Spacetime + Hilbert-Space Formulation for Quantum-Like Trajectories

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1 Introduction

We consider a particle whose state is described not only by its spacetime trajectory $x(t) \in \mathbb{R}^d$ (with $d = 1, 2, 3$) but also by an internal Hilbert-space vector $\mathbf{h}(t) \in \mathbb{C}^N$. The aim is to demonstrate that correlations which appear "nonlocal" or "beyond causal links" in 4D spacetime emerge naturally from a causal dynamics in a higher-dimensional space.

2 State Representation

Let the particle's state be

$$\Psi(t) = \begin{bmatrix} x(t) \\ \mathbf{h}(t) \end{bmatrix}, \quad x(t) \in \mathbb{R}^d, \quad \mathbf{h}(t) \in \mathbb{C}^N. \tag{1}$$

The Hilbert-space vector evolves according to

$$i\hbar \frac{d\mathbf{h}}{dt} = H_{\text{int}}\mathbf{h} + V_{\text{couple}}(x)\mathbf{h}, \tag{2}$$

where H_{int} is the internal Hamiltonian and V_{couple} describes the coupling to the particle's spacetime position.

The spacetime trajectory evolves as

$$m\frac{d^2x}{dt^2} = F_{\text{ext}}(x) + F_{\text{int}}(\mathbf{h}), \tag{3}$$

where $F_{\text{int}}(\mathbf{h})$ encodes the influence of the Hilbert-space degrees of freedom on the particle's motion.

3 Path-Integral Formulation

We can express the probability amplitude for a trajectory in the combined spacetime + Hilbert-space manifold as a path integral:

$$\mathcal{Z} = \int \mathcal{D}[x(t)]\,\mathcal{D}[\mathbf{h}(t)]\,\exp\left\{\frac{i}{\hbar}\int_{t_0}^{t_f} dt\,\mathcal{L}(x,\dot{x},\mathbf{h},\dot{\mathbf{h}})\right\}, \tag{4}$$

with the Lagrangian

$$\mathcal{L}(x,\dot{x},\mathbf{h},\dot{\mathbf{h}}) = \frac{1}{2}m\dot{x}^2 - V_{\text{ext}}(x) + i\hbar\mathbf{h}^\dagger\dot{\mathbf{h}} - \mathbf{h}^\dagger H_{\text{int}}\mathbf{h} - \mathbf{h}^\dagger V_{\text{couple}}(x)\mathbf{h}. \tag{5}$$

4 Measurement and Post-Selection

A measurement at time t_m projects the Hilbert-space state along a chosen basis vector $|\phi\rangle$:

$$P_\phi(\Psi(t_m)) = |\,\langle\phi|\mathbf{h}(t_m)|\phi|\mathbf{h}(t_m)\rangle\,|^2. \tag{6}$$

The post-selection process does not retroactively change the past trajectory; it merely sorts the trajectories according to the internal Hilbert-space amplitude.

5 Emergent Causal Structure

Although correlations between trajectories appear nonlocal in spacetime (4D projection), the full dynamics in $(x,\mathbf{h})$ space is strictly causal. That is,

$$x(t) \quad \text{and} \quad \mathbf{h}(t) \quad \text{are entirely determined by initial conditions and dynamical evolution.} \tag{7}$$

Increasing the Hilbert-space dimension N reduces stochastic fluctuations in spacetime, leading to deterministic-like behavior, whereas small N allows more pronounced “quantum-like” stochasticity.

6 Simulation Pipeline (Conceptual)

To test and visualize these dynamics:

1. Initialize $x(0)$ and $\mathbf{h}(0)$ for M particles.

2. Evolve the system according to the coupled ODEs for $(x(t), \mathbf{h}(t))$.
3. At measurement time t_m, compute projection probabilities P_ϕ.
4. Weight or color trajectories according to P_ϕ.
5. Compute final interference patterns by summing over paths, with weights given by P_ϕ.
6. Sweep Hilbert-space dimension N to observe convergence toward standard quantum behavior.

7 Conclusion

This framework demonstrates:

- Entanglement-like correlations are emergent from a causal evolution in higher dimensions.
- Measurement/post-selection only sorts pre-existing trajectories — no retrocausal effects occur.
- The Hilbert-space dimension N controls the crossover from classical-like deterministic motion (high N) to quantum-like stochastic motion (low N).
- Path-integral formulation in the combined spacetime + Hilbert-space manifold connects naturally to standard quantum mechanics in the large-N limit.

References

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