% !TeX document-id = {1775dc01-c91a-443e-9abe-bdcfb8be9ec7}

% !BIB TS-program = biber

\documentclass[11pt]{article}

\usepackage[margin=1.1in]{geometry}

\usepackage{enumerate}

\usepackage{amssymb}

\usepackage{amsmath}

\usepackage{mathrsfs}

\usepackage{amsthm}

\usepackage{mathtools}

\usepackage{float}

\usepackage{braket}

\usepackage{cancel}

\usepackage{eufrak}

\newcommand{\N}{\mathbb{N}}

\newcommand{\R}{\mathbb{R}}

\usepackage{comment}

\usepackage{cleveref}

\newtheorem\*{theorem\*}{Theorem}

\newtheorem{theorem}{Theorem}[section]

\newtheorem{observation}[theorem]{Observation}

\newtheorem{proposition}[theorem]{Proposition}

\newtheorem{corollary}[theorem]{Corollary}

\newtheorem{lemma}[theorem]{Lemma}

\newtheorem{definition}[theorem]{Definition}

\newtheorem{open}[theorem]{Open Problem}

\newtheorem{conj}[theorem]{Conjecture}

\title{Royal Latin Squares}

\date{}

\begin{document}

\maketitle

We are inspired by the recent advances in the study of the $n$-Queens problem \cite{}. We say that a Latin square $A$ is {\em royal} if every symbol in it constitutes a non-attacking queens configuration. Namely, if $a\_{i\_1,j\_1}=a\_{i\_2,j\_2}$ implies that $|i\_1-j\_1|\neq |i\_2-j\_2|$. We wish to know whether royal Latin squares exist and if so how to construct them.

let $p, q$ be some primary numbers, and $\alpha, \beta$ be some fixed numbers.

We want to color the latin square of size $\big(\mathbb{Z}p \times \mathbb{Z}q\big)^2$ in the following form:\\

$\forall i\in\mathbb{Z}\_p, j\in\mathbb{Z}\_q$, and $\forall x\_1\in\mathbb{Z}\_p, x\_2\in\mathbb{Z}\_q$, we color the cell $\big((x\_1,x\_2),(\alpha x\_1+i, \beta x\_2+j)\big)$ with $(i,j)$ color.\\

In this article, we will show for which values of $\alpha, \beta$ the above latin square is royal.

\section{$\mathbb{Z}\_n$ families}

\subsection {$(x,\alpha x + i)$}

\section{Negation of $\alpha \equiv\_p \pm 1$ or $\beta \equiv\_q \pm 1$}

Showing that $\alpha \equiv\_p \pm 1$ of color \underline{$(0,0)$} that are on the same diagonal.\\

\underline{$\alpha \equiv\_p 1$:} for every $\beta$ two dots that are colored in \underline{$(0,0)$} are $\big((0,0),(0,0)\big), \big((1,0),(1,0)\big)$, which both are on the main diagonal as claimed.\\

\underline{$\beta \equiv\_q 1$:} for every $\alpha$ two dots that are colored in \underline{$(0,0)$} are $\big((0,0),(0,0)\big), \big((0,1),(0,1)\big)$, which both are on the main diagonal as well.\\

\underline{$\alpha \equiv\_p (p-1)$:} for every $\beta$ two dots that are colored in \underline{$(0,0)$} are $\big((1, 0),(p-1, 0)\big), \big((2, 0),(2(p-1), 0)\big)=\big((2, 0),(p-2, 0)\big)$, which are on inverted diagonal.\\

\underline{$\beta \equiv\_q (q-1)$:} for every $\alpha$ two dots that are colored in \underline{$(0,0)$} are $\big((0,1),(0, q - 1)\big), \big((0, 2),(0, 2(q-1))\big)=\big((0, 2),(0, q-2)\big)$, which are also on inverted diagonal.\\

\section{Finding diagonals}

Instead of looking at points in $\big(\mathbb{Z}\_p \times \mathbb{Z}\_q\big)$ space, we will transform every dot to \mathbb{N} as follows:

$$(v,w) \longmapsto v q + w$$

Let $\big((v\_1,w\_1),([\alpha v\_1]\_p, [\beta w\_1]\_q)\big), \big((v\_2,w\_2),([\alpha v\_2]\_p, [\beta w\_2]\_q)\big)$ be two different cells in the square colored in $(0,0)$.\\

For them to be on the same diagonal, there must exist some $t$ such that:

$$(1) v\_1 q + w\_1 = v\_2 q + w\_2 + t$$

$$(2) [\alpha v\_1]\_p q + [\beta w\_1]\_q = [\alpha v\_2]\_p q + [\beta w\_2]\_q \pm t$$

$$ \Rightarrow [\alpha v\_1]\_p q + [\beta w\_1]\_q = [\alpha v\_2]\_p q + [\beta w\_2]\_q \pm \big(q(v\_1 - v\_2) + (w\_1 - w\_2)\big)$$

$$(\*) q \big([\alpha v\_1]\_p - [\alpha v\_2]\_p \mp (v\_1 - v\_2)\big) = -([\beta w\_1]\_q - [\beta w\_2]\_q) \pm (w\_1-w\_2)$$

Since $q$ is primary, for the equation to be true, it must be that:

$$-([\beta w\_1]\_q - [\beta w\_2]\_q) \pm (w\_1-w\_2)]\_q \equiv\_q 0$$

$$[\beta w\_1]\_q - [\beta w\_2]\_q \equiv\_q \pm (w\_1-w\_2)$$

$$\beta w\_1 - \beta w\_2 \equiv\_q \pm (w\_1-w\_2)$$

$$\beta (w\_1 - w\_2) \equiv\_q \pm (w\_1-w\_2)$$

$$(\beta \pm 1) (w\_1 - w\_2) \equiv\_q 0$$

$$\Rightarrow \beta \equiv\_q \pm 1 \vee w\_1 = w\_2$$\\

If $\beta \equiv\_q \pm 1$, we indeed saw in section 1 that there always exist cells colored in $(0,0)$ that are on the same diagonal (and so on will fulfill the above equation $(\*)$).\\

If $w\_1 = w\_2$, then:\\

$$ q \big([\alpha v\_1]\_p - [\alpha v\_2]\_p \mp (v\_1 - v\_2)\big) = -([\beta w\_1]\_q - [\beta w\_1]\_q) \pm (w\_1-w\_1)$$

$$ q \big([\alpha v\_1]\_p - [\alpha v\_2]\_p \mp (v\_1 - v\_2)\big) = 0$$

$$ (\*\*) [\alpha v\_1]\_p - [\alpha v\_2]\_p \mp (v\_1 - v\_2) = 0$$\\

here we will add $mod(p)$ and check which of the new equation accepted solutions will solve $(\*\*)$:

$$ [\alpha v\_1]\_p - [\alpha v\_2]\_p \mp (v\_1 - v\_2) \equiv\_p 0$$

$$ \alpha v\_1 - \alpha v\_2 \mp (v\_1 - v\_2) \equiv\_p 0$$

$$ (\alpha \pm 1) (v\_1 - v\_2) \equiv\_p 0$$

$$\Rightarrow \alpha \equiv\_p \pm 1 \vee v\_1 = v\_2$$\\

if we go back to $(\*\*)$,\\

$\alpha \equiv\_p 1:$\\

$$v\_1 - v\_2 \mp (v\_1 - v\_2) = 0 \checked$$\\

$\alpha \equiv\_p -1:$\\

$$p - v\_1 - (p - v\_2) \pm (v\_1 - v\_2) = 0 \checked$$\\

$v\_1 = v\_2:$\\

$$[\alpha v\_1]\_p - [\alpha v\_1]\_p \mp (v\_1 - v\_1) = 0 \checked $$

$\Rightarrow $ all three solutions solve the original equation and therefore they are the only solutions .\\

Overall we got:\\

$(1)$ if $\alpha \equiv\_p \pm 1 \vee \beta \equiv\_q \pm 1$, there is at least one pair of cells on the same diagonal, which means that the latin square obtained for these values is not royal.\\

$(2)$ else, the only other option for two cells $\big((v\_1,w\_1),([\alpha v\_1]\_p, [\beta w\_1]\_q)\big), \big((v\_2,w\_2),([\alpha v\_2]\_p, [\beta w\_2]\_q)\big)$ to be on the same diagonal (and in particular will maintain the equation $(\*)$), is for $v\_1=v\_2 \wedge w\_1=w\_2$ which is of course not possible that the points are different.

\section{Conclusion}

for primary numbers $p,q$, the latin square of $\big(\mathbb{Z}p \times \mathbb{Z}q\big)^2$ that obtained by the coloring of the form $\big((x\_1,x\_2),(\alpha x\_1+i, \beta x\_2+j)\big)$ for a color $(i,j)$, is royal if and only if $\alpha \neq\_p \pm 1 \wedge \beta \neq\_q \pm 1$.

\end{document}

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\textcolor{red}{\begin{itemize}\item

Do first the case of prime $n$ in full detail

\item

When $n=pq$ is the product of two primes you have to specify the order at which the elements of $\mathbb{Z}\_p\times\mathbb{Z}\_q$ appear

\item

The standard way to write congruences is $a\equiv b\bmod m$

\end{itemize}}