Category theory course Lecture 4 ITI9200, Spring 2020

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1 Functors

1.1 Morphisms of categories

Definition 1 (Functor). Let C and D be two categories; we define a functor $F: C \to D$ as a pair (F_0, F_1) consisting of the following data:

- F_0 is a function $C_o \to \mathcal{D}_o$ sending an object $C \in C_o$ to an object $FC \in \mathcal{D}_o$;
- F_1 is a family of functions $F_{AB}: \mathcal{C}(A,B) \to \mathcal{D}(FA,FB)$, one for each pair of objects $A,B \in \mathcal{C}_o$, sending each arrow $f:A \to B$ into an arrow $Ff:FA \to FB$, and such that:
 - $F_{AA}(1_A) = 1_{FA};$
 - $F_{AC}(g \circ_{\mathcal{C}} f) = F_{BC}(g) \circ_{\mathcal{D}} F_{AB}(f).$

```
-- | All instances of the `Functor` type-class must satisfy two laws.
-- These laws are not checked by the compiler.
-- * The law of identity
-- `forall x. (id <$> x) ~ x`
-- 
-- * The law of composition
-- `forall f g x.(f . g <$> x) ~ (f <$> (g <$> x))`

class Functor k where
-- Pronounced, eff-map.

(<$>) :: (a -> b) -> k a -> k b
```

If $F: \mathcal{C} \to \mathcal{D}$ is a functor, it corresponds to a data constructor f:: * -> * (this is the F_0 part, a function on objects from the collection of object \mathcal{C}_o of \mathcal{C} to the collection of objects \mathcal{D}_o^{-1}) and a function on morphisms

```
<$> :: (a -> b) -> f a -> f b
```

If you recall how the associativity of \rightarrow works, it is evident that this is a function from the type of maps $a \rightarrow b$ to the type of maps $f a \rightarrow f b$; or rather, this is a function that given a function $u :: a \rightarrow b$, and an "element" of type f a, yields an element x :: f b.

Remark 1. There is a category whose objects are (small) categories, and whose morphisms $\mathcal{C} \to \mathcal{D}$ are functors. The identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ of a category \mathcal{C} , and the composition $G \circ F : \mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ of two functors are defined in the obvious way; all category axioms follow.²

1.2 Examples of functors

As you can imagine, both Haskell and mathematics are crawling with examples of functors;

Example 1 (Examples of functors).

1. Let's rule out a few edge examples: for every category C, there is a unique functor $F: \varnothing \to C$, where \varnothing is the empty category with no objects an no morphisms; for every category C, there is an obvious

¹Although in Haskell, $\mathcal{C} = \mathcal{D}$ is always the same nameless category *.

²The reader might now wonder if there is a category of Haskell types; unfortunately there is no such thing, but better languages have a better behaviourin this respect (for example Agda or Idris).

definition of the identity functor $1_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$, whose correspondences on objects and on arrows $1_{\mathcal{C},AB}: \mathcal{C}(A,B) \to \mathcal{C}(A,B)$ are all identity functions.

- 2. As we have hinted in the preceding lesson, all correspondences that forget a structure, yielding for example the underlying set $U\mathbb{M}$ of a thingum \mathbb{M} , are functors Thng \rightarrow Set from their structured categories of definition to the category of sets and functions.
- 3. All correspondences that regard a structure as an example of another are -more or less tautological- examples of functors: a monoid can be seen as a one-object category, and this yields a functor B: Mon → Cat because a functor F: BM → BN between monoids is but a monoid homomorphism f: M → N; a preset can be seen as a category with at most one arrow between any two objects, and this defines a functor ^X(_): Pres → Cat, because a functor F: ^XP → ^XQ between presets is but a monotone function f: P → Q; a set can be seen as a discrete topological space A^δ or as a category having only identity arrows, and these are functors Set → Top, Cat. (Continue the list at your will.)
- 4. Let M be a monoid, and Δ be the category having objects nonempty, finite, totally ordered sets and monotone functions as morphisms. The correspondence that sends $[n] \in \Delta$ to the set $M^{[n]}$ of ordered n-tuples $[a_1|\ldots|a_n]$ of elements of M and a monotone function $f:[m] \to [n]$ to the function $M^f:M^{[n]} \to M^{[n]}$ defined by

$$M^f[a_1|\dots|a_n] = [a_{f1}|\dots|a_{fn}]$$

is a functor $\Delta^{op} \to \mathsf{Set}$. This functor is called the classifying space of the monoid.

- 5. Let Set_* be the category of pointed sets: objects are pairs (A,a) where $a \in A$ is a distinguished element, and a morphism $f: (A,a) \to (B,b)$ is a function $f: A \to B$ such that fa = b. It is easy to prove that this category is isomorphic to the coslice $*/\mathsf{Set}$ of functions $a: * \to A$.
- 6. Let ∂Set be the category so defined: objects are sets, and a morphism (f,S): A → B is a pair (S ⊆ A, f:S → B). This is called a partial function from A to B. It is evident how to define the composition of morphisms, and the identity arrow of A ∈ ∂Set. There is a functor (_)•: ∂Set → Set* defined as A → (A □ {∞}, ∞), where ∞ is an element that does not belong to A, and a function (f, S) goes to (f, S)•: A □ {∞A} → B □ {∞B}, defined sending S^c ∪ {∞A} to {∞B}.

7. Every directed graph \underline{G} , with set of vertices V and set of edges E, defines a quotient set of V by the equivalence relation \approx generated by the subset of those (A,B) for which there is an arrow $A \to B$; the symmetric and transitive closure of this relation yields a quotient V/R that is usually denoted as the set of connected components $\pi_0(\underline{G})$. This is a functor $Gph \to Set$, and if we now regard a category C as a mere directed graph we obtain a well-defined set $\pi_0(C)$ of connected components of a category.

1.3 Diagrams or: functors as pictures

Most of category theory is based on the intuition that a diagram of a certain shape in a category \mathcal{C} is a functor $F:\mathcal{J}\to\mathcal{C}$; the definition of a diagram itself is engineered to blur the distinction between a functor as a morphism of categories, and a functor as a correspondence that "pictures" a small category \mathcal{J} inside a big category \mathcal{C} . After having recalled the definition of diagram, and the definition of commutative diagram, we collect a series of examples to convince the reader that this identification of "diagrams as pictures" is fruitful and suggestive.

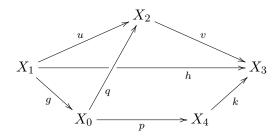
The recommended soundtrack while reading this section is, of course, Emerson, Lake & Palmer's *Pictures at an Exhibition*.

1.3.1 Diagrams, formally

Arrangements of objects and arrows in a category are called diagrams; to some extent, category theory is the art of making diagrams commute, i.e. the art of proving that two paths $X \to A_1 \to A_2 \to \cdots \to A_n \to Y$ and $X \to B_1 \to \cdots \to B_m \to Y$ result in the same arrow when they are fully composed.

Remark 2 (Diagrams and their commutation). Depicting morphisms as arrows allows to draw regions of a given category C as parts of a (possibly non planar) graph; we call a diagram such a region in C, the graph whose vertices are objects of C and whose edges are morphisms of suitable domains

and codomains. For example, we can consider the diagram



The presence of a composition rule in \mathcal{C} entails that we can meaningfully compose paths $[u_0,\ldots,u_n]$ of morphisms of \mathcal{C} . In particular, we can consider diagrams having distinct paths between a fixed source and a fixed 'sink' (say, in the diagram above, we can consider two different paths $\mathfrak{P} = [k,p,g]$ and $\mathfrak{Q} = [v,u]$); both paths go from X_1 to X_3 , and we can ask the two compositions $\circ [k,p,g] = k \circ p \circ g$ and $v \circ u$ to be the same arrow $X_1 \to X_3$; we say that a diagram commutes at $\mathfrak{P},\mathfrak{Q}$ if this is the case; we say that a diagram commutes (without mention of $\mathfrak{P},\mathfrak{Q}$) if it commutes for every choice of paths for which this is meaningful.

Searching a formalisation of this intuitive pictorial idea leads to the following:

Definition 2. A diagram is a map of directed graphs ('digraphs') $D: J \to \mathcal{C}$ where J is a digraph and \mathcal{C} is the digraph underlying a category.³ Such a diagram D commutes if for every pair of parallel edges $f, g: i \rightrightarrows j$ in J one has Df = Dg.

1.3.2 Diagrams, everywhere

• An object is a diagram: the terminal object in the category of categories is the category

$$[0] = \left\{ \bullet \bigcirc 1 \right\}$$

A functor $C: \mathbf{1} \to \mathcal{C}$ consists of an object of \mathcal{C} .

• An arrow is a diagram: let

$$[1] = \left\{ 0 \longrightarrow 1 \right\}$$

³Every small category has an underlying graph, obtained keeping objects and arrows and forgetting all compositions; there is of course a category of graphs, and regarding a category as a graph is another example of forgetful functor. Of course, making this precise means that the collection of categories and functors form a category on its own right.

be the category with two objects 0, 1 and a single nonidentity morphism $0 \to 1$. A functor $f : \mathbf{2} \to \mathcal{C}$ consists of two objects $X_i = f(i)$ for i = 0, 1, and a morphism $X_0 \to X_1$.

• A chain is a diagram: let n > 2 be a positive integer. Let

$$[n] = \left\{ 0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n \right\}$$

be the category with exactly n+1 objects and exactly one nonidentity arrow $j-1 \to j$ for $j=1,\ldots,n$. A functor $c:[n] \to \mathcal{C}$ consists of a tuple of objects $X_0 \to X_1 \to \cdots \to X_n$, and morphisms $X_{j-1} \to X_j$ for $j=1,\ldots,n$.

• A span is a diagram: let

$$\mathcal{S} = \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\}$$

be the category with three objects and arrows as depicted. A functor $F: \mathcal{S} \to \mathcal{C}$ is a diagram of the same shape, labeled in the same way...

So, you get the idea. Define a *cospan* in \mathcal{C} to be a functor F with domain the opposite category \mathcal{S}^{op} .

• A commutative square is a diagram: let

$$(00) \longrightarrow (10)$$

$$\downarrow \qquad \qquad \downarrow$$

$$(01) \longrightarrow (11)$$

be the category with four objects and nonidentity morphisms forming a commutative square. A functor F with this domain, and values in C, consists of a commutative square of objects in C.

- In a similar fashion, a commutative cube is a diagram: a cube is a morphism between the square of sources and the square of targets. Draw a picture, and formalise this statement (what is the identity cube of a square? How do you compose cubes?).
- A diagram is a diagram,..., is a diagram: generalise to *n*-dimensional cubes.

1.4 General definition of limits and colimits

Definition 3 (Cone completions of \mathcal{J}). Let \mathcal{J} be a small category; we denote $\mathcal{J}^{\triangleright}$ the category obtained adding to \mathcal{J} a single terminal object ∞ ; more in detail, $\mathcal{J}^{\triangleright}$ has objects $\mathcal{J}_o \cup \{\infty\}$, where $\infty \notin \mathcal{J}$, and it is defined by

$$\mathcal{J}^{\triangleright}(J, J') = \mathcal{J}(J, J')$$
$$\mathcal{J}^{\triangleright}(J, \infty) = \{*\}$$

and it is empty otherwise. This category is called the right cone of \mathcal{J} .

Dually, we define a category $\mathcal{J}^{\triangleleft}$, the left cone of \mathcal{J} , as the category obtained adding to J a single initial object $-\infty$; this means that $\mathcal{J}^{\triangleleft}(J,J') = \mathcal{J}(J,J')$, $\mathcal{J}^{\triangleleft}(-\infty,J) = \{*\}$, and it is empty otherwise.

Example 2. Let ω be the ordinal number obtained from the union of all finite ordinals; then, when regarded as a category, ω^{\triangleleft} is the category $\{-\infty \to 0 \to 1 \to \ldots\}$ (and thus is isomorphic to ω in an evident way), whereas ω^{\triangleright} is $\omega + 1$, in the sense of ordinal sum.

For those willing to embark in an extra exercise: what is G^{\triangleright} if G is a monoid regarded as a category?

Remark 3. The correspondences $\mathcal{J} \mapsto \mathcal{J}^{\triangleright}$ and $\mathcal{J} \mapsto \mathcal{J}^{\triangleleft}$ are functorial. As an easy exercise, define them on morphisms and prove their functoriality. Prove also that there is a natural embedding i_{\triangleright} of \mathcal{J} into $\mathcal{J}^{\triangleright}$ and one i_{\triangleleft} of \mathcal{J} into $\mathcal{J}^{\triangleleft}$ (that we invariably denote i in the following discussion).

Definition 4 (Cone of a diagram). Let \mathcal{J} be a small category, \mathcal{C} a category, and $D: \mathcal{J} \to \mathcal{C}$ a functor; all along this section, an idiosyncratic way to refer to D will be as a diagram of shape \mathcal{J} . We call a cone for D any extension of the diagram D to the left cone category of \mathcal{J} defined in , so that the diagram

$$\begin{array}{c|c}
\mathcal{J} \xrightarrow{D} \mathcal{C} \\
\downarrow i_{\triangleleft} \downarrow & \bar{D}
\end{array}$$

$$\mathcal{J}^{\triangleleft}$$

commutes.

Every such extension is thus forced to coincide with D on all objects in $\mathcal{J} \subseteq \mathcal{J}^{\triangleleft}$; the value of \bar{D} on $-\infty$ is called the base of the cone; dually, the value of an extension of D to $\mathcal{J}^{\triangleright}$ coincides with D on $\mathcal{J} \subseteq \mathcal{J}^{\triangleright}$, and $\bar{D}(\infty)$ is called the tip of the cone.

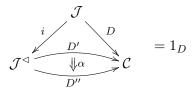
There is of course a similar definition of a *cocone* for D: it is an extension of the diagram D to the right cone category of \mathcal{J} so that the diagram

$$\begin{array}{c|c}
\mathcal{J} & \xrightarrow{D} \mathcal{C} \\
\downarrow i_{\triangleright} & \downarrow \bar{D} \\
\mathcal{J}^{\triangleright}
\end{array}$$

commutes. Exercise sheds a light on and it substantiates the quote from [?] therein: cones for D are exactly cocones for the opposite functor D^{op} .

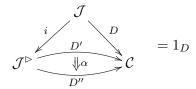
Remark 4.

• The class of cones for D forms a category Cn(D), whose morphisms are the natural transformations $\alpha: D' \Rightarrow D'': \mathcal{J} \to \mathcal{C}$ such that the right whiskering of α with $i: \mathcal{J} \to \mathcal{J}^{\triangleleft}$ coincides with the identity natural transformation of D; this means that a morphism α of this sort is a natural transformation such that



as a 2-cell $D \Rightarrow D$.

• Dually, the class of cocones for D forms a category Ccn(D), whose morphisms are the natural transformations $\alpha: D' \Rightarrow D'': \mathcal{J} \to \mathcal{C}$ such that the right whiskering of α with $i: \mathcal{J} \to \mathcal{J}^{\triangleright}$ coincides with the identity natural transformation of D; this means that a morphism α of this sort is a natural transformation such that

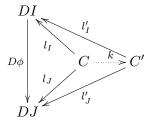


as a 2-cell $D \Rightarrow D$.

Definition 5 (Colimit, limit). The limit of a diagram $D: \mathcal{J} \to \mathcal{C}$ is the terminal object denoted $\lim_{\mathcal{J}} D$ in the category of cones for D; dually, the colimit of D is the initial object denoted 'colim $_{\mathcal{J}} D$ ' in the category of cocones for D.

It is a good idea to unwind definitions and in order to obtain the more classically explained notion of limit and colimit: the key for this unwinding operation is that a co/cone for $D: \mathcal{J} \to \mathcal{C}$ amounts to a natural family of maps from a constant object (the base) or to a constant object (the tip).

- A cone for a diagram $D: \mathcal{J} \to \mathcal{C}$ is a natural transformation from a constant functor $\Delta_c: \mathcal{J} \to \mathcal{C}$ to $D(\underline{\ })$;
- there is a category of cones for $D(_)$, where morphisms between a cone $c \to D(_)$ and a cone $C' \to D(_)$ are arrows $k: C \to C'$ such that the diagram



is commutative;

• a limit for D is a terminal object in the category of cones for D. This means that given a cone for D, there is a unique arrow k which is a morphism of cones.

Of course, a straightforward dualisation yields the definition of a cocone, and a colimit for D.

1.5 Examples of limits and colimits

We start with a classical edge example: a terminal object is the limit of the empty diagram.

Example 3. Let $J = \emptyset$ be the empty set; the limit of the unique diagram $D: \mathcal{J} \to \mathcal{C}$ is the terminal object of \mathcal{C} .

The universal property exhibited by the terminal object * of \mathcal{C} is the following: there is a unique morphism $C \to *$ for every $C \in \mathcal{C}$ (with no other condition, since \mathcal{J} is empty).

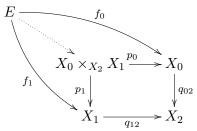
Example 4 (Product). Let \mathcal{J} be a set, and $\{X_i \mid i \in \mathcal{J}\}$ a family of objects of a category \mathcal{C} ; the product of the $\{X_i\}$'s, denoted $\prod_{i \in \mathcal{J}} X_i$, is the limit of the diagram $D: \mathcal{J} \to \mathcal{C}$, when the set \mathcal{J} is regarded as a discrete category.

The universal property exhibited by the object $\prod_{i \in \mathcal{J}} X_i$ is the following: there is a cone $\bar{p} = \{p_i : \prod X_j \to X_i \mid i \in \mathcal{J}\}$ such that $(\bar{p}(-\infty)) = \prod X_j$ and) for every other cone $\{f_i : E \to X_i \mid i \in \mathcal{J}\}$ there exists a unique dotted $\bar{f} : E \to \prod_{i \in \mathcal{J}} X_i$ such that



commutes for every $\bar{\imath} \in \mathcal{J}$.

Example 5 (Pullback). Let \mathcal{J} be the category $0 \to 2 \leftarrow 1$, and $\{X_0 \to X_2 \leftarrow X_1\}$ the corresponding diagram $X: \mathcal{J} \to \mathcal{C}$; the pullback of the diagram X, denoted $X_0 \times_{X_2} X_1$, is the limit of X; the universal property exhibited by the object $X_0 \times_{X_2} X_1$ is the following: there is a cone $X_0 \stackrel{p_0}{\leftarrow} X_0 \times_{X_2} X_1 \stackrel{p_1}{\rightarrow} X_1$ such that for every other cone $X_0 \stackrel{f_0}{\leftarrow} E \stackrel{f_1}{\rightarrow} X_1$ there exists a unique dotted $\langle f_0, f_1 \rangle$ such that



In the same notation above, when we want to stress the dependence of the pullback from the maps q_{02}, q_{12} , the object is sometimes denoted as $q_{02} \times q_{12}$ instead of $X_0 \times_{X_2} X_1$. This is not a real clash of notation, as it is possible to prove that $q_{02} \times q_{12}$ is the product (in the sense of 4 above) of q_{02}, q_{12} regarded as objects of the slice category C/X_2 .

In the category of sets, the pullback $X_0 \times_{X_2} X_1$ of a pair f_0, f_1 can be easily characterised as the subset of $X_0 \times X_1$ made by all pairs (x_0, x_1) such that $f_0(x_0) = f_1(x_1)$.

Example 6 (Equaliser). Let \mathcal{J} be the category $0 \rightrightarrows 1$, and $\{X_0 \overset{u}{\rightrightarrows} X_1\}$ the corresponding diagram $X: \mathcal{J} \to \mathcal{C}$; the equaliser of the diagram X, denoted eq(u,v), is the limit of X; the universal property exhibited by the object is the following: there is an cone $e: eq(u,v) \to X_0$ and for every other cone

 $k: E \to X_0$ there is a unique dotted $\bar{k}: E \to eq(u, v)$ such that

$$eq(u,v) \xrightarrow{e} X_0 \xrightarrow{u} X_1$$

$$\downarrow k \qquad \downarrow k$$

$$E$$

In the category of sets, the equaliser of a pair of maps u, v can be easily characterised as the subset of X_0 made by all elements such that u(x) = v(x); it is 'the largest subset of X_0 where u = v'.

1.5.1 Exercises

Exercises denoted with a star symbol are supposed to be difficult. Don't be put off, and enjoy!

- 1. Give instances of functors for (,) a, Maybe and Either. The same proof shows that the corresponding maps $Set \rightarrow Set$ are indeed functors.
- 2. Show that there is a "tautological" functor $i: \mathsf{Set} \to \mathsf{Cat}$ sending a set to itself, regarded as a discrete category. Given a set A and a category \mathcal{C} , show that there is a bijection between the set of functions $\pi_0 \mathcal{C} \to A$ and the set of functors $\mathcal{C}toiA$.
- 3. (\star) Let $F: \mathcal{C} \to \mathcal{Z}$ and $G: \mathcal{D} \to \mathcal{Z}$ be two functors; define the *comma category* of F, G as the category whose
 - objects are arrows in \mathcal{Z} of the form $FC \xrightarrow{f} GD$ (more formally, an object is a tuple $(C, D, f : \mathcal{Z}(FC, GD))$);
 - morphisms with source $f:FC\to GD$ and target $f':FC'\to GD'$ are pairs $u:C\to C', v:D\to D'$ such that the square

$$FC \xrightarrow{f} GD$$

$$Fu \downarrow Gv$$

$$FC' \xrightarrow{f'} GD'$$

is commutative. Show that (F/G) is indeed a category. Show that (F/G) has the following universal property: there is a commutative square

$$(F/G) \xrightarrow{P} \mathcal{Z}^{\rightarrow}$$

$$\downarrow J$$

$$\mathcal{C} \times \mathcal{D} \xrightarrow{F \times G} \mathcal{Z} \times \mathcal{Z}$$

and for every other commutative square

$$\begin{array}{c|c} \mathcal{X} & \xrightarrow{H} & \mathcal{Z}^{\rightarrow} \\ \downarrow & & \downarrow J \\ \mathcal{C} \times \mathcal{D} & \xrightarrow{F \times G} & \mathcal{Z} \times \mathcal{Z} \end{array}$$

there is a unique functor $\langle H,K\rangle:\mathcal{X}\to (F/G)$ such that $P\circ\langle H,K\rangle=H$ and $Q\circ\langle H,K\rangle=K$. In other words, (F/G) is the pullback of J and $F\times G$.