Dynamics and Control (UESTC 3001)

Notes prepared by: Dr Ola R. Popoola

Syllabus delivery

- Lecture 1: Root Locus Analysis
- Lecture 2: Root Locus II and Nyquist Plots
- Lecture 3: Bode Plots
- Lecture 4: Bode Plots II
- Lecture 5: Stability in Frequency Domain
- Lecture 6: Stability Examples
- Lecture 7: Compensators
- Lecture 8: Tutorials and Test Exercises

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Intended Learning Objectives

At the end of this lecture, you will be able to:

- Explain the purpose of root locus analysis as it relates to pole movement
- Analyze the stability of a closed loop system
- Design controllers to achieve desired system performances

Additional Reading Materials

- "Modern Control Systems" - R. C. Dorf, R.H. Bishop Addison Wesley
- "Modern Control Engineering" - K. Ogata. Prentice Hall International
- "Feedback Control of Dynamic Systems" - Franklin, Powell, Emami-Naeini. Addison Wesley.
- "Feedback Systems: An Introduction for Scientists and Engineers" – K. J. Åström, R. M. Murray. Princeton University Press, Princeton.
  Available at
  https://www.cds.caltech.edu/~murray/amwiki/index.php/Main_Page
- "Control Engineering: An introduction with the use of Matlab", D. P. Atherton.
  Available at http://bookboon.com/en/control-engineering-matlab-ebook

These are recommended only and are not required to pass the course.

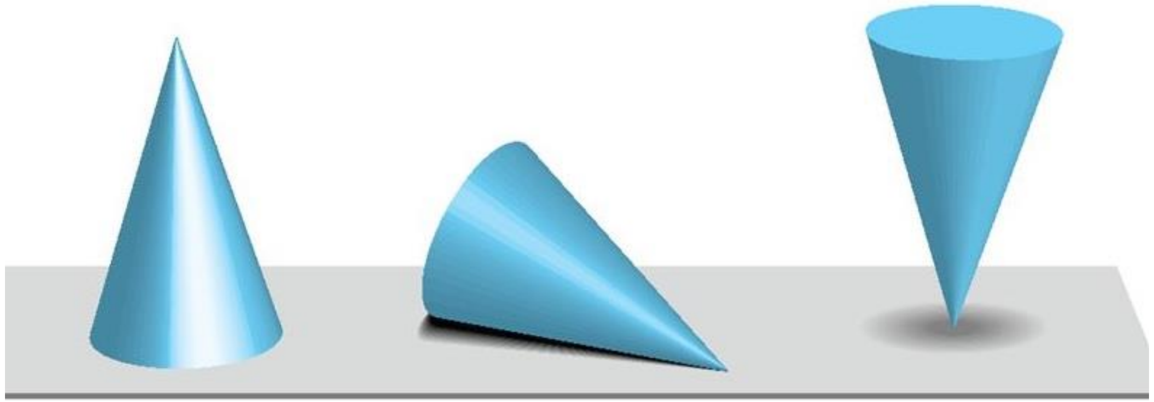
Additional Notes

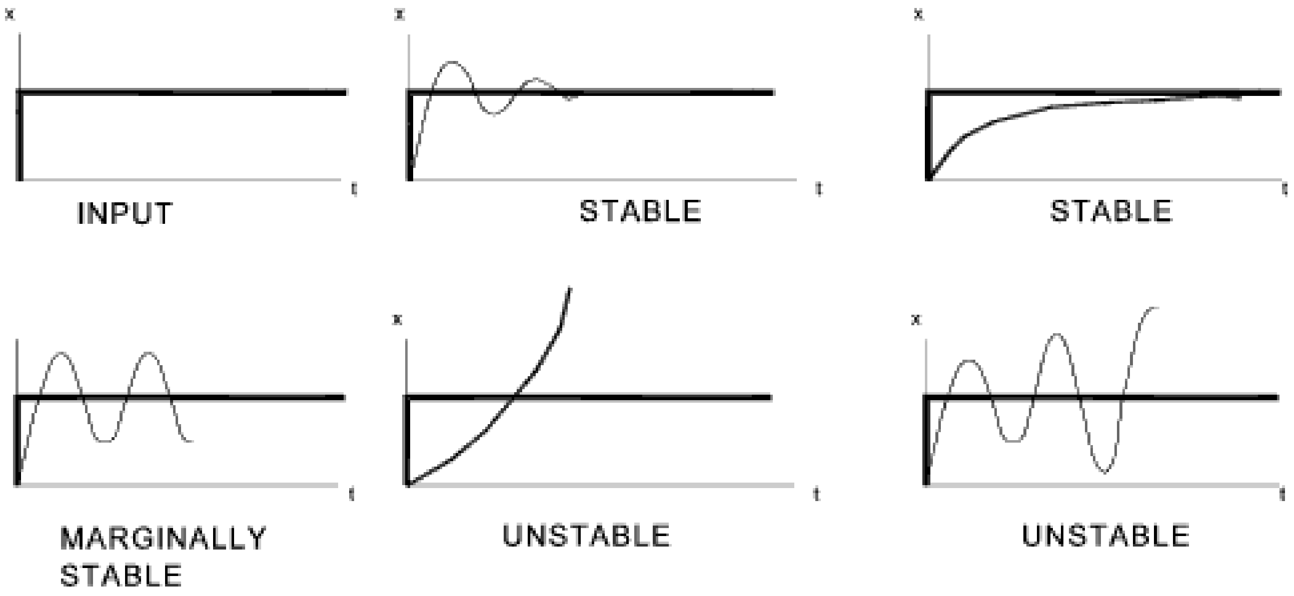
- All required notes/tutorials/exam answers may be found on Moodle.
- You are encouraged to do your own reading to complement the lecture material.
- You can also use the forum on Moodle to ask questions about course material, rather than using email.

Background Discussion -Stability

Stability in Control Systems

BIBO Stability: A system is said to be BIBO stable if, for any bounded input signal, the output of the system remains bounded over time.





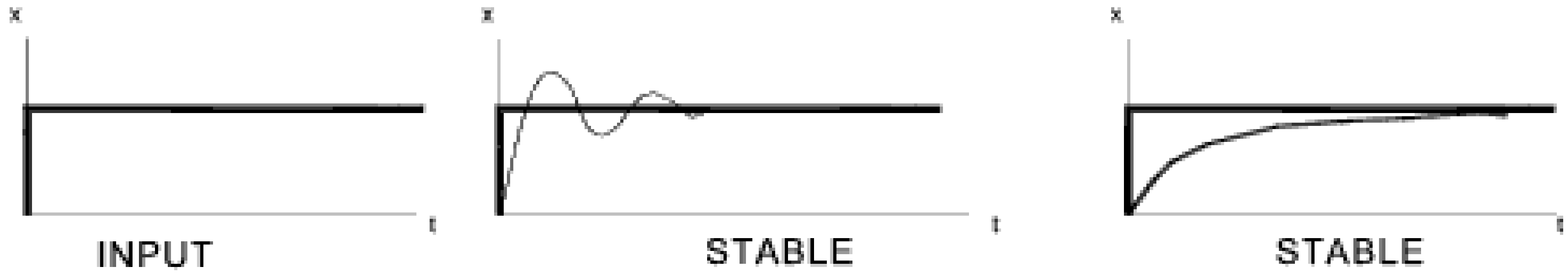
Importance of Stability

- **Performance Assurance:** A stable system ensures that the desired control objectives are achieved reliably and consistently.
- **Robustness:** Stability is often a prerequisite for robustness, which refers to the ability of a system to maintain performance in the face of uncertainties, variations, or disturbances.
- **Safety:** Stability ensures that the system operates within safe limits and avoids dangerous or unpredictable behaviours.
- **Predictability:** Stable systems exhibit predictable behaviour, allowing engineers to analyse, model, and predict their response to different inputs and conditions.
- **Ease of Implementation**: Stable systems are easier to design, implement, and tune

Stability –necessary and sufficient

Determination of Stability is important but not sufficient.

A system may be stable but with low damping, so it is undesirable.



Absolute stability is important via Route Criterion

Relative stability (peak overshoot, settling time, etc) via Root Locus

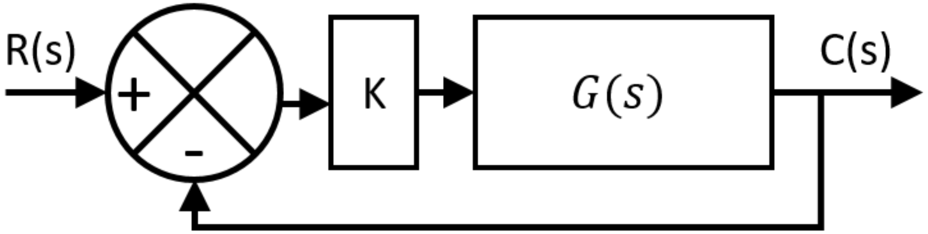
Root Locus Analysis

Importance of Root Locus Analysis

- **System stability analysis**: Determine the stability of a feedback control system. How changes in the system affect stability.
- **Design Parameter Selection**: Shows how design parameters such as damping ratio influence system's closed loop behaviour.
- **Pole placement**: Used to get values of controller parameters to put poles at desired locations
- **Understanding system's behaviour**: Provide information on the dynamic behaviour of the system by show how poles change on complex plane.
- **System optimization**: Provides information of tradeoff between stability, speed of response, and robustness

Root Locus Concept

Let us consider a simple feedback system:



$$G(s) = \frac{K}{s(s+a)}; \frac{C(s)}{R(s)} = \frac{K}{s^2 + as + K}$$

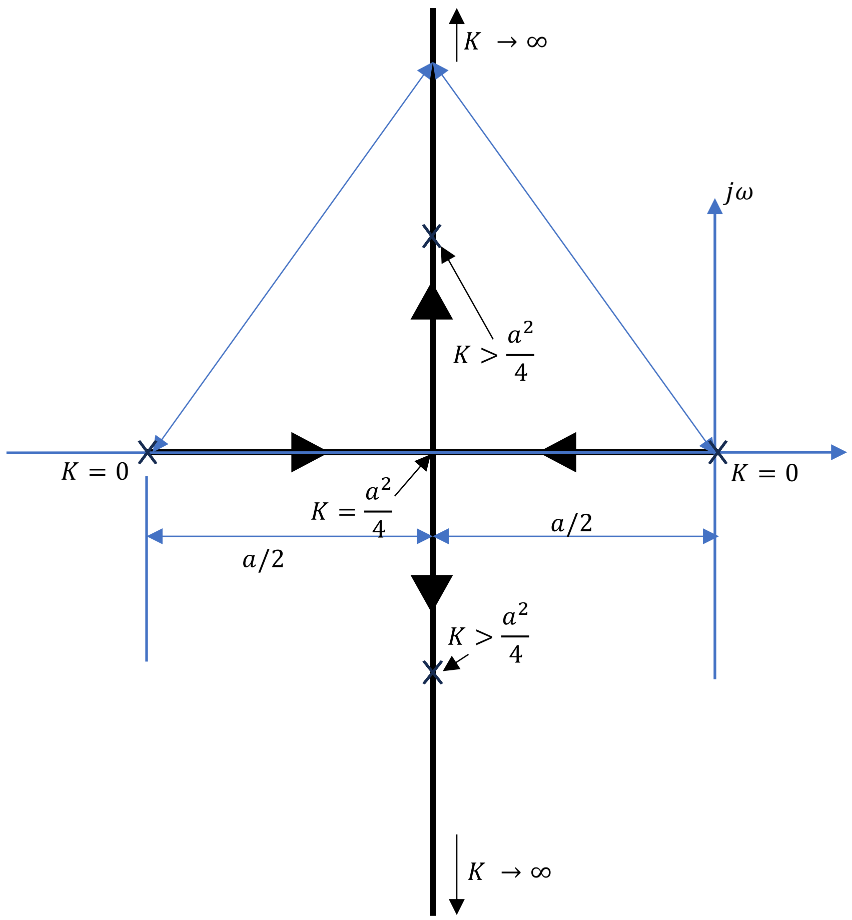
What are the open loop system poles?
What are the closed loop system poles?
What is the nature of the system for (a,K)>0?
If a is constant, what is the nature of the system as K goes from 0 to infinity?

Root Locus Concept

Given the characteristic equation $s^2 + as + K$, the roots:

$$s_1, s_2 = -\frac{a}{2} \pm \sqrt{\left[\left(\frac{a}{2}\right)^2 - K\right]}$$

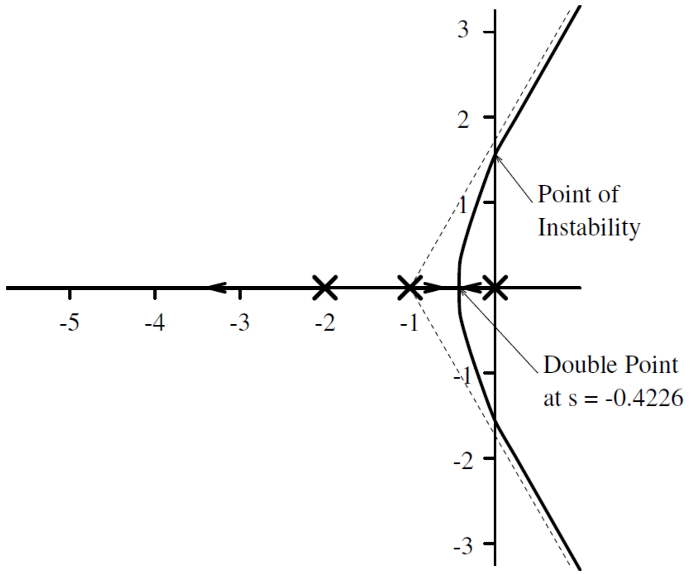
i. For $0 \le K < a^2/4$; roots are real and distinct. When $K = 0$, roots are 0,-a
ii. For $K = a^2/4$, $s_1 = s_2 = -a/2$
iii. For $a^2/4 < K < \infty$, the roots are complex conjugates with real part $-a/2$



Properties of Root Locus

The response of a system is dependent upon the position of the closed-loop poles of the transfer function describing that system.

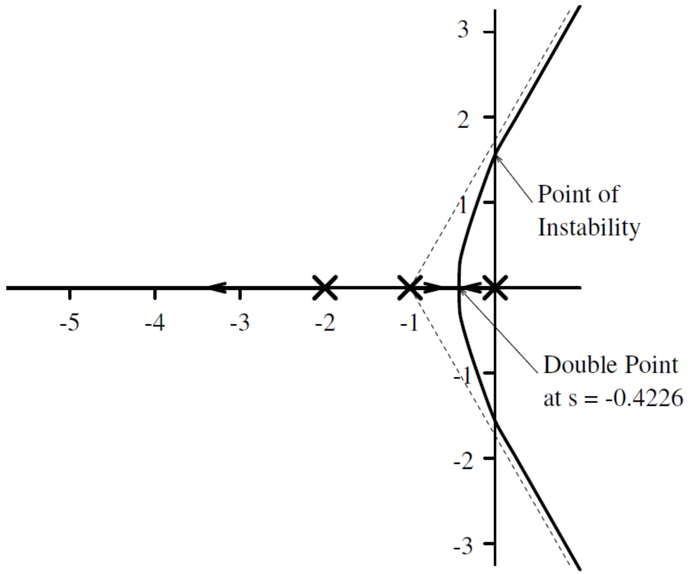
As the gain of the system varies the position of the open-loop poles will remain constant, however the position of the closed-loop poles will change.



Properties of Root Locus

The root-locus diagram is a method of studying the migration of the closed-loop poles of a system as a parameter (typically the gain of the forward-path controller) varies from 0 to ∞.

This is a graphical method in which it is seen that the locus depends upon the positions of the open-loop singularities (poles and zeros).



Root Locus Construction (RLC)

Let us consider a simple feedback system:

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$$\frac{C(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)} = \frac{KB(s)}{A(s) + KB(s)}$$

Writing $G(s) = \frac{B(s)}{A(s)}$, we can show that:
1. The closed-loop zeros are the same as the open-loop zeros,
2. The characteristic equation defines the closed loop poles

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Root Locus Construction (RLC)

The characteristic equation $\mathrm{F}(s) = 1 + KG(s) = A(s) + KB(s) = 0$

- The degree of F(s) is the same as A(s): *n*
- This means there are *n* closed-loop poles for *n*≥*m*

The equation can be rewritten as $G(s) = -\frac{1}{K}$

- The **calibration equation**: $|G(s)| = \frac{1}{K}$ giving K for given $s = j\omega$

Root Locus Construction (RLC)

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- The **calibration equation**: $|G(s)| = \frac{1}{K}$ giving K for given $s = j\omega$

$$G(s) = -\frac{1}{k} + j0$$

$$\theta = -\tan^{-1}(0) = -180 + q$$

$$q = 0,1,2,\ldots$$

- The **angle condition**: $\arg \mathrm{G}(s) = 180° + q360°$ independent of K

These two can be used to work the location of the closed loop poles as K varies

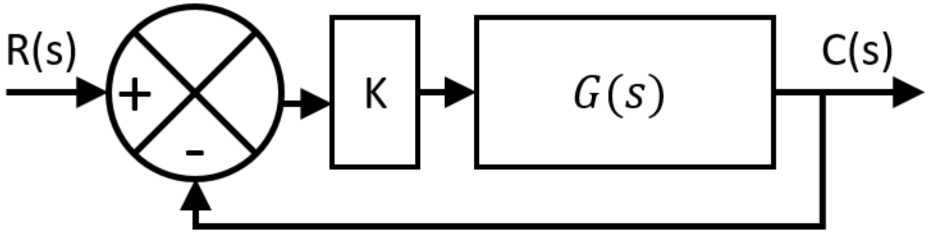
RLC –angle criterion

- Rewriting $KG(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$ with m zeros and n poles

- $\arg G(s) = \arg(s-z_1) + \arg(s-z_2) + \cdots$
$$-\arg(s-p_1) - \arg(s-p_2) - \cdots$$

- Using the angle condition $\arg \mathrm{G}(s) = 180° + q360°$
- The **angle criterion** is:

$$\sum_{j=1}^{m} \arg(s-z_j) - \sum_{i=1}^{n} \arg(s-p_i) = 180° + q360°$$



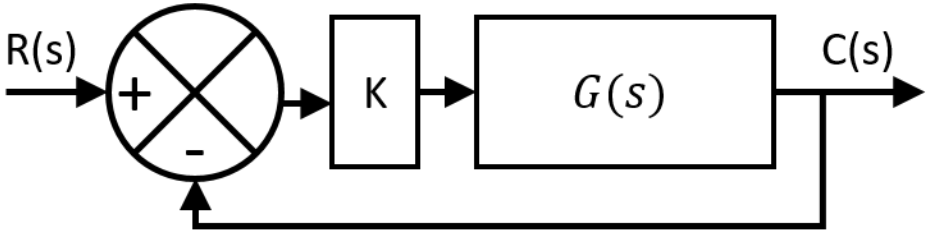
RLC –calibration equation

- $KG(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$ with *m* zeros and *n* poles

- $\mathrm{K} = \frac{1}{|G(s)|}$ and $|G(s)| = \frac{|s-z_1||s-z_2|\cdots|s-z_m|}{|s-p_1||s-p_2|\cdots|s-p_n|} = \frac{\prod_1^m distance\ to\ open\ loop\ zeros}{\prod_1^n distance\ to\ open\ loop\ poles}$

- This leads to the **calibration equation:**

$$\boldsymbol{K = \frac{\prod_1^n distance\ to\ open\ loop\ poles}{\prod_1^m distance\ to\ open\ loop\ zeros}}$$



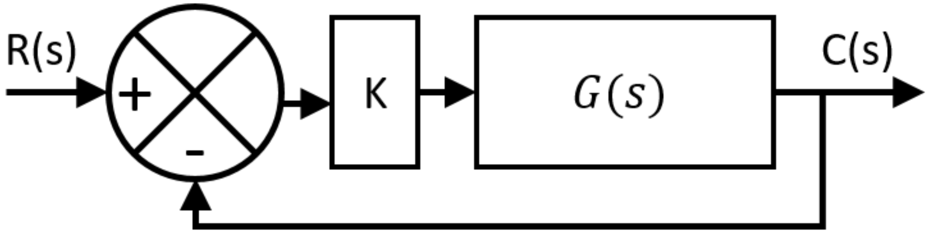
RLC –Points on the real axis

From the angle criterion is:

$$\sum_{j=1}^{m} arg(s - z_j) - \sum_{i=1}^{n} arg(s - p_i) = 180° + q360°$$

Only points on the real axis which have ODD number of open-loop SINGULARITIES (poles/zeros) to the RIGHT are on the locus.

Points with an EVEN number of open-loop SINGULARITIES to the RIGHT are not part of the locus.

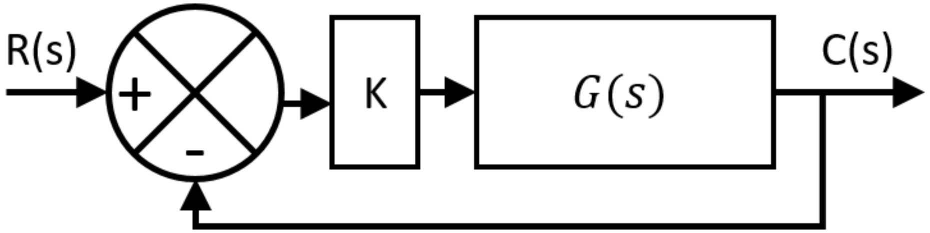


RLC –Behavior of the Root Locus for K=0

$$\mathrm{F}(s) = 1 + KG(s) = A(s) + KB(s) = 0$$

The poles of the closed-loop are the same as those of A(s), they coincide with the open-loop poles.

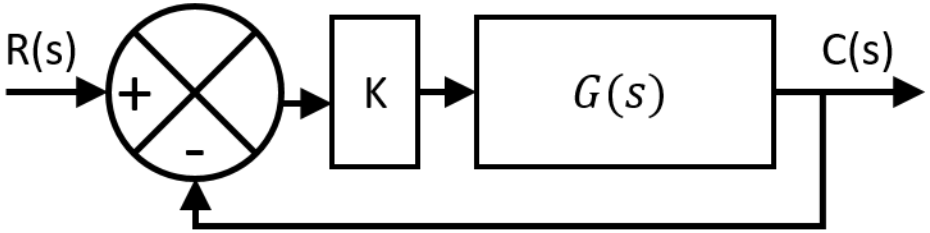
Therefore, the root locus starts at the open-loop poles for K = 0.



RLC –Behavior of the Root Locus for large K

$$\mathrm{K} = \frac{1}{|G(s)|} = \frac{|s-p_1||s-p_2|\cdots|s-p_n|}{|s-z_1||s-z_2|\cdots|s-z_m|}$$
$$= \frac{\prod_1^n distance\ to\ open\ loop\ poles}{\prod_1^m distance\ to\ open\ loop\ zeros}$$

- Then for $s = z_1, \cdots, z_m$ the value of K is infinity
- So m branches of the root locus tend to the OPEN-LOOP ZEROS



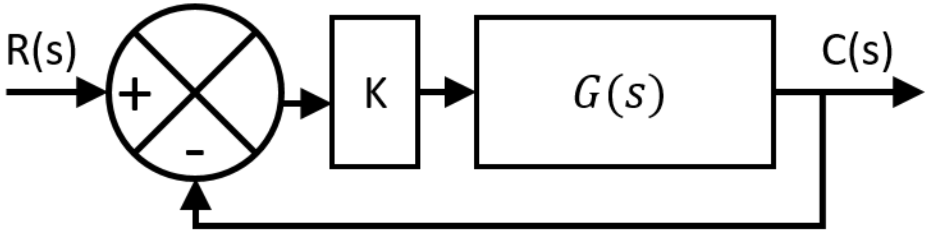
RLC –Behavior of the Root Locus for K 0-> inf

- $K = \frac{1}{|G(s)|} = \frac{|s-p_1||s-p_2|\cdots|s-p_n|}{|s-z_1||s-z_2|\cdots|s-z_m|} = \frac{\prod_1^n distance\ to\ open\ loop\ poles}{\prod_1^m distance\ to\ open\ loop\ zeros}$

- Ensures that *n-m* branches go to $|s| = \infty$ when $|K| = \infty$.
- Their directions (asymptotes) are found using the angle criterion

$$\sum_{j=1}^{m} arg(s - z_j) - \sum_{i=1}^{n} arg(s - p_i) = 180° + q360°$$

$$\begin{array}{c} K=0 \\ \text{Open loop poles} \end{array} \xrightarrow{\text{Root Locus}} \begin{array}{c} K=inf \\ \text{Open loop zeros} \end{array}$$



RLC –Behavior of the Root Locus for K 0-> inf

- Their directions (asymptotes) are found using the angle criterion

$$\sum_{j=1}^{m} arg(s - z_j) - \sum_{i=1}^{n} arg(s - p_i) = 180° + q360°$$

- The **angle** of the asymptotes is given by

$$\theta = \frac{180}{n-m} + q\frac{360}{n-m}, \mathrm{q} = 0,1,2, \ldots, \mathrm{n} - \mathrm{m} - 1$$

- The **intersection** of the asymptotes with the real axis

$$x = \frac{\sum_1^n \Re(p_i) - \sum_1^m \Re(z_j)}{n-m}$$

Proof Chapter 5 of reference book

RLC –Breakaway Points

- Two poles approaching each other on the real axis will form a double/breakaway point **for some value of K** and then become a complex conjugate pair.

- Roots of a quadratic equation: $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- if $b^2 - 4ac < 0$ then the $r \in \mathbb{C}$, otherwise $r \in \mathbb{R}$.

- It is useful to know the coordinates of the **double/breakaway point**

RLC –Breakaway Points

$F(s)=\frac{1}{K}+G(s)$ has TWO roots at $s=\alpha$ for a given value of K

$$F(s)=(s-\alpha)^2F_2(s)$$

Differentiating

$$\frac{\mathrm{d}F(s)}{\mathrm{d}s}=0+\frac{\mathrm{d}G(s)}{\mathrm{d}s}=2(s-\alpha)F_2(s)+(s-\alpha)^2\frac{\mathrm{d}F_2(s)}{\mathrm{d}s}$$

This still has a single root at $s=\alpha$. The double point can be deduced from the roots of $\frac{\mathrm{d}G(s)}{\mathrm{d}s}=0$

RLC –Summary

1. Identify number of poles (n), zeros (m), and determine the order (n − m).
2. Plot the location of the poles (x) and zeros (o) in the complex plane
3. Determine which parts of the real axis are on the locus
4. Identify the number and directions of the asymptotes
5. Find where the asymptotes meet and draw the asymptotes on the graph.
6. Identify any poles uniquely connected to zeros along the x-axis — these form one branch of the locus.
7. If there are complex poles or zeros the angles of arrival or departure need to be determined (Chapter 5 of reference book – rule 4)
8. If two poles on the axis need to become complex to go to asymptotes, they must pass through a double point found by the solutions of $\frac{\mathrm{d}G(s)}{\mathrm{d}s} = 0$.

In many cases, steps 1–6 allow a reasonable estimation of the approximate shape of the root-locus.

Mentimeter session

RLC –MATLAB Plot

If the open loop transfer function is

$$OLTF = \frac{s+1}{(s+2)^2} = \frac{s+1}{s^2+4s+4}$$

The system function is defined as follows:

```
h = tf([1 1],[1 4 4])
```

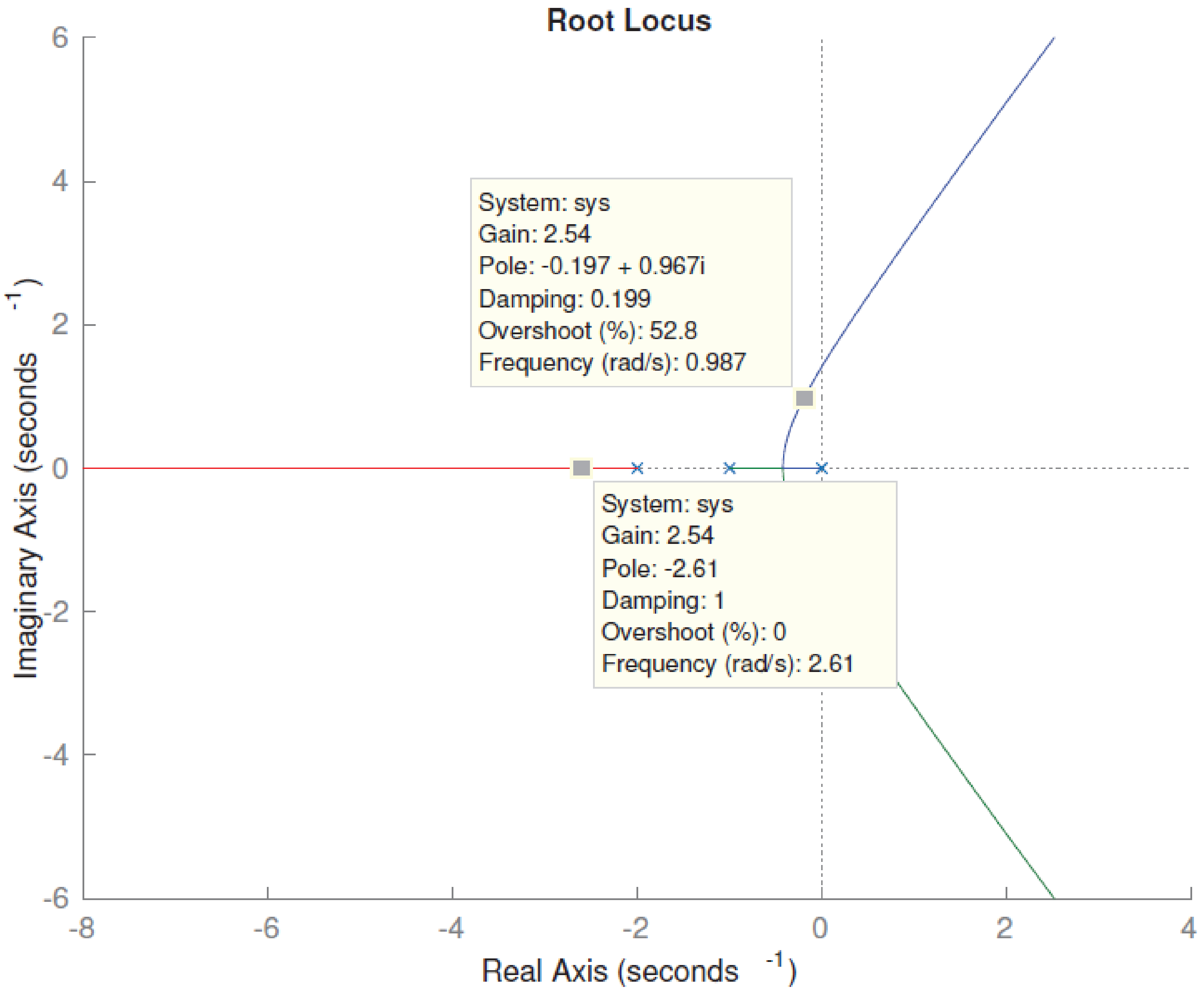
Then to plot the root locus as a function of K:

```
rlocus(h)
```

If you need to look at specific values of K

```
K = [0 0.1 0.2 10]
rlocus(h,K)
```

RLC –MATLAB Plot



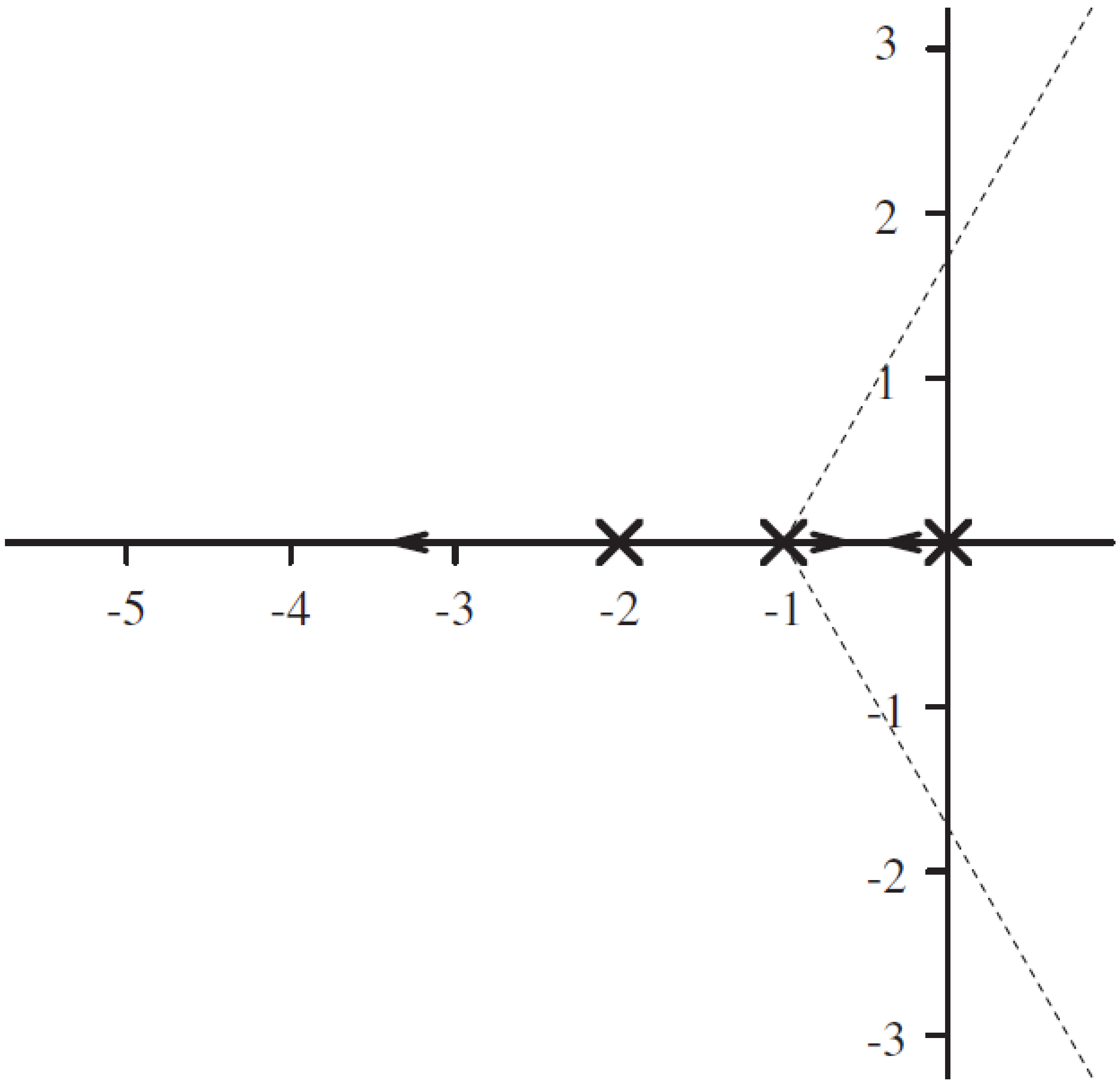
RLC –Examples

Consider the system $G(s) = \frac{K}{s(s+1)(s+2)}$

We first draw the root locus diagram using the standard rules:

- Plot the open-loop poles and zeros on paper.
- Number of open-loop poles n = 3
- Number of open-loop zeros m = 0
- The order n-m = 3 ➔ 3 asymptotes.
- Angle criterion: $\theta = \frac{180}{n-m} + q\frac{360}{n-m}$ ➔ 60, 180, 300 ➔ 180º and ±60º
- Centre of gravity: $x = \frac{\sum_1^n \Re(p_i) - \sum_1^m \Re(z_j)}{n-m} = \frac{0-1-2}{3} = -1$
- Draw the asymptotes on a graph
- Check for parts of real axis on locus. Points which have an odd number of singularities to their right are on the locus.

RLC –Examples



RLC –Examples

- Breakaway/Double point $\frac{\mathrm{d}G(s)}{\mathrm{d}s} = 0$,

$$\frac{dG}{ds} = \frac{d}{ds}\left(\frac{K}{s(s+1)(s+2)}\right) = \frac{K\left(3s^2+6s+2\right)}{\left(s(s+1)(s+2)\right)^2} = 0$$

Thus

$$3s^2+6s+2=0,\text{ or } s = \frac{-6\pm\sqrt{36-24}}{6} = -1 \pm \frac{\sqrt{12}}{6} = -1 \pm \frac{1}{\sqrt{3}}$$

$$s = -0.4226,\text{ or } s = -1.5774$$

In this case we know that s = -1.5774 is not on the locus and so can be disregarded.

RLC –Examples