Dynamics and Control (UESTC 3001)

Notes prepared by: Dr Ola R. Popoola

Dynamics and Control (UESTC 3001)

- Lecture 1: Root Locus Analysis
- Lecture 2: Root Locus II and Nyquist Plots
- Lecture 3: Bode Plots
- Lecture 4: Bode Plots II
- Lecture 5: Stability in Frequency Domain
- Lecture 6: Stability cont. and Stability Examples
- Lecture 7: Compensators
- Lecture 8: Tutorials and Test Exercises

Notes prepared by: Dr Ola R. Popoola

Intended Learning Objectives

At the end of this lecture, you will be able to:

- Stability in frequency domain
- Phase and Gain Margins with Bode plots
- Cauchy’s principle of argument
- State the Nyquist stability Criterion
- Determine the margins of a system using Nyquist plot
- [Optional] Constant m, n circles

Stability in Frequency Domain

Stability Margins

- **What are stability margins?**
- **Why are stability margins important?**

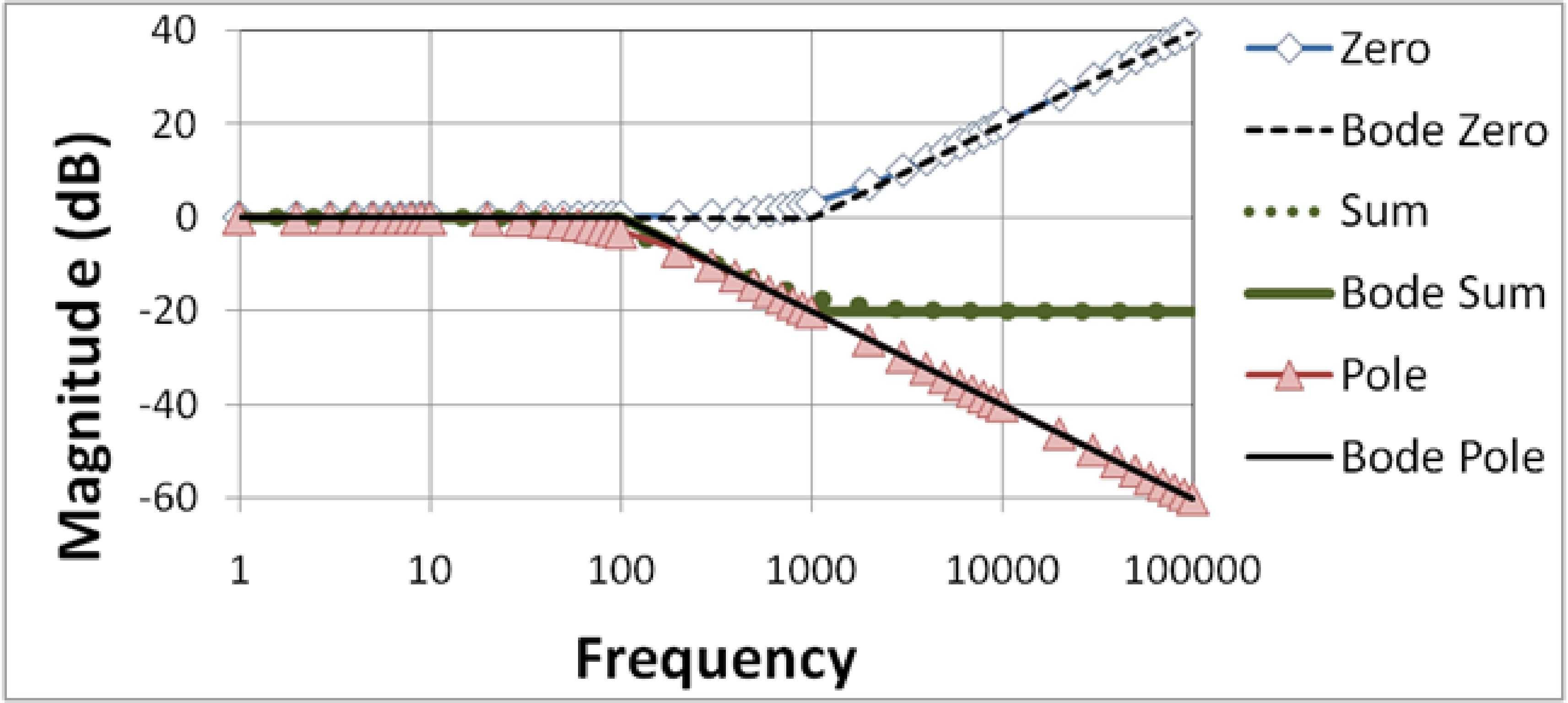
Stability Margins

- Two types of margins in Frequency domain analysis are the gain margins and the phase margin
- The stability margins measure the system's buffet against instability to assure of robustness
- They indicate tolerance of system to parameter variations
- They offer design validation by verifying the control system's reliability under different conditions
- The help prevent against unexpected system failures

Stability Margins using Bode plot

- **Given an open loop system G($j\omega$), the closed loop transfer function can be written as:**

$$G_{cl}(j\omega) = \frac{G(j\omega)}{1+G(j\omega)} \text{ and } |G_{cl}(j\omega)| = \frac{|G(j\omega)|}{|1+G(j\omega)|}$$



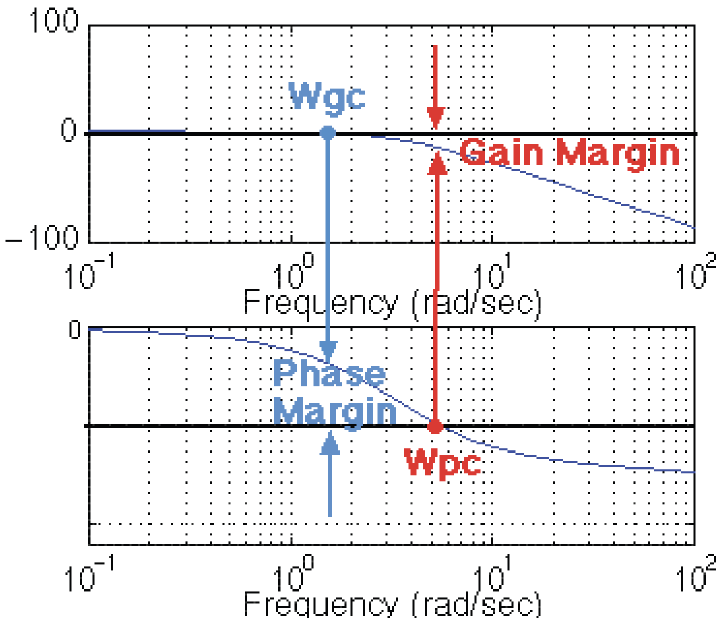
Stability Margins using Bode Plot

- $20\log|\boldsymbol{G_{cl}}(\boldsymbol{j\omega})| = 20\log|\boldsymbol{G}(\boldsymbol{j\omega})| - \mathbf{20}\log|\mathbf{1} + \boldsymbol{G}(\boldsymbol{j\omega})|$
- By focusing on the contribution of the new term: $-\mathbf{20}\log|\mathbf{1} + \boldsymbol{G}(\boldsymbol{j\omega})|$, specifically as $\mathbf{1} + \boldsymbol{G}(\boldsymbol{j\omega}) \rightarrow \mathbf{0}$,
- We can write

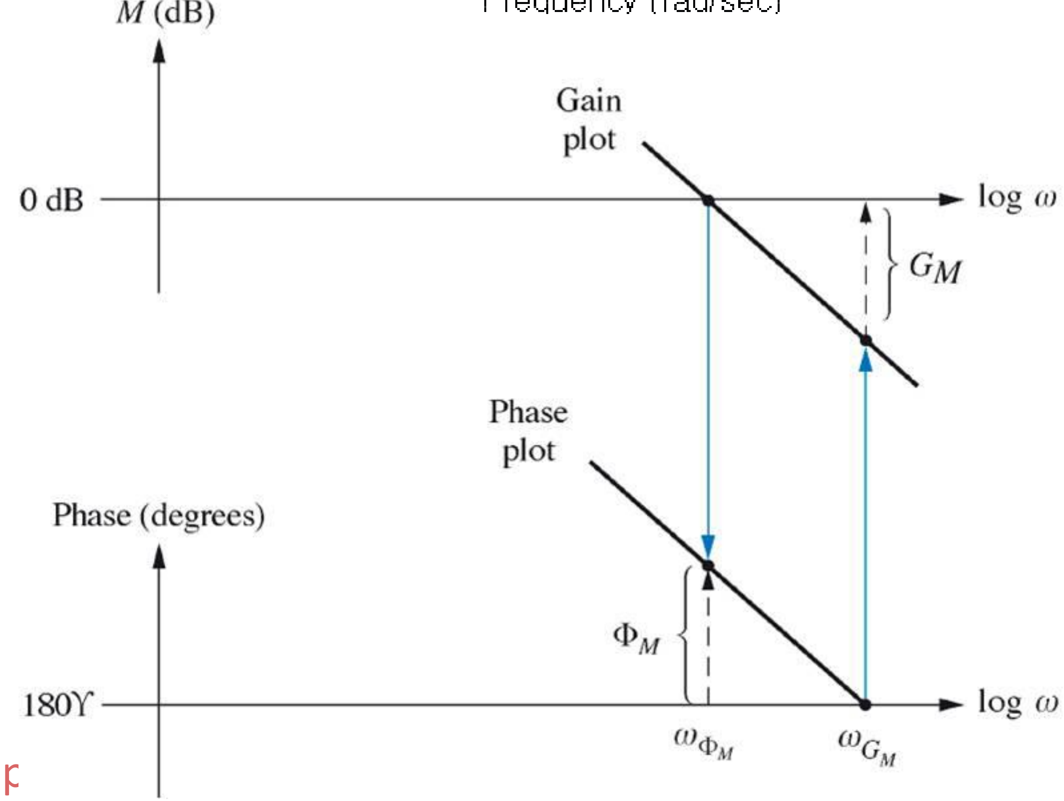
$$\lim_{\mathbf{1}+\boldsymbol{G}(\boldsymbol{j\omega}) \rightarrow \mathbf{0}} -\mathbf{20}\log|\mathbf{1} + \boldsymbol{G}(\boldsymbol{j\omega})| = \infty; \text{ which implies instability}$$

- Since instability occurs when $\mathbf{1} + \boldsymbol{G}(\boldsymbol{j\omega}) = \mathbf{0}$, we must avoid
  - $|\mathrm{G}(j\omega)| = 1$
  - $< G(j\omega) = -180$ deg
- Stability margins measure how far the system is from ($|\mathrm{G}(j\omega)| = 1$, $< G(j\omega) = -180$ deg)

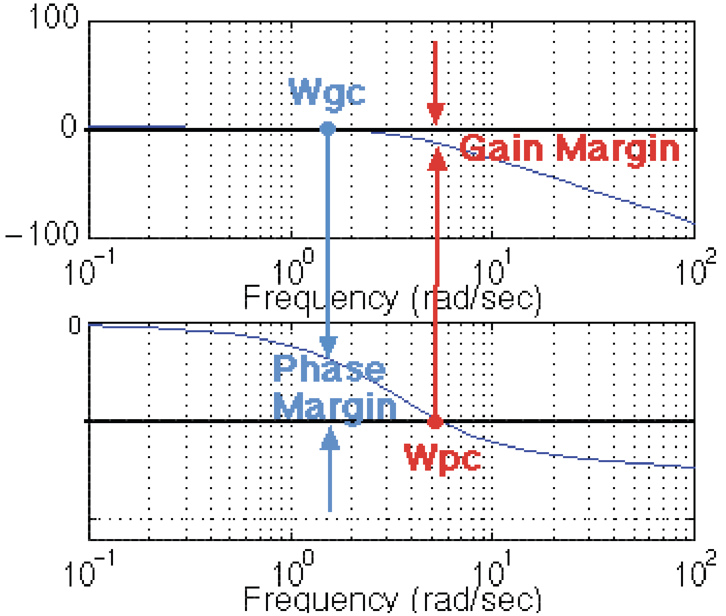
Phase Margin



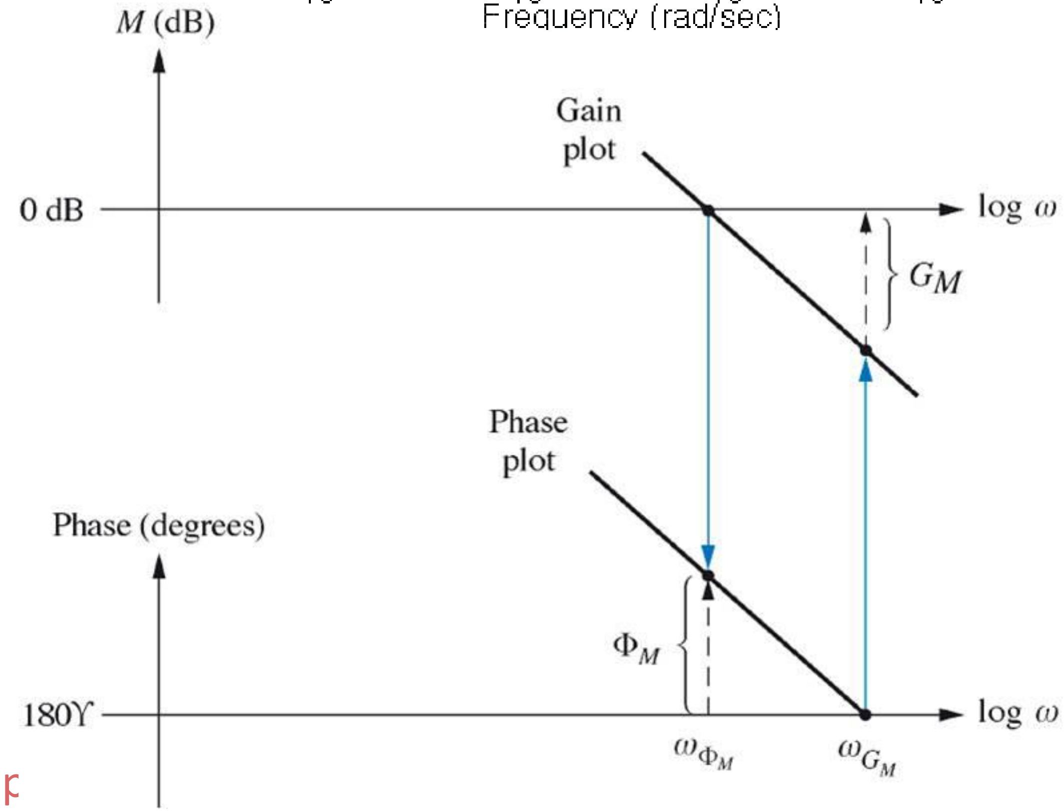
- The phase margin, ϕ, is that amount of additional phase lag that could be added at the gain crossover frequency required to bring the system to the verge of closed loop instability.
- The gain crossover frequency is the frequency at which the magnitude of the open loop transfer function, $|G(j\omega)|$, is unity
- It is computed by determining the phase , θ, when $|G(j\omega)| = 1$ and adding this phase to 180 deg
- $Phase\ Margin, \phi = 180^o + \theta$



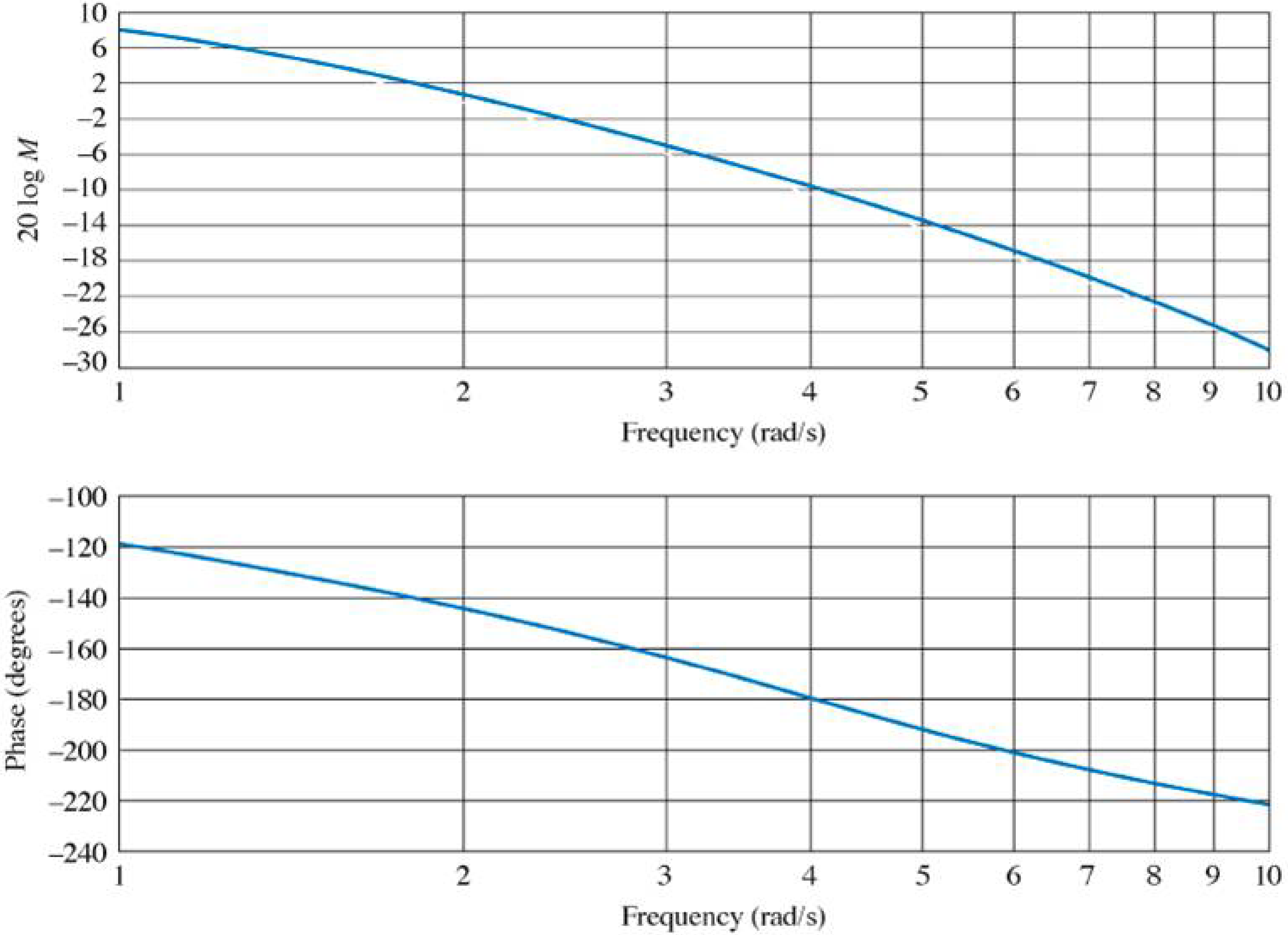
Gain Margins



- The gain margin is defined as the factor by which you would have to increase the gain of a system to make it unstable.
- The phase crossover frequency is the frequency at which the phase angle of the open-loop transfer function is -180
- It is computed by determining the magnitude , λ, when , $arg\{G(j\omega)\} = -180^o$, and defined as follows:
- Gain Margin = $\frac{1}{\lambda}$



Stability Margins using Bode Plot-Example



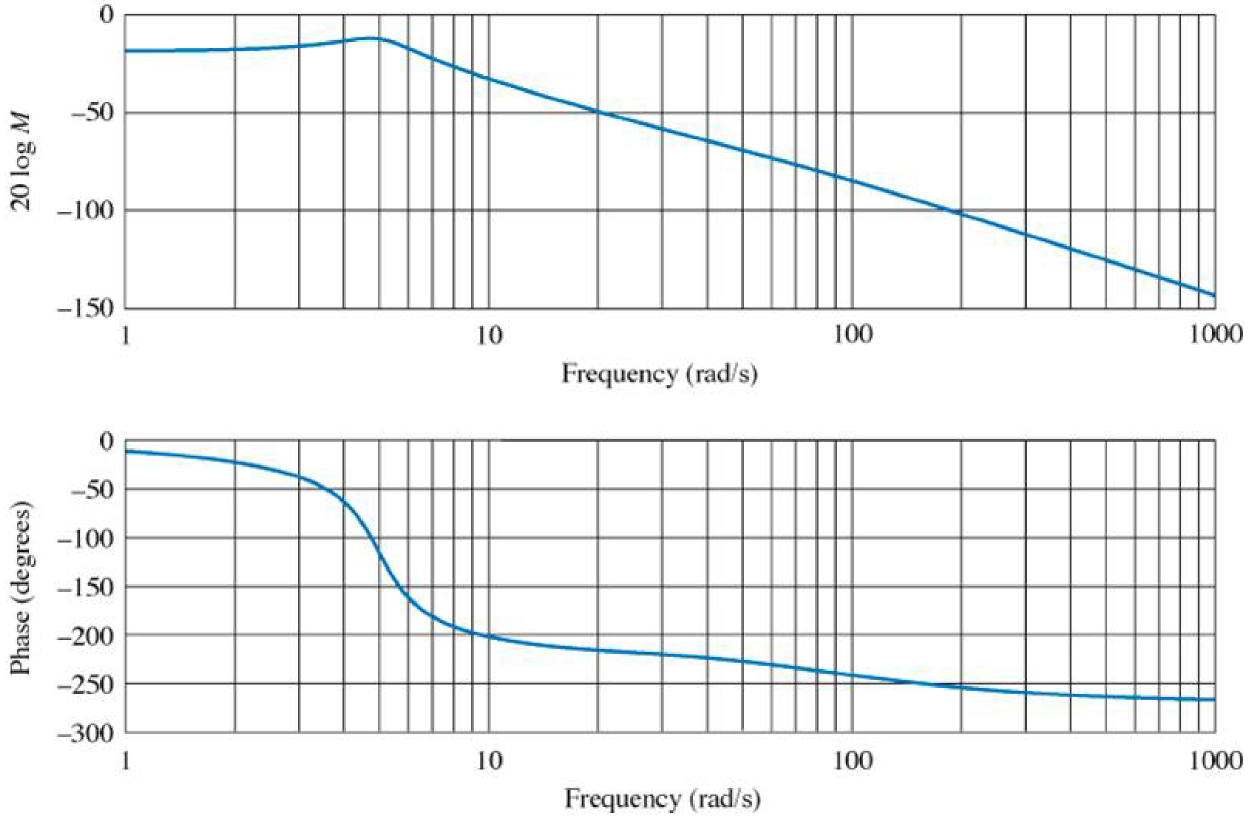
(b)

PM = 35

GM = 10dB

Stability Margins using Bode Plot-Example

- Sometimes the margins are undefined
  - There is no crossover at 0dB
  - There is no crossover at -180 deg



Comments on System Margins

- It is important to note once again that these measures of stability are valid for open-loop stable systems only.
- A large gain margin or large phase margin indicates a very stable feedback system but usually a very sluggish one.
- A gain margin close to unity or a phase margin close to zero corresponds to a highly oscillatory system.
- Usually, in practice, a GM of about 6 db or a PM about 30-35 deg results in a reasonably good degree of relative stability.
- In most cases a good GM guarantees a good PM and vice versa with some other “special” situations where this does not hold.

Frequency domain stability

- By applying Cauchy's principle of argument to the open-loop system transfer function, we will get information about stability of closed-loop system.
- Nyquist stability is important as it can also be used to determine the relative degree of system stability.
- For a SISO feedback system, the closed-loop transfer function is given by:

$$M(s) = \frac{G(s)}{1 + H(s)G(s)}$$

- The closed-loop system poles are obtained by solving:

$$1 + H(s)G(s) = 0 = \Delta(s)$$

Frequency domain stability

- **Using the complex function,**

$$\boldsymbol{D(s) = 1 + H(s)G(s)}$$

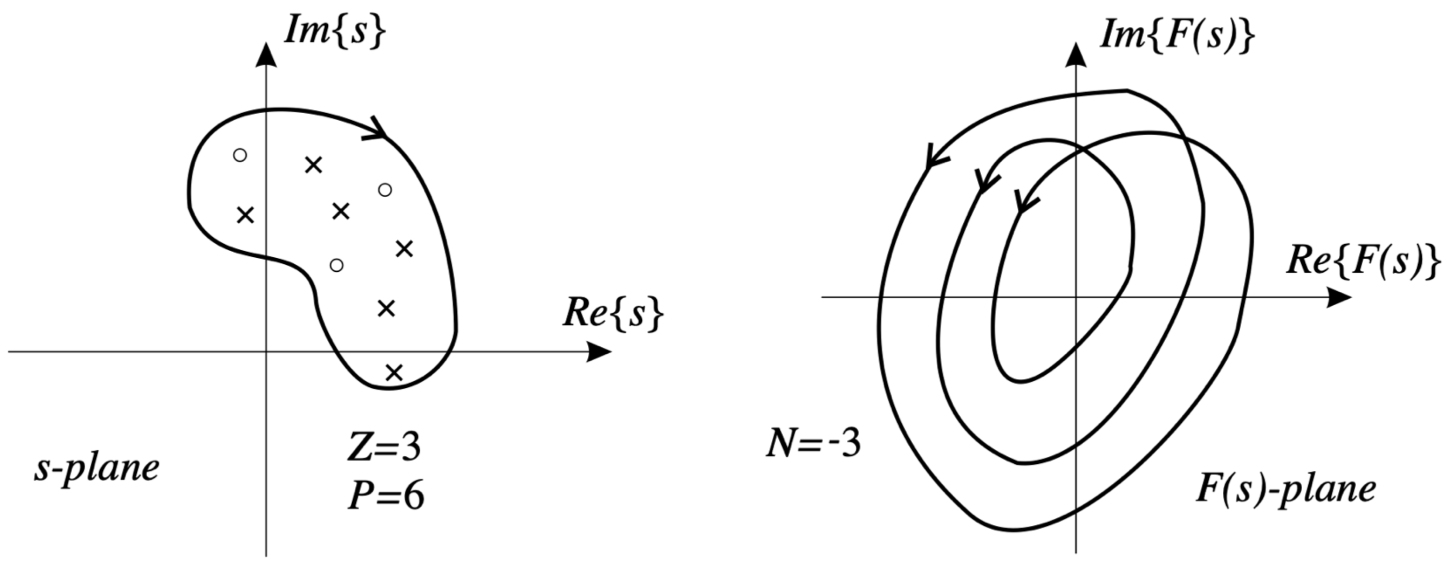
- **The poles of D(s) are the open-loop control system poles since they are contributed by the poles of $\boldsymbol{H(s)G(s)}$.**

- **Nyquist stability test is obtained by applying Cauchy principle of argument to the complex function D(s).**

- **What is the Cauchy principle of argument?**

Cauchy's Principle of Argument

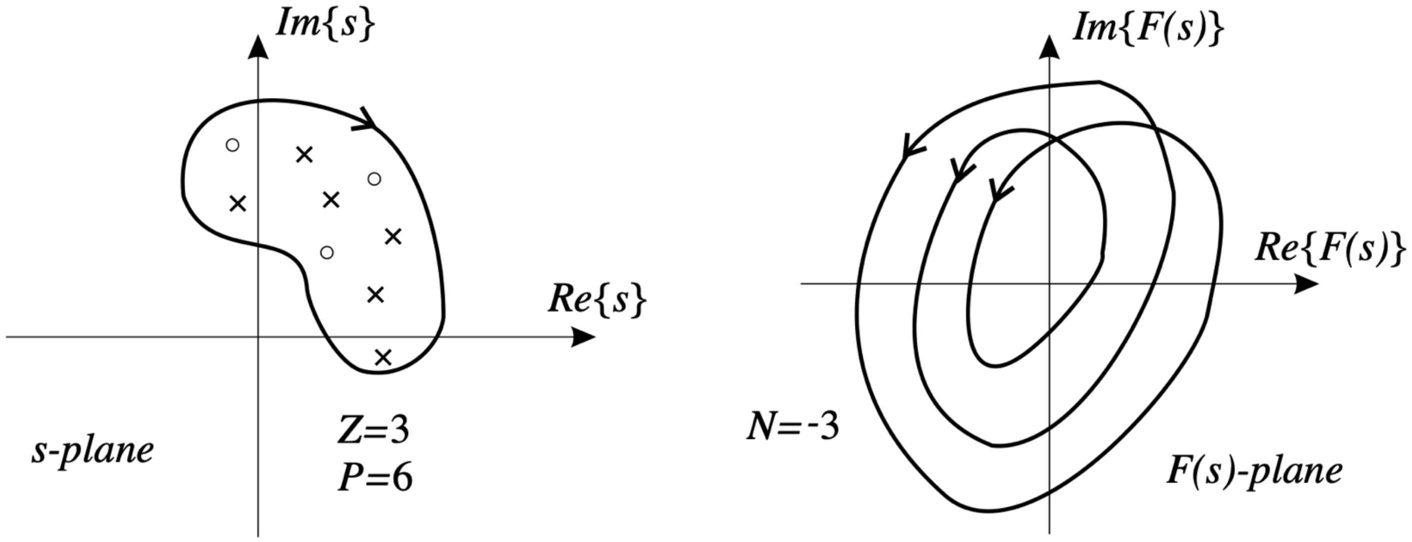
- Let F(s) be an analytic function in a closed region of the complex plane *s*, at a finite number of points called the poles of F(s)
- If F(s) is analytic at every point of the contour, then as *s* travels around the contour in the s-plane, in the clockwise direction, the function F(s) encircles the origin in the $(Re\{F(s)\}, Im\{F(s)\})$-plane in the same direction N times, with N-given by $N = Z - P$



Cauchy's Principle of Argument

$$arg\{F(s)\} = (Z - P)2\pi = 2\pi N$$

- **This justifies the use of the term "the principle of Argument".**
- **Example: Imagine a function F(s), with one pole and no zero, inside the contour C, as you move around C, once, the argument of F(s) increases by 2π because there is one zero contributing (2π) and no poles.**



Nyquist Stability Criterion

- It states that the number of unstable closed-loop poles is equal to the number of unstable open-loop poles plus the number of encirclements of the origin of the Nyquist plot of the complex function D(s)

- The above can be modified slightly, instead of plotting the function $D(s) = 1 + H(s)G(s)$, we write D'(s)=G(s)H(s) and count the encirclement of D'(s) around (-1,j0).

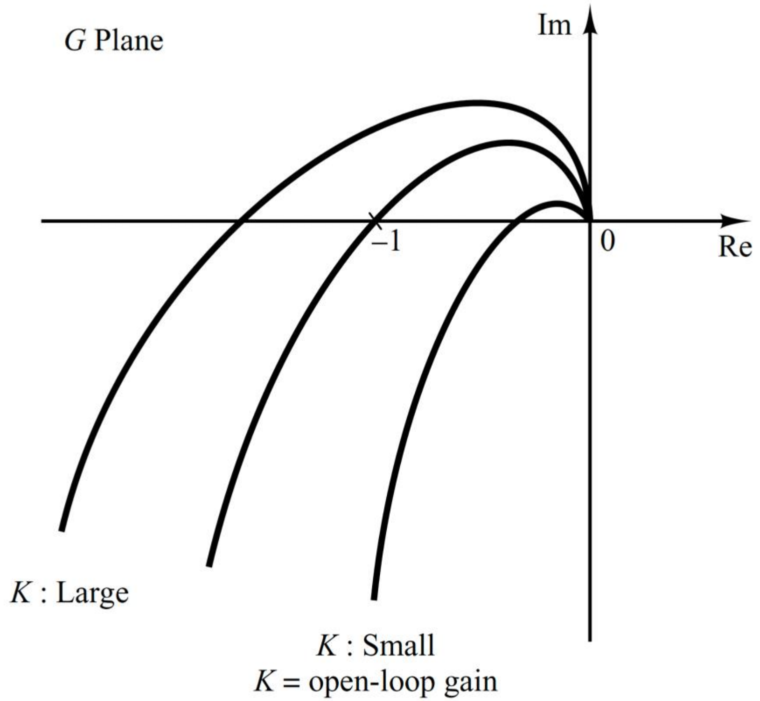
- Modified criterion becomes: The number of unstable closed loop poles is equal to the number of unstable open loop poles plus the number of encirclements N of the point (-1,j0) of the Nyquist plot of G(s)H(s).

Stability Margins

- **Consider a feedback system with the following open-loop transfer function**

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(j\omega T_1 + 1)(j\omega T_2 + 1)}$$

Let us investigate the stability of the system for various values of K



Stability Margins

Given the significance of the point of intersection of the polar plot with the negative real axis, how do we estimate this?

The point of intersection of the polar plot with the negative real axis can be determined by setting the imaginary part of $G(j\omega)H(j\omega)$ equal to zero

Set the frequency at the point of intersection to be ω_2, and using the above, estimate ω_2

Stability Margins

- The point of intersection of the polar plot with the negative real axis can be determined by setting the imaginary part of $G(j\omega)H(j\omega)$ equal to zero

$$G(j\omega)H(j\omega) = \mathrm{u} + \mathrm{jv} = \frac{-\mathrm{K}(\mathrm{T}_1 + \mathrm{T}_2) - \mathrm{jK}\left(\frac{1}{\omega}\right)\left(1 - \omega^2 T_1 T_2\right)}{1 + \omega^2\left(T_1^2 + T_2^2\right) + \omega^4 T_1^2 T_2^2}$$

If we set the frequency at the point of intersection to be ω_2, we have:

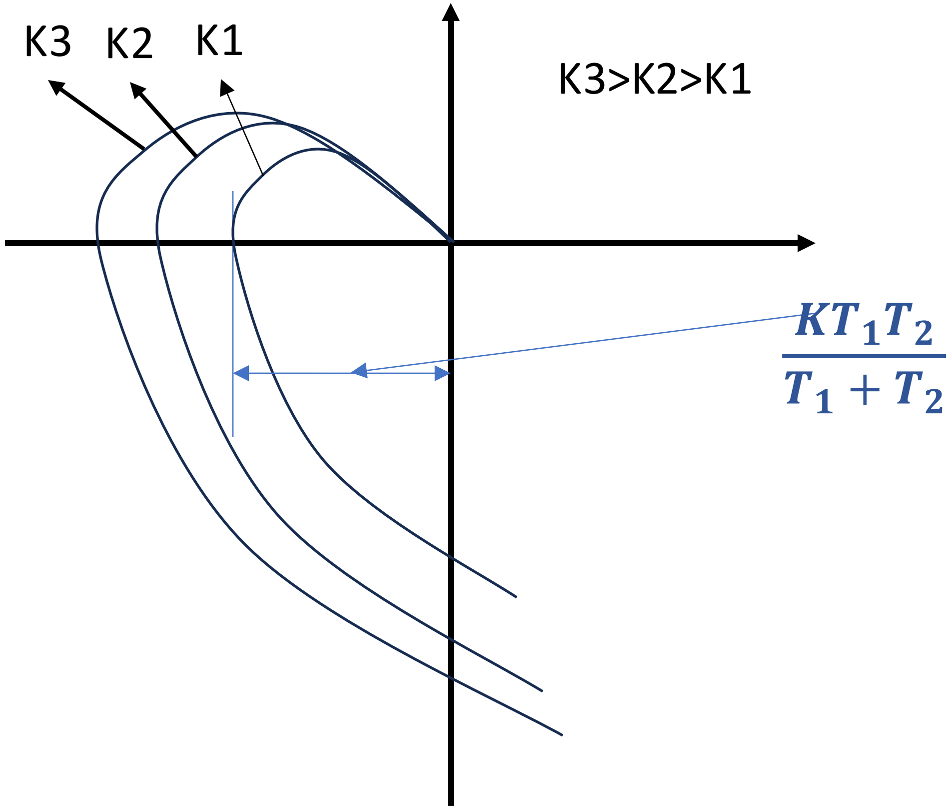
$$\mathrm{v} = \frac{-\mathrm{K}\left(\frac{1}{\omega_2}\right)\left(1 - \omega_2^2 T_1 T_2\right)}{1 + \omega_2^2\left(T_1^2 + T_2^2\right) + \omega_2^4 T_1^2 T_2^2} = 0$$

This gives $\omega_2 = \frac{1}{\sqrt{T_1 T_2}}$

Stability Margins

- The magnitude of the real part at frequency ω_2 is given by:

$$\mathbf{u} = \frac{-\mathrm{K}(\mathrm{T}_1 + \mathrm{T}_2)}{1 + \omega^2\left(T_1^2 + T_2^2\right) + \omega^4 T_1^2 T_2^2} = -\frac{KT_1T_2}{T_1 + T_2}$$



For the system to be stable,

$$\frac{KT_1T_2}{T_1+T_2} < 1 \text{ or } K < \frac{T_1+T_2}{T_1T_2}$$

Recall: Margins

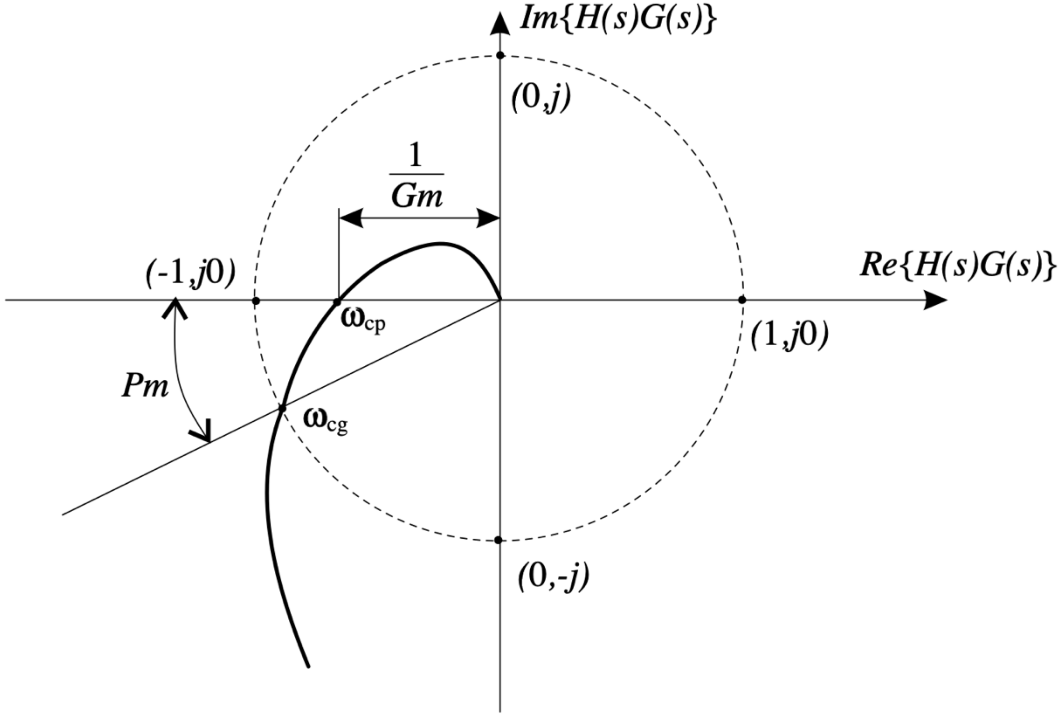
- The gain margin is defined as the factor by which you would have to increase the gain of a system to make it unstable.
- It is computed by determining the magnitude , λ, when , $arg\{G(j\omega)\} = -180^o$, and defined as follows: Gain Margin $= \frac{1}{\lambda}$

- The phase margin, ϕ, is that amount of additional phase lag that could be added at the gain crossover frequency required to bring the system to the verge of closed loop instability.
- It is computed by determining the phase , θ, when $|G(j\omega)| = 1$ and adding this phase to 180 deg. $Phase\ Margin, \phi = 180^o + \theta$

Nyquist Margins

$$Pm = 180^o + arg\{G(j\omega_{gc}), H(j\omega_{gc})$$

$$Gm[dB] = 20\,log\frac{1}{|G(j\omega_{pc})H(j\omega_{pc})|}[dB]$$



Correlation between PM and Damping Factor ζ

- Consider a unity feedback second order system with an open-loop transfer function

$$G(s)H(s) = \frac{K}{s(\tau s + 1)} = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

where $\omega_n = \sqrt{\frac{K}{\tau}}$ and $2\zeta\omega_n = \frac{1}{\tau}$

Replacing s by $j\omega$ for obtaining a polar plot, we have

$$G(j\omega)H(j\omega) = \frac{\omega_n^2}{j\omega(j\omega + 2\zeta\omega_n)}$$

Finding the phase margin can show that for $\zeta \leq 0.7, \zeta \approx 0.01\phi$, where ϕ is the phase margin.

Frequency domain specifications

- Resonance peak M_r: This is the maximum value of M, the magnitude of the closed loop frequency response.
- Resonance frequency ω_r: The frequency at which the resonance peak occurs.
- Bandwidth: The range of frequencies for which the system gain is more than -3 db
- Cut-off rate: It is the slope of the log-magnitude curve near the cur-off frequency
- Gain and phase margins: recently discussed
- Graphical techniques to determine M_r and ω_r require that constant M-contours be drawn on the complex plane and constant N contours discussed next.

Constant-M circles

Constant-M circles

- Consider any point $G(j\omega) = x + jy$, on the polar plot of $G(j\omega)$. The closed-loop frequency response is:

$$T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1 + G(j\omega)} = \frac{x + jy}{1 + x + jy} = Me^{j\alpha}$$

The magnitude M is given by

$$M = \frac{|x + jy|}{|1 + x + jy|} = \left[\frac{x^2 + y^2}{(1 + x)^2 + y^2}\right]^{\frac{1}{2}}$$

Rearranging and rewriting in form of the equation of a circle, we get the centre

$$x_0 = -\frac{M^2}{M^2 - 1}; y_o = 0; r_o = \frac{M}{M^2 - 1}$$

Constant-M circles

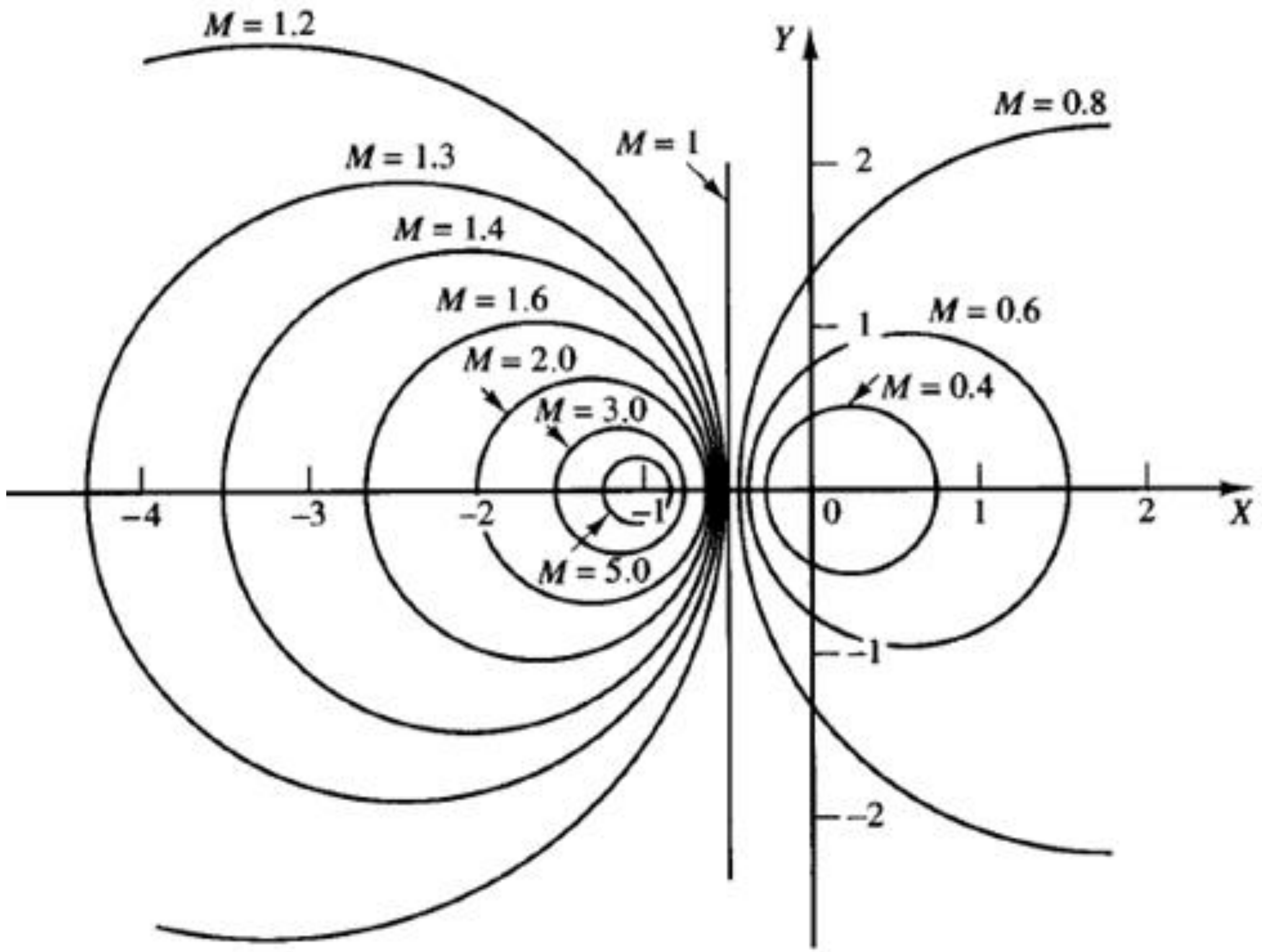
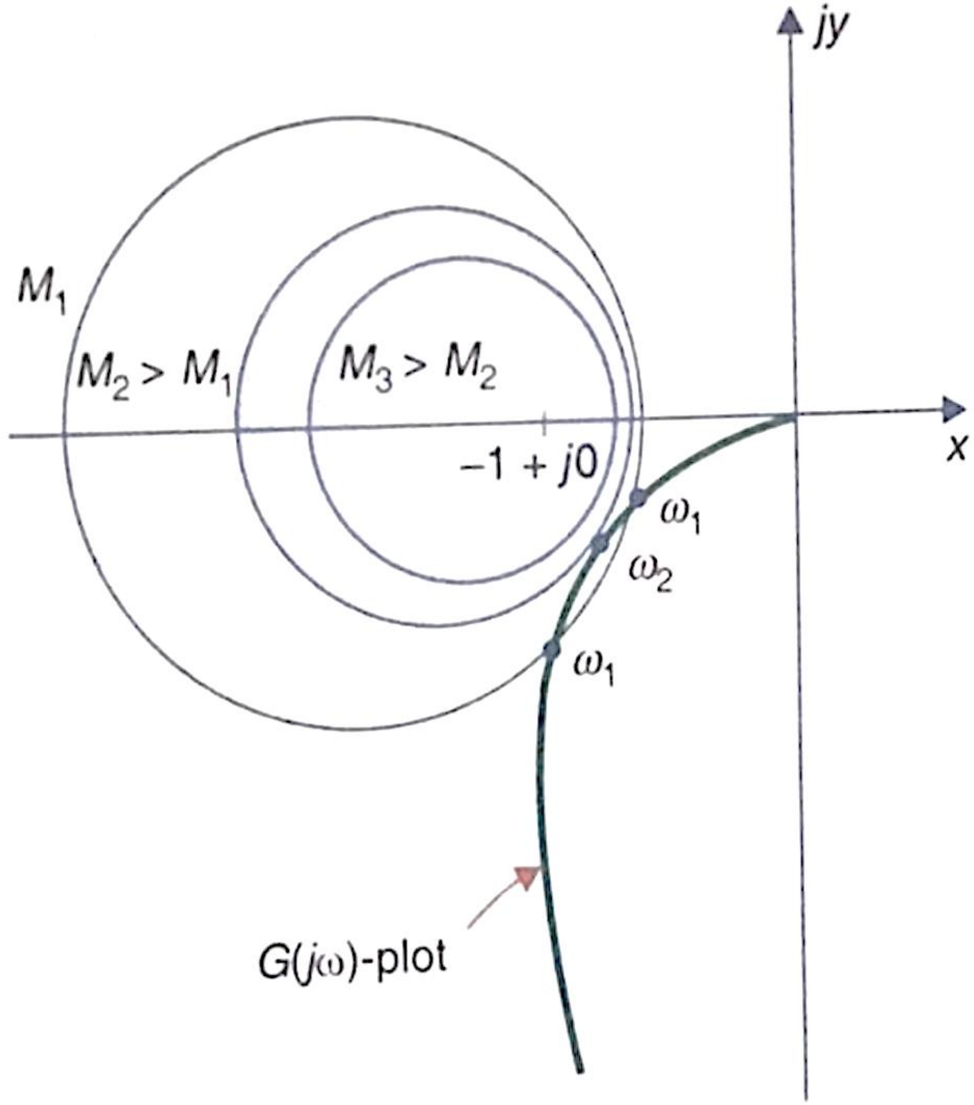


Figure 1



$$M_r = M_2$$

Figure 2

Constant-N circles

Constant-N circles

- **Recall:**

$$T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)} = \frac{x+jy}{1+x+jy} = Me^{j\alpha}$$

- **The phase angle of $T(j\omega)$ is given by**

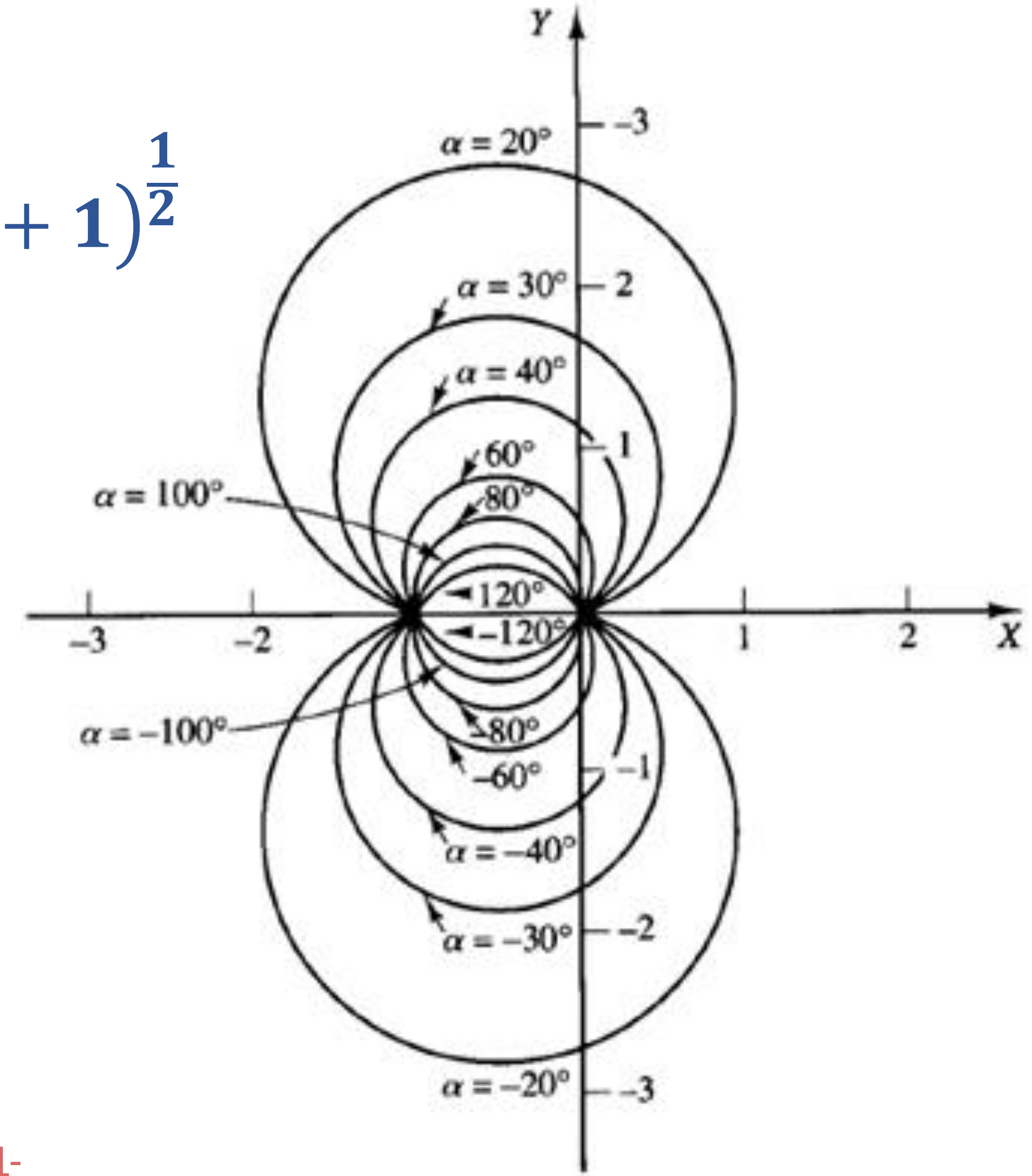
$$\angle T(j\omega) = \alpha = \angle\left(\frac{x+jy}{1+x+jy}\right)$$

$$= tan^{-1}\left(\frac{y}{x}\right) - tan^{-1}\left(\frac{y}{1+x}\right)$$

Solving and rearranging, you get $x_o = -\frac{1}{2}, y_o = \frac{1}{2N}, r_o = \frac{1}{2N}\left(N^2+1\right)^{\frac{1}{2}}$

Constant-N circles

$$x_o = -\frac{1}{2}, y_o = \frac{1}{2N}, r_o = \frac{1}{2N}\left(N^2+1\right)^{\frac{1}{2}}$$



Non-unity feedback systems

- **For a non-unity feedback system, the closed loop transfer function is given by**

$$T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1 + G(j\omega)H(j\omega)} = \frac{1}{H(j\omega)}\left[\frac{G(j\omega)H(j\omega)}{1 + G(j\omega)H(j\omega)}\right]$$

$$= \frac{1}{H(j\omega)}\left[\frac{G_o(j\omega)}{1 + G_o(j\omega)}\right] = \frac{1}{H(j\omega)}T_o(j\omega)$$

Where $G_o(j\omega) = G(j\omega)H(j\omega)$ **and** $T_o(j\omega) = \frac{G_o(j\omega)}{1+G_o(j\omega)}$