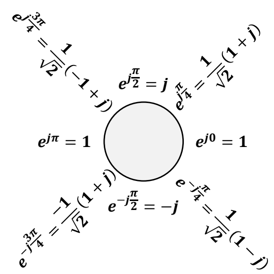
UESTC Student ID ____________________ UOG Student ID ______________

System Properties (Z-Transform)

Causality	ROC is the exterior of a circle including infinity
Causality (Rational $H(s)$)	ROC is the exterior of a circle outside the outermost pole + 分子 z 阶数不能比分母大
Stability	ROC includes the unit circle

Causal LTI system with rational $H(s)$: all poles lies inside the unit circle + 分子 z 阶数不能比分母大



Nyquist Sampling Theorem

Let $x(t)$ be a band-limited signal within $\pm\omega_M$, then $x(t)$ is uniquely determined by its samples $x(nT)$ if $\omega_s > 2\omega_M$, where $\omega_s=\frac{2\pi}{T}$.

Nyquist rate: $2\omega_M$

诱导公式

$\sin(2k\pi+\alpha)=\sin\alpha$
$\cos(2k\pi+\alpha)=\cos\alpha$
$\sin(-\alpha)=-\sin\alpha$
$\cos(-\alpha)=\cos\alpha$
$\sin(\pi-\alpha)=\sin\alpha$
$\cos(\pi-\alpha)=-\cos\alpha$
$\sin(\pi+\alpha)=-\sin\alpha$
$\cos(\pi+\alpha)=-\cos\alpha$
$\sin\left(\frac{\pi}{2}-\alpha\right)=\cos\alpha$
$\cos\left(\frac{\pi}{2}-\alpha\right)=\sin\alpha$
$\sin\left(\frac{\pi}{2}+\alpha\right)=\cos\alpha$
$\cos\left(\frac{\pi}{2}+\alpha\right)=-\sin\alpha$
$\mathrm{tg}(k\pi+\alpha)=\mathrm{tg}\,\alpha$
$\mathrm{tg}(-\alpha)=-\mathrm{tg}\,\alpha$
$\mathrm{tg}(\pi-\alpha)=-\mathrm{tg}\,\alpha$
$\mathrm{tg}(\pi+\alpha)=\mathrm{tg}\,\alpha$
$\mathrm{tg}\left(\frac{\pi}{2}-\alpha\right)=\mathrm{ctg}\,\alpha$
$\mathrm{tg}\left(\frac{\pi}{2}+\alpha\right)=-\mathrm{ctg}\,\alpha$

Euler's formula 欧拉公式

$$e^{j\theta}=\cos\theta+j\sin\theta$$
$$\cos\theta=\frac{e^{j\theta}+e^{-j\theta}}{2}$$
$$\sin\theta=\frac{e^{j\theta}-e^{-j\theta}}{2j}$$

和差化积公式与积化和差公式

$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
$\sin\alpha-\sin\beta=2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$
$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$
$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$
$\cos\alpha\sin\beta=\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$
$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$
$\sin\alpha\sin\beta=-\frac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)]$

Convolution 卷积

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\sum_{k=-\infty}^{\infty}x[n-k]h[k]$$
$$y(t)=\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau\int_{-\infty}^{+\infty}x(t-\tau)h(\tau)d\tau$$

一个函数的t换成τ，另一个函数的t换成$t-\tau$；如果结果以原点为界，记得写成$u[n]u(t)$的形式；$x(\tau)$的图像正常，$h(t-\tau)$的图像关于y轴翻转后，原点为t

Z 变换特殊峰的判断

信号为正，零点有峰
信号为负，无穷有峰

Group Delay, Phase Delay

Phase delay: $\tau_p(\omega_0)=-\frac{\theta(\omega_0)}{\omega_0}$

Group delay: $\tau_g(\omega)=-\frac{d\theta(\omega)}{d\omega}$

$$y_a(t)=a\left(t-\tau_g(\Omega_C)\right)\cos\Omega_C\left(t-\tau_p(\Omega_C)\right)$$

Hilbert Transform

$\frac{1}{\pi t}\overset{\mathcal{F}}{\leftrightarrow}-j\cdot\mathrm{sign}\,\omega$, $\cos\omega_0 t\overset{h(t)}{\longrightarrow}\sin\omega_0 t$

Sinc Function

$$\mathrm{sinc}\,\theta=\frac{\sin\pi\theta}{\pi\theta}$$

$2N$-Point DFT using a single N-Point DFT

$v[n]$长$2N$，$g[n]=v[2n],h[n]=v[2n+1]$

则$V[k]=G[\langle k\rangle_N]+W_{2N}^{k}H[\langle k\rangle_N]$

系统性质的判据	输入输出关系	单位冲激响应
Memoryless	Output at $t=t_0$ depends only on the value of input at $t=t_0$	$h(t)=0$ for $t\neq 0$
Invertible	There exist an inverse system	$\exists h_1[n]\ s.t.\ h[n]*h_1[n]=\delta[n]$
Causality	Output at this time only depends on values of the input at the present time and in the past	$h[n]=0$ for $n<0$
Stability	BIBO	$\int_{-\infty}^{\infty}\lvert h(\tau)\rvert d\tau<\infty$
Time-invariance	Let $x_1(t)=x(t-t_0)$ Check $y_1(t)=?\,y(t-t_0)$	Must satisfy
Linearity	Let $x_1(t),x_2(t),x_3(t)=x_1(t)+x_2(t)$ Check $y_3(t)=?\,ay_1(t)+by_2(t)$	Must satisfy

圆周卷积算线性卷积

1. 补 0 法：两序列都补到$M+N-1$长，算圆周卷积即可
2. Overlap-add：将$x[n]$切段，每段与$h[n]$用补 0 法算圆周卷积，最后相加时要有$N-1$长度的 overlap
3. Overlap-save：相加时不重叠，而是切断时重叠$N-1$长度；最前面补$N-1$个 0，相加时每段舍去前$N-1$

Z-变换对快速检查单

Signal	Transform	ROC	Pole
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert>\lvert a\rvert$	a
$a^n u[-n-1]$	$-\frac{1}{1-az^{-1}}$	$\lvert z\rvert<\lvert a\rvert$	a
$(-a)^n u[n]$	$\frac{1}{1+az^{-1}}$	$\lvert z\rvert>\lvert a\rvert$	$-a$
$(-a)^n u[-n-1]$	$-\frac{1}{1+az^{-1}}$	$\lvert z\rvert<\lvert a\rvert$	$-a$
$na^n u[n]$	$\frac{1}{(1-az^{-1})^2}$	$\lvert z\rvert>\lvert a\rvert$	$a(\times 2)$

Time-Domain Sampling 时域采样

$x_a(t)\overset{CTFT}{\longleftrightarrow}X_a(j\Omega)$, $x(t)=x_a(t)p(t)\overset{CTFT}{\longleftrightarrow}X(j\Omega)=\frac{1}{T}\sum_{k=-\infty}^{\infty}X_a\big(j(\Omega-k\Omega_s)\big)$

$x[n]=x(t)|_{t=nT}\overset{F}{\leftrightarrow}X(e^{j\omega})=X(j\Omega)|_{\Omega=\frac{\omega}{T}}=\frac{1}{T}\sum_{k=-\infty}^{\infty}X_a\left(j\frac{\omega-2k\pi}{T}\right)$

采样后信号的频率响应幅值变为$1/T$倍，因此恢复信号时要通过一个增益为T的 BPF，BPF 的截止频率最好为$\frac{\omega_s}{2}$；Ω轴上的Ω_s对应ω轴上的2π

连续离散公式：时域$t=nT$，频域$\omega=\Omega T$，任何地方$f=1/T$

时域采样导致频域周期化：$F_{out}=F_{in}\pm kF_s$，$\Omega_{out}=\Omega_{in}\pm k\Omega_s$

带通采样

$\Omega_H=M(\Delta\Omega)\Rightarrow$

$\Omega_T=2(\Delta\Omega)=\frac{2\Omega_H}{M}$

恢复使用 gain 为T的

$\Omega_L\le|\Omega|\le\Omega_H$的 BPF

Fourier-Domain Sampling 频域采样

频域采N个样（不关心$x[n]$周期是不是N），时域信号就以N为周期

$$y[n]=\sum_{n=-\infty}^{\infty}x[n+mN],0\le n\le N-1$$

多画几个周期找规律！

Geometric Series 等比数列

通项公式：$a_n=a_1q^{n-1}$

两项关系：$a_n=a_mq^{n-m}$

求和公式：$S_n=a_1\frac{a-q^n}{1-q}=\frac{a_1-a_nq}{1-q}$

信号的分解

奇偶分解	$x_{ev}(t)=\frac{1}{2}\big(x(t)+x(-t)\big)$ $x_{od}(t)=\frac{1}{2}\big(x(t)-x(-t)\big)$	$X_{ev}[k]=\frac{1}{2}\big(X[k]+X[\langle -k\rangle_N]\big)$ $X_{od}[k]=\frac{1}{2}\big(X[k]-X[\langle -k\rangle_N]\big)$
实虚分解	$x_{re}(t)=\frac{1}{2}\big(x(t)+x^*(t)\big)$ $x_{im}(t)=\frac{1}{2j}\big(x(t)-x^*(t)\big)$	$X_{re}[k]=\frac{1}{2}\big(X[k]+X^*[k]\big)$ $X_{im}[k]=\frac{1}{2}\big(X[k]-X^*[k]\big)$
共轭对称分解	$x_{cs}(t)=\frac{1}{2}\big(x(t)+x^*(-t)\big)$ $x_{ca}(t)=\frac{1}{2}\big(x(t)-x^*(-t)\big)$	$X_{cs}[k]=\frac{1}{2}\big(X[k]+X^*[\langle -k\rangle_N]\big)$ $X_{ca}[k]=\frac{1}{2}\big(X[k]-X^*[\langle -k\rangle_N]\big)$

信号的范数、能量、功率

$\|x\|_p=\left(\sum_{n=-\infty}^{\infty}|x[n]|^p\right)^{\frac{1}{p}}$, $\|x\|_\infty=|x|_{max}$

$\|X\|_p=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|^p\right)^{\frac{1}{p}}$, $\|X\|_\infty=\max|X(e^{j\omega})|$

$MSE=\frac{1}{N}\sum_{i=0}^{N-1}(|y[n]-x[n]|)^2=\frac{1}{N}(\|y[n]-x[n]\|_2)^2$

Total energy: $\epsilon_x=\sum_{n=-\infty}^{\infty}|x[n]|^2$ (Energy signal: finite energy)

Average power: $P_x=\lim_{K\to\infty}\frac{1}{2K+1}\sum_{k=-K}^{K}|x[n]|^2$ (Power signal: finite power)

Passive system: $\epsilon_y\le\epsilon_x<\infty$, lossless system: $\epsilon_y=\epsilon_x<\infty$

Circular Time-Shift/Reversal

原：0 1 2 3 4
移：3 4 0 1 2
反：0 4 3 2 1

DFT Matrix Relation

$\boldsymbol{x}=D_N^{-1}\boldsymbol{X}$, $\boldsymbol{X}=D_N\boldsymbol{x}$

$$D_N=\begin{bmatrix}W_N^0&W_N^0&W_N^0\\W_N^0&W_N^1&W_N^2\\W_N^0&W_N^2&W_N^4\end{bmatrix},\ D_N^{-1}=\frac{1}{N}\begin{bmatrix}W_N^0&W_N^0&W_N^0\\W_N^0&W_N^{-1}&W_N^{-2}\\W_N^0&W_N^{-2}&W_N^{-4}\end{bmatrix},\ D_4=\begin{bmatrix}1&1&1&1\\1&-j&-1&j\\1&-1&1&-1\\1&j&-1&-j\end{bmatrix}$$

离散三角信号的周期性

$\omega N=2k\pi\Rightarrow N=\frac{2k\pi}{\omega}$

离散三角信号的周期性小结论

1. 周期N必须是整数
2. 角频率相差$2k\pi$的信号完全相同
3. 任何数列的最大频率为π（folding frequency），且对称频率（0.6π和1.4π）信号相同

DFT Geometric Symmetry

Geometric symmetry: $x[n]=x[N-1-n],\theta(\omega)=-\frac{N-1}{2}\omega+\beta$ (type1 odd, type2 even)

Geometric anti-symmetry: $x[n]=-[N-1-n],\theta(\omega)=-\frac{N-1}{2}\omega+\frac{\pi}{2}+\beta$ (type3 odd, type4 even)，中心系数为 0

变换

CTFT	DTFT	DFT	Z-transform
时域连续非周期 频域连续非周期	时域离散非周期 频域连续周期	时域离散周期 频域离散周期	
$X_a(j\Omega)=\int_{-\infty}^{\infty}x_a(t)e^{-j\Omega t}dt$ $x_a(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_a(j\Omega)e^{j\Omega t}d\Omega$	$X(e^{j\omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$ $x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega$	$X[k]=\sum_{n=0}^{N-1}x[n]W_N^{kn}$ $x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-kn}$	$G(z)=\sum_{n=-\infty}^{\infty}g[n]z^{-n}$ $g[n]=\frac{1}{2\pi j}\oint_C G(z)z^{n-1}dz$
$\int_{-\infty}^{\infty}\lvert x_a(t)\rvert^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X_a(j\Omega)\rvert^2d\Omega$	$\sum_{n=-\infty}^{\infty}\lvert x[n]\rvert^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\lvert X(e^{j\omega})\rvert d\omega$	$\sum_{n=0}^{N-1}\lvert x[n]\rvert^2=\frac{1}{N}\sum_{k=0}^{N-1}\lvert X[k]\rvert^2$	
	$x[-n]\leftrightarrow X(e^{-j\omega})$ $x^*[-n]\leftrightarrow X^*(e^{j\omega})$ $x^*[n]\leftrightarrow X^*(e^{-j\omega})$	$x[\langle -n\rangle_N]\leftrightarrow X[\langle -k\rangle_N]$ $x[\langle -n\rangle_N]\leftrightarrow X^*[k]$ $x^*[n]\leftrightarrow X^*[\langle -k\rangle_N]$	$x[-n]\leftrightarrow X(z^{-1})$ with inverted R

Theorems

Property	Signal	DTFT	DFT	Z-Transform	ROC
Linearity	$ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$	$aX[k]+bY[k]$	$aX_1(z)+bX_2(z)$	At least $R_1\cap R_2$
Time Shifting	$x[n-n_0], x[\langle n-n_0\rangle_N]$	$e^{-j\omega n_0}X(e^{j\omega})$	$W_N^{kn_0}X[k]$	$z^{-n_0}X(z)$	R except for possible origin
Frequency Shifting (Z-Domain Scaling)	$e^{j\omega_0 n}x[n], W_N^{-k_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$	$X[\langle k-k_0\rangle_N]$	$X(e^{-j\omega_0}z)$	R
Time Expansion	$x_{(k)}[n]=x\left[\frac{n}{k}\right]$ if $n=mk$ else 0	$X(e^{jk\omega})$		$X(z^k)$	$R^{\frac{1}{k}}$
Convolution	$x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$	$X[k]Y[k]$	$X(z)Y(z)$	At least $R_1\cap R_2$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	$\frac{1}{N}\sum_{m=0}^{N-1}X[m]Y[\langle k-m\rangle_N]$	$\frac{1}{2\pi j}\oint_C X(v)H\left(\frac{z}{v}\right)v^{-1}dv$	At least R_1R_2
Differencing in Time	$x[n]-x[n-1]$	$(1-e^{-j\omega})X(e^{j\omega})$		$(1-z^{-1})X(z)$	At least $R\cap\{\lvert z\rvert>0\}$
Accumulation	$\sum_{k=-\infty}^{n}x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$		$\frac{1}{1-z^{-1}}X(z)$	At least $R\cap\{\lvert z\rvert>1\}$
Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$		$-z\frac{dX(z)}{dz}$	R

DTFT Transform Pairs

Signal	Transform
$\delta[n]$	1
$\mu[n]$	$\frac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{\infty}\pi\delta(\omega+2\pi k)$
1	$\sum_{k=-\infty}^{\infty}2\pi\delta(\omega+2\pi k)$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty}2\pi\delta(\omega-\omega_0+2\pi k)$
$\alpha^n\mu[n],(\lvert\alpha\rvert<1)$	$\frac{1}{1-\alpha e^{-j\omega}}$
$(n+1)\alpha^n\mu[n],(\lvert\alpha\rvert<1)$	$\frac{1}{(1-\alpha e^{-j\omega})^2}$
$h_{LP}[n]=\frac{\sin\omega_c n}{\pi n}$	$H_{LP}(e^{j\omega})=\begin{cases}1, & 0\le\lvert\omega\rvert\le\omega_c\\0, & \omega_c<\lvert\omega\rvert\le\pi\end{cases}$
$\cos\omega_0 n$	$\pi\sum_{l=-\infty}^{\infty}\{\delta(\omega-\omega_0-2\pi l)+\delta(\omega+\omega_0-2\pi l)\}$
$\sin\omega_0 n$	$\frac{\pi}{j}\sum_{l=-\infty}^{l=\infty}\{\delta(\omega-\omega_0-2\pi l)-\delta(\omega+\omega_0-2\pi l)\}$

Z-Transform Pairs

Signal	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert>-1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert<-1$
$\delta[n-m]$	z^{-m}	All z, except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert>\lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert<\lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert>\lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert<\lvert a\rvert$
$[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert>1$
$[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert>1$
$[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert>r$
$[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert>r$

P-Series Test

$\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges for $p\ge 1$

Initial Value Theorem

If $x[n]=0$ for $n<0$, $x[0]=\lim_{z\to\infty}X(z)$.

DFT 关键公式

$$\frac{1}{N}\sum_{n=0}^{N}e^{j\frac{2\pi}{N}(k-l)n}=\begin{cases}1, & k=l+rn\\0, & k\ne l\end{cases}$$

$$W_N=e^{-j\frac{2\pi}{N}},\quad \cos\left[\frac{2\pi}{N}rn\right]=\frac{1}{2}(W_N^{rn}+W_N^{-rn})$$

$$\frac{1}{1-\alpha e^{-j\omega}}=\frac{1}{1-\alpha e^{-j\omega}}\cdot\frac{1-\alpha e^{j\omega}}{1-\alpha e^{j\omega}}$$
$$=\frac{1-\alpha\cos\omega-j\alpha\sin\omega}{1-2\alpha\cos\omega+\alpha^2}$$
$$=\frac{1}{\sqrt{1-2\alpha\cos\omega+\alpha^2}}\measuredangle\arctan\left(-\frac{\alpha\sin\omega}{1-\alpha\cos\omega}\right)$$

DFT Transform Pairs

Signal	Transform
$\delta[n]$	1
$\delta[n-m]$	W_N^{km}
$\cos\left(\frac{2\pi}{N}rn\right)$	$\begin{cases}\frac{N}{2}, & k=r \text{ or } N-r\\0, & otherwise\end{cases}$

Basic BD Pairs

$$H(z)=\frac{1}{1-\frac{1}{4}z^{-1}}$$

$$H(z)=\frac{1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$