UESTC Student ID ____________________   UOG Student ID _______________

Continuous-time periodic square wave 连续时间方波

$$x(t)=\begin{cases}1 & |t|<T_1\\ 0 & T_1<|t|<\frac{T}{2}\end{cases}\qquad a_k=\frac{1}{T}\int_{-T_1}^{T_1}e^{-jk\omega_0 t}\,dt=\begin{cases}\frac{2T_1}{T} & k=0\\ -\frac{e^{-jk\omega_0T_1}-e^{jk\omega_0T_1}}{jk\omega_0T} & k\neq 0\end{cases}=\begin{cases}\frac{2T_1}{T} & k=0\\ \frac{\sin(k\omega_0T_1)}{k\pi} & k\neq 0\end{cases}$$

等比数列

通项公式：$a_n=a_1q^{n-1}$

两项关系：$a_n=a_mq^{n-m}$

求和公式：$S_n=a_1\frac{1-q^n}{1-q}=\frac{a_1-a_nq}{1-q}$

Discrete-time periodic square wave 离散时间方波

$x[n]=1 \text{ for } -N_1\le n\le N_1$

When $k=0,\pm N,\pm 2N,\dots,\quad a_k=\frac{1}{N}\sum_{n=<N>}x[n]e^{-jk\omega_0 n}=\frac{1}{N}\sum_{n=-N_1}^{N_1}e^{-jk\frac{2\pi}{N}n}=\frac{1}{N}\sum_{n=-N_1}^{N_1}1=\frac{2N_1+1}{N}$

When $k\neq 0,\pm N,\pm 2N,\dots,\quad a_k=\frac{1}{N}\sum_{n=<N>}x[n]e^{-jk\omega_0 n}=\frac{1}{N}\sum_{n=-N_1}^{N_1}e^{-jk\omega_0 n}=\frac{1}{N}e^{jk\frac{2\pi}{N}N_1}\frac{1-e^{-jk\frac{2\pi}{N}(2N_1+1)}}{1-e^{-jk\frac{2\pi}{N}}}=\frac{1}{N}\frac{\sin\left(\frac{2k\pi\left(N_1+\frac{1}{2}\right)}{N}\right)}{\sin\left(\frac{k\pi}{N}\right)}$

化简技巧$1-e^{-jk\frac{2\pi}{N}}=e^{-jk\frac{2\pi}{2N}}\left(e^{jk\frac{2\pi}{2N}}-e^{-jk\frac{2\pi}{2N}}\right)$

Even and odd decomposition

$$\mathcal{Ev}\{x[t]\}=\frac{1}{2}\{x(t)+x(-t)\}$$

$$\mathcal{Od}\{x[t]\}=\frac{1}{2}\{x(t)-x(-t)\}$$

奇分解围绕原点旋转除 2，偶分解围绕 y 轴变换除 2

系统性质的判据	输入输出关系	单位冲激响应
Memoryless	Output at $t=t_0$ depends only on the value of input at $t=t_0$	$h(t)=0$ for $t\neq 0$
Invertible	There exist an inverse system	$\exists h_1[n]\ s.t.\ h[n]*h_1[n]=\delta[n]$
Causality	Output at this time only depends on values of the input at the present time and in the past	$h[n]=0$ for $n<0$
Stability	BIBO	$\int_{-\infty}^{\infty}\lvert h(\tau)\rvert d\tau<\infty$
Time-invariance	Let $x_1(t)=x(t-t_0)$ Check $y_1(t)=?\,y(t-t_0)$	Must satisfy
Linearity	Let $x_1(t),x_2(t),x_3(t)=x_1(t)+x_2(t)$ Check $y_3(t)=?\,ay_1(t)+by_2(t)$	Must satisfy

Convolution 卷积

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\sum_{k=-\infty}^{\infty}x[n-k]h[k]$$

$$y(t)=\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau\int_{-\infty}^{+\infty}x(t-\tau)h(\tau)d\tau$$

一个函数的t换成τ，另一个函数的t换成$t-\tau$；如果结果以原点为界，记得写成$u[n]u(t)$的形式；$x(\tau)$的图像正常，$h(t-\tau)$的图像关于y轴翻转后，原点为t

Singularity functions

$u_k(t)$求导；$u_{-k}(t)$积分；$u_1(t)$是 unit doublet

Impulse train

$$x(t)=\sum_{k\in\mathbb{R}}\delta(t-kT)\overset{FS}{\leftrightarrow}a_k=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\delta(t)e^{-jk\omega_0 0}\,dt=\frac{1}{T}$$

Fourier Series 傅里叶级数

$x(t)=e^{st}\xrightarrow{LTI}y(t)=H(s)e^{st}$ $\sum_k a_ke^{s_kt}\xrightarrow{LTI}\sum_k a_kH(s_k)e^{s_kt}$	$x[n]z^n\xrightarrow{LTI}y[n]=H(z)z^n$ $\sum_k a_kz_k^n\xrightarrow{LTI}\sum_k a_kH(z_k)z_k^n$
$H(s)=\int_{-\infty}^{\infty}h(t)e^{-st}\,dt$ $H(j\omega)=\int_{-\infty}^{\infty}h(t)e^{-j\omega t}\,dt$	$H(z)=\sum_{k=-\infty}^{\infty}h[k]z^{-k}$ $H(e^{j\omega})=\sum_{n=-\infty}^{\infty}h[n]e^{-j\omega n}$
$x(t)=\sum_{k=-\infty}^{\infty}a_ke^{jk\omega_0t}$ $a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0t}\,dt$	$x[n]=\sum_{k=<N>}a_ke^{jk\omega_0n}$ $a_k=\frac{1}{N}\sum_{n=<N>}x[n]e^{-jk\omega_0n}$

Properties of convolution 卷积的性质

$x(t)*\delta(t)=x(t)$

$x(t)*\delta(t-t_0)=x(t-t_0)$

$x(t-t_1)*\delta(t-t_2)=x(t-t_1-t_2)$

$x(t)*\delta'(t)=x'(t)$

$x(t)*u(t)=x^{-1}(t)=\int_{-\infty}^{t}x(\tau)d\tau$

$x(t)*h(t)=x'(t)*h^{-1}(t)$

Euler's formula 欧拉公式

$e^{j\theta}=\cos\theta+j\sin\theta$

$\cos\theta=\frac{e^{j\theta}+e^{-j\theta}}{2}$

$\sin\theta=\frac{e^{j\theta}-e^{-j\theta}}{2j}$

诱导公式

$\sin(2k\pi+\alpha)=\sin\alpha$

$\cos(2k\pi+\alpha)=\cos\alpha$

$\sin(-\alpha)=-\sin\alpha$

$\cos(-\alpha)=\cos\alpha$

$\sin(\pi-\alpha)=\sin\alpha$

$\cos(\pi-\alpha)=-\cos\alpha$

$\sin(\pi+\alpha)=-\sin\alpha$

$\cos(\pi+\alpha)=-\cos\alpha$

$\sin\left(\frac{\pi}{2}-\alpha\right)=\cos\alpha$

$\cos\left(\frac{\pi}{2}-\alpha\right)=\sin\alpha$

$\sin\left(\frac{\pi}{2}+\alpha\right)=\cos\alpha$

$\cos\left(\frac{\pi}{2}+\alpha\right)=-\sin\alpha$

$\mathrm{tg}(k\pi+\alpha)=\mathrm{tg}\,\alpha$

$\mathrm{tg}(-\alpha)=-\mathrm{tg}\,\alpha$

$\mathrm{tg}(\pi-\alpha)=-\mathrm{tg}\,\alpha$

$\mathrm{tg}(\pi+\alpha)=\mathrm{tg}\,\alpha$

$\mathrm{tg}\left(\frac{\pi}{2}-\alpha\right)=\mathrm{ctg}\,\alpha$

$\mathrm{tg}\left(\frac{\pi}{2}+\alpha\right)=-\mathrm{ctg}\,\alpha$

TABLE 17.1 The method of undetermined coefficients for selected equations of the form

$$ay''+by'+cy=G(x).$$

If $G(x)$ has a term that is a constant multiple of . . .	And if	Then include this expression in the trial function for y_p.
e^{rx}	r is not a root of the auxiliary equation	Ae^{rx}
	r is a single root of the auxiliary equation	Axe^{rx}
	r is a double root of the auxiliary equation	Ax^2e^{rx}
$\sin kx$, $\cos kx$	ki is not a root of the auxiliary equation	$B\cos kx+C\sin kx$
px^2+qx+m	0 is not a root of the auxiliary equation	Dx^2+Ex+F
	0 is a single root of the auxiliary equation	Dx^3+Ex^2+Fx
	0 is a double root of the auxiliary equation	$Dx^4+Ex^3+Fx^2$

和差化积公式与积化和差公式

$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$

$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$

自变量变换

Systematic approach: $x(t) \rightarrow x(t+\beta) \rightarrow x(\alpha t+\beta)$

Real, even, odd properties of a signal

If $x(t)$ is real, then from the synthesis equation:

$$\sum_{k\in\mathbb{R}} a_k e^{jk\omega_0 t} = \sum_{k\in\mathbb{R}} a_k^* e^{-jk\omega_0 t} = \sum_{-k\in\mathbb{R}} a_{-k}^* e^{jk\omega_0 t} \Rightarrow a_k = a_{-k}^*$$

If $x(t)$ is real and even, then $x(t) = x(-t) \Rightarrow a_k = a_{-k}$; $a_{-k} = a_{-k}^* \Rightarrow a_k = a_k^*$

Hence a_k is real and even.

Similarly, if $x(t)$ is real and odd, then $a_k = -a_{-k}$ and $a_k = -a_k^*$, a_k is purely imaginary and odd.

N^th order linear constant-coefficient differential equation

Expression: $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$

Solution: $y(t) = y_p(t) + y_h(t)$, where $y_h(t) = A_1 e^{s_1 t} + \cdots + A_N e^{s_N t}$

Causal and LTI ⇔ Initial rest (i.e. if $x(t) = 0$ for $t < t_0$ then $y(t) = 0$ for $t < t_0$)

N^th order linear constant-coefficient difference equation

Expression: $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$

Solution: $y[n] = y_p[n] + y_h[n]$, where $y_h[n] = A_1 z_1^n + \cdots + A_N z_N^n$

Causal and LTI ⇔ Initial rest (i.e. if $x[n] = 0$ for $n < n_0$ then $y[n] = 0$ for $n < n_0$)

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l}x(t)\\y(t)\end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases}a_k = a^*_{-k}\\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\}\\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\}\\ \lvert a_k\rvert = \lvert a_{-k}\rvert\\ \sphericalangle a_k = -\sphericalangle a_{-k}\end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases}x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}]\\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}]\end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T \lvert x(t)\rvert^2 dt = \sum_{k=-\infty}^{+\infty} \lvert a_k\rvert^2$$

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l}x[n]\\y[n]\end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l}a_k\\b_k\end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases}x[n/m], & \text{if } n \text{ is a multiple of } m\\ 0, & \text{if } n \text{ is not a multiple of } m\end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k$ (viewed as periodic with period mN)
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k]$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases}a_k = a^*_{-k}\\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\}\\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\}\\ \lvert a_k\rvert = \lvert a_{-k}\rvert\\ \sphericalangle a_k = -\sphericalangle a_{-k}\end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases}x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}]\\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}]\end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} \lvert x[n]\rvert^2 = \sum_{k=\langle N\rangle} \lvert a_k\rvert^2$$