J}^{\triangleright}$ and $\mathcal{J} \mapsto \mathcal{J}^{\triangleleft}$ are functorial. As an easy exercise, define them on morphisms and prove their functoriality. Prove also that there is a natural embedding $i_{\triangleright}$ of $\mathcal{J}$ into $\mathcal{J}^{\triangleright}$ and one $i_{\triangleleft}$ of $\mathcal{J}$ into $\mathcal{J}^{\triangleleft}$ (that we invariably denote i in the following discussion).*

Definition 4 (Cone of a diagram)**.** *Let $\mathcal{J}$ be a small category, $\mathcal{C}$ a category, and $D : \mathcal{J} \to \mathcal{C}$ a functor; all along this section, an idiosyncratic way to refer to D will be as a* diagram *of shape $\mathcal{J}$. We call a* cone *for D any extension of the diagram D to the left cone category of $\mathcal{J}$ defined in , so that the diagram*

$$\begin{tikzcd} \mathcal{J} \arrow[r, "D"] \arrow[d, "i_{\triangleleft}"'] & \mathcal{C} \\ \mathcal{J}^{\triangleleft} \arrow[ur, dotted, "\bar{D}"'] & \end{tikzcd}$$

commutes.

Every such extension is thus forced to coincide with D on all objects in $\mathcal{J} \subseteq \mathcal{J}^{\triangleleft}$; the value of $\bar{D}$ on $-\infty$ is called the base of the cone; dually, the value of an extension of D to $\mathcal{J}^{\triangleright}$ coincides with D on $\mathcal{J} \subseteq \mathcal{J}^{\triangleright}$, and $\bar{D}(\infty)$ is called the *tip* of the cone.

There is of course a similar definition of a *cocone* for D: it is an extension of the diagram D to the right cone category of $\mathcal{J}$ so that the diagram

$$\begin{tikzcd} \mathcal{J} \arrow[r, "D"] \arrow[d, "i_{\triangleright}"'] & \mathcal{C} \\ \mathcal{J}^{\triangleright} \arrow[ur, dotted, "\bar{D}"'] & \end{tikzcd}$$

commutes. Exercise sheds a light on and it substantiates the quote from [**?**] therein: cones for D are exactly cocones for the opposite functor D^{op}.

Remark 4.

- *The class of cones for D forms a category $Cn(D)$, whose morphisms are the natural transformations $\alpha : D' \Rightarrow D'' : \mathcal{J} \to \mathcal{C}$ such that the right whiskering of α with $i : \mathcal{J} \to \mathcal{J}^{\triangleleft}$ coincides with the identity natural transformation of D; this means that a morphism α of this sort is a natural transformation such that*

$$\begin{tikzcd} & \mathcal{J} \arrow[dl, "i"'] \arrow[dr, "D"] & \\ \mathcal{J}^{\triangleleft} \arrow[rr, bend left, "D'"] \arrow[rr, bend right, "D''"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & \mathcal{C} \end{tikzcd} \quad = 1_D$$

  as a 2-cell $D \Rightarrow D$.

- *Dually, the class of cocones for D forms a category $Ccn(D)$, whose morphisms are the natural transformations $\alpha : D' \Rightarrow D'' : \mathcal{J} \to \mathcal{C}$ such that the right whiskering of α with $i : \mathcal{J} \to \mathcal{J}^{\triangleright}$ coincides with the identity natural transformation of D; this means that a morphism α of this sort is a natural transformation such that*

$$\begin{tikzcd} & \mathcal{J} \arrow[dl, "i"'] \arrow[dr, "D"] & \\ \mathcal{J}^{\triangleright} \arrow[rr, bend left, "D'"] \arrow[rr, bend right, "D''"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & \mathcal{C} \end{tikzcd} \quad = 1_D$$

  as a 2-cell $D \Rightarrow D$.

Definition 5 (Colimit, limit)**.** *The* limit *of a diagram $D : \mathcal{J} \to \mathcal{C}$ is the terminal object denoted '$\lim_{\mathcal{J}} D$' in the category of cones for D; dually, the* colimit *of D is the initial object denoted '$\operatorname{colim}_{\mathcal{J}} D$' in the category of cocones for D.*

It is a good idea to unwind definitions and in order to obtain the more classically explained notion of limit and colimit: the key for this unwinding operation is that a co/cone for $D : \mathcal{J} \to \mathcal{C}$ amounts to a natural family of maps from a constant object (the base) or to a constant object (the tip).

- A *cone* for a diagram $D : \mathcal{J} \to \mathcal{C}$ is a natural transformation from a constant functor $\Delta_c : \mathcal{J} \to \mathcal{C}$ to $D(\_)$;
- there is a category of cones for $D(\_)$, where morphisms between a cone $c \to D(\_)$ and a cone $C' \to D(\_)$ are arrows $k : C \to C'$ such that the diagram

$$\begin{tikzcd}
DI \arrow[dd, "D\phi"'] & & \\
& C \arrow[ul, "l_I"'] \arrow[dl, "l_J"] \arrow[r, dotted, "k"] & C' \arrow[ull, "l'_I"'] \arrow[dll, "l'_J"] \\
DJ & &
\end{tikzcd}$$

  is commutative;
- a limit for D is a terminal object in the category of cones for D. This means that given a cone for D, there is a unique arrow k which is a morphism of cones.

Of course, a straightforward dualisation of diagram (**??**) yields the definition of a cocone, and a colimit for D.

1.5 Examples of limits and colimits

1.5.1 in monoids and Mon

1.5.2 in posets and Pos

1.5.3 in Set, and other algebraic categories

1.5.4 Exercises

Exercises denoted with a star symbol are supposed to be difficult. Don't be put off, and enjoy!

1.