Dynamics and Control (UESTC 3001)

Notes prepared by: Dr Ola R. Popoola

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- Lecture 1: Root Locus Analysis
- Lecture 2: Root Locus II and Nyquist Plots
- Lecture 3: Bode Plots
- Lecture 4: Bode Plots II
- Lecture 5: Stability in Frequency Domain
- Lecture 6: Stability cont. and Stability Examples
- Lecture 7: Compensators
- Lecture 8: Tutorials and Test Exercises

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Intended Learning Objectives

At the end of this lecture, you will work through:

- Stability cont.
- Non-unity feedback systems
- Examples on stability regions
- Examples on stability margins

Stability Margins

- The point of intersection of the polar plot with the negative real axis can be determined by setting the imaginary part of $G(j\omega)H(j\omega)$ equal to zero

$$G(j\omega)H(j\omega) = \mathrm{u} + \mathrm{jv} = \frac{-\mathrm{K}(\mathrm{T}_1 + \mathrm{T}_2) - \mathrm{jK}\left(\frac{1}{\omega}\right)\left(1 - \omega^2 T_1 T_2\right)}{1 + \omega^2\left(T_1^2 + T_2^2\right) + \omega^4 T_1^2 T_2^2}$$

If we set the frequency at the point of intersection to be ω_2, we have:

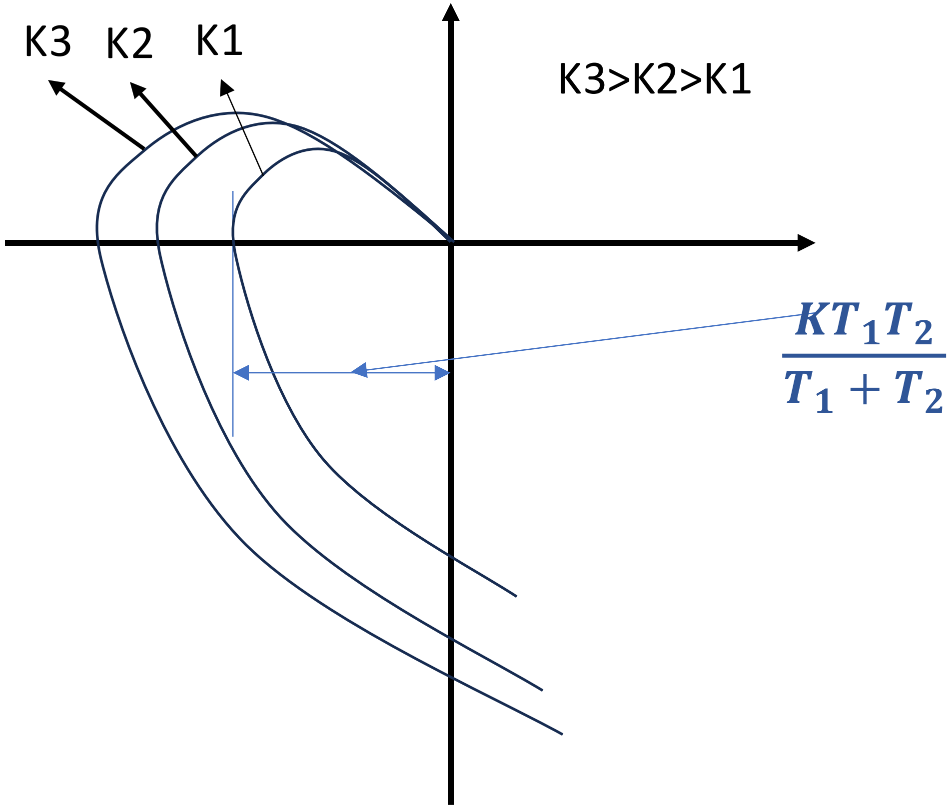
$$\mathrm{v} = \frac{-\mathrm{K}\left(\frac{1}{\omega_2}\right)\left(1 - \omega_2^2 T_1 T_2\right)}{1 + \omega_2^2\left(T_1^2 + T_2^2\right) + \omega_2^4 T_1^2 T_2^2} = 0$$

This gives $\omega_2 = \frac{1}{\sqrt{T_1 T_2}}$

Stability Margins

- The magnitude of the real part at frequency ω_2 is given by:

$$\mathbf{u} = \frac{-\mathrm{K}(\mathrm{T}_1 + \mathrm{T}_2)}{1 + \omega^2\left(T_1^2 + T_2^2\right) + \omega^4 T_1^2 T_2^2} = -\frac{KT_1T_2}{T_1 + T_2}$$



For the system to be stable,

$$\frac{KT_1T_2}{T_1+T_2} < 1 \text{ or } K < \frac{T_1+T_2}{T_1T_2}$$

Recall: Margins

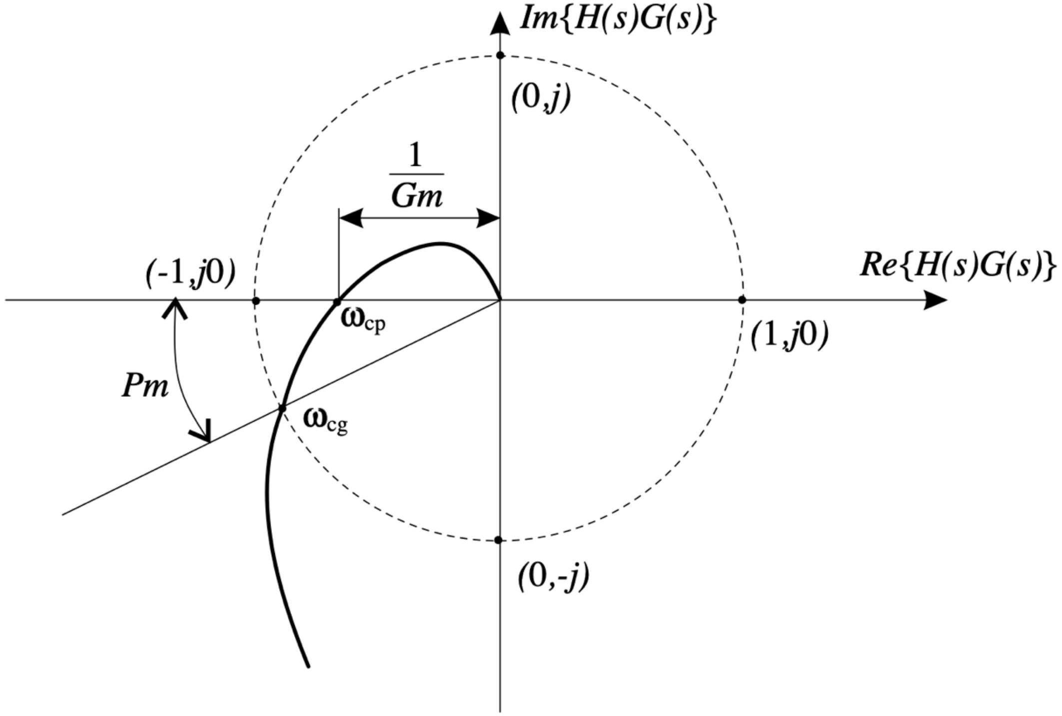
- The gain margin is defined as the factor by which you would have to increase the gain of a system to make it unstable.
- It is computed by determining the magnitude , λ, when , $arg\{G(j\omega)\} = -180^o$, and defined as follows: Gain Margin = $\frac{1}{\lambda}$

- The phase margin, ϕ, is that amount of additional phase lag that could be added at the gain crossover frequency required to bring the system to the verge of closed loop instability.
- It is computed by determining the phase , θ, when $|G(j\omega)| = 1$ and adding this phase to 180 deg. $Phase\ Margin, \phi = 180^o + \theta$

Nyquist Margins

$$Pm = 180^o + arg\{G(j\omega_{gc}), H(j\omega_{gc})$$

$$Gm[dB] = 20\,log\frac{1}{|G(j\omega_{pc})H(j\omega_{pc})|}[dB]$$



Correlation between PM and Damping Factor ζ

- **Consider a unity feedback second order system with an open-loop transfer function**

$$G(s)H(s) = \frac{K}{s(\tau s + 1)} = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

where $\omega_n = \sqrt{\frac{K}{\tau}}$ **and** $2\zeta\omega_n = \frac{1}{\tau}$

Replacing s **by** $j\omega$ **for obtaining a polar plot, we have**

$$G(j\omega)H(j\omega) = \frac{\omega_n^2}{j\omega(j\omega + 2\zeta\omega_n)}$$

Finding the phase margin can show that for $\zeta \le 0.7, \zeta \approx 0.01\phi$**, where** ϕ **is the phase margin.**

Non-unity feedback systems

Non-unity feedback systems

- For a non-unity feedback system, the closed loop transfer function is given by

$$T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1 + G(j\omega)H(j\omega)} = \frac{1}{H(j\omega)}\left[\frac{G(j\omega)H(j\omega)}{1 + G(j\omega)H(j\omega)}\right]$$

$$= \frac{1}{H(j\omega)}\left[\frac{G_o(j\omega)}{1 + G_o(j\omega)}\right] = \frac{1}{H(j\omega)}T_o(j\omega)$$

Where $G_o(j\omega) = G(j\omega)H(j\omega)$ and $T_o(j\omega) = \frac{G_o(j\omega)}{1+G_o(j\omega)}$

Stability in Frequency Domain -Examples

Stability Regions or Point of Instability

Methods of determining point of instability

- Three methods of determining the crossing point into the right half plane (i.e. points of instability).
- Use Route-Hurwitz to determine the value of K for which the system is unstable and use the magnitude condition to determine the position.
- Use the angle criterion of root locus directly and determine the crossing point. Then use the magnitude condition to determine the value of gain, K, for which the system becomes unstable.
- By substituting s= jω in the transfer function and then using the Nyquist stability criterion to determine the frequency at which instability occurs. This frequency is the value of ω.

Example 1a

- Using Routh-Hurwitz criterion, find the value of K for which the system below is stable and determine the point of instability.

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

$$F(s) = s^3 + 3s^2 + 2s + K = 0$$

Route table:

s^3	1	2
s^2	3	K
s^1	-1/3(K-6)	0
s^0	K	0

Example 1a

- **Using Routh-Hurwitz criterion, find the value of K for which the system below is stable and determine the point of instability.**

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

- **Show that K=6, at the point of instability**
- **Determine the point of instability by using:**

$$K = \frac{\Pi_1^n \text{ distance to open loop poles}}{\Pi_1^m \text{distance to open loop zeros}}$$

$$K = x\sqrt{x^2+1}\sqrt{x^2+4} = 6$$

$$x^2 = 2$$

Example 1b

- **Using the root locus angle criterion, find the value of K for which the system below is stable and determine the point of instability.**

$$\sum angles\ from\ zeros\ -\sum angles\ from\ poles\ -180 \pm k360^o$$

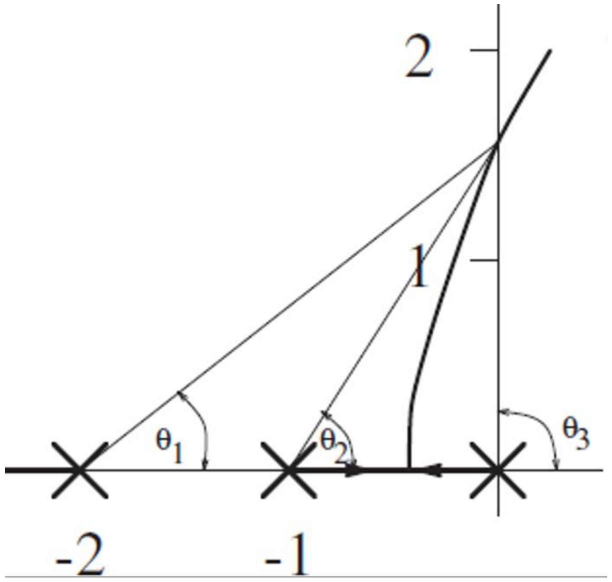
$$-\theta_1 - \theta_2 - \theta_3 = -tan^{-1}x - tan^{-1}\left(\frac{x}{2}\right) - 90^o = -180^o$$

$$tan^{-1}x + tan^{-1}\left(\frac{x}{2}\right) = 90^o$$

$$\tan^{-1}A + \tan^{-1}B = \tan^{-1}\left\{\frac{A+B}{1-AB}\right\}$$

$$\tan^{-1}\left\{\frac{3x/2}{1-x^2/2}\right\} = 90^\circ$$

$$1 - x^2/2 = 0 \quad \Rightarrow \quad x = \pm\sqrt{2}$$

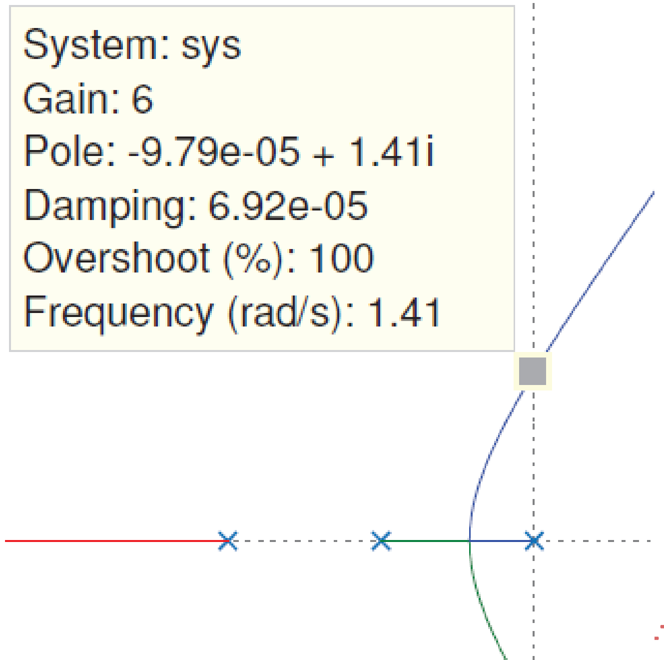


Example 1b

- Using the root locus angle criterion, find the value of K for which the system below is stable and determine the point of instability.

Recall, for points on the locus,

- $K = \dfrac{\Pi_1^n \text{ distance to open loop poles}}{\Pi_1^m \text{distance to open loop zeros}}$
- $K = \sqrt{2} \times \sqrt{2+1} \times \sqrt{2+4} = 6$



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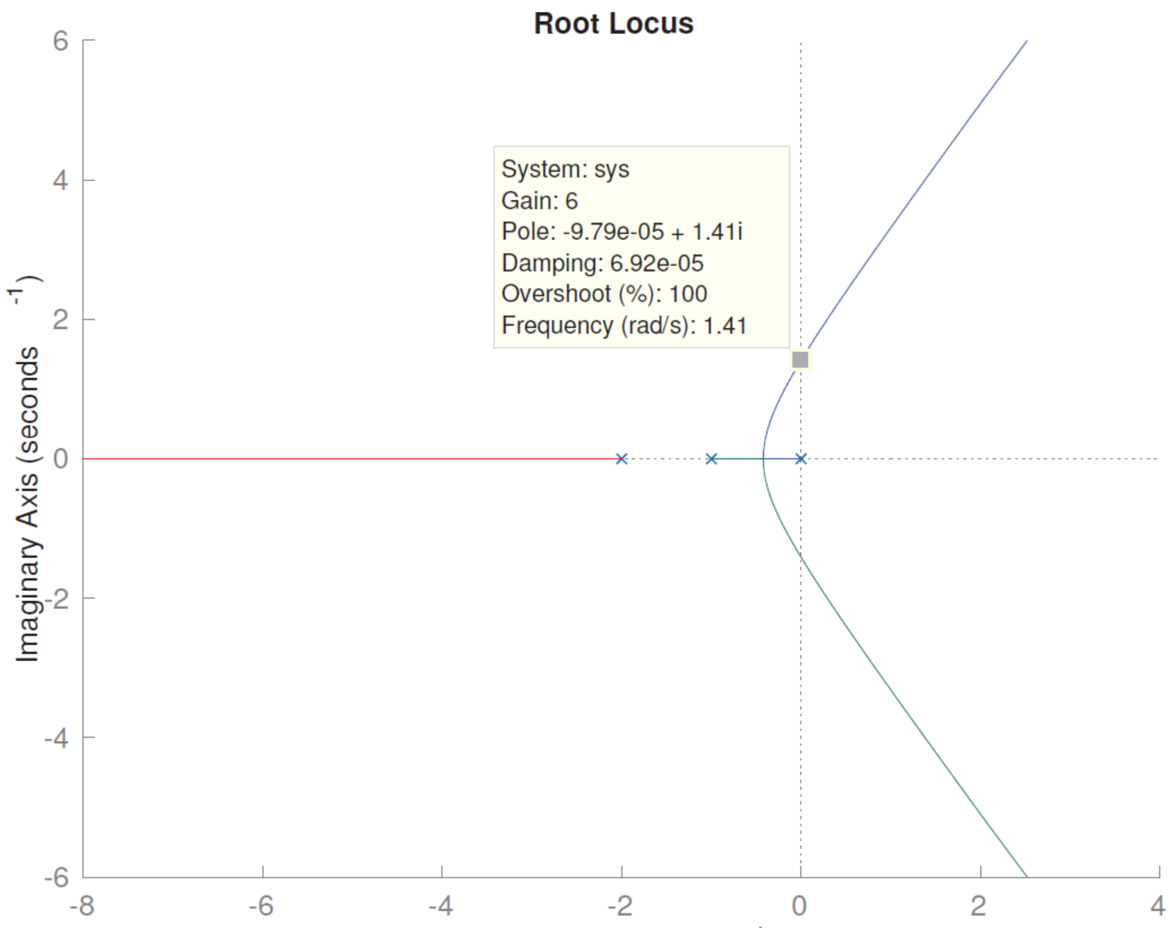


Example 1c

- **Using Nyquist criterion, find the value of K for which the system below is stable and determine the point of instability.**

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

- **Two techniques can be used to determine the crossing point of the negative real axis, which both give the same solution:**

a. Find the frequency at which $\mathrm{Im}\{G(j\omega)\} = 0$

b. Find the frequency at which the phase $\arg\{G(j\omega)\} = -180^o$

After the frequency is determined, calculate the magnitude .

Example 1c

- **Using Nyquist criterion, with the magnitude**

$$|G(j\omega)| = \frac{K/2}{\omega\sqrt{1+\omega^2}\sqrt{1+\omega^2/4}}$$

$$\arg G(j\omega) = -90^\circ - \tan^{-1}\omega - \tan^{-1}\omega/2 = -90^\circ - \tan^{-1}\frac{3\omega/2}{1-\omega^2/2}$$

$$G(j\omega) = \frac{K}{j\omega(j\omega+1)(j\omega+2)}; img\big(G(j\omega)\big) = 0;$$

$$\frac{K/2(-j\omega + \frac{j\omega^3}{2})}{\omega^2(\omega^2+1)\left(\frac{\omega^2}{4}+1\right)} = 0$$

equating this to zero implies

$$-\omega + \omega^3/2 = 0 \quad \Rightarrow \quad \omega = 0 \text{ or } \omega = \sqrt{2}$$

Example 1c

- **Using Nyquist criterion, with the phase**

Using $\arg G(j\omega) = -180^\circ$ gives

$$\tan^{-1} \frac{3\omega/2}{1-\omega^2/2} = -90^\circ$$

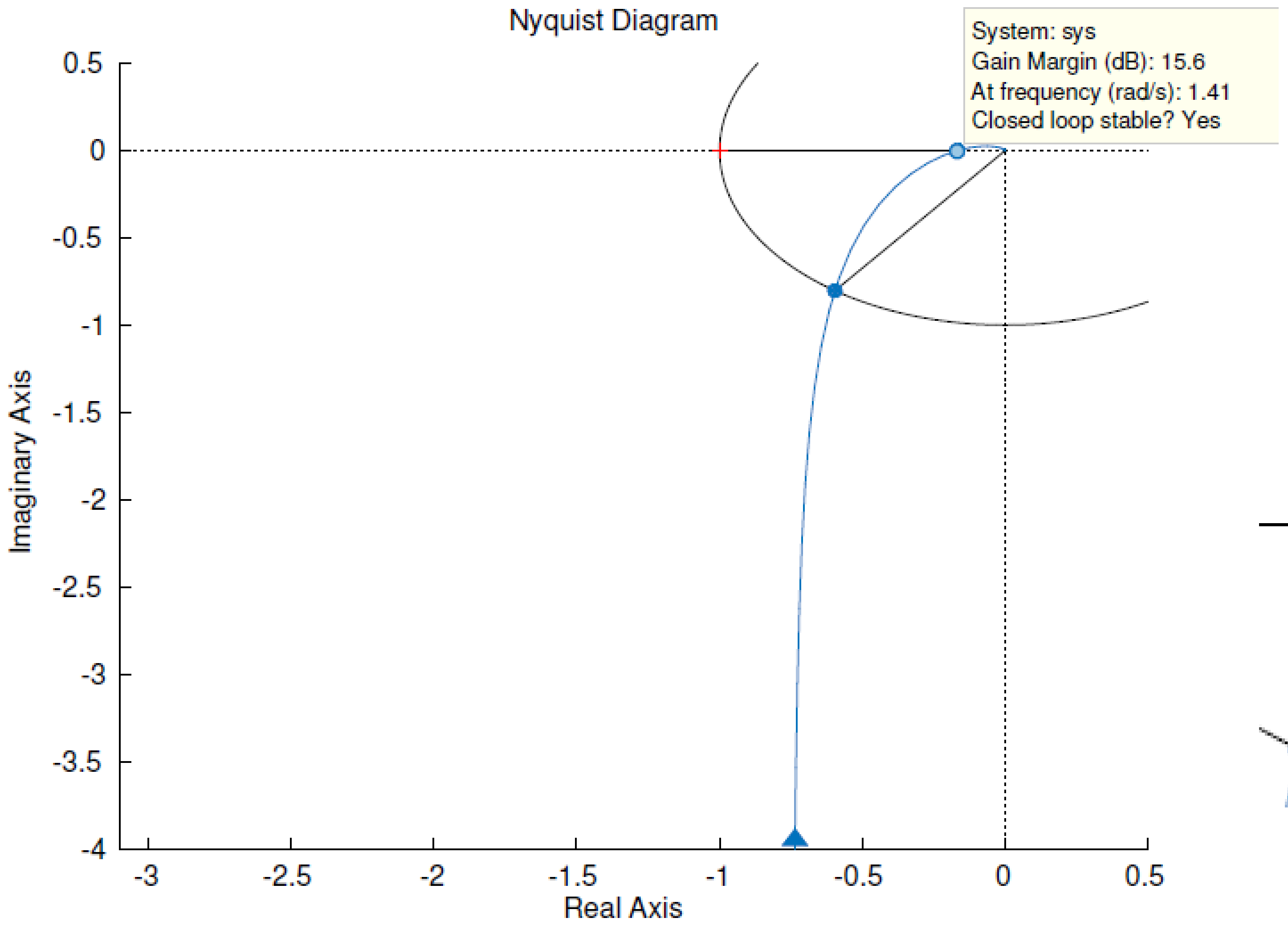
which implies that

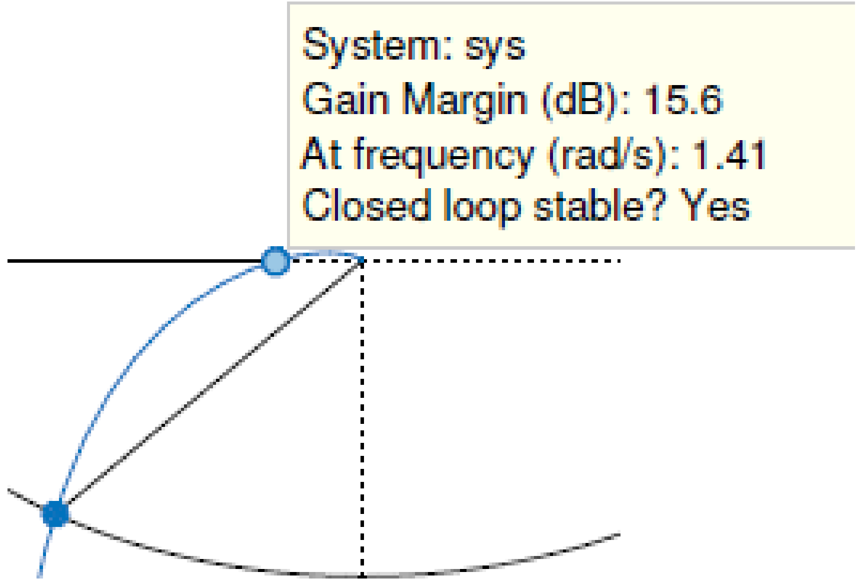
$$\frac{3\omega/2}{1-\omega^2/2} = \infty$$

and thus $1-\omega^2/2 = 0$, or $\omega = \pm\sqrt{2}$. The crossing point of the locus therefore follows as above.

Example 1c

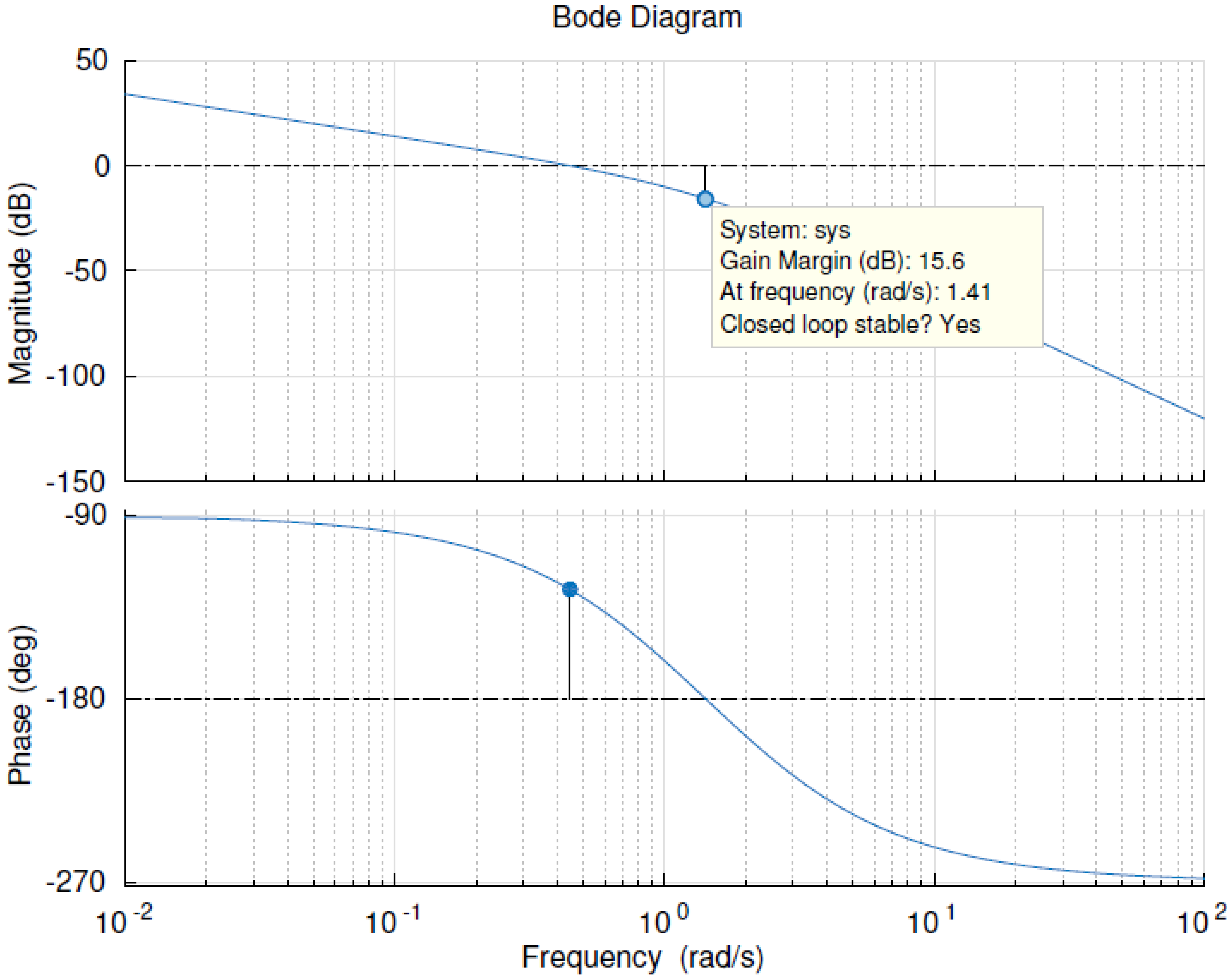
- **Nyquist plot of frequency response:**

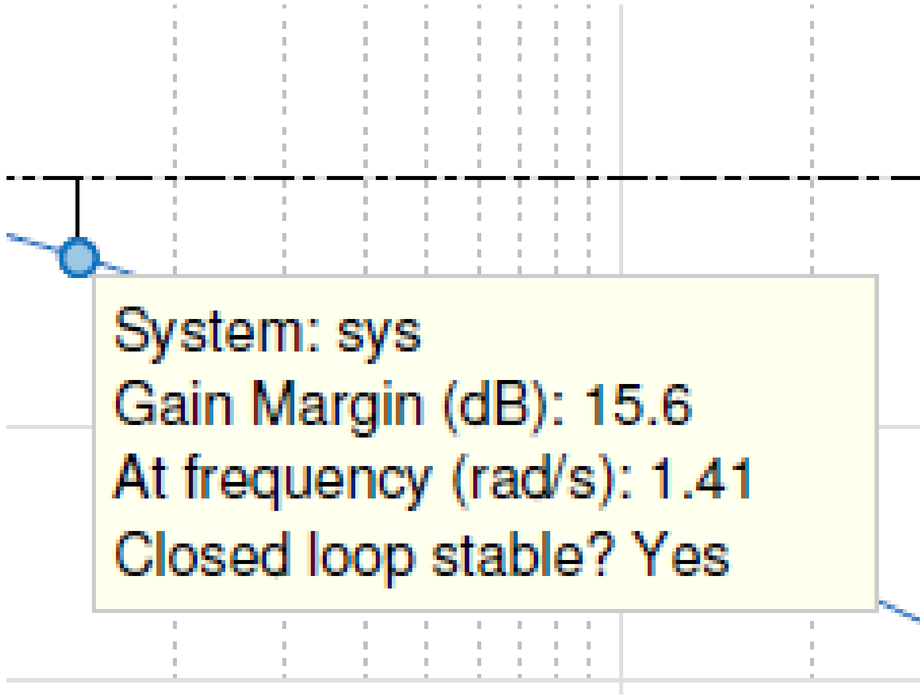




Example 1c

- **Bode plot of frequency response:**





Example 2

- For an open-transfer function below, with K=2, determine if the closed-loop transfer function is stable and find the critical value for K

$$G(s)H(s) = \frac{K}{s(s+1)(2s+1)}$$

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(j\omega+1)(2j\omega+1)} = \frac{K}{-3\omega^2 + j\omega(1-2\omega^2)}$$

For stability, the Nyquist plot should not encircle the -1+j0 point, by setting imaginary part of $G(j\omega)H(j\omega)$ as $0, \omega = \pm\frac{1}{\sqrt{2}}$,

by substitution, $G(j\omega)H(j\omega) = -\frac{2K}{3}$, by equating this to -1, critical value is 1.5 and stable range is 0 < K < 1.5

Example 3

- Consider a unity feedback system with open loop transfer function below, for marginal stability find the radian frequency of oscillation

$$G(s) = \frac{K}{(s^2 + 2s + 2)(s + 2)}$$

- Show that:

$$G(j\omega)H(j\omega) = \frac{4(1-\omega^2) - j\omega(6-\omega^2)}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2}$$

Set imaginary part to 0, find ω

$\omega = \sqrt{6}$ and show the real part yields $\left(\frac{1}{20}\right)\angle 180^o$.

This closed loop system is stable if the magnitude of the frequency response is less than 1 at 180 deg. Hence K<20 is stable, K>20 is unstable, and K=20 is marginally stable. Frequency of oscillation is $\sqrt{6}$

Example 4

- Consider a unity feedback system with transfer function:

$$G(s) = \frac{Ke^{-0.8s}}{s+1}$$

Using Nyquist plot, determine the critical value of K for stability

- For this system,

$$G(j\omega) = \frac{Ke^{-j0.8\omega}}{j\omega+1} = \frac{\big(Kcos(0.8\omega) - jKsin(0.8\omega)\big)(1-j\omega)}{1+\omega^2}$$

$$= \frac{K}{1+\omega^2}[cos(0.8\omega) - \omega sin(0.8\omega) - j(sin(0.8\omega) + \omega cos(0.8\omega))]$$

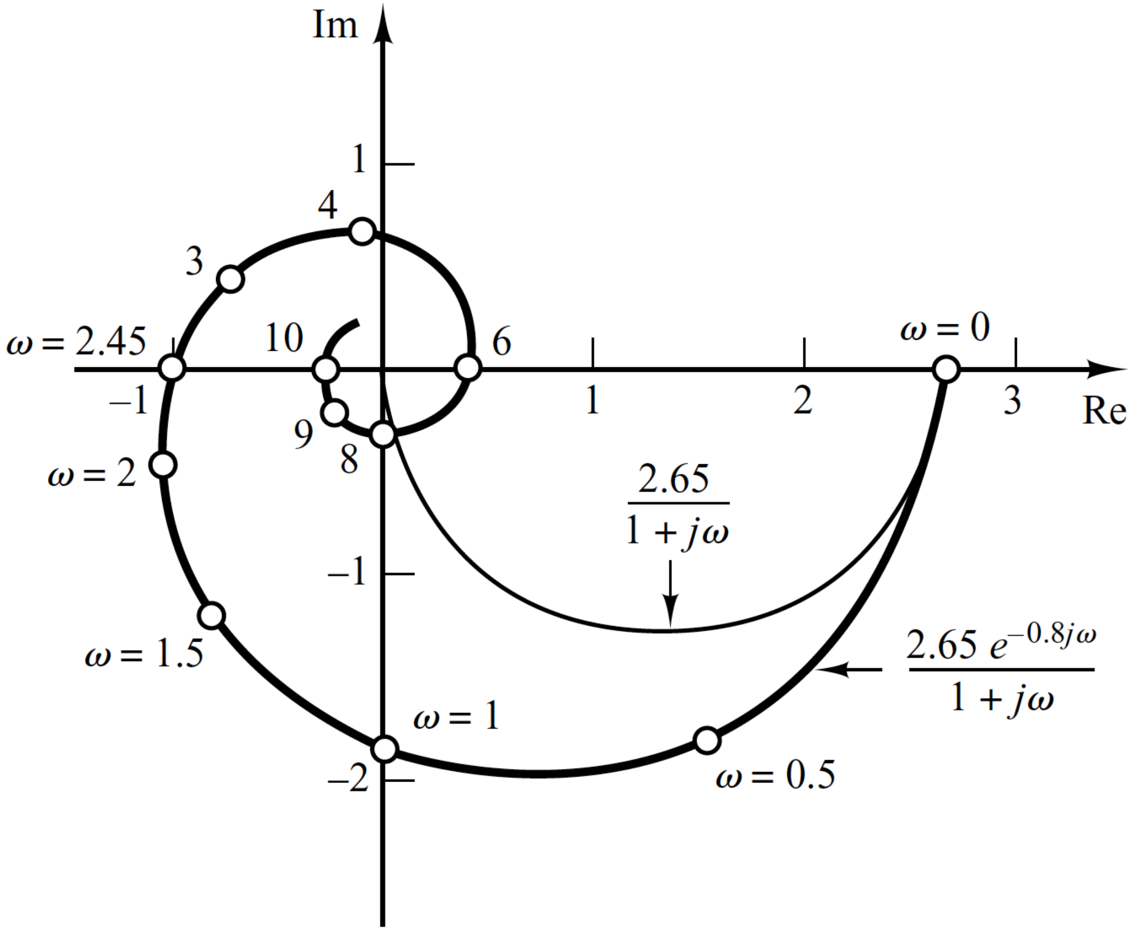
When imaginary part is zero, show $\omega = 2.4482$

Substitute this into $G(j\omega)$ and show that K – 2.65

Example 3

- **Consider a unity feedback system with transfer function:**

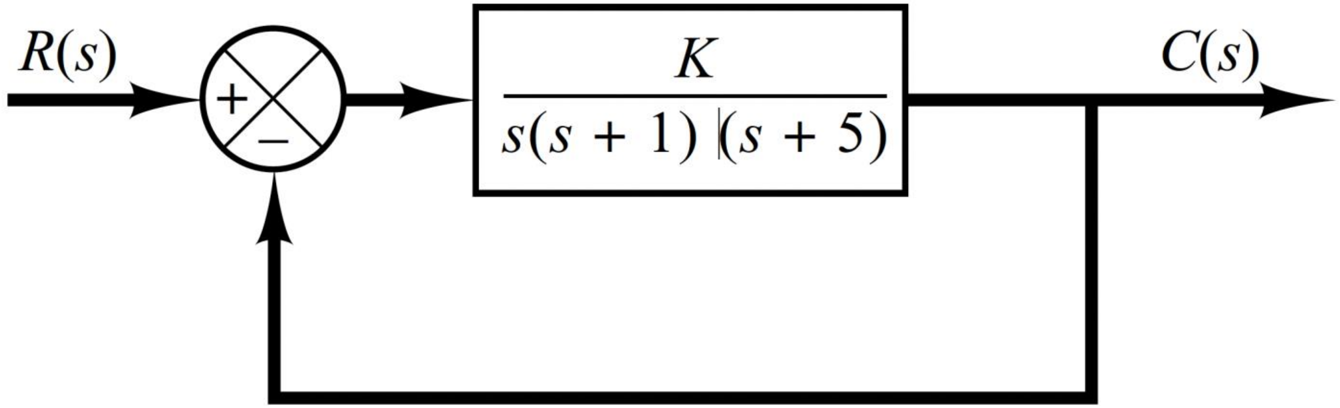
$$G(s) = \frac{Ke^{-0.8s}}{s+1}$$



Stability Margins

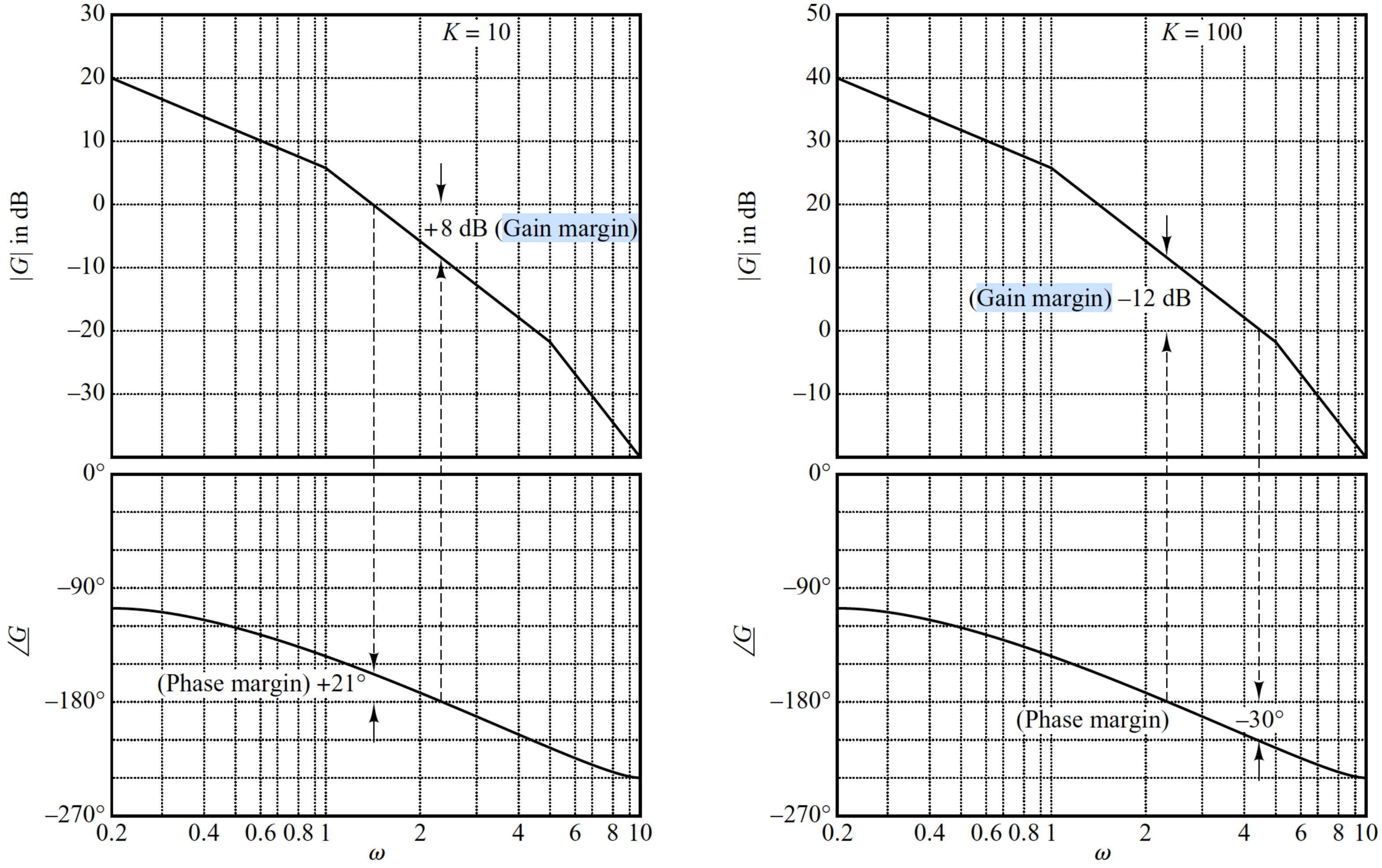
Example 1

- **By drawing the bode plots, obtain the phase and the gain margins for the system below for the two cases where K=10 and K=100.**



Example 1

- **By drawing the bode plots, obtain the phase and the gain margins for the system below for the two cases where K=10 and K=100.**

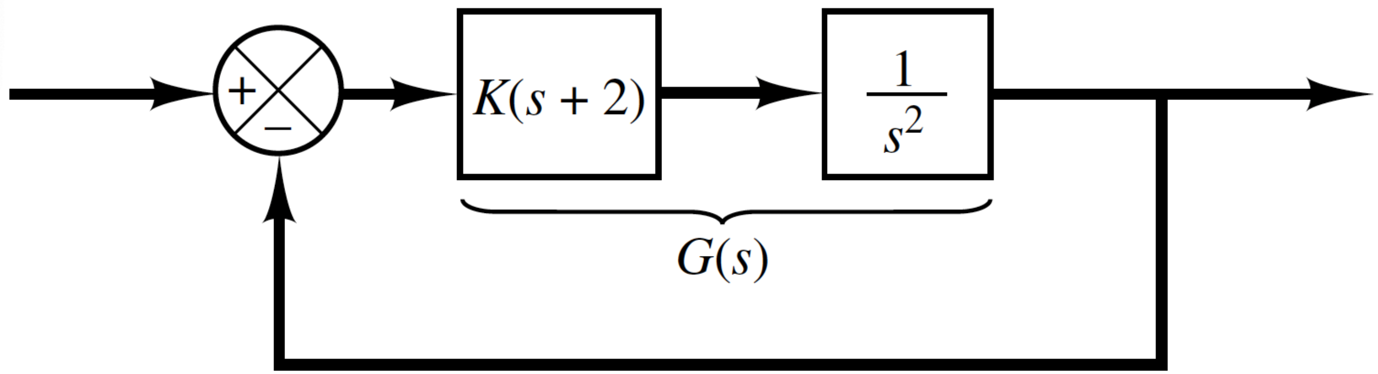


(a) (b)

- **At K=10**
  - **GM = 8 dB**
  - **PM = 21 deg**
  - **Stable**
- **At K=100**
  - **GM = -12 dB**
  - **PM = -30 deg**
  - **Unstable**

Example 2

For the space-vehicle system below, draw the bode plot and determine the value of K so that the phase margin is 50 deg. What is the gain margin with this value of gain K?



$$G(j\omega) = \frac{K(j\omega + 2)}{j\omega^2}$$

$$\angle G(j\omega) = \angle(j\omega + 2) - 2\angle(j\omega) = tan^{-1}\left(\frac{\omega}{2}\right) - 180^o$$

For gain margin to be 50 deg, this means $\angle G(j\omega_c) = -130^o$ where ω_c is gain cross over frequency

Example 2

For the space-vehicle system below, draw the bode plot and determine the value of K so that the phase margin is 50 deg. What is the gain margin with this value of gain K?

$$tan^{-1}\left(\frac{\omega_c}{2}\right) = 50^o$$

$$\omega_c = 2.3835\, rad/sec$$

Since the phase curve does not cross -180 deg line, GM is infinite, the magnitude of $G(j\omega)$ must be equal to 0db at ω= 2.3835

$$\frac{K(j\omega + 2)}{j\omega^2} = 1\, at\, \omega = 2.3835$$

K = 1.8259

Example 3

- **For a standard second order transfer function, write the bandwidth as a function of damping factor and natural frequency.**

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The bandwidth ω_b (3dB point) is determined by

$$\left|\frac{C(j\omega_b)}{R(j\omega_b)}\right| = -3dB = 0.707$$

$$\left|\frac{C(j\omega_b)}{R(j\omega_b)}\right| = \frac{\omega_n^2}{(j\omega_b)^2 + 2\zeta\omega_n(j\omega_b) + \omega_n^2} = 0.707$$

$$\omega_n^4 = 0.5\left[\left(\omega_n^2 - \omega_b^2\right)^2 + 4\zeta^2\omega_n^2\omega_b^2\right]$$

Example 3

$$\omega_n^4 = 0.5\left[\left(\omega_n^2 - \omega_b^2\right)^2 + 4\zeta^2\omega_n^2\omega_b^2\right]$$

Dividing by ω_n^4, and solving for $\left(\frac{\omega_b}{\omega_n}\right)^2$, yields

$$\left(\frac{\omega_b}{\omega_n}\right)^2 = -2\zeta^2 + 1 \pm \sqrt{4\zeta^4 - 4\zeta^2 + 2}$$

Since $\left(\frac{\omega_b}{\omega_n}\right)^2 > 0$

$$\omega_b^2 = \omega_n^2\left(-2\zeta^2 + 1 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right)$$

$$\omega_b = \omega_n\left(-2\zeta^2 + 1 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right)^{\frac{1}{2}}$$

Exercise 1

For a unity feedback system with transfer function below, plot a Nyquist diagram and use your Nyquist diagram to find the range of gain K for stability.

$$G(s) = \frac{K}{(s+2)(s+4)(s+6)}$$

Ans: 480 > K > -48

Exercise 2

Find the gain margin and phase margin of the system if K=6

$$G(s) = \frac{K}{(s+2)(s+4)(s+6)}$$

Gm = 38.1 dB

Pm = inf

Exercise 3

For a unity feedback system with transfer function below,

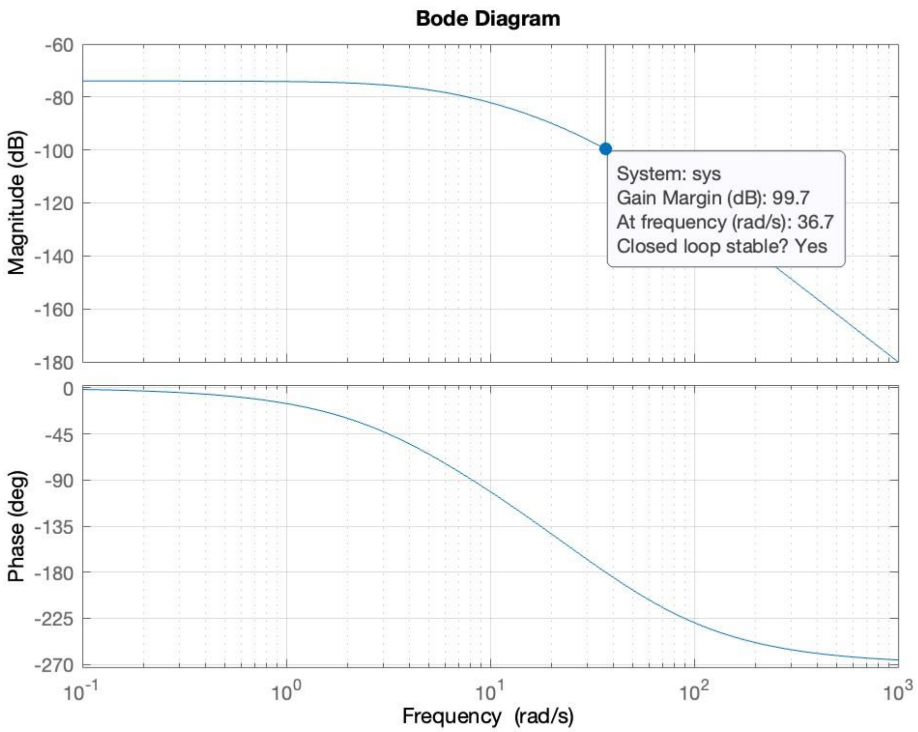
Draw Bode magnitude and phase plots

Find the range of K for stability from your plot

Evaluate the gain margin, phase margin, 0 dB frequency, and 180 deg frequency from your Bode plot for K=10,000

$$G(s) = \frac{K}{(s+5)(s+20)(s+50)}$$

Exercise 3



96250>K>-5000

Gm = 99.7 dB

Pm = inf

Wcg = 36.7

Wcp = does not exist