Category theory course Lecture 4 ITI9200, Spring 2020

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	Functors: morphisms of categories

0.1 Functors: morphisms of categories

Si prende un foglio di esercizi di data61 che ti costringono a implementare istanze di Functor per aggeggi di uso comune (le liste, gli alberi, Maybe, Either etcetera) e fargli vedere prima come si scrive in hs, e poi come si scrive in CT, per far loro capire che sono la stessa cosa

Definition 1 (Functor). Let C and D be two categories; we define a functor $F: C \to D$ as a pair (F_0, F_1) consisting of the following data:

- F_0 is a function $C_o \to \mathcal{D}_o$ sending an object $C \in C_o$ to an object $FC \in \mathcal{D}_o$;
- F_1 is a family of functions $F_{AB}: \mathcal{C}(A,B) \to \mathcal{D}(FA,FB)$, one for each pair of objects $A,B \in \mathcal{C}_o$, sending each arrow $f:A \to B$ into an arrow $Ff:FA \to FB$, and such that:
 - $-F_{AA}(1_A) = 1_{FA};$ $-F_{AC}(g \circ_{\mathcal{C}} f) = F_{BC}(g) \circ_{\mathcal{D}} F_{AB}(f).$

```
-- | All instances of the `Functor` type-class must satisfy two laws.
-- These laws are not checked by the compiler.
-- * The law of identity
-- `forall x. (id <$> x) ~ x`
--
-- * The law of composition
-- `forall f g x.(f . g <$> x) ~ (f <$> (g <$> x))`
class Functor k where
-- Pronounced, eff-map.
(<$>) :: (a -> b) -> k a -> k b
```

If $F: \mathcal{C} \to \mathcal{D}$ is a functor, it corresponds to a data constructor f:: * -> * (this is the F_0 part, a function on objects from the collection of object \mathcal{C}_o of \mathcal{C} to the collection of objects \mathcal{D}_o^{-1}) and a function on morphisms

```
<$> :: (a -> b) -> f a -> f b
```

If you recall how the associativity of \rightarrow works, it is evident that this is a function from the type of maps $a \rightarrow b$ to the type of maps $f a \rightarrow f b$; or rather, this is a function that given a function $u :: a \rightarrow b$, and an "element" of type f a, yields an element x :: f b.

0.2 Examples of functors

As you can imagine, both Haskell and mathematics are crawling with examples of functors;

Example 1 (Examples of functors).

- 1. Let's rule out a few edge examples: for every category C, there is a unique functor $F: \varnothing \to C$, where \varnothing is the empty category with no objects an no morphisms; for every category C, there is an obvious definition of the identity functor $1_C: C \to C$, whose correspondences on objects and on arrows $1_{C,AB}: C(A,B) \to C(A,B)$ are all identity functions.
- 2. As we have hinted in the preceding lesson, all correspondences that forget a structure, yielding for example the underlying set UM of a thingum M, are functors Thng \rightarrow Set from their structured categories of definition to the category of sets and functions.

¹Although in Haskell, C = D is always the same nameless category *.

- 3. All correspondences that regard a structure as an example of another are -more or less tautological- examples of functors: a monoid can be seen as a one-object category, and this yields a functor B: Mon → Cat because a functor F: BM → BN between monoids is but a monoid homomorphism f: M → N; a preset can be seen as a category with at most one arrow between any two objects, and this defines a functor ^X(_): Pres → Cat, because a functor F: ^XP → ^XQ between presets is but a monotone function f: P → Q; a set can be seen as a discrete topological space A^δ or as a category having only identity arrows, and these are functors Set → Top, Cat. (Continue the list at your will.)
- 4. Let M be a monoid, and Δ be the category having objects nonempty, finite, totally ordered sets and monotone functions as morphisms. The correspondence that sends $[n] \in \Delta$ to the set $M^{[n]}$ of ordered n-tuples $[a_1|\ldots|a_n]$ of elements of M and a monotone function $f:[m] \to [n]$ to the function $M^f:M^{[n]} \to M^{[n]}$ defined by

$$M^f[a_1|\dots|a_n] = [a_{f1}|\dots|a_{fn}]$$

is a functor $\Delta^{op} \to \mathsf{Set}$. This functor is called the classifying space of the monoid.

- 5. Let Set_* be the category of pointed sets: objects are pairs (A,a) where $a \in A$ is a distinguished element, and a morphism $f: (A,a) \to (B,b)$ is a function $f: A \to B$ such that fa = b. It is easy to prove that this category is isomorphic to the coslice */Set of functions $a: * \to A$.
- 6. Let $\partial \mathsf{Set}$ be the category so defined: objects are sets, and a morphism $(f,S):A\to B$ is a pair $(S\subseteq A,f:S\to B)$. This is called a partial function from A to B. It is evident how to define the composition of morphisms, and the identity arrow of $A\in \partial \mathsf{Set}$. There is a functor $(_)_{\bullet}:\partial \mathsf{Set}\to \mathsf{Set}_*$ defined as $A\mapsto (A\sqcup \{\infty\},\infty)$, where ∞ is an element that does not belong to A, and a function (f,S) goes to $(f,S)_{\bullet}:A\sqcup \{\infty_A\}\to B\sqcup \{\infty_B\}$, defined sending $S^c\cup \{\infty_A\}$ to $\{\infty_B\}$.
- 7. Every directed graph \underline{G} , with set of vertices V and set of edges E, defines a quotient set of V by the equivalence relation \approx generated by the subset of those (A, B) for which there is an arrow $A \to B$; the symmetric and transitive closure of this relation yields a quotient V/R that is usually denoted as the set of connected components $\pi_0(\underline{G})$. This is a functor $\mathsf{Gph} \to \mathsf{Set}$, and if we now regard a category $\mathcal C$ as a mere directed graph we obtain a well-defined set $\pi_0(\mathcal C)$ of connected components of a category.

0.3 Diagrams or: functors as pictures

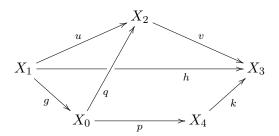
Most of category theory is based on the intuition that a diagram of a certain shape in a category \mathcal{C} is a functor $F: \mathcal{J} \to \mathcal{C}$; the definition of a diagram itself is engineered to blur the distinction between a functor as a morphism of categories, and a functor as a correspondence that "pictures" a small category \mathcal{J} inside a big category \mathcal{C} . After having recalled the definition of diagram, and the definition of commutative diagram, we collect a series of examples to convince the reader that this identification of "diagrams as pictures" is fruitful and suggestive.

The recommended soundtrack while reading this section is, of course, Emerson, Lake & Palmer's *Pictures at an Exhibition*.

0.3.1 Diagrams, formally

Arrangements of objects and arrows in a category are called diagrams; to some extent, category theory is the art of making diagrams commute, i.e. the art of proving that two paths $X \to A_1 \to A_2 \to \cdots \to A_n \to Y$ and $X \to B_1 \to \cdots \to B_m \to Y$ result in the same arrow when they are fully composed.

Remark 1 (Diagrams and their commutation). Depicting morphisms as arrows allows to draw regions of a given category C as parts of a (possibly non planar) graph; we call a diagram such a region in C, the graph whose vertices are objects of C and whose edges are morphisms of suitable domains and codomains. For example, we can consider the diagram



The presence of a composition rule in C entails that we can meaningfully compose paths $[u_0, \ldots, u_n]$ of morphisms of C. In particular, we can consider diagrams having distinct paths between a fixed source and a fixed 'sink' (say, in the diagram above, we can consider two different paths $\mathfrak{P} = [k, p, g]$ and $\mathfrak{Q} = [v, u]$); both paths go from X_1 to X_3 , and we can ask the two compositions $\circ [k, p, g] = k \circ p \circ g$ and $v \circ u$ to be the same arrow $X_1 \to X_3$; we say that a diagram commutes at $\mathfrak{P}, \mathfrak{Q}$ if this is the case; we say that

a diagram commutes (without mention of $\mathfrak{P}, \mathfrak{Q}$) if it commutes for every choice of paths for which this is meaningful.

Searching a formalisation of this intuitive pictorial idea leads to the following:

Definition 2. A diagram is a map of directed graphs ('digraphs') $D: J \to \mathcal{C}$ where J is a digraph and \mathcal{C} is the digraph underlying a category.² Such a diagram D commutes if for every pair of parallel edges $f, g: i \rightrightarrows j$ in J one has Df = Dg.

0.3.2 Diagrams, everywhere

- An object is a diagram
- A morphism is a diagram
- A chain is a diagram
- A span is a diagram
- A cospan is a diagram
- A square is a diagram
- A cube is a diagram
- A diagram is a diagram is a diagram,..., is a diagram

0.4 General definition of limits and colimits

Definition 3 (Cone completions of \mathcal{J}). Let \mathcal{J} be a small category; we denote $\mathcal{J}^{\triangleright}$ the category obtained adding to \mathcal{J} a single terminal object ∞ ; more in detail, $\mathcal{J}^{\triangleright}$ has objects $\mathcal{J}_o \cup \{\infty\}$, where $\infty \notin \mathcal{J}$, and it is defined by

$$\mathcal{J}^{\triangleright}(J, J') = \mathcal{J}(J, J')$$
$$\mathcal{J}^{\triangleright}(J, \infty) = \{*\}$$

and it is empty otherwise. This category is called the right cone of \mathcal{J} .

Dually, we define a category $\mathcal{J}^{\triangleleft}$, the left cone of \mathcal{J} , as the category obtained adding to J a single initial object $-\infty$; this means that $\mathcal{J}^{\triangleleft}(J,J') = \mathcal{J}(J,J')$, $\mathcal{J}^{\triangleleft}(-\infty,J) = \{*\}$, and it is empty otherwise.

²Every small category has an underlying graph, obtained keeping objects and arrows and forgetting all compositions; there is of course a category of graphs, and regarding a category as a graph is another example of forgetful functor. Of course, making this precise means that the collection of categories and functors form a category on its own right.

Example 2. Let ω be the ordinal number obtained from the union of all finite ordinals; then, when regarded as a category, ω^{\triangleleft} is the category $\{-\infty \to 0 \to 1 \to \dots\}$ (and thus is isomorphic to ω in an evident way), whereas ω^{\triangleright} is $\omega + 1$, in the sense of ordinal sum.

For those willing to embark in an extra exercise: what is G^{\triangleright} if G is a monoid regarded as a category?

Remark 2. The correspondences $\mathcal{J} \mapsto \mathcal{J}^{\triangleright}$ and $\mathcal{J} \mapsto \mathcal{J}^{\triangleleft}$ are functorial. As an easy exercise, define them on morphisms and prove their functoriality. Prove also that there is a natural embedding i_{\triangleright} of \mathcal{J} into $\mathcal{J}^{\triangleright}$ and one i_{\triangleleft} of \mathcal{J} into $\mathcal{J}^{\triangleleft}$ (that we invariably denote i in the following discussion).

Definition 4 (Cone of a diagram). Let \mathcal{J} be a small category, \mathcal{C} a category, and $D: \mathcal{J} \to \mathcal{C}$ a functor; all along this section, an idiosyncratic way to refer to D will be as a diagram of shape \mathcal{J} . We call a cone for D any extension of the diagram D to the left cone category of \mathcal{J} defined in , so that the diagram

$$\begin{array}{c|c}
\mathcal{J} & \xrightarrow{D} \mathcal{C} \\
\downarrow i_{\triangleleft} & \downarrow \bar{D} \\
\mathcal{J}^{\triangleleft}
\end{array}$$

commutes.

Every such extension is thus forced to coincide with D on all objects in $\mathcal{J}\subseteq\mathcal{J}^{\lhd}$; the value of \bar{D} on $-\infty$ is called the base of the cone; dually, the value of an extension of D to $\mathcal{J}^{\triangleright}$ coincides with D on $\mathcal{J}\subseteq\mathcal{J}^{\triangleright}$, and $\bar{D}(\infty)$ is called the tip of the cone.

There is of course a similar definition of a *cocone* for D: it is an extension of the diagram D to the right cone category of \mathcal{J} so that the diagram

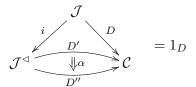
$$\begin{array}{c|c}
\mathcal{J} & \xrightarrow{D} & \mathcal{C} \\
\downarrow i_{\triangleright} & & \bar{D} \\
\mathcal{J}^{\triangleright} & & & \\
\end{array}$$

commutes. Exercise sheds a light on and it substantiates the quote from [?] therein: cones for D are exactly cocones for the opposite functor D^{op} .

Remark 3.

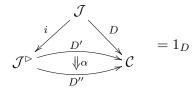
• The class of cones for D forms a category Cn(D), whose morphisms are the natural transformations $\alpha: D' \Rightarrow D'': \mathcal{J} \to \mathcal{C}$ such that the

right whiskering of α with $i: \mathcal{J} \to \mathcal{J}^{\triangleleft}$ coincides with the identity natural transformation of D; this means that a morphism α of this sort is a natural transformation such that



as a 2-cell $D \Rightarrow D$.

• Dually, the class of cocones for D forms a category Ccn(D), whose morphisms are the natural transformations $\alpha: D' \Rightarrow D'': \mathcal{J} \to \mathcal{C}$ such that the right whiskering of α with $i: \mathcal{J} \to \mathcal{J}^{\triangleright}$ coincides with the identity natural transformation of D; this means that a morphism α of this sort is a natural transformation such that



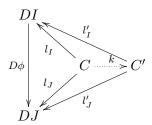
as a 2-cell $D \Rightarrow D$.

Definition 5 (Colimit, limit). The limit of a diagram $D: \mathcal{J} \to \mathcal{C}$ is the terminal object denoted $\lim_{\mathcal{J}} D$ in the category of cones for D; dually, the colimit of D is the initial object denoted 'colim $_{\mathcal{J}} D$ ' in the category of cocones for D.

It is a good idea to unwind definitions and in order to obtain the more classically explained notion of limit and colimit: the key for this unwinding operation is that a co/cone for $D: \mathcal{J} \to \mathcal{C}$ amounts to a natural family of maps from a constant object (the base) or to a constant object (the tip).

- A cone for a diagram $D: \mathcal{J} \to \mathcal{C}$ is a natural transformation from a constant functor $\Delta_c: \mathcal{J} \to \mathcal{C}$ to $D(\underline{\ })$;
- there is a category of cones for $D(_)$, where morphisms between a cone $c \to D(_)$ and a cone $C' \to D(_)$ are arrows $k: C \to C'$ such

that the diagram



is commutative;

• a limit for D is a terminal object in the category of cones for D. This means that given a cone for D, there is a unique arrow k which is a morphism of cones.

Of course, a straightforward dualisation of diagram (??) yields the definition of a cocone, and a colimit for D.

- 0.5 Examples of limits and colimits
- 0.5.1 in monoids and Mon
- 0.5.2 in posets and Pos
- 0.5.3 in Set, and other algebraic categories