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System Properties (Laplace Transform)

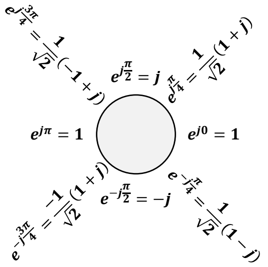
Causality	ROC is a right-half plane
Causality (Rational $H(s)$)	ROC is the right-half plane to the right of the rightmost pole
Anticausality	ROC is a left-half plane (to the left of the leftmost pole)
Stability	ROC includes the entire $j\omega$-axis

Causal LTI system with rational $H(s)$: all poles lie in the left-half of the s-plane

System Properties (Z-Transform)

Causality	ROC is the exterior of a circle including infinity
Causality (Rational $H(s)$)	ROC is the exterior of a circle outside the outermost pole + 分子 z 阶数不能比分母大
Stability	ROC includes the unit circle

Causal LTI system with rational $H(s)$: all poles lies inside the



Z-Transform Z 变换

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

Euler's formula 欧拉公式

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$$

Nyquist Sampling Theorem

Let $x(t)$ be a band-limited signal within $\pm\omega_M$, then $x(t)$ is uniquely determined by its samples $x(nT)$ if $\omega_s > 2\omega_M$, where $\omega_s = \frac{2\pi}{T}$.

Nyquist rate: $2\omega_M$

Z 变换特殊峰的判断

信号为正，零点有峰

信号为负，无穷有峰

诱导公式

$\sin(2k\pi+\alpha) = \sin\alpha$

$\cos(2k\pi+\alpha) = \cos\alpha$

$\sin(-\alpha) = -\sin\alpha$

$\cos(-\alpha) = \cos\alpha$

$\sin(\pi-\alpha) = \sin\alpha$

$\cos(\pi-\alpha) = -\cos\alpha$

$\sin(\pi+\alpha) = -\sin\alpha$

$\cos(\pi+\alpha) = -\cos\alpha$

$\sin\left(\frac{\pi}{2}-\alpha\right) = \cos\alpha$

$\cos\left(\frac{\pi}{2}-\alpha\right) = \sin\alpha$

$\sin\left(\frac{\pi}{2}+\alpha\right) = \cos\alpha$

$\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$

$\mathrm{tg}(k\pi+\alpha) = \mathrm{tg}\,\alpha$

$\mathrm{tg}(-\alpha) = -\mathrm{tg}\,\alpha$

$\mathrm{tg}(\pi-\alpha) = -\mathrm{tg}\,\alpha$

$\mathrm{tg}(\pi+\alpha) = \mathrm{tg}\,\alpha$

$\mathrm{tg}\left(\frac{\pi}{2}-\alpha\right) = \mathrm{ctg}\,\alpha$

$\mathrm{tg}\left(\frac{\pi}{2}+\alpha\right) = -\mathrm{ctg}\,\alpha$

Fourier Transform 傅里叶变换

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

时域频域卷积相乘关系

$$y(t) = h(t) * x(t) \overset{\mathcal{F}}{\leftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)$$

$$r(t) = s(t)p(t) \overset{\mathcal{F}}{\leftrightarrow} R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P\big(j(\omega-\theta)\big)d\theta$$

Convolution 卷积

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau$$

一个函数的t换成τ，另一个函数的t换成$t-\tau$；如果结果以原点为界，记得写成$u[n]u(t)$的形式；$x(\tau)$的图像正常，$h(t-\tau)$的图像关于y轴翻转后，原点为t

Laplace Transform 拉普拉斯变换

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

Sinc Function

$$\mathrm{sinc}\,\theta = \frac{\sin\pi\theta}{\pi\theta}$$

等比数列

通项公式：$a_n = a_1 q^{n-1}$

两项关系：$a_n = a_m q^{n-m}$

求和公式：$S_n = a_1\frac{a-q^n}{1-q} = \frac{a_1 - a_n q}{1-q}$

Other Transform Pairs

$u_n(t) = \frac{d^n}{dt^n}\delta(t) \overset{\mathcal{L}}{\leftrightarrow} s^n, u^{-n}(t) = \frac{1}{s^n}$

$e^{-at}\cos\omega_0 t\, u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{s+a}{(s+a)^2+\omega_0^2}$

$e^{-at}\sin\omega_0 t\, u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{\omega_0}{(s+a)^2+\omega_0^2}$

$\left(\frac{\sin t}{\pi t}\right)^2 \overset{\mathcal{F}}{\leftrightarrow}$三角形底$-2$到$2$，高$\frac{1}{\pi}$

Even and odd decomposition

$$\mathcal{E}v\{x[t]\} = \frac{1}{2}\{x(t) + x(-t)\}$$

$$\mathcal{O}d\{x[t]\} = \frac{1}{2}\{x(t) - x(-t)\}$$

奇分解围绕原点旋转除 2，偶分解围绕 y 轴变换除 2

Fourier Series 傅里叶级数

$x(t) = e^{st} \xrightarrow{LTI} y(t) = H(s)e^{st}$ $\sum_k a_k e^{s_k t} \xrightarrow{LTI} \sum_k a_k H(s_k)e^{s_k t}$	$x[n]z^n \xrightarrow{LTI} y[n] = H(z)z^n$ $\sum_k a_k z_k^n \xrightarrow{LTI} \sum_k a_k H(z_k)z_k^n$
$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$ $H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt$	$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$ $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$
$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ $a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$ $a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n}$

和差化积公式与积化和差公式

$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$

$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$

采样与恢复的过程

$$p(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT) \overset{\mathcal{F}}{\leftrightarrow} \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega - k\frac{2\pi}{T}\right) = \omega_s\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$$

$$p(t)x(t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2\pi}X(j\omega) * P(j\omega) = \frac{\omega_s}{2\pi}\sum_{k=-\infty}^{\infty} X\big(j(\omega-k\omega_s)\big) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\big(j(\omega-k\omega_s)\big)$$

采样后信号的频率响应幅值变为$1/T$倍，因此恢复信号时要通过一个增益为T的BPF，BPF 的截止频率最好为$\frac{\omega_s}{2}$

Hilbert Transform

$\frac{1}{\pi t} \overset{\mathcal{F}}{\leftrightarrow} -j\cdot\mathrm{sign}\,\omega$, $\cos\omega_0 t \xrightarrow{h(t)} \sin\omega_0 t$

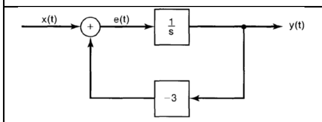
Quick Reference Checklist 快速检查单

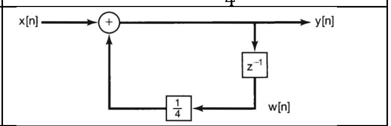
Signal	Transform	ROC	Pole
$e^{-at}u(t)$	$\frac{1}{s+a}$	$Re\{s\} > -a$	$-a$
$e^{-at}u(-t)$	$-\frac{1}{s+a}$	$Re\{s\} < -a$	$-a$
$e^{at}u(t)$	$\frac{1}{s-a}$	$Re\{s\} > a$	a
$e^{at}u(-t)$	$-\frac{1}{s-a}$	$Re\{s\} < a$	a
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$	a
$a^n u[-n-1]$	$-\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$	a
$(-a)^n u[n]$	$\frac{1}{1+az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$	$-a$
$(-a)^n u[-n-1]$	$-\frac{1}{1+az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$	$-a$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$Re\{s\} > -a$	$-a(\times 2)$
$na^n u[n]$	$\frac{1}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$	$a(\times 2)$

Basic BD Pairs

$$H(s) = \frac{1}{s+3}$$

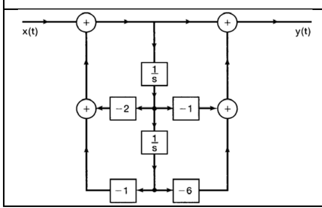
$$H(z) = \frac{1}{1-\frac{1}{4}z^{-1}}$$

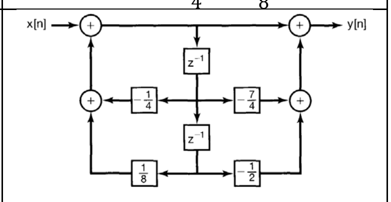




$$H(s) = \frac{s^2-s-6}{s^2+2s+1}$$

$$H(z) = \frac{1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$





Parseval's Relation (Aperiodic Sig.)
$\int_{-\infty}^{\infty}|x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2 d\omega$

Initial/Final Value Theorem
$x(0^+) = \lim_{s\to\infty} sX(s)$, $\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

Initial Value Theorem
If $x[n] = 0$ for $n < 0$, $x[0] = \lim_{z\to\infty} X(z)$.

Group Delay
$\tau(\omega) = -\frac{d}{d\omega}\{\sphericalangle H(j\omega)\}$

Properties of Fourier Transform 傅里叶变换的性质

Property	Aperiodic signal	Fourier transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X\big(j(\omega - \omega_0)\big)$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\theta)Y\big(j(\omega-\theta)\big)d\theta$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ Re\{X(j\omega)\} = Re\{X(-j\omega)\} \\ Im\{X(j\omega)\} = -Im\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition for Real Signals	$x_e(t) = Ev\{x(t)\}$ $x_o(t) = Od\{x(t)\}$ $x(t)$ real	$Re\{X(j\omega)\}$ $jIm\{X(j\omega)\}$

Basic Fourier Transform Pairs

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0,\ otherwise$
$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0,\ otherwise$
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0,\ otherwise$
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1,\ a_k = 0,$ $k \neq 0$
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$ And $x(t + T) = x(t)$	$\sum_{k=-\infty}^{\infty}\frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi}\operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{\infty}\delta(t - nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	
$\delta(t)$	1	
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	
$\delta(t - t_0)$	$e^{-j\omega t_0}$	
$e^{-at}u(t), Re\{a\} > 0$	$\frac{1}{a + j\omega}$	
$te^{-at}u(t), Re\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t), Re\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	

Properties of Laplace Transform 拉普拉斯变换的性质

Property	Signal	Laplace Transform	ROC
	$x(t)$	$X(s)$	
	$x_1(t)$	$X_1(s)$	
	$x_2(t)$	$X_2(s)$	
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
Shifting in the s-domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Differentiation in the s-domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in the Time Domain	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap Re\{s\} > 0$

Laplace Transform Pairs

#	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$1/s$	$\mathcal{Re}\{s\} > 0$
3	$-u(-t)$	$1/s$	$\mathcal{Re}\{s\} < 0$
6	$e^{-at}u(t)$	$\frac{1}{s + a}$	$\mathcal{Re}\{s\} > -a$
7	$-e^{-at}u(-t)$	$\frac{1}{s + a}$	$\mathcal{Re}\{s\} < -a$
8	$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s + a)^n}$	$\mathcal{Re}\{s\} > -a$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$	$\frac{1}{(s + a)^n}$	$\mathcal{Re}\{s\} < -a$
10	$\delta(t - T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\mathcal{Re}\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$\mathcal{Re}\{s\} > 0$

Properties of Z-Transform

Property	Signal	z-Transform	ROC
	$x[n]$	$X(z)$	R
	$x_1[n]$	$X_1(z)$	R_1
	$x_2[n]$	$X_2(z)$	R_2
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least $R_1 \cap R_2$
Tiime shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R except for possible origin
Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
	$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R
Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R
Time expansion	$x_{(k)}[n] = \begin{cases} x[r], n = rk \\ 0, n \neq rk \end{cases}$	$X(z^k)$	$R^{\frac{1}{k}}$
Conjugation	$x^*[n]$	$X^*(z^*)$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least $R_1 \cap R_2$
First difference	$x[n] - x[n - 1]$	$(1 - z^{-1})X(z)$	At least $R \cap \{\lvert z\rvert > 0\}$
Accumuation	$\sum_{k=-\infty}^{n}x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least $R \cap \{\lvert z\rvert > 1\}$
Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R

Z-Transform Pairs

Signal	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert > -1$
$-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert < -1$
$\delta[n - m]$	z^{-m}	All z, except 0 or ∞
$a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n - 1]$	$\frac{1}{1 - az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n - 1]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$[\cos\omega_0 n]u[n]$	$\frac{1 - [\cos\omega_0]z^{-1}}{1 - [2\cos\omega_0]z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
$[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1 - [2\cos\omega_0]z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
$[r^n\cos\omega_0 n]u[n]$	$\frac{1 - [r\cos\omega_0]z^{-1}}{1 - [2r\cos\omega_0]z^{-1} + r^2 z^{-2}}$	$\lvert z\rvert > r$
$[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1 - [2r\cos\omega_0]z^{-1} + r^2 z^{-2}}$	$\lvert z\rvert > r$