% !TeX document-id = {1775dc01-c91a-443e-9abe-bdcfb8be9ec7}

% !BIB TS-program = biber

\documentclass[11pt]{article}

\usepackage[margin=1.1in]{geometry}

\usepackage{enumerate}

\usepackage{amssymb}

\usepackage{amsmath}

\usepackage{mathrsfs}

\usepackage{amsthm}

\usepackage{mathtools}

\usepackage{float}

\usepackage{braket}

\usepackage{cancel}

\usepackage{eufrak}

\usepackage{xcolor}

\newcommand{\N}{\mathbb{N}}

\newcommand{\R}{\mathbb{R}}

\usepackage{comment}

\setlength{\parindent}{0em}

\usepackage{cleveref}

\newtheorem\*{theorem\*}{Theorem}

\newtheorem{theorem}{Theorem}[section]

\newtheorem{observation}[theorem]{Observation}

\newtheorem{proposition}[theorem]{Proposition}

\newtheorem{corollary}[theorem]{Corollary}

\newtheorem{lemma}[theorem]{Lemma}

\newtheorem{definition}[theorem]{Definition}

\newtheorem{open}[theorem]{Open Problem}

\newtheorem{conj}[theorem]{Conjecture}

\title{Royal Latin Squares}

\date{}

\begin{document}

\maketitle

We are inspired by the recent advances in the study of the $n$-Queens problem \cite{}. We say that a Latin square $A$ is {\em royal} if every symbol in it constitutes a non-attacking queens configuration. Namely, if

\begin{equation}\label{eq:RLS}

a\_{i\_1,j\_1}=a\_{i\_2,j\_2} \Rightarrow |i\_1-j\_1| \neq |i\_2-j\_2|

\end{equation}

We wish to know whether royal Latin squares (henceforth RLS) exist and if so how to construct them.

\section{RLS of prime order}

\begin{theorem}

For every prime $n\ge 5$ an RLS exists.

\end{theorem}

\begin{proof}

In what follows every integer should be taken $\bmod n$. We pick $\alpha \not \equiv 0,\pm 1 \bmod ~n$, and create the following

array whose rows and columns are indexed by elements of $\mathbb{Z}\_n$ and so are its entries. For every $i,x\in\mathbb{Z}\_n$

we put the symbol $i$ in position $(x, \alpha x + i)$. Our claim is that this is, in fact a RLS.

\textcolor{red}{Need to show: every element of $\mathbb{Z}\_n$ appears exactly once in every row and every column.

We need to show that the condition in equation (\ref{eq:RLS}) holds. Say a few words why we need to assume $\alpha\neq\pm 1$. Also, what happens if $n$ is not prime?}\\

\textcolor{blue}{For which values of $n$ do we know how to construct an RLS? What happens with other values of $n$. Is it true that one exists for every large enough $n$?}

\end{proof}

Let n be prime , , and $A$ be a latin square which , the symbol $i$ appears in .

\begin{lemma}

For $\alpha \not \equiv 0, \pm 1 \bmod n$ $A$ is {\em royal}.

\end{lemma}

For proving this $lemma$ 1.1, we will present two arguments:

\begin{lemma}

$A$ is latin square well defined.

\end{lemma}

\begin{proof}

In every row, each symbol appears in a different column.\\

Assume the symbol $i$ appears more then once in the same column, which means that there exist $x \neq y$ such that $$\alpha x + i \equiv \alpha y + i\bmod n$$

$$\alpha x \equiv \alpha y\bmod n \rightarrow \alpha (x - y) \equiv 0 \bmod n$$

For $\alpha \not \equiv 0 \bmod n$ the equation never takes place, giving a legal latin square.

\end{proof}

As we mentioned above, in order for a Latin square to be {\em royal}, for every symbol $i$,

$$\forall x \neq y: |x-y| \not \equiv |(\alpha x + i)-(\alpha y + i)| \bmod n = |\alpha x-\alpha y| \bmod n$$. That is, in order to check the diagonals it is enough to check only for the symbol $0$.

\begin{lemma}

$A$ is a {\em royal} latin square.

\end{lemma}

\begin{proof}

For proving {\em royalty}, suppose in contribution that the symbol $0$ appears more than once on some diagonal. Then, there exists $(x, \alpha x \bmod n), (y, \alpha y \bmod n)$, a fixed number $t$ such that \\

$$(1) x = y + t, (2)\alpha x \equiv \alpha y \pm t \bmod n $$

$$(\alpha \pm 1) (x - y) \equiv 0 \bmod n$$

$\Rightarrow$ For $\alpha \not \equiv \pm 1 \bmod n$

there is no diagonal the symbol $0$ appears in more then once, meaning the symbol constitutes a non-attacking queens configuration. Leading that every symbol constitutes a non-attacking queens configuration as necessary.

\end{proof}

\section{$\mathbb{Z}\_p \times \mathbb{Z}\_q$}

let $p, q$ be some prime numbers, and $\alpha, \beta$ be some fixed numbers.\\

Define the order at which the elements of $\mathbb{Z}\_p\times\mathbb{Z}\_q$ appear as follows:

$$(0,0), (0,1), \dots (0,q-1), (1,0), (1,1), \dots (p-1, q-1)$$

That is, running on the second element first.\\

Let $B$ be a latin square of size $\big(\mathbb{Z}p \times \mathbb{Z}q\big)^2$ filled in the following form:\\

$\forall i\in\mathbb{Z}\_p, j\in\mathbb{Z}\_q$, and $\forall x\_1\in\mathbb{Z}\_p, x\_2\in\mathbb{Z}\_q$, the symbol $(i,j)$ will appear in $\big((x\_1,x\_2),(\alpha x\_1+i, \beta x\_2+j)\big)$.\\

Instead of looking at points in $\big(\mathbb{Z}\_p \times \mathbb{Z}\_q\big)$ space, we will transform every dot to \mathbb{N} as follows:

$$(v,w) \longmapsto v q + w$$

\begin{lemma}

For $\alpha \not \equiv \pm 1 \bmod p \wedge \beta \not \equiv \pm 1 \bmod q$ $B$ is {\em royal}.

\end{lemma}

\begin{proof}

We will show that the symbol $(0,0)$ constitutes a non-attacking queens configuration by supposing in contribution that there exists $\big((v\_1,w\_1),([\alpha v\_1]\_p, [\beta w\_1]\_q)\big), \big((v\_2,w\_2),([\alpha v\_2]\_p, [\beta w\_2]\_q)\big)$ two different cells on the same diagonal with symbol $(0,0)$.\\

For them to be on the same diagonal, there must exist some $t$ such that:

$$(1) v\_1 q + w\_1 = v\_2 q + w\_2 + t$$

$$(2) [\alpha v\_1]\_p q + [\beta w\_1]\_q = [\alpha v\_2]\_p q + [\beta w\_2]\_q \pm t$$

$$ \Rightarrow [\alpha v\_1]\_p q + [\beta w\_1]\_q = [\alpha v\_2]\_p q + [\beta w\_2]\_q \pm \big(q(v\_1 - v\_2) + (w\_1 - w\_2)\big)$$

$$(\*) q \big([\alpha v\_1]\_p - [\alpha v\_2]\_p \mp (v\_1 - v\_2)\big) = -([\beta w\_1]\_q - [\beta w\_2]\_q) \pm (w\_1-w\_2)$$

Since $q$ is prime, for the equation to be true, it must be that:

$$-([\beta w\_1]\_q - [\beta w\_2]\_q) \pm (w\_1-w\_2)]\_q \equiv 0\bmod q$$

$$[\beta w\_1]\_q - [\beta w\_2]\_q \equiv \pm (w\_1-w\_2) \bmod q$$

$$\beta w\_1 - \beta w\_2 \equiv \pm (w\_1-w\_2) \bmod q$$

$$\beta (w\_1 - w\_2) \equiv \pm (w\_1-w\_2) \bmod q$$

$$(\beta \pm 1) (w\_1 - w\_2) \equiv 0 \bmod q$$

$$\Rightarrow \beta \equiv \pm 1 \bmod q \vee w\_1 = w\_2$$\\

If $\beta \not \equiv \pm 1 \bmod q$, it must be that $w\_1 = w\_2$, then substitute in $(\*)$:\\

$$ q \big([\alpha v\_1]\_p - [\alpha v\_2]\_p \mp (v\_1 - v\_2)\big) = -([\beta w\_1]\_q - [\beta w\_1]\_q) \pm (w\_1-w\_1)$$

$$ q \big([\alpha v\_1]\_p - [\alpha v\_2]\_p \mp (v\_1 - v\_2)\big) = 0$$

$$ (\*\*) [\alpha v\_1]\_p - [\alpha v\_2]\_p \mp (v\_1 - v\_2) = 0$$\\

here we will add $mod(p)$ and check which of the new equation accepted solutions will solve $(\*\*)$:

$$ [\alpha v\_1]\_p - [\alpha v\_2]\_p \mp (v\_1 - v\_2) \equiv 0 \bmod p$$

$$ \alpha v\_1 - \alpha v\_2 \mp (v\_1 - v\_2) \equiv 0 \bmod p$$

$$ (\alpha \pm 1) (v\_1 - v\_2) \equiv 0 \bmod p$$

$$\Rightarrow \alpha \equiv \pm 1 \bmod p \vee v\_1 = v\_2$$\\

if we go back to $(\*\*)$,\\

$\alpha \equiv 1 \bmod p:$\\

$$v\_1 - v\_2 \mp (v\_1 - v\_2) = 0 $$\\

$\alpha \equiv -1 \bmod p:$\\

$$p - v\_1 - (p - v\_2) \pm (v\_1 - v\_2) = 0 $$\\

$v\_1 = v\_2:$\\

$$[\alpha v\_1]\_p - [\alpha v\_1]\_p \mp (v\_1 - v\_1) = 0 \checked $$

$\Rightarrow $ all three solutions solve the original equation $(\*\*)$ and therefore they are it's only solutions .\\

Overall we got that if $\alpha \not \equiv \pm 1 \bmod p \wedge \beta \not \equiv \pm 1 \bmod q$ then for a symbol to appear more then once on the same diagonal it must be that ${v\_1=v\_2$, $w\_1=w\_2$.\\

$\Rightarrow B$ is {\em royal} if $\alpha \not \equiv \pm 1 \bmod p$ and $ \beta \not \equiv \pm 1 \bmod q$.

\end{proof}

\section{Conclusion}

for prime numbers $p,q$, the latin square of $\big(\mathbb{Z}p \times \mathbb{Z}q\big)^2$ that obtained by the coloring of the form $\big((x\_1,x\_2),(\alpha x\_1+i, \beta x\_2+j)\big)$ for a color $(i,j)$, is {\em royal} if and only if $\alpha \not \equiv \pm 1 \bmod p \wedge \beta \not \equiv \pm 1 \bmod q$.

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