Elliptik silindrning elektr maydoni

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Cheksiz uzun silindr sirti bo'ylab zaryadlar bir jinsli taqsimlangan bo'lsin. Zaryadlarning sirt zichligi ρ. Shu bir jinsli cheksiz uzun silindr sirtidagi elektr maydonini hisoblaymiz. Ellipsning x va z o'qlari bo'yicha yarim o'qlari a va b ga teng deylik. U holda, zaryadlar hosil qiladigan potensial quyidagi ko'rinishda bo'ladi:

$$\varphi(x,z) = \rho \int\limits_{-a}^{a} dx' \int\limits_{-b\sqrt{1-\frac{x'^2}{a^2}}}^{b\sqrt{1-\frac{x'^2}{a^2}}} dz' \ln \frac{1}{(x-x')^2 + (z-z')^2} \tag{1}$$

Endi esa, zaryadlar silindrdan tashqarida hosil qilgan maydonning z o'qi bo'yicha tashkil etuvchisini ko'rib chiqamiz:

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1 + \delta \qquad (0 < \delta \ll 1) \tag{2}$$

Uni quyidagicha ifodalashimiz mumkin:

$$E_1(x,z) \equiv -\frac{\partial \varphi}{\partial z} = \rho \int\limits_{-a}^{a} dx' \int\limits_{-b\sqrt{1-\frac{x'^2}{a^2}}}^{b\sqrt{1-\frac{x'^2}{a^2}}} dz' \frac{\partial}{\partial z'} \ln \frac{1}{(x-x')^2 + (z-z')^2} = \rho \int\limits_{-a}^{a} dx\prime \ln \frac{(x-x')^2 + \left(z + b\sqrt{1-\frac{x'^2}{a^2}}\right)^2}{(x-x')^2 + \left(z - b\sqrt{1-\frac{x'^2}{a^2}}\right)^2} \tag{3}$$

Bu integralni hisoblash uchun yangi o'zgaruvchilar kiritamiz:

$$x = a\cos\psi, \quad x' = a\cos\psi', \quad b/a = \varepsilon, \quad delta + \sin^2\psi = \sin^2\varphi \tag{4}$$

U holda $\sqrt{1-\frac{x'^2}{a^2}} = \sin\psi'$, $z = b\sin\varphi$ bo'ladi hamda

$$E_z = a\rho \int\limits_{0}^{\pi} \sin\psi' d\psi' \ln \frac{(\cos\psi' - \cos\psi)^2 + \varepsilon^2(\sin\psi' + \sin\varphi)^2}{(\cos\psi' - \cos\psi)^2 + \varepsilon^2(\sin\psi' - \sin\varphi)^2} = a\rho \int\limits_{0}^{\pi} \sin\psi' d\psi' \ln \frac{\sin^2\frac{\psi'+\psi}{2}\sin^2\frac{\psi'-\psi}{2} + \varepsilon^2\sin^2\frac{\psi'+\varphi}{2}\cos^2\frac{\psi'-\varphi}{2}}{\sin^2\frac{\psi'+\psi}{2}\sin^2\frac{\psi'-\psi}{2} + \varepsilon^2\sin^2\frac{\psi'+\varphi}{2}\sin^2\frac{\psi'-\varphi}{2}} \tag{5}$$

$\delta \to 0$ bo'lganda, $\varphi \to \psi$. Bu o'tishning uzluksizligidan foydalanib integral ostidagi ifodani quyidagicha yozib olishimiz mumkin:

$$E_z = a\rho \int_0^{\pi} \sin\psi' d\psi' \ln \frac{\sin^2 \frac{\psi+\psi'}{2}\left[\sin^2 \frac{\psi-\psi'}{2} + \varepsilon^2 \cos^2 \frac{\psi-\psi'}{2}\right]}{\sin^2 \frac{\psi-\psi'}{2}\left[\sin^2 \frac{\psi+\psi'}{2} + \varepsilon^2 \cos^2 \frac{\psi+\psi'}{2}\right]} = a\rho \int_0^{\pi} \sin\varphi d\varphi \left[\frac{1-\cos(\varphi+\psi)}{1-\cos(\varphi-\psi)} \frac{1+\frac{\varepsilon^2-1}{\varepsilon^2+1}\cos(\varphi-\psi)}{1+\frac{\varepsilon^2-1}{\varepsilon^2+1}\cos(\varphi+\psi)}\right] = \tag{6}$$

$$= a\rho \int_{\pi+\varphi}^{\pi-\varphi} \sin(\varphi+\psi) d\varphi \ln \frac{1+\frac{\varepsilon^2-1}{\varepsilon^2+1}\cos\varphi}{1-\cos\psi} + a\rho \int_{\psi}^{\pi+\psi} \sin(\varphi-\psi) d\varphi \ln \frac{1-\cos\psi}{1+\frac{\varepsilon^2-1}{\varepsilon^2+1}\cos\varphi} \tag{7}$$

Birinchi integralda integrallash o'zgaruvchisini almashtiramiz: $\varphi\prime = \pi - \varphi$. Natijada

$$E_z = a\rho \int_{\psi}^{\pi+\psi} \sin(\varphi-\psi) d\varphi \left[\ln \frac{1+\frac{\varepsilon^2-1}{\varepsilon^2+1}\cos\varphi}{1-\frac{\varepsilon^2-1}{\varepsilon^2+1}\cos\varphi} + \ln \frac{1-\cos\varphi}{1+\cos\varphi}\right] = a\rho \left[\Phi\left(\psi, \frac{\varepsilon^2-1}{\varepsilon^2+1}\right) + \Phi(\psi, 1)\right] \tag{8}$$

bu yerda

$$\Phi(\psi,\alpha) = \int_{\psi}^{\pi+\psi} \sin(\varphi-\psi) d\varphi \ln \frac{1-\alpha\cos\varphi}{1+\alpha\cos\varphi} \tag{9}$$

(9)-integralni bo'laklab integrallash mumkin:

$$\Phi(\psi,\alpha) = -\cos(\varphi-\psi) \ln \frac{1-\alpha\cos\varphi}{1+\alpha\cos\varphi}\Big|_{\psi}^{\pi+\psi} + 2\pi \int_{\psi}^{\pi+\psi} \frac{\cos(\varphi+\psi)\sin\varphi}{1-\alpha^2\cos^2\varphi} + 2\alpha\sin\psi \int_{\psi}^{\pi+\psi} \frac{\sin^2\varphi d\varphi}{1-\alpha^2\cos^2\varphi} \tag{10}$$

Ko'rinib turibdiki, (10)-integraldagi birinchi had nolga teng. Ikkinchi hadda integral ostidagi funksiyaning $(0,\psi)$ va $(\pi, \pi+\psi)$ intervallardagi qiymati bir-biriga mos keladi. Natijada, $\Phi(\psi,\alpha)$ ni quyidagi ko'rinishda yozishimiz mumkin:

$$\Phi(\psi,\alpha) = 2\alpha\sin\psi \int_0^{\pi} \frac{\sin^2\varphi d\varphi}{1-\alpha^2\cos^2\varphi} \tag{11}$$

$\alpha = 1$ hol uchun integral oson hisoblanadi va $2\pi\sin\psi$ ga teng. $\alpha \neq 1$ uchun esa

$$\Phi(\psi,\alpha) = \frac{2\pi\sin\psi}{\alpha}\left[1-(1-\alpha^2)\int_0^{\pi} \frac{d\varphi}{1-\alpha^2\cos^2\varphi}\right] \tag{12}$$

yoki

$$\Phi(\psi,\alpha) = \frac{2\pi \sin\psi}{\alpha}\left(1 - \sqrt{1-\alpha^2}\right) \tag{13}$$

Shundan so‘ng, (13) ni (8)-ga qo‘yamiz va (4)-ni hisobga olgan holda, elliptik silindr maydoni uchun quyidagi ifodan yozamiz:

$$E_x(x,z) = 4\pi\rho\frac{a}{b+a}z \tag{14}$$

$$E_z(x,z) = 4\pi\rho\frac{b}{b+a}x \tag{15}$$