Materi 8

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Recurrence Relations

Tujuan Instruksional Khusus

- Memahami konsep *recurrent relation*

Definisi

Recurrence Relation untuk baris $\{a_n\}$ adalah sebuah persamaan yang mengekspresikan a_n dalam satu atau lebih term baris sebelumnya, yaitu a_0, a_1, ... , a_{n-1}

Sebuah baris dikatakan sebagai solusi dari Recurrence Relation bila semua term memenuhi Recurrence Relation

Initial conditions pada Baris

Initial conditions adalah term yang mengawali term pertama yang dihasilkan oleh recurrence relation

Contoh:

$a_n = a_{n-1} - a_{n-2}$ for n = 2, 3, 4 … i.c. n = 0, 1

$a_0 = 3,\ a_1 = 5$

$a_2 = a_1 - a_0 = 2,\ \ a_3 = a_2 - a_1 = 2 - 5 = -3$

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Contoh: fibonacci(n)

$fib_n = fib_{n-1} + fib_{n-2}$ for n = 2, 3, 4 …

i.c. n = 0, 1: $fib_0 = 0$, $fib_1 = 1$

Contoh: compound interest

$P_n = P_{n-1} + (0.06)P_{n-1}$ for n = 1, 2, 3, 4 …

i.c. n = 0 : $P_0 = 10.000.000$

Contoh: Tower of Hanoi

Sejumlah piringan dengan diameter berbeda-beda harus dipindahkan satu per satu dari tiang-1 ke tiang-3 dengan bantuan tiang-2. Selanjutnya piringan dengan ukuran lebih kecil tidak boleh berada di bawah piringan yang lebih besar.

Banyaknya pemindahan n piringan dinotasikan dengan H_n.

Algoritma:

Pindahkan (n-1) piringan teratas dari tiang-1 ke tiang-2; H_{n-1}

Pindahkan 1 piringan terbesar dari tiang-1 ke tiang-3; 1

Pindahkan (n-1) piringan teratas dari tiang-2 ke tiang-3; H_{n-1}

Maka : $H_n = H_{n-1} + 1 + H_{n-1}$

$= 2\,H_{n-1} + 1$ dengan i.c. $H_1 = 1$

$$H_n = 2\,H_{n-1} + 1 \qquad \& \qquad H_1 = 1$$

$$H_n = 2\,H_{n-1} + 1$$

$$= 2\,(2H_{n-2} + 1) + 1 = 2^2H_{n-2} + 2 + 1$$

$$= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3H_{n-3} + 2^2 + 2 + 1$$

$$= 2^3(2H_{n-4} + 1) + 2^2 + 2 + 1 = 2^4H_{n-4} + 2^3 + 2^2 + 2 + 1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$n - i = 1$$

$$i = n - 1$$

$$H_n = 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} \ldots. + 2^3 + 2^2 + 2 + 1$$

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$$= 2^{n-1} + 2^{n-2} + 2^{n-3} \ldots. + 2^3 + 2^2 + 2 + 1 = 2^n - 1$$

Contoh Soal

Selesaikan reccurence relation berikut

1. $x(n) = x(n-1) + 5$; untuk $x(1) = 0$
2. $X(n) = 3x(n-1)$; untuk $x(1) = 4$
3. $X(n) = x(n-1) + n$; $n>0$; $x(0) = 0$
4. $X(n) = x(n/2) + n$; $n>1$, $x(1) = 1$, $n \bmod 2 = 0$; (hint : $n=2^k$)

Solving Recurrence Relations

Sub-bab 7.2

Teorema

Teorema 1: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ **dan i.c.** a_0 **dan** a_1

$r^2 - c_1 r - c_2 = 0$ memiliki <u>dua akar r_1 dan r_2 yang berbeda;</u>

deret $\{a_n\}$ adalah solusi dari r.r. $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

jika dan hanya jika $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

untuk $n = 0, 1, 2, \ldots$ dan α_1, α_2 adalah konstanta

Teorema 2: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ **dan** **i.c.** a_0 **&** a_1

$r^2 - c_1 r - c_2 = 0$ **memiliki <u>akar kembar r_0</u>**

deret $\{a_n\}$ adalah solusi dari r.r. $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

jika dan hanya jika $a_n = \alpha_1 r_0^{\,n} + \alpha_2\, n\, r_0^{\,n}$

untuk $n = 0, 1, 2, \ldots$ **dan** α_1, α_2 **adalah konstanta**

Contoh:

$a_n = 6\,a_{n-1} - 9a_{n-2}$

$a_0 = 1,\ a_1 = 6$

Akar kembar dari $r^2 - 6r + 9 = 0$ adalah $r_0 = 3$

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n \quad = \alpha_1 \times 3^n + \alpha_2 \times n \times 3^n$$

$$\alpha_1 = a_0 = 1$$

$$\alpha_2 = (a_1 - a_0\, r_0\,) \,/\, r_0 = (6 - 1 \times 3) \,/\, 3 = 1$$

$$a_n = 1 \times 3^n + 1 \times n \times 3^n$$

$$= 3^n + n \times 3^n = (n+1)\, 3^n$$

Theorem 3:

$c_1, c_2, \ldots c_k$ are real numbers

$r^k - c_1 r^{k-1} - c_2 r^{k-2} \ \ldots - c_k = 0$ has k distinct roots $r_1, r_2, \ldots, r_k$

A sequence $\{a_n\}$ is a solution of the r.r. $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$

iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \ \ldots + \alpha_k r_k^n$ for $n = 0, 1, 2, \ldots$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants

Contoh:

$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ dengan *i.c.* $a_0 = 2, a_1 = 5, a_2 = 15$

$r^3 - 6r^2 + 11r - 6 = 0 \rightarrow r_1 = 1, r_2 = 2, r_3 = 3$

$a_0 = \alpha_1 r_1^0 + \alpha_2 r_2^0 + \alpha_3 r_3^0 \rightarrow 2 = \alpha_1 + \alpha_2 + \alpha_3$

$a_1 = \alpha_1 r_1^1 + \alpha_2 r_2^1 + \alpha_3 r_3^1 \rightarrow 5 = \alpha_1 + 2\alpha_2 + 3\alpha_3$ $\qquad \alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 2$

$a_2 = \alpha_1 r_1^2 + \alpha_2 r_2^2 + \alpha_3 r_3^2 \rightarrow 15 = \alpha_1 + 4\alpha_2 + 9\alpha_3$

Jadi $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n = 1 - 2^n + 2 \times 3^n$

Theorem 4:

$c_1, c_2, \ldots c_k$ are real numbers

$r^k - c_1 r^{k-1} - c_2 r^{k-2} \ \ldots - c_k = 0$ has t distinct roots $r_1, r_2, \ldots, r_t$

with multiplicities $m_1, m_2, \ldots, m_t$ respectively and $m_1 + m_2 + \ldots + m_t = k$

Then a sequence $\{a_n\}$ is a solution of the r.r. $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$

iff

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \ldots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n$$

$$+ (\alpha_{2,0} + \alpha_{2,1} n + \ldots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n$$

$$+ \ldots + (\alpha_{t,0} + \alpha_{t,1} n + \ldots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \le i \le t$ and $0 \le j \le m_i - 1$

Penjelasan:

Jika akar-akar karakteristik adalah 2, 2, 2, 5, 5, 9, maka

$r_1 = 2$, $m_1 = 3$; $r_2 = 5$, $m_2 = 2$; $r_3 = 9$, $m_3 = 1$ ($t = 3$, $k = 6$, $m_1 + m_2 + m_3 = 6$)

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Contoh:

$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ ***i.c.*** $a_0 = 1,\ a_1 = -2,\ a_2 = -1$

$c_1, = -3,\ c_2 = -3,\ c_3 = -1$

$r^3 + 3r^2 + 3r + 1 = 0 \rightarrow (r+1)^3 = 0 \rightarrow r = -1$ (dengan *multiplicity* $m_1 = 3$)

$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2)\, r^n = (\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2)\, r^n$

$a_0 = (\alpha_{1,0} + \alpha_{1,1} \times 0 + \alpha_{1,2} \times 0) \times (-1)^0 \rightarrow \quad 1 = \alpha_{1,0}$

$a_1 = (\alpha_{1,0} + \alpha_{1,1} \times 1 + \alpha_{1,2} \times 1) \times (-1)^1 \rightarrow -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2}$

$a_2 = (\alpha_{1,0} + \alpha_{1,1} \times 2 + \alpha_{1,2} \times 4) \times (-1)^2 \rightarrow -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}$

$\alpha_{1,0} = 1,\ \alpha_{1,1} = 3,\ \alpha_{1,2} = -2$

Jadi $a_n = (\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2)\, r^n = (1 + 3n - 2n^2)(-1)^n$

Tugas

- Ex 8.2 no. 4 (b.d.f), 12, 41